ZJU-UIUC INSTITUTE

Zhejiang University – University of Illinois at Urbana-Champaign Institute

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ECE-210 Analog Signal Processing Spring 2022
Homework #12: Submission Deadline 18th May(10:00 PM)

1. A system is described by an impulse response $h(t) = \delta(t-2) - \delta(t+2)$
   Sketch the system response $y(t) = h(t) * f(t)$ to the following inputs:
   (a) $f(t) = u(\frac{t-2}{2})$
   (b) $f(t) = \triangle(\frac{t+2}{2})$

2. Determine the Fourier transform of the following signals —Simplify the results as much as possible. Sketch the result if it is real valued.
   (a) $f(t) = 4\cos(4t) + 3\sin(5t)$
   (b) $x(t) = \sin^2(6t)$
   (c) $y(t) = e^t u(-t) * \cos(2t)$
   (d) $z(t) = [2 + 3\cos(2t)]\, e^{-t}\, u(t)$

3. Determine the inverse Fourier transform of the following:
   (a) $F(w) = 3\pi[\delta(2w-2) - \delta(2w+2)] + 4\pi\delta(w)$
   (b) $A(w) = 2\pi\sin(5w)$
   (c) $B(w) = \sum_{n=-\infty}^{\infty} 2\pi \frac{1}{1+n^2}\delta(2w-3n)$
   (d) $C(w) = \frac{8}{jw-2} + 4\pi\delta(w)$

4. (a) Show that the following LTI systems with impulse responses:

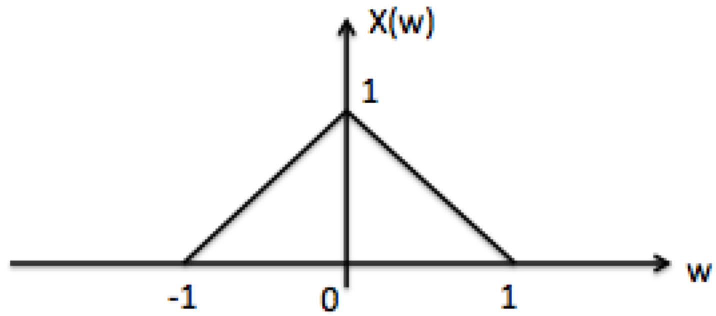
$$h_1(t) = u(t)$$
$$h_2(t) = -2\delta(t) + 5e^{-2t}\, u(t)$$
$$h_3(t) = 2te^{-t}\, u(t)$$

   All have the same response to $x(t) = \cos(t)$
   (b) Find the impulse response of another LTI system with the same response to $x(t) = \cos(t)$
   (This problem illustrates the fact that the response to cos cannot be used to specify an LTI uniformly)

5. (a) Let $x(t)$ have the Fourier transform $\chi(w)$, and let $p(t)$ be periodic with fundamental frequency w_0 and Fourier series representation

$$p(t) = \sum_{n=-\infty}^{+\infty} P_n\, e^{jnw_0 t}$$

   Determine an expression for the Fourier transform of $y(t) = x(t)\, p(t)$
   (b) Suppose that $\chi(w)$ is as depicted in the following figure:



   Sketch the spectrum of $y(t) = x(t)\, p(t)$ found in part(a) for each of the following choices of $p(t)$:
   (1) $p(t) = \cos(\frac{t}{2})$
   (2) $p(t) = \cos(t)$
   (3) $p(t) = \cos(2t)$

(4) $p(t) = \sin(t)\,\sin(2t)$
(5) $p(t) = \cos(2t) - \cos(t)$
(6) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$
(7) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n)$
(8) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 4\pi n)$

6. The input and the output of a causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{d\,y(t)}{dt} + 8\,y(t) = 2\,x(t)$$

(a) Find the impulse response of this system.
(b) What is the response of this system if $x(t) = t\,e^{-2t}\,u(t)$?
(c) Repeat part(a) for the causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{d\,y(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2\,x(t)$$

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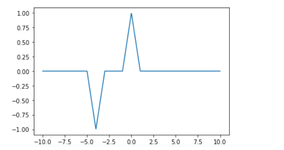
1. A system is described by an impulse response $h(t) = \delta(t-2) - \delta(t+2)$
   Sketch the system response $y(t) = h(t) * f(t)$ to the following inputs:
   (a) $f(t) = u(\frac{t-2}{2})$
   (b) $f(t) = \triangle(\frac{t+2}{2})$

a) $y(t) = (\delta(t-2) - \delta(t+2)) * u(\frac{t-2}{2})$
$= \delta(t-2) * u(\frac{t-2}{2}) - \delta(t+2) * u(\frac{t-2}{2})$
$= u(\frac{(t-2)-2}{2}) - u(\frac{(t+2)-2}{2}) = u(\frac{t}{2}-2) - u(\frac{t}{2})$

b) $y(t) = (\delta(t-2) - \delta(t+2)) * \triangle(\frac{t+2}{2})$
$= \triangle(\frac{t}{2}) - \triangle(\frac{t}{2}+2)$

−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0

```
[17]: plt.plot(t,(delta(t/2,1)-delta(t/2+2,1)))
[17]: [<matplotlib.lines.Line2D at 0x7fab2d5b32e0>]
```



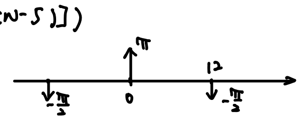
2. Determine the Fourier transform of the following signals —Simplify the results as much as possible.
   Sketch the result if it is real valued.
   (a) $f(t) = 4\cos(4t) + 3\sin(5t)$
   (b) $x(t) = \sin^2(6t)$
   (c) $y(t) = e^t u(-t) * \cos(2t)$
   (d) $z(t) = [2 + 3\cos(2t)]\, e^{-t}\, u(t)$

(c) $\mathcal{F}\{y(t)\} = \mathcal{F}(e^{t}u(-t)) \cdot \mathcal{F}(\cos(2t))$
$= \frac{1}{1-jw} \cdot \pi[\delta(w-2) + \delta(w+2)]$

(d) $\mathcal{F}\{z(t)\} = 2\mathcal{F}\{e^{-t}u(t)\} + 3\mathcal{F}\{e^{-t}u(t)\cos(2t)\}$
$= \frac{2}{1+jw} + \frac{3(1+jw)}{(1+jw)^2+4}$

(a) $\mathcal{F}\{f(t)\} = 4\mathcal{F}\{\cos(4t)\} + 3\mathcal{F}\{\sin(5t)\}$
$= 4(\pi[\delta(w-4) + \delta(w+4)]) + 3(j\pi[\delta(w+5) - \delta(w-5)])$

(b) $\mathcal{F}\{x(t)\} = \mathcal{F}\{\frac{1-\cos(12t)}{2}\} = \mathcal{F}\{\frac{1}{2}\} - \frac{1}{2}\mathcal{F}\{\cos(12t)\}$
$= \pi\delta(w) - \frac{1}{2}\pi[\delta(w-12) + \delta(w+12)]$



3. Determine the inverse Fourier transform of the following:
   (a) $F(w) = 3\pi[\delta(2w-2) - \delta(2w+2)] + 4\pi\delta(w)$
   (b) $A(w) = 2\pi\sin(5w)$
   (c) $B(w) = \sum_{n=-\infty}^{\infty} 2\pi \frac{1}{1+n^2}\delta(2w-3n)$
   (d) $C(w) = \frac{8}{jw-2} + 4\pi\delta(w)$

(b) $F(t) \Leftrightarrow 2\pi f(-w)$
$f(-w) = \sin(5w) \Rightarrow F(t) = j\pi[\delta(t-5) - \delta(t+5)]$

(c) $B(w) = \sum_{n=-\infty}^{\infty} 2\pi \frac{1}{2(1+n^2)}\delta(w - \frac{3n}{2}) = \sum_{n=-\infty}^{\infty} \frac{1}{2(1+n^2)} e^{j\frac{3n}{2}t}$

(a) $F(w) = \frac{3}{2}[2\pi \cdot (\frac{1}{2}\delta(w-1) - \frac{1}{2}\delta(w+1))] + 2\cdot(2\pi\delta(w))$
$= \frac{3}{j2}[j\pi(\delta(w-1) - \delta(w+1))] + 2(2\pi\delta(w))$
$\Rightarrow \mathcal{F}^{-1}\{F(w)\} = -\frac{3j}{2}\sin(t) + 2$

(d) $C(w) = -8(\frac{1}{2-jw}) + 2\cdot 2\pi\delta(w)$
$= -8e^{2t}u(-t) + 2$

4. (a) Show that the following LTI systems with impulse responses:

$$h_1(t) = u(t)$$
$$h_2(t) = -2\delta(t) + 5e^{-2t}\,u(t)$$
$$h_3(t) = 2te^{-t}\,u(t)$$

   All have the same response to $x(t) = \cos(t)$
   (b) Find the impulse response of another LTI system with the same response to $x(t) = \cos(t)$
   (This problem illustrates the fact that the response to cos cannot be used to specify an LTI uniformly)

a) h_1: $H(w) = \mathcal{F}\{u(t)\} = e^{-0t}\cdot u(t) = \frac{1}{jw}$
$X(w) = \mathcal{F}\{\cos(t)\} = \pi[\delta(w-1) + \delta(w+1)]$
$\Rightarrow Y(w) = \frac{\pi}{jw}[\delta(w-1) + \delta(w+1)] \Rightarrow Y(w) = j\pi[\delta(w+1) - \delta(w-1)]$
$\Rightarrow y(t) = \sin(t)$

h_3: $H(w) = \mathcal{F}\{2te^{-t}u(t)\} = \frac{2}{(1+jw)^2}$
$X(w) = \pi[\delta(w-1) + \delta(w+1)]$
$Y(w) = \frac{2\pi}{(1+w^2)^2}(1-jw)^2[\delta(w-1) + \delta(w+1)]$
$= \frac{\pi}{2}(-2j)\delta(w-1) + \frac{\pi}{2}(2j)\delta(w+1)$
$= \pi j[-\delta(w-1) + \delta(w+1)]$
$\Rightarrow y(t) = \sin(t)$

a) $h_2(t)$: $H(w) = \mathcal{F}\{-2\delta(t) + 5e^{-2t}u(t)\}$
$= -2 + \frac{5}{2+jw}$
$X(w) = \pi[\delta(w-1) + \delta(w+1)]$

$\Rightarrow Y(w) = -2\pi[\delta(w-1) + \delta(w+1)] + \frac{\pi\cdot 5(2-jw)}{4+w^2}[\delta(w-1) + \delta(w+1)]$
$= -2\pi[\delta(w-1) + \delta(w+1)] + \frac{(10-5jw)\cdot\pi}{4+w^2}\delta(w-1) + \frac{(10-5jw)\pi}{4+w^2}\delta(w+1)$
$= -2\pi[\delta(w-1) + \delta(w+1)] + (2-j)\pi\delta(w-1) + (2+j)\pi\delta(w+1)$
$= j\pi[\delta(w-1) - \delta(w+1)] \Rightarrow y(t) = \sin(t)$

(b) $\pi[\delta(w-1)+\delta(w+1)]\cdot X(w) = j\pi[-\delta(w-1)+\delta(w+1)]$

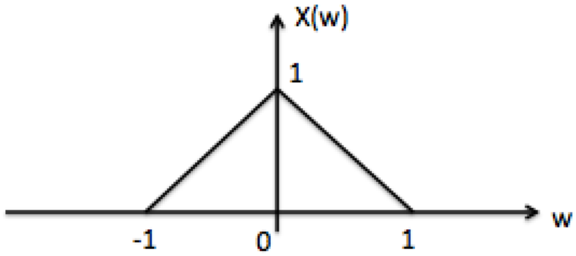
$\Rightarrow X(w)$ can be $-jw$

5. (a) Let $x(t)$ have the Fourier transform $\chi(w)$, and let $p(t)$ be periodic with fundamental frequency w_0 and Fourier series representation

$$p(t) = \sum_{n=-\infty}^{+\infty} P_n\, e^{jnw_0 t}$$

Determine an expression for the Fourier transform of $y(t) = x(t)\,p(t)$
(b) Suppose that $\chi(w)$ is as depicted in the following figure:

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t)\cdot p(t)\}$$
$$= \sum_{n=-\infty}^{+\infty} \mathcal{F}\{P_n e^{jnw_0 t}\} * \mathcal{F}\{x(t)\} / 2\pi$$
$$= \sum_{n=-\infty}^{\infty} P_n \cdot 2\pi\delta(w-nw_0) * X(w) / 2\pi$$
$$= \sum_{n=-\infty}^{\infty} P_n \cdot X(w-nw_0)$$




```
[14]: plt.plot(t,(delta(t+1/2,1)+delta(t-1/2,1))/2)
[14]: [<matplotlib.lines.Line2D at 0x7fe102c62bb0>]
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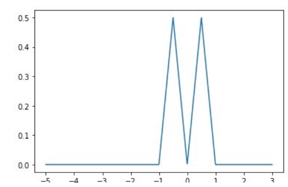
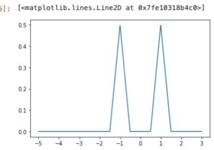


Sketch the spectrum of $y(t) = x(t)\,p(t)$ found in part(a) for each of the following choices of $p(t)$:
(1) $p(t) = \cos(\frac{t}{2})$
(2) $p(t) = \cos(t)$
(3) $p(t) = \cos(2t)$
(4) $p(t) = \sin(t)\,\sin(2t)$
(5) $p(t) = \cos(2t) - \cos(t)$
(6) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-\pi n)$
(7) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-2\pi n)$
(8) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-4\pi n)$

(1) $Y = \mathcal{F}\{p(t)\cdot x(t)\} = \frac{1}{2\pi} P(w) * X(w) = \frac{1}{2\pi}\cdot\pi[\delta(w-\frac{1}{2})+\delta(w+\frac{1}{2})] * \Delta(w) = \frac{1}{2}(\delta(w-\frac{1}{2})*\Delta(w) + \delta(w+\frac{1}{2})*\Delta(w))$
$= \frac{1}{2}(\Delta(w-\frac{1}{2})+\Delta(w+\frac{1}{2}))$

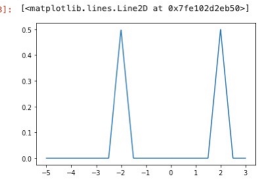
(2) $Y = \mathcal{F}\{p(t)\cdot x(t)\} = \frac{1}{2\pi} P(w) * X(w) = \frac{1}{2}(\delta(w-1)*\Delta(w) + \delta(w+1)*\Delta(w))$
$= \frac{1}{2}(\Delta(w-1)+\Delta(w+1))$

```
[15]: plt.plot(t,(delta(t+1,1)+delta(t-1,1))/2)
[15]: [<matplotlib.lines.Line2D at 0x7fe10318b4c0>]
```



(3) $Y = \mathcal{F}\{p(t) * y(t)\} = \frac{1}{2}(\Delta(w-2)+\Delta(w+2))$

```
[13]: plt.plot(t,(delta(t+2,1)+delta(t-2,1))/2)
[13]: [<matplotlib.lines.Line2D at 0x7fe102d2eb50>]
```

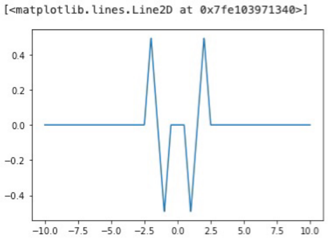


(4) $p(t) = \frac{e^{jt}-e^{-jt}}{2j}\cdot\frac{e^{j2t}-e^{-j2t}}{2j} = -\frac{1}{4}(e^{jt}-e^{-jt})\cdot(e^{j2t}-e^{-j2t})$
$= -\frac{1}{4}(e^{j3t} - e^{-jt} - e^{jt} + e^{-j3t}) = -\frac{1}{4}\cos 3t + \frac{1}{4}\cos t$

$\Rightarrow Y = \mathcal{F}\{x(t)\cdot p(t)\} = -\frac{1}{4}(\frac{1}{2}(\Delta(w-3)+\Delta(w+3)) + \frac{1}{4}(\frac{1}{2}(\Delta(w-1)+\Delta(w+1)))$

(5) $Y = \mathcal{F}\{x(t)\cdot p(t)\} = \frac{1}{2}[\Delta(w-2)+\Delta(w+2)] - \frac{1}{2}[\Delta(w-1)+\Delta(w+1)]$

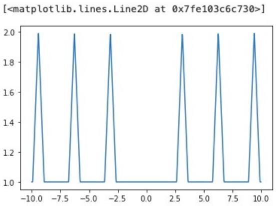
```
plt.plot(t,(1)*(delta(t+2,1)+delta(t-2,1))/2+(-1)*((1/2)*(delta(t+1,1)+delta(t-1,1)))
[<matplotlib.lines.Line2D at 0x7fe103971340>]
```



(6) $\sum_{n=-\infty}^{+\infty} \delta(t-\pi n) \Leftrightarrow \sum_{n=-\infty}^{+\infty} e^{j\pi n w}$

$Y = \mathcal{F}\{x(t)\cdot p(t)\} = \sum_{n=-\infty}^{+\infty} e^{j\pi n w} * \Delta(w)$
$= 2\sum_{n=1}^{\infty} \cos(n\pi\cdot w) * \Delta(w) + 1 * \Delta(w)$
$= \sum_{n=1}^{\infty} [\Delta(w-n\pi)+\Delta(w+n\pi)]$
$+ \int_{-\infty}^{\infty} \Delta(\tau) f(t-\tau)\, d\tau = 1 + \sum_{n=1}^{\infty}[\Delta(w-n\pi)+\Delta(w+n\pi)]$

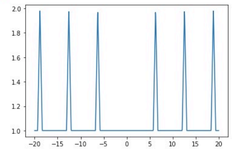
```
[27]: def Y(t,n):
          a = np.zeros(len(t))
          a += 1
          for i in range(1,n+1):
              a += delta(t+np.pi*i,1)+ delta(t-np.pi*i,1)
          return a
      plt.plot(t,Y(t,10))
[27]: [<matplotlib.lines.Line2D at 0x7fe103c6c730>]
```



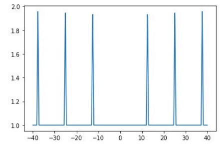
(7) $Y = 1 + \sum_{n=1}^{\infty} [\Delta(w-2n\pi)+\Delta(w+2n\pi)]$

(8) $Y = 1 + \sum_{n=1}^{\infty} [\Delta(w-4n\pi)+\Delta(w+4n\pi)]$

```
[30]: def Y(t,n):
          a = np.zeros(len(t))
          a += 1
          for i in range(1,n+1):
              a += delta(t+2*np.pi*i,1) + delta(t-2*np.pi*i,1)
          return a
      plt.plot(t,Y(t,10))
[30]: [<matplotlib.lines.Line2D at 0x7fe100bed940>]
```



```
[32]: def Y(t,n):
          a = np.zeros(len(t))
          a += 1
          for i in range(1,n+1):
              a += delta(t+4*np.pi*i,1) + delta(t-4*np.pi*i,1)
          return a
      plt.plot(t,Y(t,10))
[32]: [<matplotlib.lines.Line2D at 0x7fe100801910>]
```



6. The input and the output of a causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{d\,y(t)}{dt} + 8\,y(t) = 2\,x(t)$$

(a) Find the impulse response of this system.
(b) What is the response of this system if $x(t) = t\,e^{-2t}\,u(t)$?
(c) Repeat part(a) for the causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{d\,y(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2\,x(t)$$

(a) $\mathcal{L}\{L[y]\} = \mathcal{L}\{2x(t)\}$

$s^2Y - sy(0) - y'(0) + 6sY - 6y(0) + 8Y = 2X$

$\Rightarrow Y(s^2 - 6s + 8) = 2X + (6+s)y(0) + y'(0)$

$\Rightarrow Y = \dfrac{2X + (6+s)\,y(0) + y'(0)}{(s^2 - 6s + 8)}$

$y(t) = \mathcal{L}^{-1}\left\{\dfrac{2X + (6+s)y(0) + y'(0)}{(s^2 - 6s + 8)}\right\}$

(b) $\because x(t) = te^{-2t}u(t) = \dfrac{1}{(s+2)^2}$ $y(0)=0$, $y'(0)=0$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)(s-4)(s+2)^2}\right\} = \mathcal{L}^{-1}\left\{\left(-\dfrac{1}{32}\right)\dfrac{1}{s-2} + \dfrac{1}{72}\dfrac{1}{s-4} + \dfrac{1}{24}\dfrac{1}{(s+2)^2} + \dfrac{5}{288}\dfrac{1}{s+2}\right\}$

$\Rightarrow y(t) = -\frac{1}{32}e^{2t}u(t) + \frac{1}{72}e^{4t}u(t) + \frac{1}{24}te^{-2t}u(t) + \frac{5}{288}e^{-2t}u(t)$

(c) $s^2Y - sy(0) - y'(0) + \sqrt{2}(sY - y(0)) + Y = 2s^2X - 2sx(0) - 2x'(0) - 2(sX - x(0))$

$\Rightarrow (s^2 + \sqrt{2}s + 1)Y = (sy(0) + y'(0) + \sqrt{2}y(0)) - 2sx(0) - 2x'(0) - 2x(0) + (2s^2 + 2s)X$

$\Rightarrow Y = \dfrac{(s+\sqrt{2})y(0) + y'(0)}{s^2 + \sqrt{2}s + 1} - \dfrac{2(s+1)x(0) + 2x'(0)}{s^2 + \sqrt{2}s + 1} + \dfrac{2s^2 + 2s}{s^2 + \sqrt{2}s + 1}X$

$y(t) = \mathcal{L}^{-1}\{Y\}$