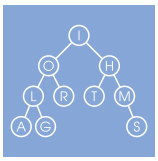


Module: Linear Programming (Week 2 out of 5)
Course: Advanced Algorithms and Complexity (Course 5 out of 6)
Specialization: Data Structures and Algorithms

Programming Assignment 2: Linear Programming

Revision: April 6, 2018

Introduction

Welcome to your second programming assignment of the Advanced Algorithms and Complexity class! In this programming assignment, you will be practicing reducing real-world problems to linear programming and implementing algorithms to solve them.

Recall that starting from this programming assignment, the grader will show you only the first few tests (see the questions 5.4 and 5.5 in the FAQ section).

Learning Outcomes

Upon completing this programming assignment you will be able to:

1. Implement Gaussian Elimination, brute-force algorithm for Linear Programming and Simplex Method.
2. Design and implement efficient algorithms for the following computational problems:
   (a) inferring energy values of ingredients from the menu with calorie counts;
   (b) optimal diet problem;
   (c) online advertisement allocation problem.

Passing Criteria: 2 out of 3

Passing this programming assignment requires passing at least 2 out of 3 code problems from this assignment. In turn, passing a code problem requires implementing a solution that passes all the tests for this problem in the grader and does so under the time and memory limits specified in the problem statement.

1 Problem: Infer Energy Values of Ingredients

Problem Introduction

In this problem, you will apply Gaussian Elimination to infer the energy values of ingredients given a restaurant menu with calorie counts and ingredient lists provided for each item.

Problem Description

Task. You're looking into a restaurant menu which shows for each dish the list of ingredients with amounts and the estimated total energy value in calories. You would like to find out the energy values of individual ingredients (then you will be able to estimate the total energy values of your favorite dishes).

Input Format. The first line of the input contains an integer n — the number of dishes in the menu, and it happens so that the number of different ingredients is the same. Each of the next n lines contains description $a_1, a_2, \ldots, a_n, E$ of a single menu item. a_i is the amount of i-th ingredient in the dish, and E is the estimated total energy value of the dish. If the ingredient is not used in the dish, the amount will be specified as $a_i = 0$; **beware that although the amount of any ingredient in any real menu would be positive, we will test that your algorithm works even for negative amounts** $a_i < 0$.

Constraints. $0 \le n \le 20$; $-1000 \le a_i \le 1000$.

Output Format. Output n real numbers — for each ingredient, what is its energy value. These numbers can be non-integer, so output them with at least 3 digits after the decimal point.

Your output for a particular test input will be accepted if all the numbers in the output are considered correct. The amounts and energy values are of course approximate, and the computations in real numbers on a computer are not always precise, so each of the numbers in your output will be considered correct if either absolute or relative error is less than 10^{-2}. That is, if the correct number is 5.245000, and you output 5.235001, your number will be considered correct, but 5.225500 will not be accepted. Also, if the correct number is 1001, and you output 1000, your answer will be considered correct, because the relative error will be less than 10^{-2}, but if the correct answer is 0.1, and you output 0.05, your answer will not be accepted, because in this case both the absolute error (0.05) and the relative error (0.5) are more than 10^{-2}. **Note that we ask you to output at least 3 digits after the decimal point, although we only require precision of 10^{-2}, intentionally: if you output only 2 digits after the decimal point, your answer can be rejected while being correct because of the rounding issues. The easiest way to avoid this mistake is to output at least 3 digits after the decimal point.**

Time Limits.

language	C	C++	Java	Python	C#	Haskell	JavaScript	Ruby	Scala
time (sec)	1	1	1.5	5	1.5	2	5	5	3

Memory Limit. 512MB.

Sample 1.

Input:

```
0
```

Output:

```

```

There are no dishes in the menu — you don't need to output anything in this case.

Sample 2.

Input:

```
4
1 0 0 0 1
0 1 0 0 5
0 0 1 0 4
0 0 0 1 3
```

Output:

```
1.000000 5.000000 4.000000 3.000000
```

Explanation:
This is an easy test. Each dish contains just one component, and the amount used is exactly 1, so the energy value of each ingredient is just equal to the energy value of the whole dish in which it is used.

Sample 3.

Input:

```
2
1 1 3
2 3 7
```

Output:

```
2.000000 1.000000
```

Explanation:
You can see that the numbers match: $1 \cdot 2.0 + 1 \cdot 1.0 = 3$ and $2 \cdot 2.0 + 3 \cdot 1.0 = 7$. **If you output** 1.994000 **and** 1.009000 **instead of** 2.000000 **and** 1.000000 **respectively, your answer will still be accepted, but don't forget to output at least 3 digits after the decimal point!**

Sample 4.

Input:

```
2
5 -5 -1
-1 -2 -1
```

Output:

```
0.200000 0.400000
```

Explanation:
Beware that there will be tests with negative amounts and negative total energy values, although this is impossible in reality! Also note that the answers can be non-integer! You can check that the numbers match: $5 \cdot 0.2 + (-5) \cdot 0.4 = -1$ and $(-1) \cdot 0.2 + (-2) \cdot 0.4 = -1$.

Starter Files

The starter solutions for this problem read the data from the input, pass it to a blank procedure and output the result. They also contain some convenience functions and data structures. You need to change the main procedure to implement Gaussian Elimination if you are using `C++`, `Java`, or `Python3`. For other programming languages, you need to implement a solution from scratch. Filename: `energy_values`

What To Do

Implement the Gaussian Elimination algorithm from the lectures.

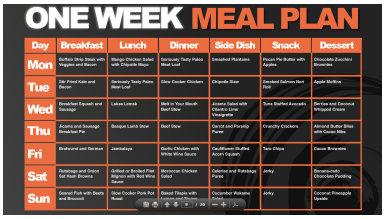
Need Help?

Ask a question or see the questions asked by other learners at this forum thread.

2 Problem: Optimal Diet Problem

Problem Introduction

In this problem, you will implement an algorithm for solving linear programming with only a few inequalities and apply it to determine the optimal diet.



Problem Description

Task. You want to optimize your diet: that is, make sure that your diet satisfies all the recommendations of nutrition experts, but you also get maximum pleasure from your food and drinks. For each dish and drink you know all the nutrition facts, cost of one item, and an estimation of how much you like it. Your budget is limited, of course. The recommendations are of the form "total amount of calories consumed each day should be at least 1000" or "the amount of water you drink in liters should be at least twice the amount of food you eat in kilograms", and so on. You optimize the total pleasure which is the sum of pleasure you get from consuming each particular dish or drink, and that is proportional to the amount $amount_i$ of that dish or drink consumed.

The budget restriction and the nutrition recommendations can be converted into a system of linear inequalities like $\sum_{i=1}^{m} cost_i \cdot amount_i \le Budget$, $amount_i \ge 1000$ and $amount_i - 2 \cdot amount_j \ge 0$, where $amount_i$ is the amount of i-th dish or drink consumed, $cost_i$ is the cost of one item of i-th dish or drink, and $Budget$ is your total budget for the diet. Of course, you can only eat a non-negative amount $amount_i$ of i-th item, so $amount_i \ge 0$. The goal to maximize total pleasure is reduced to the linear objective $\sum_{i=1}^{m} amount_i \cdot pleasure_i \to \max$ where $pleasure_i$ is the pleasure you get after consuming one unit of i-th dish or drink (some dishes like fish oil you don't like at all, so $pleasure_i$ can be negative). Combined, all this is a linear programming problem which you need to solve now.

Input Format. The first line of the input contains integers n and m — the number of restrictions on your diet and the number of all available dishes and drinks respectively. The next $n + 1$ lines contain the coefficients of the linear inequalities in the standard form $Ax \le b$, where $x = amount$ is the vector of length m with amounts of each ingredient, A is the $n \times m$ matrix with coefficients of inequalities and b is the vector with the right-hand side of each inequality. Specifically, i-th of the next n lines contains m integers $A_{i1}, A_{i2}, \ldots, A_{im}$, and the next line after those n contains n integers $b_1, b_2, \ldots, b_n$. These lines describe n inequalities of the form $A_{i1} \cdot amount_1 + A_{i2} \cdot amount_2 + \cdots + A_{im} \cdot amount_m \le b_i$. The last line of the input contains m integers — the pleasure for consuming one item of each dish and drink $pleasure_1, pleasure_2, \ldots, pleasure_m$.

Constraints. $1 \le n, m \le 8$; $-100 \le A_{ij} \le 100$; $-1\,000\,000 \le b_i \le 1\,000\,000$; $-100 \le cost_i \le 100$.

Output Format. If there is no diet that satisfies all the restrictions, output "No solution" (without quotes). If you can get as much pleasure as you want despite all the restrictions, output "Infinity" (without quotes). If the maximum possible total pleasure is bounded, output two lines. On the first line, output "Bounded solution" (without quotes). On the second line, output m real numbers — the optimal $amount$s for each dish and drink. Output all the numbers with at least 15 digits after the decimal point.

The amounts you output will be inserted into the inequalities, and all the inequalities will be checked. An inequality $L \le R$ will be considered satisfied if actually $L \le R + 10^{-3}$. The total pleasure of your solution will be calculated and compared with the optimal value. Your output will be accepted if all

the inequalities are satisfied and the total pleasure of your solution differs from the optimal value by at most 10^{-3}. **We ask you to output at least 15 digits after the decimal point, although we will check the answer with precision of only 10^{-3}. This is because in the process of checking the inequalities we will multiply your answers with coefficients from the matrix A and with the coefficients of the vector *pleasure*, and those coefficients can be pretty large, and computations with real numbers on a computer are not always precise. This way, the more digits after the decimal point you output for each amount — the less likely it is that your answer will be rejected because of precision issues.**

Time Limits.

language	C	C++	Java	Python	C#	Haskell	JavaScript	Ruby	Scala
time (sec)	1	1	2	30	1.5	2	30	30	4

Memory Limit. 512MB.

Sample 1.

Input:

```
3 2
-1 -1
1 0
0 1
-1 2 2
-1 2
```

Output:

```
Bounded solution
0.000000000000000 2.000000000000000
```

Explanation:

Here we have only two items, and we know that $(-1) \cdot amount_1 + (-1) \cdot amount_2 \leq -1 \Rightarrow amount_1 + amount_2 \geq 1$ from the first inequality, and also that $amount_1 \leq 2$ and $amount_2 \leq 2$ from the second and the third inequalities. We also know that all amounts are non-negative. We want to maximize $(-1) \cdot amount_1 + 2 \cdot amount_2$ under those restrictions — that is, we don't like dish or drink number 1, and we twice as much like dish or drink number 2. It is optimal then to consume as few as possible of the first item and as much as possible of the second item. It turns out that we can avoid consuming the first item at all and take the maximum possible amount 2 of the second item, and all the restrictions will be satisfied! Clearly, this is a diet with the maximum possible total pleasure! **Note that integers 0 and 2 in the output are printed with 15 digits after the decimal point. Don't forget to print at least 15 digits after the decimal point, as the answers to some tests will be non-integer, and you don't want to get your answer rejected only because of some rounding problems.**

Sample 2.

Input:

```
2 2
1 1
-1 -1
1 -2
1 1
```

Output:

```
No solution
```

Explanation:
The first inequality gives $amount_1 + amount_2 \leq 1$ and the second inequality gives $(-1) \cdot amount_1 + (-1) \cdot amount_2 \leq -2 \Rightarrow amount_1 + amount_2 \geq 2$. But $amount_1 + amount_2$ cannot be less than 1 and more than 2 simultaneously, so there is no solution in this case.

Sample 3.

Input:

```
1 3
0 0 1
3
1 1 1
```

Output:

```
Infinity
```

Explanation:
The restrictions in this case are only that all amounts are non-negative (these restrictions are always there, because you cannot consume negative amount of a dish or a drink) and that $amount_3 \leq 3$. There is no restriction on how much to consume of items 1 and 2, and each of them has positive *pleasure* value, so you can take as much of items 1 and 2 as you want and receive as much total pleasure as you want. In this case, you should output "Infinite" (without quotes).

Starter Files

The starter solutions for this problem read the data from the input, pass it to a blank procedure and output the result. You need to implement this procedure if you are using `C++`, `Java`, or `Python3`. For other programming languages, you need to implement a solution from scratch. Filename: `diet`

What To Do

There are at most 8 inequalities (16 if you count in the inequalities $amount_i \geq 0$) with at most 8 variables. You can use this fact and the fact that the optimal solution is always in a vertex of the polyhedron corresponding to the linear programming problem. At least m of the inequalities become equalities in each vertex of the polyhedron. If there are n regular inequalities, m variables and m inequalities of the form $amount_i \geq 0$, you need to take each possible subset of size m out of all the $n + m$ inequalities, solve the system of linear equations where each equation is one of the selected inequalities changed to equality, check whether this solution satisfies all the other inequalities, and in the end select the solution with the largest value of the total pleasure out of those which satisfy all inequalities. The running time of this algorithm is $O(2^{n+m}(m^3 + mn))$, which is good enough to pass. 2^{n+m} is to go through all the subsets of the inequalities (although you will need only subsets of size m), m^3 is for Gaussian Elimination and mn is to check a solution of a system of linear equations against all the inequalities. Various ways to traverse all the subsets of some set are described here and here.

The only case that you would miss this way is the case when the correct answer is "Infinity". It is guaranteed that in all test cases in this problem, if a solution is bounded, then

$$amount_1 + amount_2 + \cdots + amount_m \leq 10^9 .$$

Thus, to distinguish between bounded and unbounded cases, just add this inequality to the initial problem. If the resulting program has the last inequality among those m that define the vertex, output "Infinity", otherwise output "Bounded solution" and the solution of the augmented problem on the second line.

Need Help?

Ask a question or see the questions asked by other learners at this forum thread.

3 Advanced Problem: Online Advertisement Allocation

We strongly recommend you start solving advanced problems only when you are done with the basic problems (for some advanced problems, algorithms are not covered in the video lectures and require additional ideas to be solved; for some other advanced problems, algorithms are covered in the lectures, but implementing them is a more challenging task than for other problems).

Problem Introduction

Online and mobile advertising is one of the most profitable businesses in the world. Google and Facebook are generating many billions of dollars of revenue each year, and around 90% of their revenues come from advertisement. In this problem you will help an online advertising system like Google AdSense or Yandex Direct to allocate the ad impressions in its Advertising Network so as to maximize revenue while satisfying all the advertisers' requirements.

Problem Description

Task. You have n clients, they are advertisers, and each of them wants to show their ads to some number of internet users specified in the contract (or more) next month. Your online advertising network has m placements overall on all the sites connected to the network. You know how many users each advertiser wants to reach, how many users will see each of the m ad placements next month, and how much each advertiser is willing to pay for one user who sees their ad through each particular ad placement (different placements can be on different sites attracting different types of users, and each advertiser is more interested in the visitors of some sites than the others). You can show different ads of different advertisers in the same ad placement throughout the next month or show always the same ad of the same advertiser, but the total number of users that will see some ad in that placement is estimated and fixed. You want to maximize your total revenue which is the sum of amounts each advertiser will pay you for all the users who have seen their ads.

If we denote by x_{ij} the number of users who have seen an ad of advertiser i in the ad placement j, then all the restrictions can be written as linear equalities and inequalities in x_{ij}. For example, if the total number of users that will see ad placement j is S_j, then we add an equality $\sum_{i=1}^{m} x_{ij} = S_j$. If the i-th advertiser wants to show the ad to at least U_i users, we add an inequality $\sum_{j=1}^{n} x_{ij} \geq U_i \Leftrightarrow \sum_{j=1}^{n} (-1) \cdot x_{ij} \leq -U_i$. Of course, each x_{ij} is non-negative: $x_{ij} \geq 0$. If advertiser i wishes to pay c_{ij} cents for each user who sees her advertisement through ad placement j, then the goal to maximize the total revenue is given by linear objective $\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \to \max$. This leads to a linear programming problem which you need to solve. This time it will contain more variables and inequalities, because the number of advertisers and the number of different ad placements can be large.

Input Format. You are given the ad allocation problem reduced to a linear programming problem of the form $Ax \leq b, x \geq 0, \sum_{i=1}^{q} c_i x_i \to \max$, where A is a matrix $p \times q$, b is a vector of length p, c is a vector of length q and x is the unknown vector of length q.

The first line of the input contains integers p and q — the number of inequalities in the system and the number of variables respectively. The next $p + 1$ lines contain the coefficients of the linear inequalities in the standard form $Ax \leq b$. Specifically, i-th of the next p lines contains q integers $A_{i1}, A_{i2}, \ldots, A_{iq}$,

and the next line after those p contains p integers $b_1, b_2, \ldots, b_p$. These lines describe p inequalities of the form $A_{i1} \cdot x_1 + A_{i2} \cdot x_2 + \cdots + A_{iq} \cdot x_q \le b_i$. The last line of the input contains q integers — the coefficients c_i of the objective $\sum_{i=1}^{q} c_i x_i \to \max$.

Constraints. $1 \le n, m \le 100$; $-100 \le A_{ij} \le 100$; $-1\,000\,000 \le b_i \le 1\,000\,000$; $-100 \le c_i \le 100$.

Output Format. If there is no allocation that satisfies all the requirements, output "No solution" (without quotes). If you can get as much revenue as you want despite all the requirements, output "Infinity" (without quotes). If the maximum possible revenue is bounded, output two lines. On the first line, output "Bounded solution" (without quotes). On the second line, output q real numbers — the optimal values of the vector x (recall that $x = x_{ij}$ is how many users will see the ad of advertiser i through the placement j, but we changed the numbering of variables to $x_1, x_2, \ldots, x_q$). Output all the numbers with at least 15 digits after the decimal point. Your solution will be accepted if all the inequalities are satisfied and the answer has absolute error of at most 10^{-3}. **See the previous problem output format description for the explanation of what this means and why do we ask to output at least 15 digits after the decimal point.**

Time Limits.

language	C	C++	Java	Python	C#	Haskell	JavaScript	Ruby	Scala
time (sec)	1	1	1.5	6	1.5	2	6	6	3

Memory Limit. 512MB.

Sample 1.

Input:

```
3 2
-1 -1
1 0
0 1
-1 2 2
-1 2
```

Output:

```
Bounded solution
0.000000000000000 2.000000000000000
```

Explanation:

Here we have only two variables, and we know that $(-1)\cdot x_1 + (-1)\cdot x_2 \le -1 \Rightarrow x_1 + x_2 \ge 1$ from the first inequality, and also that $x_1 \le 2$ and $x_2 \le 2$ from the second and the third inequalities. We also know that all amounts are non-negative. We want to maximize $(-1) \cdot x_1 + 2 \cdot x_2$ under those restrictions. It is optimal to minimize x_1 and maximize x_2. It turns out that we can set $x_1 = 0$ (the minimum possible) and $x_2 = 2$ (the maximum possible), and all the requirements will be satisfied. **Note that integers 0 and 2 in the output are printed with 15 digits after the decimal point. Don't forget to print at least 15 digits after the decimal point, as the answers to some tests will be non-integer, and you don't want to get your answer rejected only because of some rounding problems.**

Sample 2.

Input:

```
2 2
1 1
-1 -1
1 -2
1 1
```

Output:

```
No solution
```

Explanation:
The first inequality gives $x_1 + x_2 \leq 1$ and the second inequality gives $(-1) \cdot x_1 + (-1) \cdot x_2 \leq -2 \Rightarrow x_1 + x_2 \geq 2$. But $x_1 + x_2$ cannot be less than 1 and more than 2 simultaneously, so there is no solution in this case.

Sample 3.

Input:

```
1 3
0 0 1
3
1 1 1
```

Output:

```
Infinity
```

Explanation:
The restrictions in this case are only that all amounts are non-negative (these restrictions are always there, because you cannot show an ad to negative number of users) and that $x_3 \leq 3$. There is no upper bound on x_1 and x_2, and both c_1 and c_2 are positive, so you can set x_1 and x_2 big enough and generate as much revenue as you want. In this case, you should output "Infinite" (without quotes).

Starter Files

The starter solutions for this problem read the data from the input, pass it to a blank procedure and output the result. You need to implement this procedure if you are using `C++`, `Java`, or `Python3`. For other programming languages, you need to implement a solution from scratch. Filename: `ad_allocation`

What To Do

You will need to implement the Simplex Method from the lectures to solve this problem.

Need Help?

Ask a question or see the questions asked by other learners at this forum thread.