
Zhejiang University – University of Illinois at Urbana-Champaign Institute

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ECE-210 Analog Signal Processing Spring 2022
Homework #11: Solution

1. Let $f(t) = rect(t - \frac{1}{2})$, $h(t) = \Delta(\frac{t-1}{2})$, and let $y(t) = f(t) * h(t)$.

   (a) Determine the value of t_I, the first instant in time when $y(t)$ is non-zero.
   (b) Determine the value of t_F, the last instant in time when $y(t)$ is non-zero.
   (c) Determine the values of $y(0), y(\frac{1}{2}), y(1), y(\frac{3}{2})$.

   Solution:

   (a) In the convolution formula we choose to flip $h(t)$, writing

   $$y(t) = h(t) * f(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau.$$

   $t_I = 0$ is the first instant in time when $y(t)$ is non-zero.

   (b) If we keep moving the triangle function rightward, when $t > 3$, there is no more overlapping. Thus, $t_F = 3$ is the last instant in time when $y(t)$ is non-zero.

   (c) If we choose to flip $h(t)$, we get:

   $$\begin{aligned} y(0) &= 0, \\ y(\frac{1}{2}) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}, \\ y(1) &= 1 \times 1 \times \frac{1}{2} = \frac{1}{2}, \\ y(\frac{3}{2}) &= 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 2 = \frac{3}{4}. \end{aligned}$$

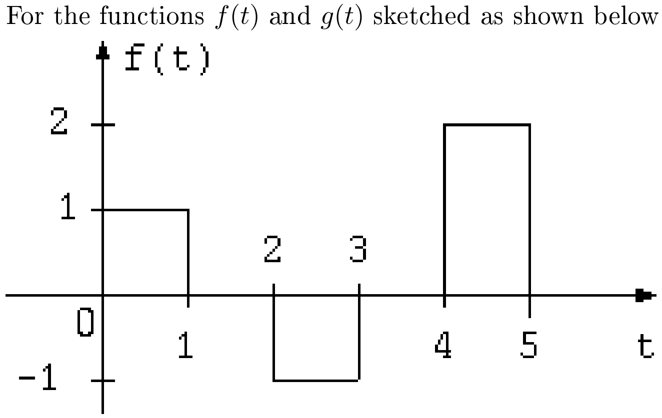
2. Let $f(t) = 3u(2-t)$ and $h(t) = e^{2t}u(-t)$, and let $y(t) = f(t) * h(t)$. Determine $y(t)$ for all $-\infty < t < \infty$. Determine $y(t)$ for all $-\infty < t < \infty$.
   Solution:
   We are going to use the derivative property and calculate $\frac{d}{dt}c(t)$ and then integrate it to get $c(t)$.

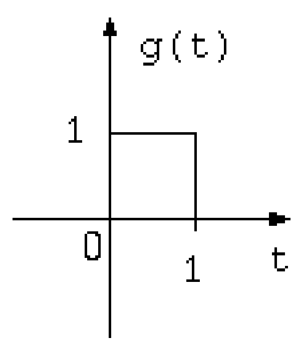
   $$y(t) \quad = \quad h(t) * f(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau$$

   for every region of t. Furthermore, we know that for $t < 10$, $c(t) = 0$. Therefore,

   $$y(t) = \begin{cases} 0 & \text{for } t \in [2, \infty), \\ \int_{t-2}^{0} \left(3 \times e^{2\tau}\right) d\tau, & \text{for } t \in [-\infty, 2), \end{cases} = \begin{cases} 0, & \text{for } t \in [2, \infty), \\ \frac{3}{2}(1 - e^{2(t-2)}), & \text{elsewhere} \end{cases}$$

3. For the functions $f(t)$ and $g(t)$ sketched as shown below:





(a) Determine $x(t) = g(t) * g(t)$ by direct integration and sketch the result.

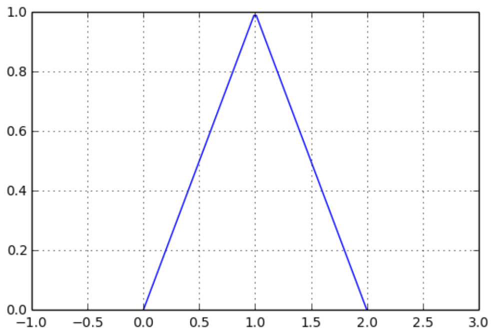
(b) Determine $y(t) = f(t) * g(t)$ using appropriate properties of convolution and the result of part (a). Sketch the result.

(c) Determine $z(t) = f(t) * f(t-1)$ using appropriate properties of convolution. Sketch the result.

Solution:

(a) Given $x(t) = g(t) * g(t)$ and $g(t) = \mathrm{rect}(t - \frac{1}{2})$

$$x(t) = \begin{cases} 0, & \text{for } t \in [-\infty, 0)\,, \\ t, & \text{for } t \in [0, 1)\,, \\ 2-t, & \text{for } t \in [1, 2)\,, \\ 0, & \text{elsewhere} \end{cases}$$



(b) We can express f(t) as:

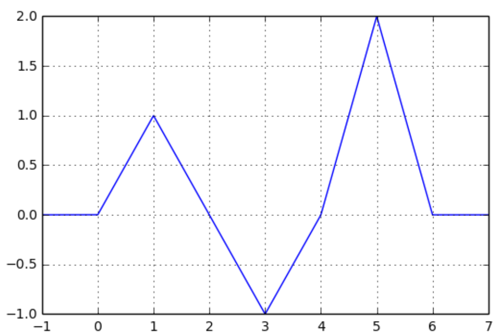
$$f(t) = g(t) - g(t-2) + 2g(t-4)$$

then,

$$y(t) = f(t) * g(t) = g(t) * g(t) - g(t-2) * g(t) + 2g(t-4) * g(t)$$

which becomes:

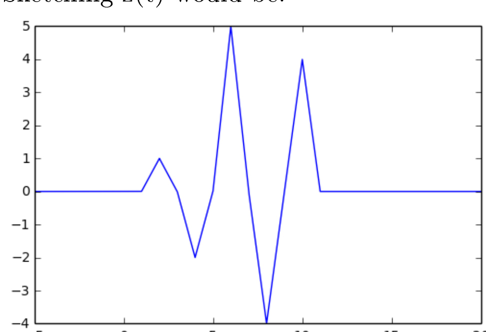
$$y(t) = x(t) - x(t-2) + 2x(t-4)$$



(c) Similarly, we can express $z(t)$ as

$$z(t) = f(t) * f(t-1) = y(t-1) - y(t-3) + 2y(t-5)$$

Sketching z(t) would be:



4. Given $h(t) = u(t)$ and $f(t) = 2\triangle(\frac{t}{2})$,

   (a) Determine $y(t) = h(t) * f(t)$ and sketch the result.

   (b) Determine $z(t) = h(t) * \frac{df}{dt}$ using appropriate properties of convolution, and sketch the result.

   Solution:

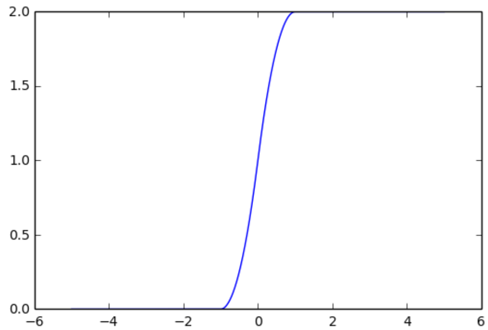
   (a) $y(t) = h(t) * f(t)$, which gives

$$y(t) = \begin{cases} 0, & \text{for } t \in [-\infty, -1)\,, \\ \int_{-1}^{t}(2+2\tau)d\tau, & \text{for } t \in [-1, 0)\,, \\ \int_{0}^{t}(2-2\tau)d\tau + \int_{-1}^{0}(2+2\tau)d\tau, & \text{for } t \in [0, 1)\,, \\ \int_{0}^{1}(2-2\tau)d\tau + \int_{-1}^{0}(2+2\tau)d\tau, & \text{for t} \in [1, \infty), \end{cases}$$

   which gives

$$y(t) = \begin{cases} 0, & \text{for } t \in [-\infty, -1)\,, \\ (t+1)^2, & \text{for } t \in [-1, 0)\,, \\ 2t - t^2 + 1, & \text{for } t \in [0, 1)\,, \\ 2, & \text{for t} \in [1, \infty), \end{cases}$$

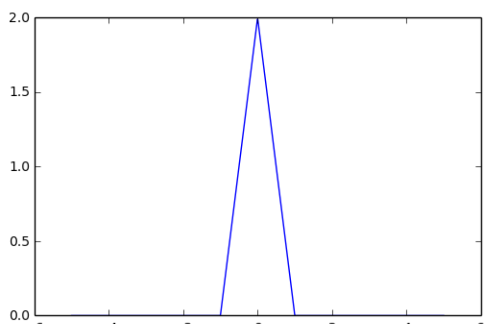
   Sketching y(t):



   (b) $z(t) = h(t) * \frac{df(t)}{dt}) = \frac{dy(t)}{dt}$, so the output is:

$$z(t) = \begin{cases} 0, & \text{for } t \in [-\infty, -1)\,, \\ 2t + 2, & \text{for } t \in [-1, 0)\,, \\ -2t + 2, & \text{for } t \in [0, 1)\,, \\ 0, & \text{for t} \in [1, \infty), \end{cases}$$

   Sketching z(t):



5. Given $f(t) = u(t)$, $g(t) = 2tu(t)$, and $q(t) = f(t-1) * g(t)$, determine $q(4)$.

   Solution:

$$q(t) \quad = \quad f(t-1) * g(t) = \int g(\tau)u(t-\tau+1)d\tau$$

In order to make the integral valid, we have $0 < \tau < t-1$, so

$$q(t) = \int_0^{t-1} g(\tau)u(t-\tau+1)d\tau = \int_0^{t-1} 2\tau d\tau = (t-1)^2.$$

Thus,

$$q(4) = (4-1)^2 = 9$$

6. Given $f(t) = u(-t)$, $h(t) = tu(-t)$, and $y(t) = f(t) * h(t)$, determine $y(-4)$ and $y(4)$.
Solution:

$$y(t) = f(t) * h(t) = \int h(\tau)f(t-\tau)d\tau$$

Similar to problem 5, we have:

$$y(t) = \int_{-t}^{0} \tau d\tau = -\frac{1}{2}t^2$$

Thus,

$$y(-4) = -8.$$

When $t > 0$, the input is 0 and $y(t) = 0$. Then, we have:

$$y(4) = 0.$$

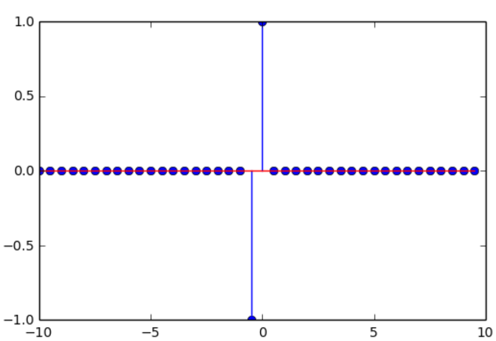
7. Simplify the following expressions involving the impulse and/or shifted impulse and sketch the results:

(a) $g(t) = \cos(2\pi t)(\frac{du}{dt} + \delta(t+0.5))$.
(b) $a(t) = \int_{-\infty}^{t} \delta(\tau+1)d\tau + \mathrm{rect}(\frac{t}{6})\delta(t-2)$.
(c) $b(t) = \delta(t-3) * u(t)$.
(d) $f(t) = (1+t^3)(\delta(t) - 2\delta(t-2))$.
(e) $y(t) = \int_{-1}^{\infty}(\tau^2+1)\delta(\tau+2)d\tau$.
(f) $c(t) = \triangle(\frac{t}{4}) * (\delta(t) - \delta(t+2))$.

Solution:

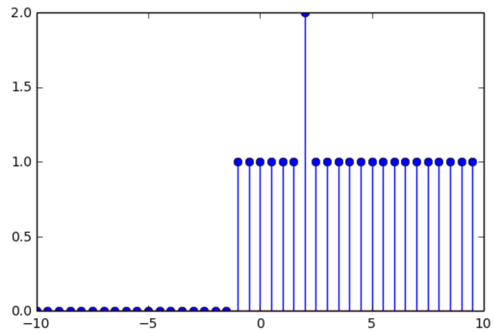
(a) $g(t) = \cos(2\pi t)(\frac{du}{dt} + \delta(t+0.5))$.

$$g(t) = \cos(0)\delta(t) + \cos(-\pi)\delta(t+0.5) = \delta(t) - \delta(t+0.5)$$



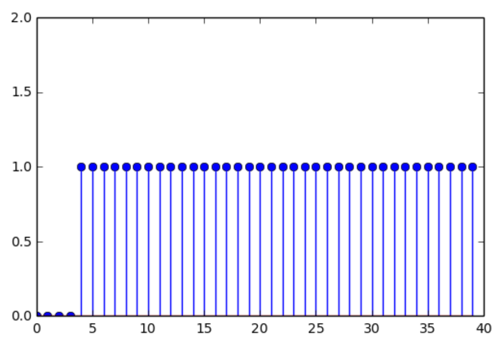
(b) $a(t) = \int_{-\infty}^{t} \delta(\tau+1)d\tau + \mathrm{rect}(\frac{t}{6})\delta(t-2)$

$$a(t) = u(t+1) + rect(\frac{1}{3})\delta(t-2) = u(t+1) + \delta(t-2)$$



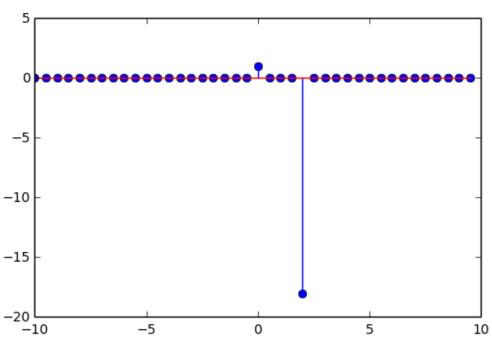
(c) $b(t) = \delta(t-3) * u(t)$

$$b(t) = u(t-3)$$



(d) $f(t) = (1+t^3)(\delta(t) - 2\delta(t-2))$.

$$f(t) = \delta(t) - 18\delta(t-2)$$



(e) $y(t) = \int_{-1}^{\infty}(\tau^2+1)\delta(\tau+2)d\tau$.

$$y(t) = 0$$

(f) $c(t) = \triangle(\frac{t}{4}) * (\delta(t) - \delta(t+2))$.

$$c(t) = \triangle(\frac{t}{4}) - \triangle(\frac{t+2}{4})$$