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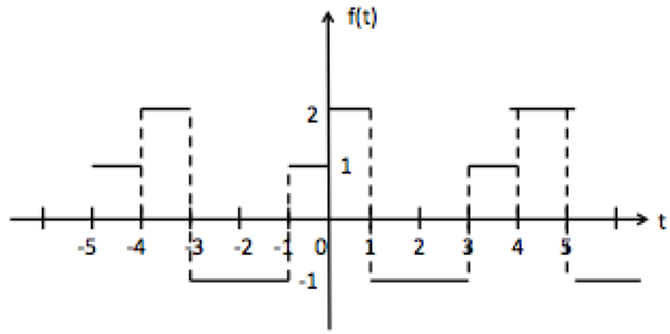
ECE-210 Analog Signal Processing Spring 2022
Homework #8: Solution

1. The function $f(t)$ is periodic with period $T = 4s$. Between t=0 and 4s, the function is described by:

$$f(t) = \begin{cases} 2, & 0 < t < 1s \\ -1, & 1 < t < 3s \\ 1, & 3 < t < 4s \end{cases}$$

(a) Plot $f(t)$ between $t = -5s$ and $t = 7s$.
(b) Determine the exponential Fourier coefficients F_n of $f(t)$ for $n = 0$, $n = \pm 1s$, and $n = \pm 2s$.
(c) Using the result of part(b), determine the compact-form Fourier coefficients C_0, $C_1$and C_2.

Solution:
(a)



(b)

$$\begin{aligned} F_n &= \frac{1}{T}\int_T f(t)e^{-jnw_0t}dt \\ &= \frac{1}{4}\int_0^1 2e^{-j\frac{\pi}{2}nt}dt + \frac{1}{4}\int_1^3 -1e^{-j\frac{\pi}{2}nt}dt + \frac{1}{4}\int_3^4 1e^{-j\frac{\pi}{2}nt}dt \\ &= \frac{2j}{n\pi}[\frac{1}{2}(e^{-j\frac{\pi}{2}n}-1) - \frac{1}{4}(e^{-j\frac{3}{2}\pi n} - e^{-j\frac{\pi}{2}n}) + \frac{1}{4}(e^{-j2\pi n} - e^{-j\frac{3}{2}\pi n})] \end{aligned}$$

Then plug in the number for coefficient

$$\begin{aligned} F_0 &= \frac{1}{4}\int_0^1 2dt + \frac{1}{4}\int_1^3 -1dt + \frac{1}{4}\int_3^4 1dt = \frac{1}{4} \\ F_1 &= \frac{2j}{\pi}[\frac{1}{2}(-j-1) - \frac{1}{4}(j+j) + \frac{1}{4}(1-j)] = \frac{5-j}{2\pi} \\ F_{-1} &= \frac{2j}{-\pi}[\frac{1}{2}(j-1) - \frac{1}{4}(-j-j) + \frac{1}{4}(1+j)] = \frac{5+j}{2\pi} \\ F_2 &= \frac{2j}{\pi}[\frac{1}{2}(-1-1) - \frac{1}{4}(-1+1) + \frac{1}{4}(1+1)] = \frac{-j}{2\pi} \\ F_{-2} &= \frac{2j}{-\pi}[\frac{1}{2}(-1-1) - \frac{1}{4}(-1+1) + \frac{1}{4}(1+1)] = \frac{j}{2\pi} \end{aligned}$$

(c) The formula to compact-form Fourier coefficents is $C_n = 2\,|F_n|$

$$\begin{aligned} C_0 &= 2\,|F_0| = \frac{1}{2} \\ C_1 &= 2\,|F_1| = \frac{\sqrt{26}}{\pi} \\ C_2 &= 2\,|F_2| = \frac{1}{\pi} \end{aligned}$$

2. Consider an LTI system whose frequency response is

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t} dt = \frac{\sin(4\,\omega)}{\omega}$$

If the input to this system is a periodic signal

$$f(t) = \begin{cases} +1, & 0 < t < 4s \\ -1, & 4 < t < 8s \end{cases}$$

with period $T = 8s$.
Determine the corresponding system output $y(t)$

Solution:
$T = 8$ and $w_0 = \frac{2\pi}{T} = \frac{\pi}{4}$

$$\begin{aligned} F_{n=0} &= \frac{1}{8}(\int_0^4 1dt + \int_4^8 -1dt) = 0 \\ F_{n\neq 0} &= \frac{j}{2n\pi}(2e^{-j\pi n} - 2) \end{aligned}$$

The zero-crossing for H(w) is $w = \frac{n}{4}\pi$ $(n \neq 0)$ and the foundamental freuquency for the input is $\frac{\pi}{4}$, therefore the LTI system will eliminate all the harmonics but the DC. However, the $F_0 = 0$ and this implies there is no output. $y(t) = 0$

3. Determine the Fourier series representations for the following signals:
(a) A periodic signal $x(t)$ with period of $T = 2s$ and

$$x(t) = e^{-t}\ for - 1 < t < 1s$$

Solution:

$$\begin{aligned} F_n &= \frac{1}{2}\int_{-1}^{1} e^{-t}e^{-jn\pi t}dt \\ &= \frac{1}{2}\int_{-1}^{1} e^{-(jn\pi+1)t}dt \\ &= -\frac{1}{2(1+jn\pi)}(e^{-(jn\pi+1)} - e^{jn\pi+1}) \end{aligned}$$

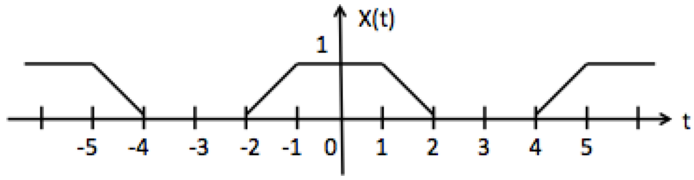
(b) A periodic signal $x(t)$ with period $T = 4s$ and

$$x(t) = \begin{cases} \sin(\pi\, t), & 0 \leq t \leq 2s \\ 0, & 2 < t \leq 4s \end{cases}$$

Solution:

$$\begin{aligned} F_n &= \frac{1}{4}\int_0^2 \frac{1}{2j}(e^{j\pi t} - e^{-j\pi t})e^{-jn\frac{\pi}{2}t}dt \\ &= \frac{1}{8j}\int_0^2 (e^{j\pi t(1-\frac{n}{2})} - e^{-j\pi t(1+\frac{n}{2})})dt \\ &= \frac{1}{8j}\left[\frac{1}{j\pi\,(1-\frac{n}{2})}e^{j\pi(1-\frac{n}{2})t} - (-\frac{1}{j\pi\,(1+\frac{n}{2})})e^{-j\pi(1+\frac{n}{2})t}\right]\Bigg|_0^2 \\ &= \frac{((-1)^n - 1)}{\pi\,(n^2-4)} \end{aligned}$$

(c)



Solution:
From the plot, we get $T = 6$ and $w_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

Let $g(t) = \frac{dx(t)}{dt} = \begin{cases} 1, & -2 \le t < -1 \\ 0, & -1 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & 2 \le t < 4 \end{cases}$

$$\begin{aligned}
G_{n\neq 0} &= \frac{1}{T}\int_T g(t)e^{-jnw_0t}dt \\
&= \frac{1}{6}(\int_{-2}^{-1} e^{-jnw_0t}dt + \int_1^2 -e^{-jnw_0t}dt) \\
&= \frac{1}{6}(\frac{e^{-jnw_0t}}{-jnw_0}\bigg|_{-2}^{-1} + \frac{-e^{-jnw_0t}}{-jnw_0}\bigg|_1^2) \\
&= -\frac{1}{6jn\frac{\pi}{3}}(e^{jn\frac{\pi}{3}} - e^{j2n\frac{\pi}{3}} - e^{-j2n\frac{\pi}{3}} + e^{-jn\frac{\pi}{3}}) \\
&= -\frac{1}{jn\pi}(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}) \\
X_{n\neq 0} &= \frac{G_{n\neq 0}}{jnw_0} \\
&= \frac{3}{n^2\pi^2}(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}) \\
X_{n=0} &= \frac{1}{6}\int_T f(t)dt = \frac{1}{2}
\end{aligned}$$

Therefore $x(t) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{3}{n^2\pi^2}(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3})e^{jn\frac{\pi}{3}t}$

4. Let

$$x(t) = \begin{cases} t, & 0 \le t \le 1s \\ 2-t, & 1 \le t \le 2s \end{cases}$$

be a periodic signal with fundamental period $T = 2s$ and Fourier coefficients X_k.
(a) Determine the value of X_0.
(b) Determine the Fourier series representation of $\frac{dx(t)}{dt}$
(c) Use the result of part (b) and the differential property of the Fourier series to help determine the Fourier series coefficients of $x(t)$.

Solution:

(a)

$$
\begin{aligned}
X_0 =& \frac{1}{2}\int_0^1 t\,dt + \frac{1}{2}\int_1^2 (2-t)dt \\
=& \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 2\,t - \frac{1}{2}t^2 \mid_1^2 \\
=& \frac{1}{4} + \frac{1}{2}(4-2-2+\frac{1}{2}) \\
=& \frac{1}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
G_n =& \frac{1}{2}\int_0^1 e^{-jn\pi t}dt + \frac{1}{2}\int_1^2 -1e^{-jn\pi t}dt \\
=& \frac{j}{2n\pi}(e^{-jn\pi}-1) - \frac{j}{2n\pi}(1-e^{jn\pi}) \\
=& \frac{j}{2n\pi}(e^{jn\pi}+e^{-jn\pi}-2)
\end{aligned}
$$

(c) Using the differential property

$$
\begin{aligned}
G_n &= F_n jnw_0 \\
F_n &= \frac{1}{jnw_0}G_n \\
&= \frac{j}{2n\pi}(e^{jn\pi}+e^{-jn\pi}-2)\times\frac{1}{jnw_0} \\
&= \frac{\cos(n\pi)-1}{n^2\pi^2}
\end{aligned}
$$

5. Let the signal $f(t) = \sin^4(t)$ be the input of an LTI system with frequency response $H(\omega) = 2\,e^{-j\omega\pi/2}$ for $\omega\epsilon[-2,\,2]$ rad/s and zero elsewhere. Obtain the steady-state response $y(t)$ of the system to the input $f(t)$.

Solution :

$$
\begin{aligned}
f(t) = \sin^4(t) &= (\frac{e^{jt}-e^{-jt}}{2j})^4 \\
&= \frac{1}{16}(e^{j4t}+e^{-j4t}-4e^{j2t}-4e^{-j2t}+6)
\end{aligned}
$$

Evaluate the $H(w)$ at $w = 0, 2, -2, 4, -4$, we get

$$
\begin{aligned}
H(0) &= 2 \\
H(2) &= -2 \\
H(-2) &= -2 \\
H(4) &= 0 \\
H(-4) &= 0
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \frac{1}{16}(6\times 2 + -2\times(-4e^{-j2t}) - 2\times(-4e^{-j2t})) \\
&= \frac{3}{4} + \cos(2t)
\end{aligned}
$$