ANALOG SIGNAL PROCESSING

ECE 210 & 211

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Objectives

- Causality and LTIC System
- Delay Lines
- Laplace transform
- Properties of Laplace transform
- Verification of Properties through Examples

Objectives

- **Causality and LTIC System**
- **Delay Lines**
- **Laplace transform**
- **Properties of Laplace transform**
- **Verification of Properties through Examples**

Causality of LTIC Systems

- **An LTI system (stable of not) is said to be causal if its zero-state response $h(t) * f(t)$ depends only on the *past and present* values *not future* values of $f(t)$**

- **All systems otherwise are *noncausal***

- **All practical analog LTI circuits built in the lab are obviously causal**

- ***Noncausal systems can be thought of a system generating an output before an input is applied – only possible if the input pattern is saved in the internal memory of the system***

Causality of LTIC Systems

- **Writing the LTI zero-state response in terms of convolution formula,**

$$y(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau) d\tau$$

- **In this system, the output $y(t_1)$ can depend on any $f(t)$ with $t > t_1$, only if $h(\tau)$ is nonzero for negative values of τ**

Causality of LTIC Systems

- **For example, if $h(-1)$ is non-zero then we see from the convolution formula that $y_1(t)$ should depend on $f(t-(-1))$ implies $f(t+1)$ that is a future value of the input f**

- **An LTI system with impulse response function $h(t)$ *is causal* if and only if $h(t) = 0$ *for $t < 0$***

Causality of LTIC Systems –Example 9

Question: **The zero-state response of an LTI system to an arbitrary input $f(t)$ is described by**

$$y(t) = f(t-2)$$

Find the impulse response $h(t)$ of the system and also find whether the system is causal or not.

Causality of LTIC Systems – Example 9

Solution: Since $f(t-2) = \delta(t-2) * f(t)$, the input output formula can be written as

$$y(t) = \delta(t-2) * f(t)$$

hence the impulse response of the system is

$$h(t) = \delta(t-2)$$

- Clearly, the system is causal, because the present system output is simply the system input from 2 time units prior

- The output at any instant does not depend on future values of the input – *Causal*

Causality of LTIC Systems – Example 10

Question: **The zero-state output $y(t)$ of an LTI system to unit step input $u(t)$ is described by**

$$g(t) = rect(t)$$

Find the impulse response $h(t)$ of the system and also find whether the system is causal or not.

Causality of LTIC Systems – Example 10

Solution: Since $rect(t)$ can be expressed in terms of unit step function as

$$rect(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$

and since the impulse response $h(t)$ of the system is the derivative of unit step response $g(t)$, we have

$$h(t) = \frac{d}{dt}rect(t) = u'\left(t + \frac{1}{2}\right) - u'\left(t - \frac{1}{2}\right)$$

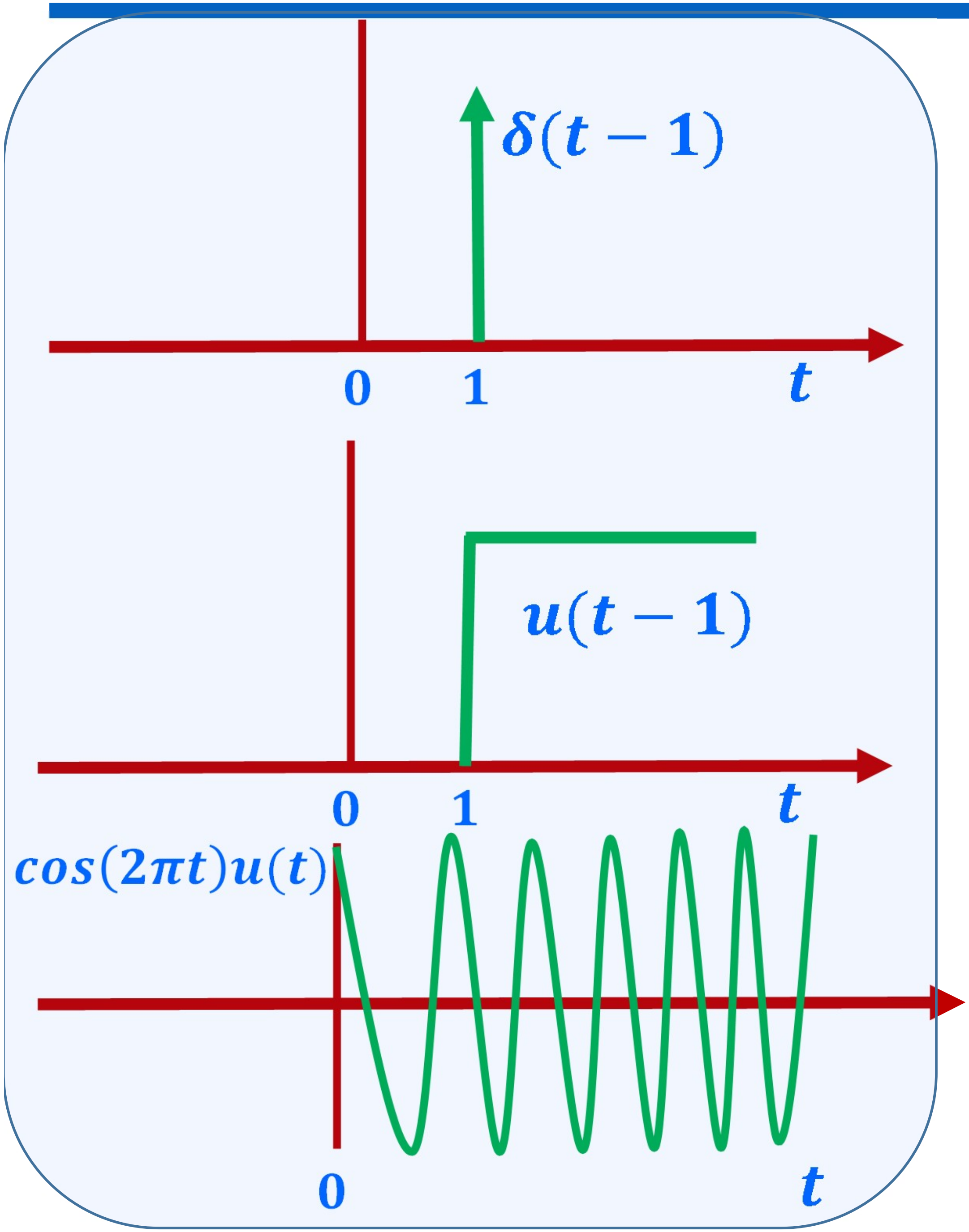
The system zero-state response to an arbitrary input $f(t)$ is

Causality of LTIC Systems – Example 10

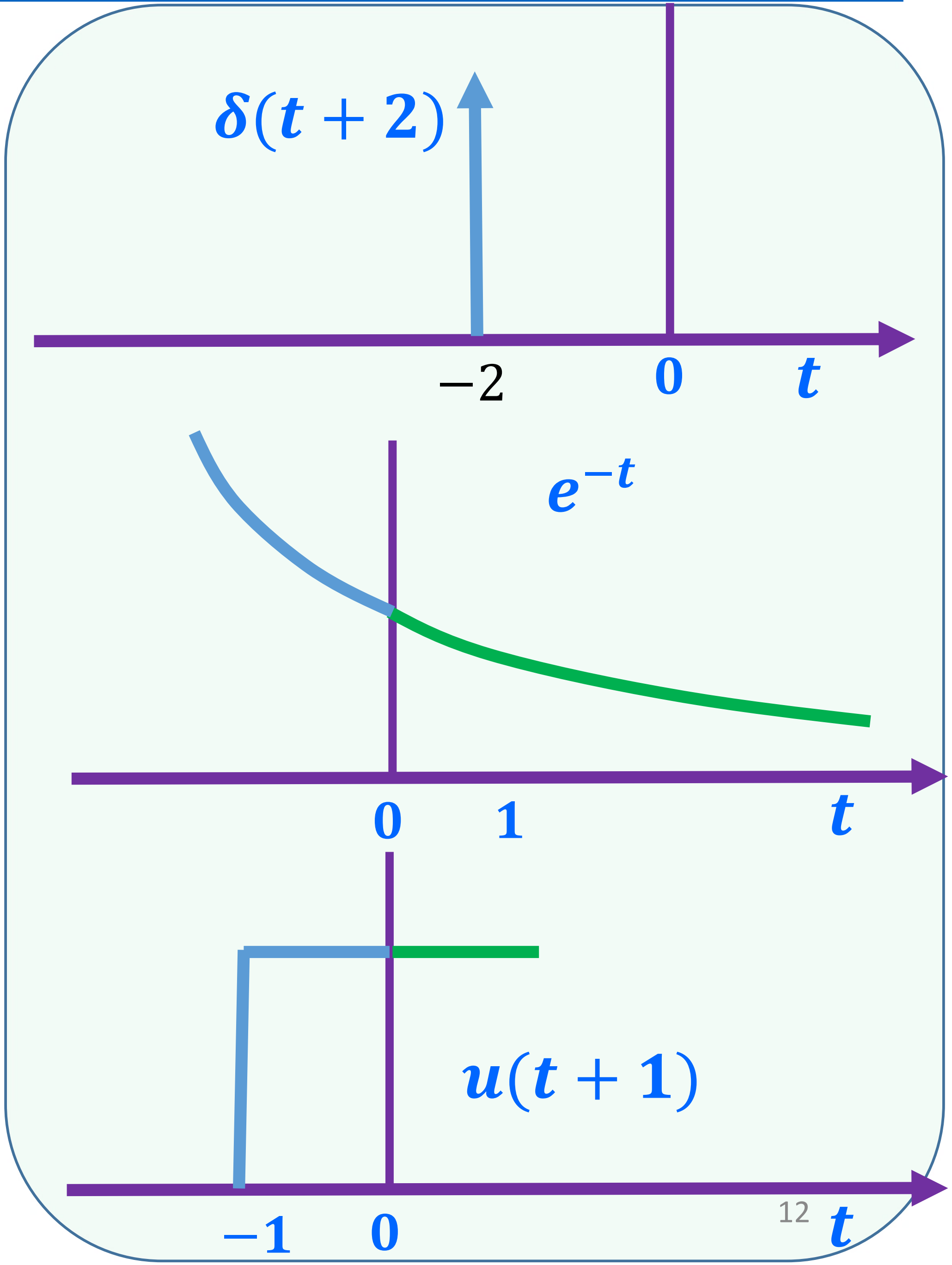
$$y(t) = \left[\delta\left(t+\frac{1}{2}\right) - \delta\left(t-\frac{1}{2}\right)\right] * f(t)$$
$$= f\left(t+\frac{1}{2}\right) - f\left(t-\frac{1}{2}\right)$$

- **Clearly, the system is noncausal, because the output is depending upon $f\left(t+\frac{1}{2}\right)$, which is an input half a time unit into future**
- **Another perspective is to note that the output $rect(t)$ *turns on* at time $t = -\frac{1}{2}$, which is earlier than $t = 0$ when the $u(t)$ *turns on – impractical***

Causal



Noncausal



Causality of LTIC Systems – Example 11

Question: **Determine whether the following system is causal:**

$$h(t) = \delta(t) + u(t+1)$$

Solution: **To solve this problem, we must determine whether the *hypothesized* $h(t)$ can be the impulse response of a causal LTI system.**
The output of an LTI system having the given $h(t)$ as its impulse response would be

$$y(t) = \big(\delta(t) + u(t+1)\big) * f(t)$$

$$y(t) = \delta(t) * f(t) + u(t+1) * f(t)$$

$$= f(t) + f(t+1) * u(t) = f(t) + \int_{-\infty}^{t} f(\tau+1)d\tau$$

Causality of LTIC Systems – Example 11

- **Clearly, the system is noncausal, because the output depends upon $f(t+1)$, which represents an input one time unit into future**

- ***The given h(t) is noncausal***

Causality of LTIC Systems – Example 12

Question: A time-varying system (of course not LTI) is described by the input-output relation given by

$$y(t) = \cos(t+5)\, f(t)$$

Find whether the system is causal or not.

Solution: It can be seen that system is causal as the output does not rely on the future values of input $f(t)$.

Causality of LTIC Systems – Example 13

Question: A system is described by the input-output relation given by

$$y(t) = f(t^2)$$

Find whether the system is causal or not.

Solution: It can be seen that the system is noncausal because for instance,

$$y(-1) = f\left(-1^2\right) = f(1)$$

showing that there are times for which the output depends on the future values of the input

Causality of LTIC Systems – Example 14

Question: **A nonlinear system is described by the input-output relation given by,**

$$y(t) = f^2(t+T)$$

Find whether the system is causal or not.

Solution: **The causality of the system depends on whether the value of T is positive or negative**

- **The system is noncausal for $T > 0$**
- **The system is causal for $T < 0$**

Causality of LTIC Systems – Example 15

Question: What type of filter is implemented by an LTI system having the impulse response

$$h(t) = \frac{\Omega}{\pi} sinc(\Omega t) cos(\omega_o t)$$

assuming $\omega_o > \Omega$? Discuss why this filter is difficult to build in lab.

Causality of LTIC Systems – Example 15

Solution: **Examining Fourier transform pair**

$$\frac{\Omega}{\pi} sinc(\Omega t) \leftrightarrow rect(\frac{\omega}{2\Omega})$$

So, use of modulation property implies that frequency response of the given system is

$$H(\omega) = \frac{1}{2} rect\left(\frac{\omega - \omega_o}{2\Omega}\right) + \frac{1}{2} rect\left(\frac{\omega + \omega_o}{2\Omega}\right)$$

This is frequency response of an ideal bandpass filter with center frequency at ω_o and bandwidth Ω- impossible to design due to noncausality.

Causality of LTIC systems – Example 16

Question: **A system is described by the input-output relation given by**

$$y(t) = f(3t)$$

Find whether the system is causal or not.
Is it time invariant? Is it LTIC?

Causality of LTIC Systems – Example 16

Solution: Since $y(1) = f(3)$, the output at $t = 1$ depends on the input at $t = 3$. Hence the system is not causal and cannot be LTIC. However, it could still be time invariant --- needs to check.

- Time invariance requires that the delayed inputs lead to equally delayed unchanged outputs

- Consider a new system input, which is delayed version of original

$$f_1(t) = f(t - t_o)$$

According to the given input-output relation, the corresponding output will be

Causality of LTIC Systems – Example 16

$$y_1(t) = f_1(3t) = f(3t - t_o)$$

because $y_1(t)$ is different from

$$y(t - t_o) = f(3(t - t_o))$$

The new output $y_1(t)$ is not a $t_o - delayed$ version of the original output so the system is time varying.

Objectives

- Causality and LTIC System
- **Delay Lines**
- Laplace transform
- Properties of Laplace transform
- Verification of Properties through Examples

Delay Lines

Consider a system

$$h(t) = K\delta(t - t_o) \quad \leftrightarrow \quad H(\omega) = Ke^{-j\omega t_o}$$

which is zero-state linear, time invariant and BIBO stable.

A system having frequency response

$$H(\omega) = Ke^{-j\omega t_o}$$

simply delays and amplitude-scales its input $f(t)$ to produce an output

$$y(t) = K\delta(t - t_o) * f(t) = Kf(t - t_o)$$

Delay Lines

- **Clearly, if $t_o \geq 0$, then the output $y(t)$ depends only on the past or present values of $f(t)$ and the system is LTIC**

- **The system shown is a *delay line* , having delay t_o and gain K**

Delay Lines – Example 17

Question: **The signal input to a coaxial line is**

$$\boldsymbol{f(t) = u(t)}$$

At the far end of the line, the output

$$\boldsymbol{y(t) = 0.2u(t-10)}$$

what is the impulse response of the system?

Delay Lines – Example 17

Solution: From the given information, we deduce that the unit-step response $g(t)$ of the system is

$$g(t) = 0.2u(t - 10)$$

Differentiating this equation on both sides,

$$h(t) = g'(t) = 0.2\delta(t - 10)$$

we can conclude that system has a gain of 0.2 and a delay of 10 seconds having an impulse response as

$$h(t) = 0.2\delta(t - 10)$$

Objectives

- Causality and LTIC System
- Delay Lines
- **Laplace transform**
- Properties of Laplace transform
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Laplace Transform

- **Consider applying an exponential input**

$$f(t) = e^{st}$$

to an LTIC system having impulse response $h(t)$

- **The zero-state response can be calculated as**

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Laplace Transform

➢ **Since for LTIC systems $h(t)$ is zero for $t < 0$, we can move the lower limit from $-\infty$ to 0, resulting in**

$$e^{st} \longrightarrow \boxed{\text{LTIC}} \longrightarrow \hat{H}(s)e^{st}$$

where

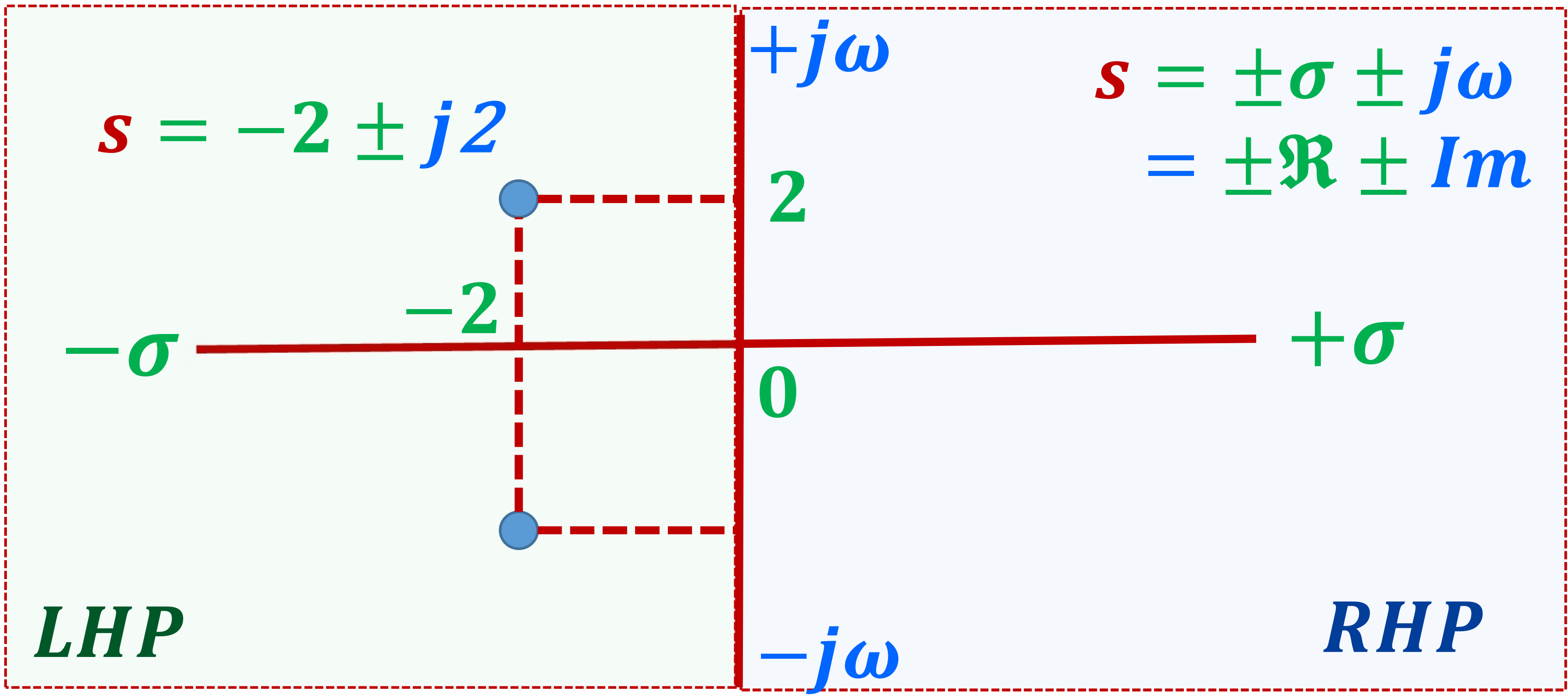
$$\hat{H}(s) = \int_{0^-}^{\infty} h(t)e^{-st}dt$$

is known as *Laplace transform* of $h(t)$ and *transfer function* of the system with impulse response $h(t)$

Laplace Transform

- Previously stated relation holds whether s is real or complex so long as the Laplace transform integral defining $\hat{H}(s)$ *converges*

- Remember that *s-plane* can be viewed as



Laplace Transform

- Notice that in some special cases with $s = j\omega$, the input-output relation becomes

$$e^{j\omega t} \longrightarrow \boxed{\text{LTIC}} \longrightarrow \hat{H}(j\omega)e^{j\omega t}$$

where

$$\hat{H}(j\omega) = \int_{0^-}^{\infty} h(t)e^{-j\omega t}dt = H(\omega)$$

is known as *system frequency response* and *Fourier transform of impulse response* $h(t)$. Notice that

$$\hat{H}(j\omega) = H(\omega)$$

assuming both $\mathcal{L}$ & $\mathcal{F}$ converge.

Laplace Transform

- **The frequency response $H(\omega)$ exists only if the system is BIBO stable**

- **However, the transfer function $\hat{H}(s)$ exists even if the system unstable – excluding some exceptions**

- **LTIC transfer function $\hat{H}(s)$ is a *generalization* of frequency response $H(\omega)$ that remains valid for many unstable systems**

- **For example $h(t) = e^{t}u(t)$ is an unstable system, but $h(t)e^{-\sigma t}$ is absolutely integrable for $\sigma > 1$. Thus, LT $\hat{H}(s)$ of $h(t) = e^{t}u(t)$ exists all s having $\sigma > 1$.**

Laplace Transform

- Laplace transform of the zero-state response

$$y(t) = h(t) * f(t)$$

of a LTIC system can be expressed as

$$\hat{Y}(s) = \hat{H}(s)\hat{F}(s)$$

if $\hat{F}(s)$ denotes the Laplace transform of a causal input signal $f(t)$.

- We will discuss Laplace transform pairs likewise Fourier transform pairs for many different causal inputs

Laplace Transform – Definition

- **Laplace transform $\hat{H}(s)$ of a signal $h(t)$ is defined as**

$$\hat{H}(s) = \int_{0^-}^{\infty} h(t)e^{-st}dt \quad \text{where } s = \sigma + j\omega$$

- **Generally, the Laplace transform integral converges for some values of s and *not for others.***

- **The region of *s-plane* containing all**

$$s = (\sigma, j\omega) = \sigma + j\omega$$

for which LT integral converges is called *region of convergence (ROC)* of the Laplace transform.

Laplace Transform – Example 1

Question: Determine the Laplace transform $\hat{H}(s)$ of $h(t) = e^{t}u(t)$ and its *ROC* .

Solution: The Laplace transform of $h(t) = e^{t}u(t)$ is

$$\hat{H}(s) = \int_{0^-}^{\infty} e^{t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{(1-s)t}dt = \left.\frac{e^{(1-s)t}}{1-s}\right|_{0}^{\infty}$$

- We must have $\sigma > 1$ for convergence, thus the *ROC* of the function $h(t)$ is described by inequality

$$\sigma = Re\{s\} > 1$$

Laplace Transform – Example 1

- For all values of *s* satisfying the inequality, the Laplace transform can be obtained as

$$\hat{H}(s) = \left.\frac{e^{(1-s)t}}{1-s}\right|_0^\infty = \frac{0-1}{1-s} = \frac{1}{s-1}$$

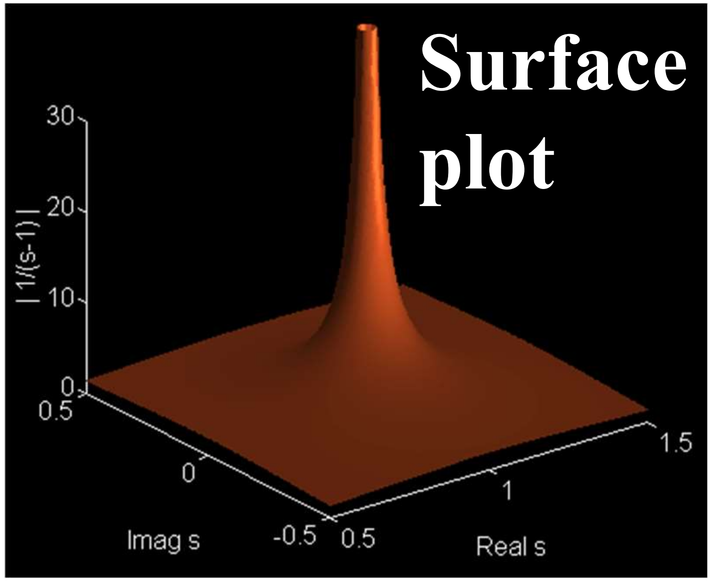
- Generating a pair:

$$\boxed{e^t u(t) \leftrightarrow \frac{1}{s-1}}$$

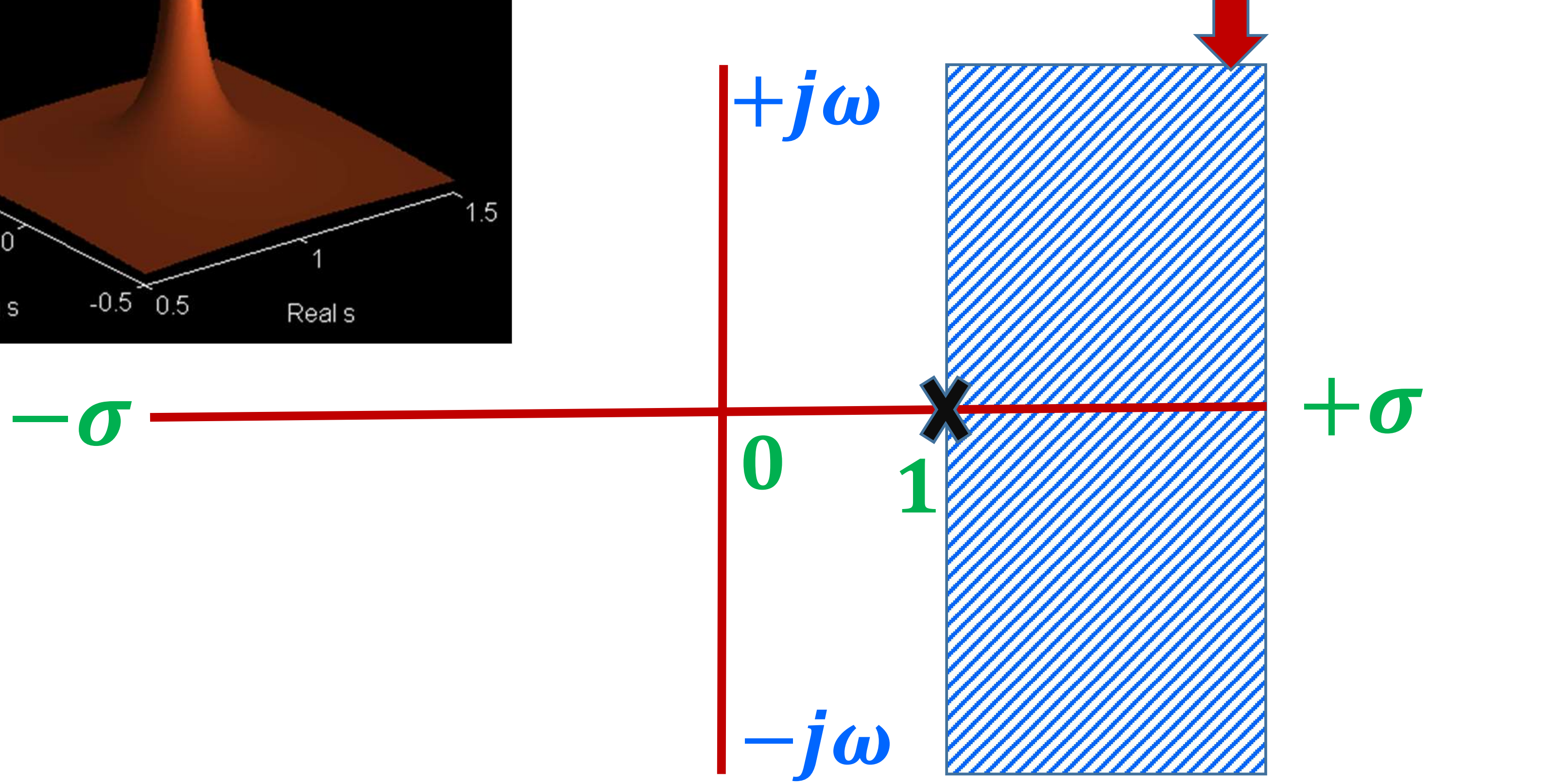
- ROC and surface plot can be shown in ***s-plane***

Laplace Transform – ROC

- The transfer function having a pole at 1,



$$\hat{H}(s) = \frac{1}{s-1}$$



Laplace Transform – Example 2

Question: Determine the Laplace transform $\hat{F}(s)$ of $f(t) = e^{-2t}u(t) - e^{-t}u(t)$ and its ROC.

Solution: The Laplace transform of

$$f(t) = e^{-2t}u(t) - e^{-t}u(t) \text{ is}$$

$$\hat{F}(s) = \int_{0^-}^{\infty} (e^{-2t} - e^{-t})u(t)e^{-st}dt$$

$$= \int_{0}^{\infty} [\, e^{-(2+s)t} - e^{-(1+s)t} \,]dt = \frac{-1}{(s+2)(s+1)}$$

Laplace transform – Example 2

- **Under the assumptions $\sigma = Re\{s\} > (-2)$ and $\sigma = Re\{s\} > (-1)$, dictated by convergence**

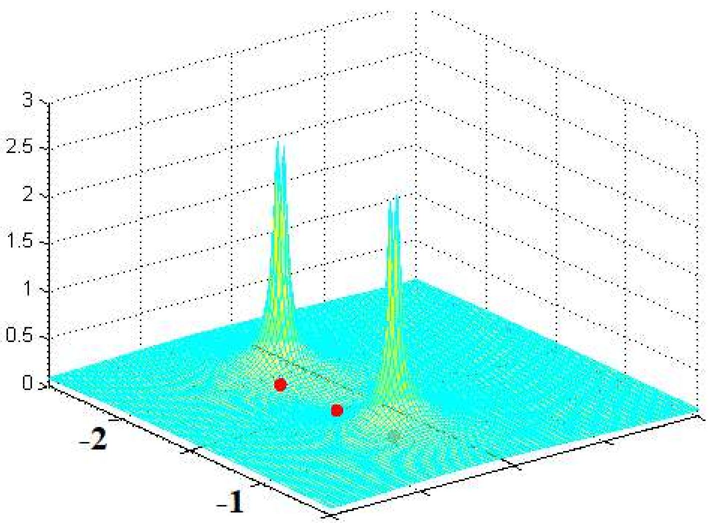
- **First condition is automatically satisfied if the second is satisfied**

- **The *ROC* consists of all complex *s* such that**

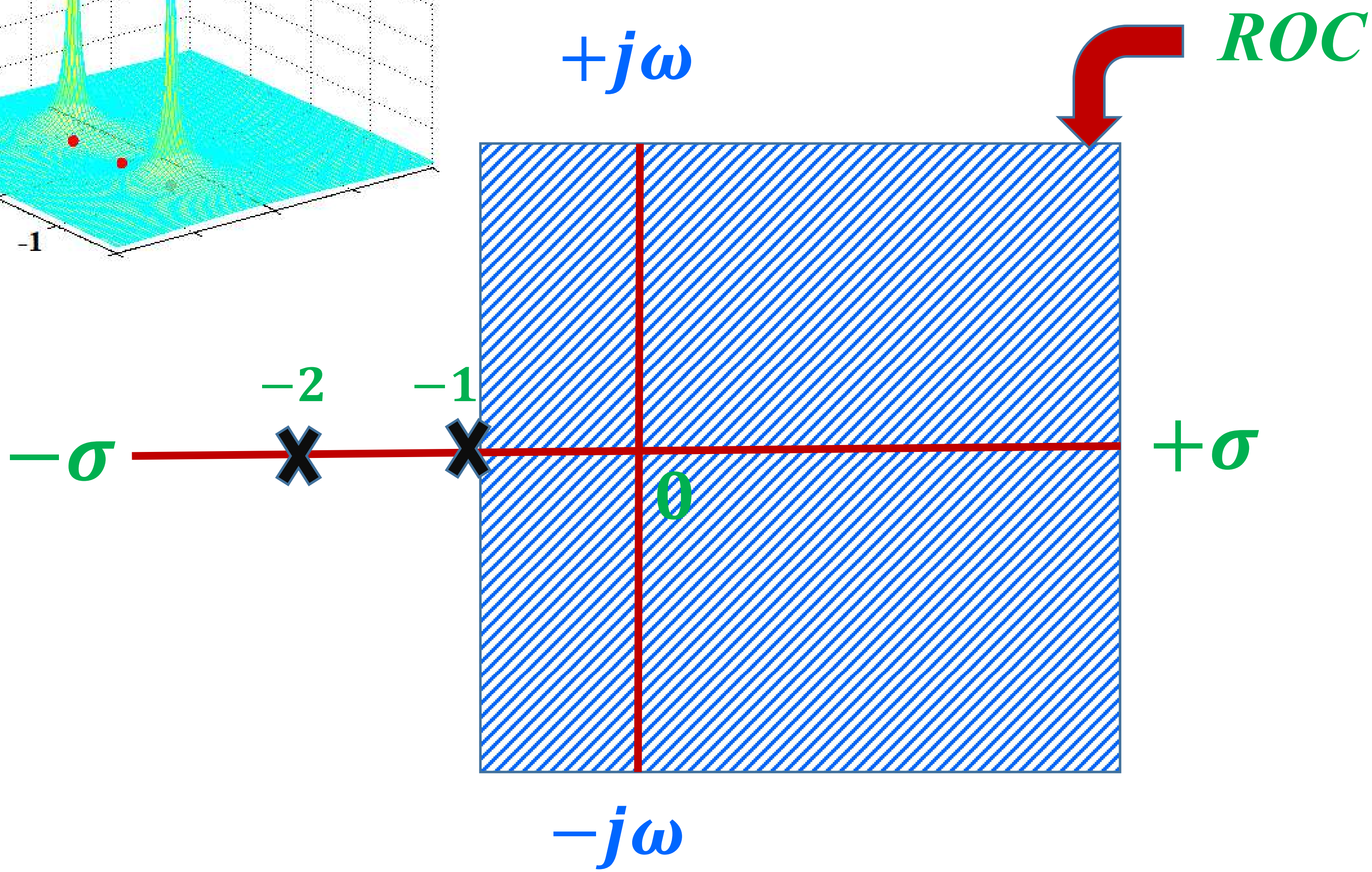
$$ROC: \sigma = Re\{s\} > (-1)$$

- **The rule of thumb is: *ROC is all s-plane to the right of the rightmost pole***

Laplace Transform – Example 2



$$\widehat{F}(s) = \frac{-1}{(s+2)(s+1)}$$



Laplace Transform – Example 3

Question: **Determine the Laplace transform $\hat{H}(s)$ of $f(t) = \delta(t)$ and its ROC .**

Solution: **In this case, using the sifting property of the impulse,**

$$\hat{H}(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-s \times 0} = 1$$

$$ROC: Entire \;\; s - plane$$

No Pole!

Laplace Transform – Example 4

Question: Using the derivative property of

$$\delta(t) * e^{st} = e^{st}$$

to determine the Laplace transform of $\delta'(t)$, the differentiator of the impulse response.

Solution: Differentiating both sides, we find that

$$\delta'(t) * e^{st} = se^{st}$$

which can be re-written as

$$\int_{-\infty}^{\infty} \delta'(\tau) e^{s(t-\tau)} d\tau = se^{st}$$

Laplace Transform – Example 4

Evaluating both sides at $t = 0$, we find that

$$\int_{-\infty}^{\infty} \delta'(\tau) e^{-\tau s} d\tau = s$$

$$ROC: Entire\ s - plane$$

A Pole at ∞

Laplace Transform – Example 5

Question: Given the Laplace transform of

$$\boldsymbol{h(t) = e^t u(t)} \quad \textbf{is} \quad \boldsymbol{\widehat{H}(s) = \frac{1}{s-1}},$$

show that the Laplace transform of $\boldsymbol{f(t) = te^t u(t)}$ is

$$\boldsymbol{\widehat{F}(s) = \frac{1}{(s-1)^2}}$$

Laplace Transform – Example 5

Solution: According to the given information,

$$\hat{H}(s) = \int_{0}^{\infty} e^{t}e^{-st}dt = \frac{1}{s-1}$$

which holds for $\{s: \sigma > 1\}$. Take the derivative with respect to s,

$$\frac{d}{ds}\left(\int_{0}^{\infty} e^{t}e^{-st}dt\right) = -\int_{0}^{\infty} te^{t}e^{-st}dt \qquad (LHS)$$

and

Laplace Transform – Example 5

$$\frac{d}{ds}\left(\frac{1}{s-1}\right) = -\frac{1}{(s-1)^2} \qquad (RHS)$$

$$\int_0^\infty te^t e^{-st} dt = \frac{1}{(s-1)^2}$$

$$ROC: \sigma = Re\{s\} > 1$$

We obtain another pair:

$$\boxed{te^t u(t) \leftrightarrow \frac{1}{(s-1)^2}}$$

Objectives

- Causality and LTIC System
- Delay Lines
- Laplace transform
- **Properties of Laplace transform**
- Verification of Properties through Examples

Laplace Transform – BIBO Stability

- Notice that the poles of Laplace transforms of *absolutely integrable* signals are confined to the left half plane (LHP)

- *It is not a coincidence*

- If a signal $h(t)$ is absolutely integrable and causal, then its FT integral is guaranteed to converge to a bounded $H(\omega) = \widehat{H}(j\omega)$

- *This requires that:* All poles of $\widehat{H}(s)$ be located within the LHP

Laplace Transform – BIBO Stability

- **BIBO stable systems must have absolutely integrable impulse response functions**

- **An LTIC system $h(t) \leftrightarrow \hat{H}(s)$ is BIBO stable *if and only if* its transfer function $\hat{H}(s)$ has all of its poles in the *LHP***

- ***Let's see stability related examples for more illustrations***

LT: BIBO Stability – Example 6

Question: **Using the common Laplace pairs and BIBO stability criterion, determine whether the two systems are BIBO stable or not.**

$$a)\ h(t) = e^{-t}u(t) + e^{2t}u(t)$$

$$b)\ g(t) = e^{-t}\cos(t)u(t)$$

LT: BIBO Stability – Example 6

Solution: **For**

$$a)\ h(t) = e^{-t}u(t) + e^{2t}u(t)$$

we have

$$\widehat{H}(s) = \frac{1}{s+1} + \frac{1}{s-2} = \frac{2s-1}{(s+1)(s-2)}$$

- **The transfer function has two poles located at -1 (within the LHP) and 2 (outside LHP)**
- **Therefore, the system is not BIBO stable**

LT: BIBO Stability – Example 6

Solution: For

$$b)\ g(t) = e^{-t} cos\,(t)u(t)$$

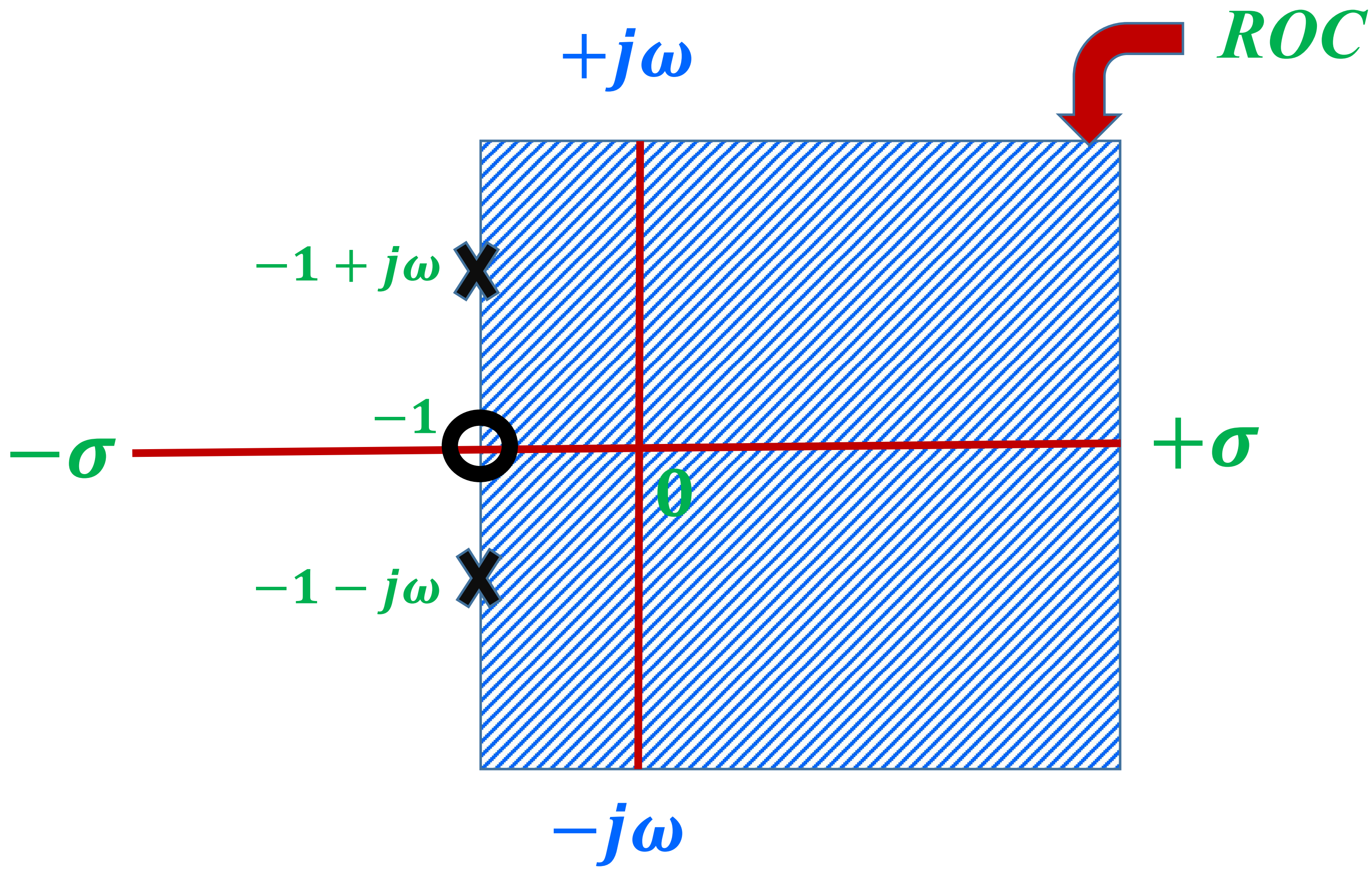
we have

$$\hat{G}(s) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{(s+1+j\omega)(s+1-j\omega)}$$

- The transfer function has two poles located at: $(-1+j\omega)$ (within the LHP) , and $(-1-j\omega)$ (within the LHP) and a zero at -1
- Therefore, the system is BIBO stable
- Zero's are locations where the system response is zero

LT: BIBO Stability – Example 6

$$\hat{G}(s) = \frac{s+1}{(s+1+j\omega)(s+1-j\omega)}$$



Linearity Property

- **The linearity property in time domain**

$$u(t) = a \cdot f(t) + b \cdot g(t)$$

Transformed to the Laplace domain

$$\mathfrak{L}\{\,a \cdot f(t) + b \cdot g(t)\,\}$$

$$= \int_{0^-}^{\infty} (a \cdot f(t) + b \cdot g(t)) * e^{-st}\mathrm{d}t$$

$$= a \underbrace{\int_{0^-}^{\infty} f(t) * e^{-st}\mathrm{d}t}_{F(s)} + b \underbrace{\int_{0^-}^{\infty} g(t) * e^{-st}\mathrm{d}t}_{G(s)}$$

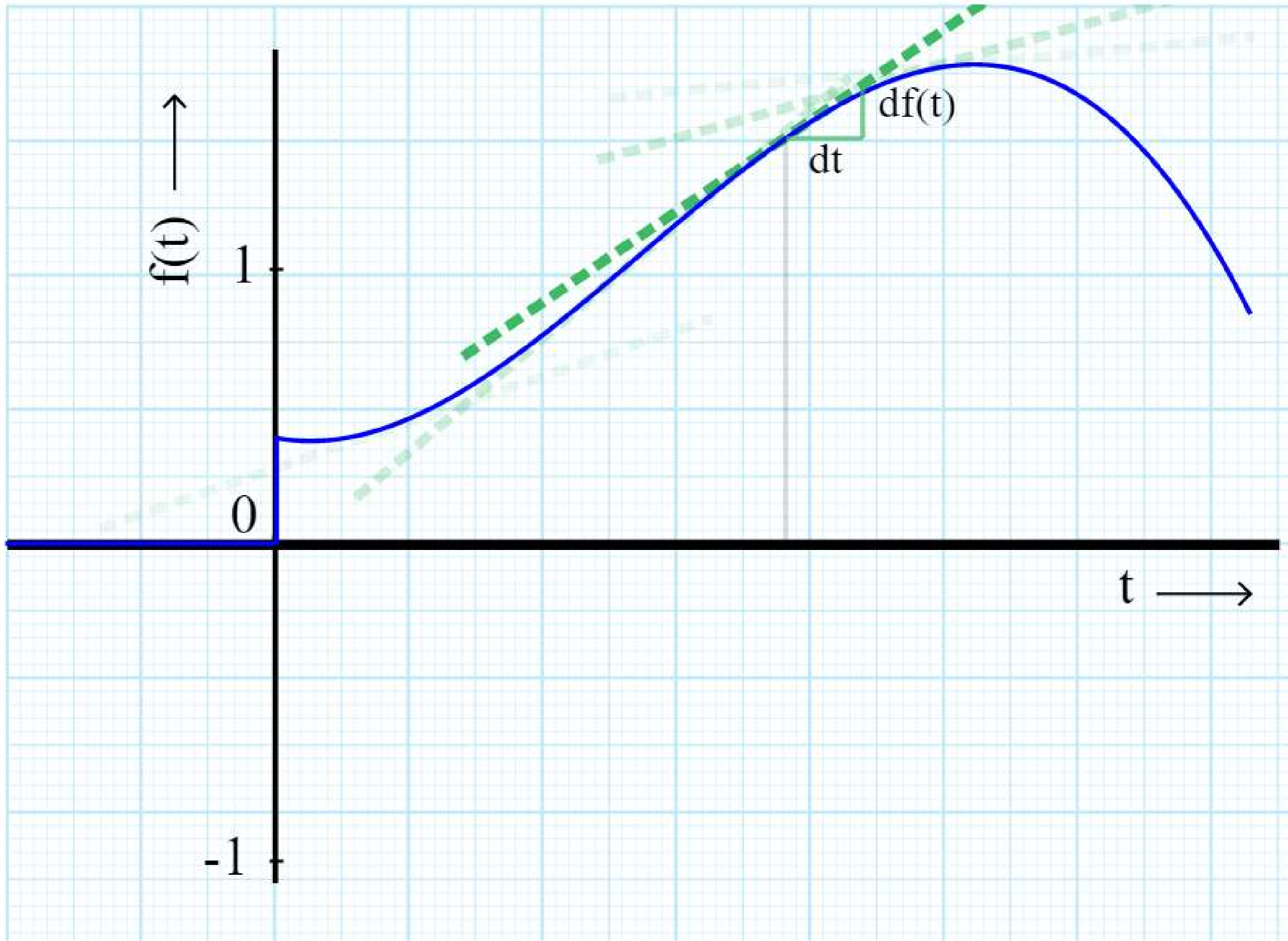
Linearity Property

which follows that

$$\boxed{a \cdot f(t) + b \cdot g(t) \overset{\mathfrak{L}}{\longleftrightarrow} a \cdot F(s) + b \cdot G(s)}$$

First Derivative Property

- We know that derivative of $f(t)$ can be represented by $\frac{df}{dt}$



Derivative Property

- **The first derivative in time is used in deriving the Laplace transform for capacitor and inductor impedance. The first derivative**

$$\frac{df(t)}{dt}$$

transformed into Laplace transform gives

$$\mathfrak{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right\} = \int_{0^-}^{\infty} e^{-st} \frac{\mathrm{d}f(t)}{\mathrm{d}t} \mathrm{d}t$$

$$= \int_{0^-}^{\infty} \underbrace{e^{-st}}_{u(t)} \underbrace{\frac{\mathrm{d}f(t)}{\mathrm{d}t}}_{v'(t)} \mathrm{d}t$$

Derivative Property

$$\mathfrak{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right\} = \left[e^{-st} f(t)\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} (-s) e^{-st} f(t) \mathrm{d}t$$

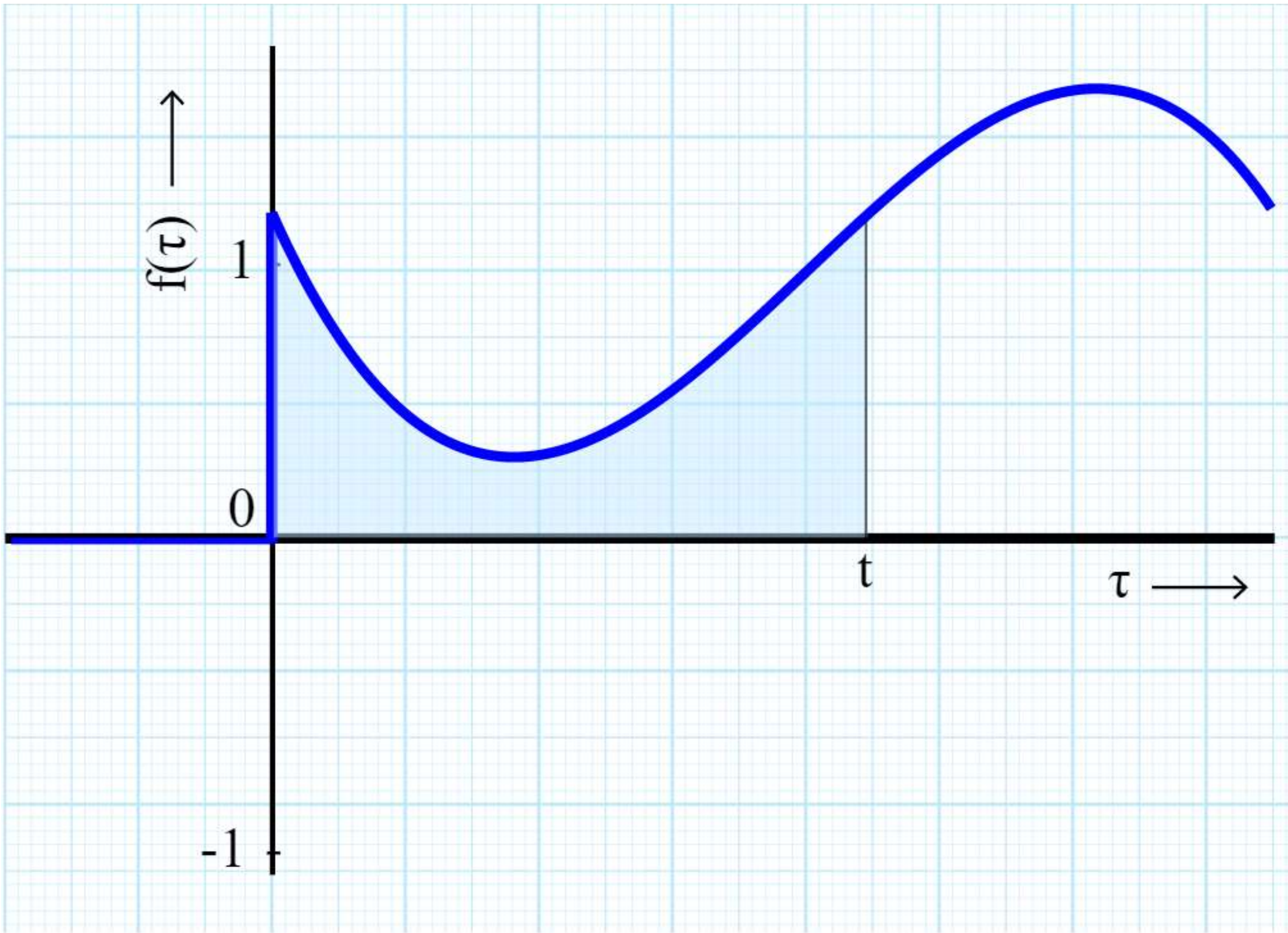
$$= \cancel{e^{-s\infty} f(\infty)} - \cancel{e^{-s0^-}} f(0^-) + \underbrace{s \int_{0^-}^{\infty} e^{-st} f(t) \mathrm{d}t}_{\mathfrak{L} f(t) = F(s)}$$

The *first term goes to zero* because *f* (∞) is finite which is a condition for the existence of the transform. The last term is simply the *definition* of the Laplace transform multiplied by *s*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} f(t) \xleftrightarrow{\mathfrak{L}} s\,F(s) - f(0^-)}$$

Integration Property

We are to determine LT of $u(t) = \int_{0^-}^{t} f(\tau)\mathrm{d}\tau$



Integration Property

$$\mathfrak{L}\left\{\int_{0^-}^{t} f(\tau)\tau\right\} = \int_{0^-}^{\infty} \underbrace{\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)}_{u(t)} \underbrace{e^{-st}}_{v'(t)}\mathrm{d}t$$

$$= \left[\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)\left(-\frac{1}{s}e^{-st}\right)\right]_{0^-}^{\infty}$$

$$-\int_{0^-}^{\infty} f(t)\left(-\frac{1}{s}e^{-st}\right)\mathrm{d}t$$

Integration Property

$$= -\frac{1}{s}\left[e^{-st}\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right]_{0^-}^{\infty}$$

$$+\frac{1}{s}\underbrace{\int_{0^-}^{\infty} f(t)e^{-st}\mathrm{d}t}_{\mathfrak{L}f(t)=F(s)}$$

$$= -\frac{1}{s}\left(e^{-s\infty}\cancel{\int_{0^-}^{\infty} f(\tau)\mathrm{d}\tau} - e^{-s0^-}\cancel{\int_{0^-}^{0^-} f(\tau)\mathrm{d}\tau}\right) + \frac{1}{s}F(s)$$

Integration Property

- The first term goes to zero because $f(\infty)$ is finite which is a condition for the existence of the transform
- In the second term, the exponential goes to one and the integral is 0 because the limits are equal
- The last term is simply the definition of the Laplace transform over s

$$\boxed{\int_{0^-}^{t} f(\tau)\tau \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s}F(s)}$$

Convolution Property

- The convolution property applies to causal systems

$$\int_{t=0^-}^{\infty} \{h(t) * f(t)\} e^{-st} dt = \int_{t=0^-}^{\infty} \left\{ \int_{-\infty}^{\infty} h(\tau) f(t-\tau) d\tau \right\} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} h(\tau) \left\{ \int_{t=0^-}^{\infty} f(t-\tau)\, e^{-st} dt \right\} d\tau$$

$$= \int_{\tau=0^-}^{\infty} h(\tau) \left\{ \int_{t=0^-}^{\infty} f(t-\tau)\, e^{-st} dt \right\} d\tau$$

$$= \int_{\tau=0^-}^{\infty} h(\tau) e^{-s\tau} \hat{F}(s) d\tau$$

$$= \hat{F}(s) \int_{\tau=0^-}^{\infty} h(\tau) e^{-s\tau} d\tau = \hat{F}(s) \hat{H}(s)$$

Objectives

- Causality and LTIC System
- Delay Lines
- Laplace transform
- Properties of Laplace transform
- **Verification of Properties through Examples**

Example 7 of Convolution Property

Problem Statement: Given $f(t) = e^{-t}u(t)$, determine $y(t) = f(t) * f(t)$ by using the time convolution property.

Solution: We know $e^{-t}u(t) \Leftrightarrow \dfrac{1}{s+1}$

By using the time-convolution property

$$\hat{Y}(s) = \hat{F}(s)F(s) = \frac{1}{(s+1)^2}$$

Since

$$te^{-t}u(t) \leftrightarrow \frac{1}{(s+1)^2}$$

we obtain $y(t) = te^{-t}u(t)$

Time Convolution Property

Problem Statement: **If $f(t) = e^{-t}$, can we take advantage of the time-convolution property to calculate $y(t) = f(t) * f(t)$?**

Solution: **Because the given $f(t)$ is not causal, the answer is No. Therefore,**

$$e^{-t} * e^{-t} \neq te^{-t}$$

Time Delay Property

- The time–delay property is guaranteed to work only for causal signals and for positive delays $t_o \geq 0$.

Assuming $f(t)$ as causal and taking Laplace transform of $f(t - t_o)$ by definition

$$\int_{t=0^-}^{\infty} f(t - t_o)e^{-st}dt = \int_{t=t_o{}^-}^{\infty} f(t - t_o)e^{-st}dt$$

$$= \int_{\tau=0^-}^{\infty} f(\tau)\, e^{-s(\tau+t_o)}d\tau$$

$$= e^{-st_o} \int_{\tau=0^-}^{\infty} f(\tau)\, e^{-s\tau}d\tau = e^{-st_o}\widehat{F}(s)$$

Time Delay Property: Example 8

Problem Statement: Using the time-delay property, determine the Laplace transform of

$$rect\left(t-\frac{1}{2}\right)$$

Solution: Since

$$rect\left(t-\frac{1}{2}\right)=u(t)-u(t-1)$$

$$u(t)-u(t-1)\Leftrightarrow\frac{1}{s}-\frac{1}{s}e^{-s}=\frac{1}{s}(1-e^{-s})$$

$$rect\left(t-\frac{1}{2}\right)\Leftrightarrow\frac{1}{s}(1-e^{-s})$$

Properties of Laplace Transform

	Name	Condition	Property
1	Multiplication	$f(t) \longrightarrow \hat{F}(s)$, constant K	$Kf(t) \longrightarrow K\hat{F}(s)$
2	Addition	$f(t) \longrightarrow \hat{F}(s)$ $g(t) \longrightarrow \hat{G}(s)$ …….	$f(t) + g(t) + \cdots \longrightarrow \hat{F}(s) + \hat{G}(s) + \cdots$
3	Time Scaling	$f(t) \longrightarrow \hat{F}(s)$, real $a > 0$	$f(at) \longrightarrow \frac{1}{a}\hat{F}\left(\frac{s}{a}\right)$
4	Time Delay	$f(t) \longrightarrow \hat{F}(s)$, $t_o \geq 0$	$f(t - t_o) \longrightarrow \hat{F}(s)e^{-st}$

Properties of Laplace Transform

	Name	Condition	Property
5	Frequency Shift	$f(t) \longrightarrow \widehat{F}(s)$	$f(t)e^{s_o t} \longrightarrow \widehat{F}(s - s_o)$
6	Time Derivative	Differentiable $f(t) \longrightarrow \widehat{F}(s)$	$f'(t) \longrightarrow s\widehat{F}(s) - f(0^-)$ $f''(t) \longrightarrow s^2\widehat{F}(s) - sf(0^-) - f'(0^-)$
7	Time Integration	$f(t) \longrightarrow \widehat{F}(s)$	$\int_{0^-}^{t} f(\tau)d\tau \longrightarrow \frac{1}{s}\widehat{F}(s)$
8	Frequency Derivative	$f(t) \longrightarrow \widehat{F}(s)$	$-tf(t) \longrightarrow \frac{d}{ds}\widehat{F}(s)$

Properties of Laplace Transform

	Name	Condition	Property
9	Time Convolution	$f(t) \longrightarrow \hat{F}(s)$ $h(t) \longrightarrow \hat{H}(s)$	$f(t) * h(t) \longrightarrow \hat{F}(s)\hat{H}(s)$
10	Frequency Convolution	$f(t) \longrightarrow \hat{F}(s)$ $g(t) \longrightarrow \hat{G}(s)$	$f(t)g(t) \longrightarrow \frac{1}{2\pi j}\hat{F}(s) * \hat{G}(s)$
11	Poles	$f(t) \longrightarrow \hat{F}(s)$	Values of s such that $\lvert\hat{F}(s)\rvert = \infty$
12	ROC	$f(t) \longrightarrow \hat{F}(s)$	Portion of the s-plane to the right of the rightmost pole $\neq \infty$

Properties of Laplace Transform

	Name	Condition	Property
13	Fourier Transform	$f(t) \longrightarrow \hat{F}(s)$	$F(\omega) = \hat{F}(j\omega)$ if and only if ROC includes $s = j\omega$
14	Final Value	Poles of $s\hat{F}(s)$ in LHP	$f(\infty) = \lim\limits_{s \to \infty} s\hat{F}(s)$
15	Initial Value	Existence of the limit	$f(0^+) = \lim\limits_{s \to \infty} s\hat{F}(s)$

Advantages of Laplace Transform

- **Signals which are not Fourier transformable may be Laplace transformable**

- **Convolution in time domain can be obtained by multiplication in the *s*-domain**

- **Integral-differential equation of a system can be easily converted into simple algebraic equations**

Properties of Laplace Transform

Relationship between Fourier & Laplace transformation

- **The Laplace transform is a superset of the Fourier transform – it is equal to it when $s = j\omega$**

- **The Laplace transform of a continuous time signal is defined by**

$$\hat{X}(s) = \int_0^{\infty} x(t)e^{-st}dt$$

- **It is basically Fourier transform of the signal $x(t)e^{-\sigma t}$, when $s = \sigma + j\omega$**

Summary

- **An LTI system (stable of not) is said to be causal if its zero-state response $h(t) * f(t)$ depends only on the *past and present* values *not future* values of $f(t)$**

- **An LTI system with impulse response function $h(t)$ *is causal* if and only if $h(t) = 0$ *for* $t < 0$**

- **Delay lines are used to model the gain/loss and delay of the transmission line to see the quality of transmission**

Summary

- **LTIC transfer function $\hat{H}(s)$ is a *generalization* of frequency response $H(\omega)$ that remains valid for many unstable systems**

- **The region in the *s-plane that* contains all $s = \sigma + j\omega$, for which LT integral converges is called the *region of convergence (ROC)* of Laplace transform**

- ***ROC is all s-plane to the right of the rightmost pole***

- **If a signal $h(t)$ is absolutely integrable and causal, then its FT integral is guaranteed to converge to a bounded $H(\omega) = \hat{H}(j\omega)$**

Further Reading

1. Ch. 10 (page 351-359), Ch. 11 (page 361-380), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.
2. Ch. 9 (page 654-691), A. V. Oppenheim, ***Signals and Systems***, 2nd ed., Prentice Hall, 1996.

Preview:

1. Ch. 11 (page 381-392), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.

Homework 12

Deadline: 10:00 PM, 18^{th} May, 2022

Thank you!