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Zhejiang University – University of Illinois at Urbana-Champaign Institute

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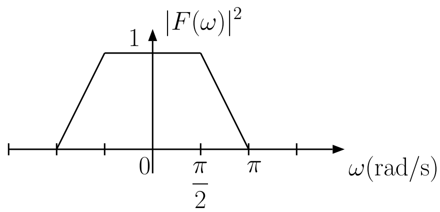
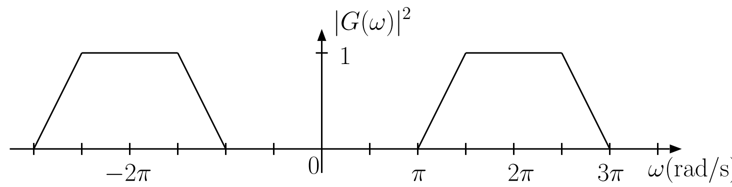
ECE-210 Analog Signal Processing Spring 2022
Homework #10: Solution

1. Determine the 3-dB bandwidth and the 95%-bandwidth of signals $f(t)$ and $g(t)$ with the following energy spectra:.

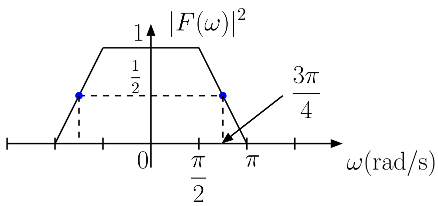




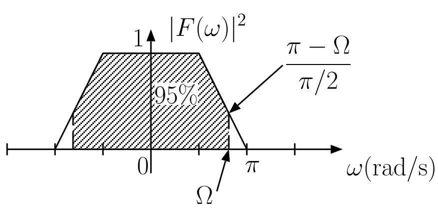
Solution:

- For a low-pass signal, the 3-dB bandwidth is the frequency where the energy spectrum $|F(\omega)|^2$ falls to one-half the energy spectral value $|F(0)|^2$ at DC. Therefore, from the graph of the energy spectra, we can determine, that the 3-dB bandwidth of the signal $f(t)$ is

$$\frac{3\pi}{4}\frac{\text{rad}}{\text{s}}.$$



Next, the 95%-bandwidth (Ω in the figure below), can be calculated by finding the total energy W, and then finding Ω such that the total energy outside $|\omega| > \Omega$ equals 5% of W.



Calculating the total energy W using the formula for the area of a trapezoid, we have

$$W = \frac{1}{2\pi}\left(\frac{2\pi+\pi}{2}\right) = \frac{3}{4} = 0.75.$$

Thus the energy outside $|\omega| > \Omega$ is $0.05W = 0.0375$. But this energy can be calculated as $\frac{1}{2\pi}$ times the combined areas of the right and left non-colored triangles shown in the figure, which is

$$\frac{1}{2\pi}(\pi-\Omega)\frac{(\pi-\Omega)}{\pi/2}.$$

Setting this quantity equal to 0.0375 yields

$$(\pi-\Omega)^2 = 0.0375\pi^2.$$

Hence,

$$\Omega = \pi\left(1-\sqrt{0.0375}\right) \approx 0.80635\pi\frac{\text{rad}}{\text{s}}.$$

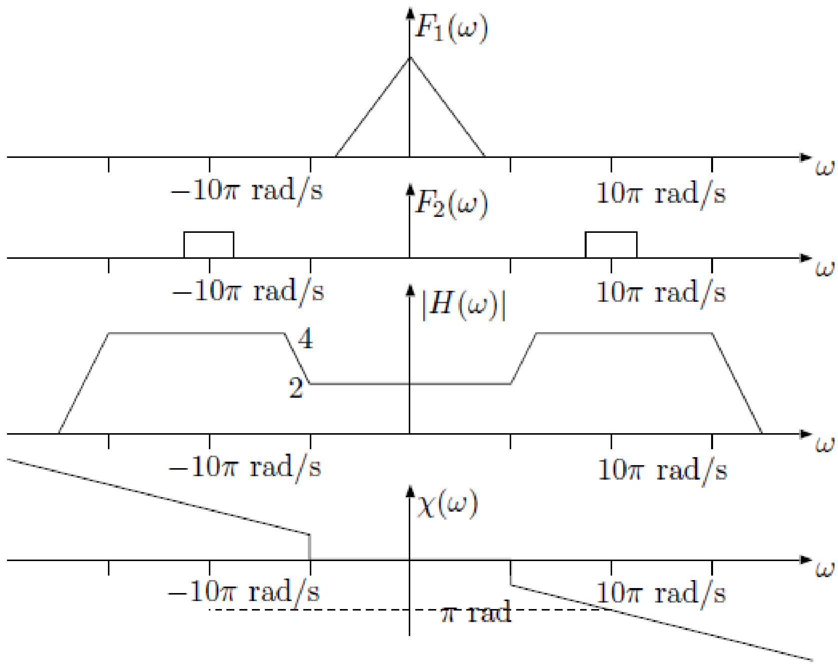
- Since the band-pass signal $g(t)$ has an energy spectrum $|G(\omega)|^2$ that is a shifted replica of $|F(\omega)|^2$, the bandwidth of $g(t)$ is twice that of $f(t)$, i.e.

$$\Omega_{3\text{dB}} = \frac{3\pi}{2}\frac{\text{rad}}{\text{s}}$$

and

$$\Omega_{95\%} \approx 1.6127\pi\frac{\text{rad}}{\text{s}}.$$

2. Let $f(t) = f_1(t) + f_2(t)$ such that $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$, and let $H(\omega) = |H(\omega)|e^{j\chi(\omega)}$. The functions $F_1(\omega)$, $F_2(\omega)$, $H(\omega)$ and $\chi(\omega)$ are given graphically below. The signal $f(t)$ is the input to an LTI system with a frequency response $H(\omega)$. Express the output $y(t)$ of the system as a superposition of scaled and/or shifted versions of $f_1(t)$ and $f_2(t)$.



Solution:

We know that in an LTI system, the input and output in the Fourier domain are related as

$$Y(\omega) = H(\omega)F(\omega).$$

With $F(\omega) = F_1(\omega) + F_2(\omega)$, we have

$$\begin{aligned} Y(\omega) &= H(\omega)\left[F_1(\omega) + F_2(\omega)\right] \\ &= \underbrace{H(\omega)F_1(\omega)}_{Y_1(\omega)} + \underbrace{H(\omega)F_2(\omega)}_{Y_2(\omega)}. \end{aligned}$$

Now, for the region where $F_1(\omega) \neq 0$, we have

$$H(\omega) = 2.$$

Therefore,

$$Y_1(\omega) = 2F_1(\omega) \longleftrightarrow y_1(t) = 2f_1(t).$$

Also, for the region where $F_2(\omega) \neq 0$, we notice a phase that is changing linearly with slope $-1/10$. Hence,

$$H(\omega) = 4e^{-j\frac{1}{10}\omega}.$$

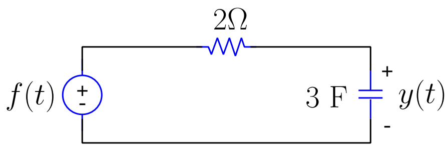
Consequently,

$$Y_2(\omega) = 4e^{-j\frac{1}{10}\omega}F_2(\omega) \longleftrightarrow y_2(t) = 4f_2\left(t - \frac{1}{10}\right).$$

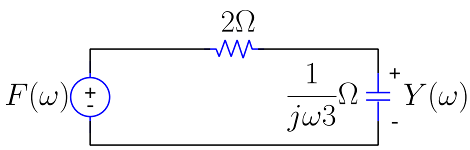
Finally, adding the two results, we obtain

$$y(t) = 2f_1(t) + 4f_2\left(t - \frac{1}{10}\right).$$

3. Determine the response $y(t)$ of the circuit shown below with an arbitrary input $f(t)$ in the form of an inverse Fourier transform and then evaluate $y(t)$ for the case $f(t) = e^{-\frac{t}{6}}u(t)\,\text{V}$.



Solution:
The equivalent circuit in the Fourier domain is



Using voltage division, we have

$$Y(\omega) = F(\omega)\frac{\frac{1}{j\omega 3}}{2 + \frac{1}{j\omega 3}} = F(\omega)\frac{\frac{1}{6}}{\frac{1}{6} + j\omega}.$$

Applying the inverse Fourier transform,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{1}{6}F(\omega)}{\frac{1}{6} + j\omega}e^{j\omega t}d\omega.$$

For the input $f(t) = e^{--t/6}u(t)$ the Fourier transform pair

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega},\ a > 0,$$

yields

$$F(\omega) = \frac{1}{\frac{1}{6} + j\omega}.$$

Substituting this $F(\omega)$ into the inverse Fourier transform, we obtain

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{1}{6}}{\left(\frac{1}{6} + j\omega\right)^2}e^{j\omega t}d\omega.$$

Finally using the Fourier transform pair

$$te^{-at}u(t) \leftrightarrow \frac{1}{(a + j\omega)^2}, a > 0,$$

gives,

$$y(t) = \frac{1}{6}te^{-\frac{1}{6}t}u(t).$$

4. Determine the Fourier transform of $g(t) = f(t + \frac{\theta}{\omega_o})\cos(\omega_o t + \theta)$ in terms of scaled and/or shifted versions of $F(\omega)$. (Hint: use time-shift property.)

Solution

$$f(t + \frac{\theta}{\omega_o})\cos(\omega_o t + \theta) = f(t + \frac{\theta}{\omega_o})\cos\left[\omega_o(t + \frac{\theta}{\omega_0})\right].$$

Using the time-shift property

$$g(t+t_0) \leftrightarrow G(\omega)e^{j\omega t_0},$$

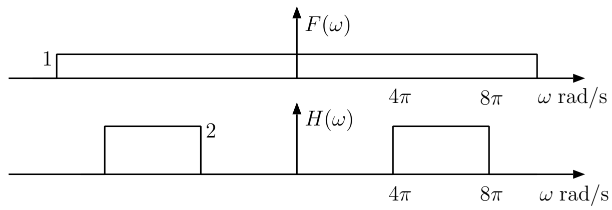
together with the modulation property

$$f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}F(\omega-\omega_o)+\frac{1}{2}F(\omega+\omega_o),$$

we obtain

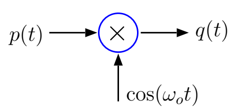
$$f(t+\frac{\theta}{\omega_o})\cos(\omega_o t+\theta) \;\leftrightarrow\; \frac{1}{2}\left[F(\omega-\omega_o)+F(\omega+\omega_o)\right]e^{j\omega\frac{\theta}{\omega_0}}.$$

5. An LTI system with frequency response $H(\omega)$ is excited with an input $f(t) \leftrightarrow F(\omega)$. $H(\omega)$ and $F(\omega)$ are plotted below:



(a) Sketch the Fourier transform $Y(\omega)$ of the system output $y(t)$ and calculate the energy W_y of $y(t)$.

(b) It is observed that output $q(t)$ of the following system equals $y(t)$ determined in part (a).



Sketch $P(\omega)$ and determine ω_o.

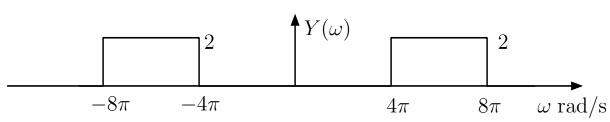
Solution

(a) The output of the system is

$$Y(\omega)=H(\omega)F(\omega),$$

but since $F(\omega)=1$, for all $H(\omega)\neq 0$, we can easily sketch

$$Y(\omega)=H(\omega).$$



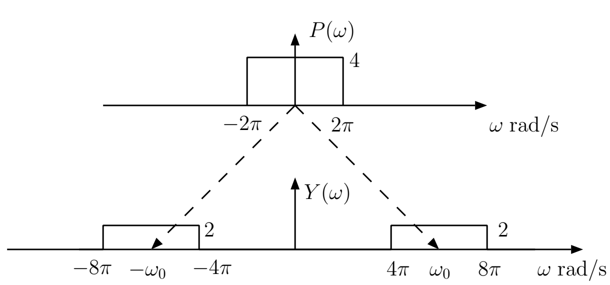
Using Parseval's theorem, the energy of the signal is

$$\begin{aligned} W_y &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2 d\omega = \frac{1}{\pi}\int_{4\pi}^{8\pi} 2^2 d\omega \\ &= \frac{4}{\pi}(8\pi-4\pi)=16. \end{aligned}$$

(b) Using $Y(\omega)=Q(\omega)$ and the modulation property, we have

$$Y(\omega)=\frac{1}{2}P(\omega-\omega_0)+\frac{1}{2}P(\omega+\omega_0).$$

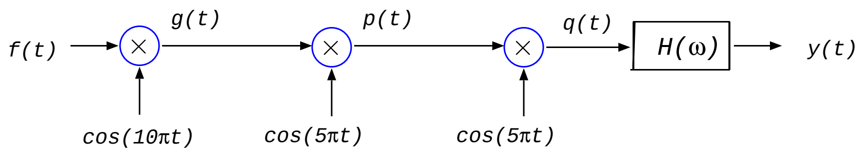
So $P(\omega)$ can be sketched as



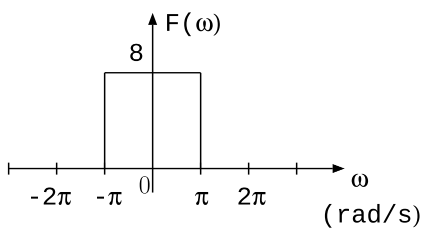
Clearly, from the graph we have

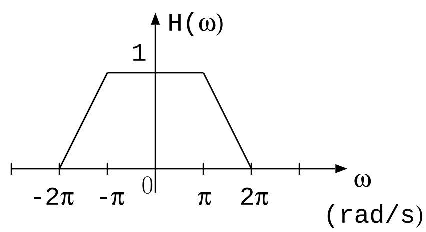
$$\omega_0 = 6\pi \frac{\text{rad}}{\text{s}}.$$

6. Consider the system



where $F(\omega)$ and $H(\omega)$ are as follows:





(a) Express $q(t)$ in terms of $p(t)$.

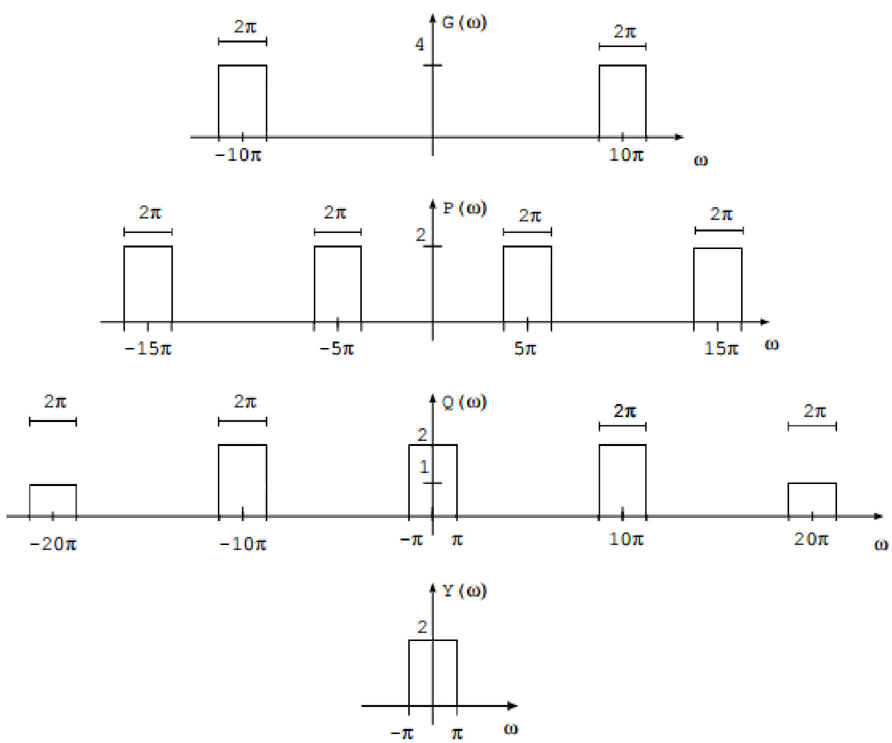
(b) Sketch the Fourier transforms $G(\omega)$, $P(\omega)$, $Q(\omega)$, and $Y(\omega)$.

(c) Express $y(t)$ in terms of $f(t)$.

Solution

(a) $q(t) = p(t)\cos(5\pi t)$.

(b) .



(c) Notice from the sketches that $Y(\omega) = \frac{1}{4}F(\omega)$ so that $y(t) = \frac{1}{4}f(t)$.