ANALOG SIGNAL PROCESSING

ECE 210 & 211

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Objectives

- **Graphical Convolution**
- **Examples of graphical convolution**
- **Impulse response**
- **Properties of impulse response**

Objectives

- Graphical Convolution
- Examples of graphical convolution
- Impulse response
- Properties of impulse response

Graphical convolution

- We know that the convolution means *"folding"*

- It is used to find the response *y(t)* of a system for an excitation *x(t)*, while knowing the impulse response *h(t)* of the system

- This is commonly known as **convolution integral**

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \qquad \text{Eq. 1}$$

or

$$y(t) = x(t) * h(t)$$

Graphical convolution

- **Eq. 1 states that the output is equal to the input convolved with the unit impulse response**

- **The convolution of two signals consists of following steps: *time-reversing of signal* , *shifting it*, *multiplying it point by point with the second signal*, and *integrating the product***

- **We can see the steps of convolution integral by explaining it in terms of mathematical signals**

Graphical convolution

1. **Folding:** **Taking the mirror image of $h(\lambda)$ about the ordinate axis to obtain $h(-\lambda)$**

2. **Displacement:** **Shift or delay $h(-\lambda)$ by *t* to obtain $h(t-\lambda)$**

3. **Multiplication:** **Find the product of $h(t-\lambda)$ and $x(\lambda)$**

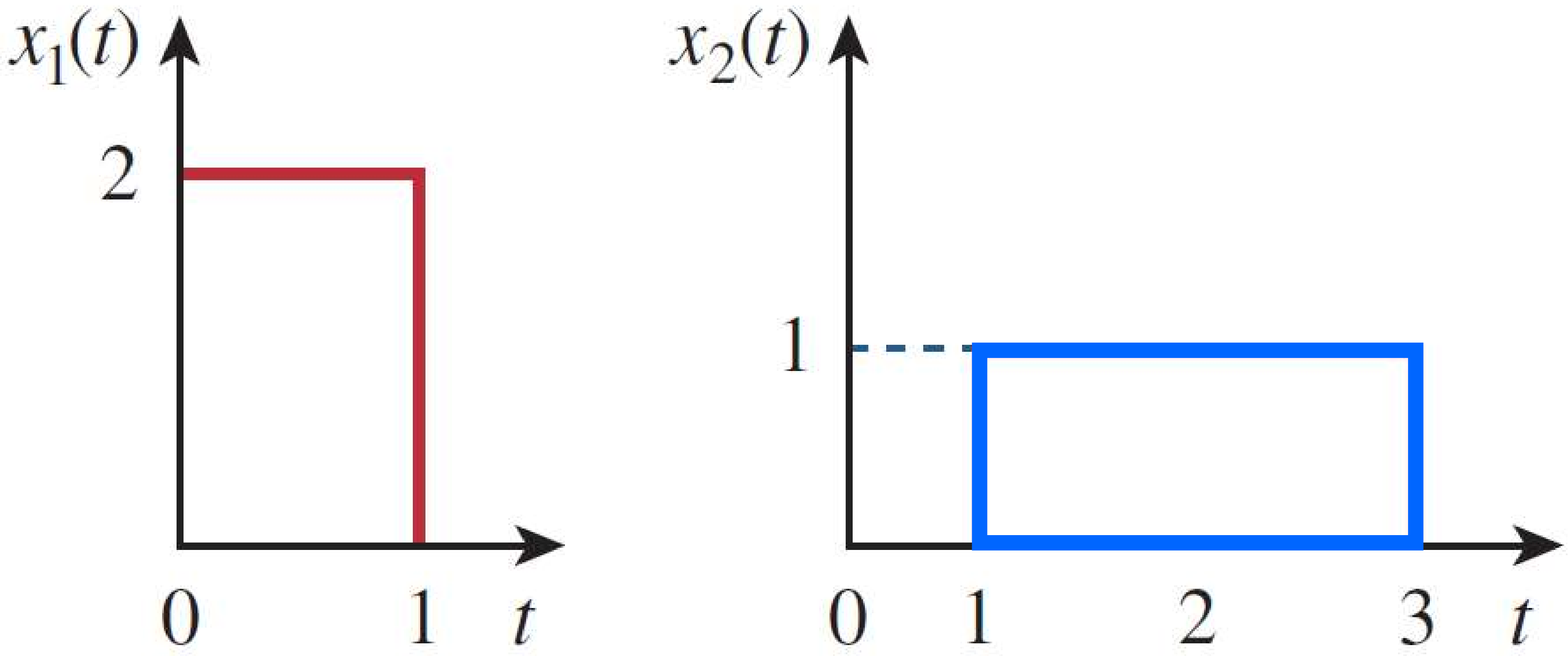
4. **Integration:** **for the given t, find the area under the product $h(t-\lambda)x(\lambda)$ for [0,*t*] to get $y(t)$ at t**

Objectives

- Graphical Convolution
- **Examples of graphical convolution**
- Impulse response
- Properties of impulse response

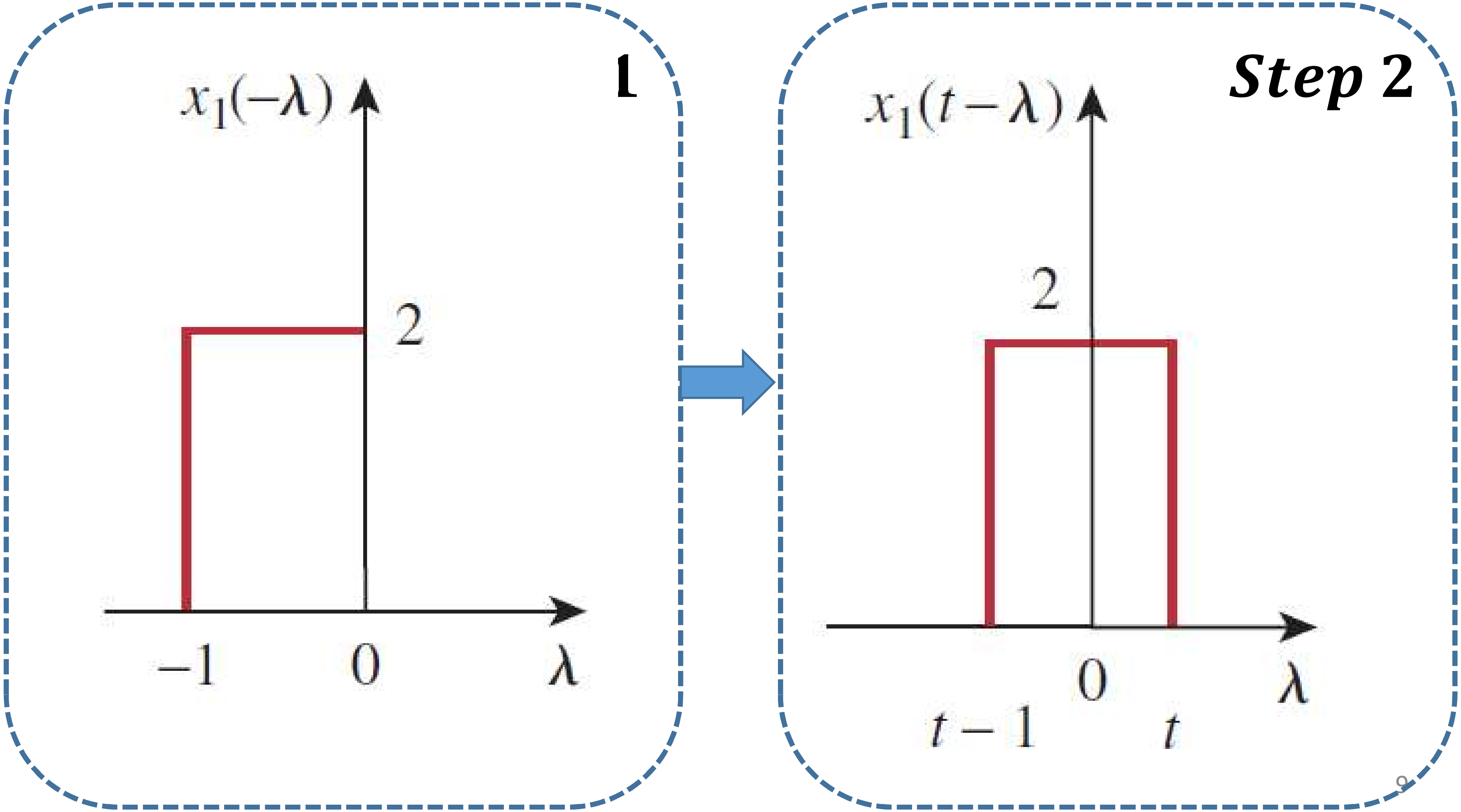
Graphical convolution – Example 1

Question: Find the convolution of two signals shown below using



Graphical convolution – Example 1

Solution: **We first fold the signal $x_1(\lambda)$ and obtain $x_1(-\lambda)$ and shift it to any t,**

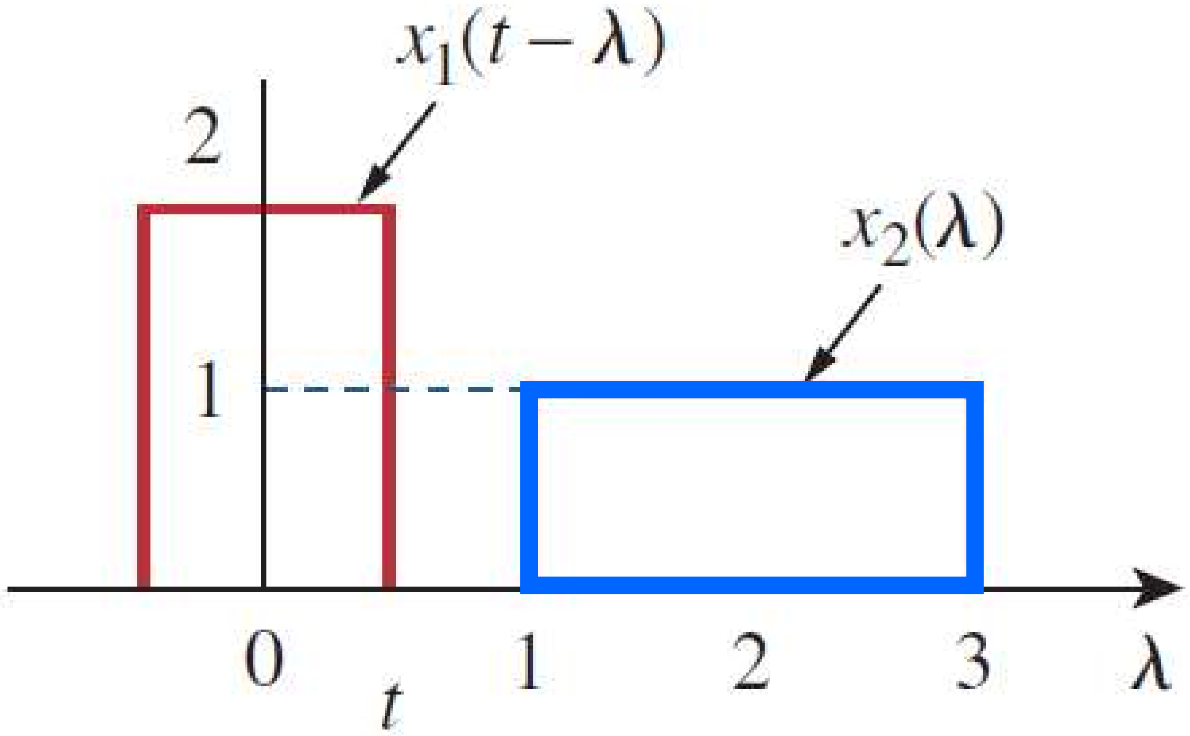


Graphical convolution – Example 1

***Step* 3**:

- **It is clear that there is no overlap area between 0 < t < 1**

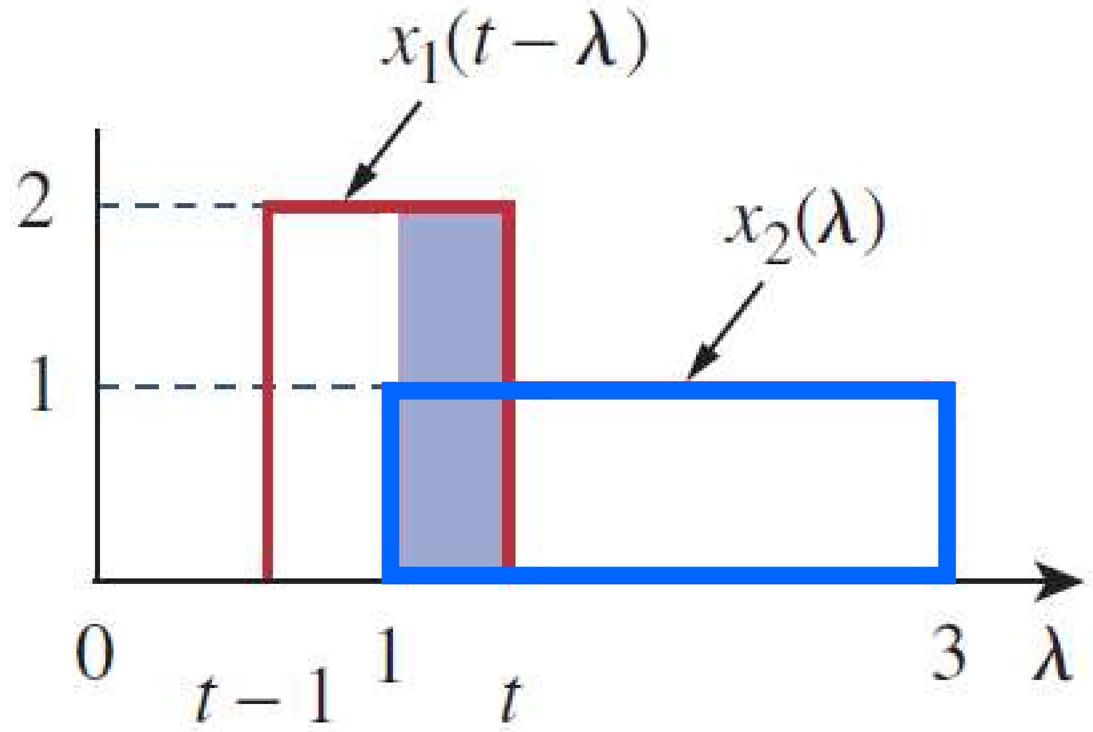
$$y(t) = x_1(t) * x_2(t) = 0, \qquad 0 < t < 1$$



Graphical convolution – Example 1

- **The two signals overlap between 1 and t for 1 < t < 2**

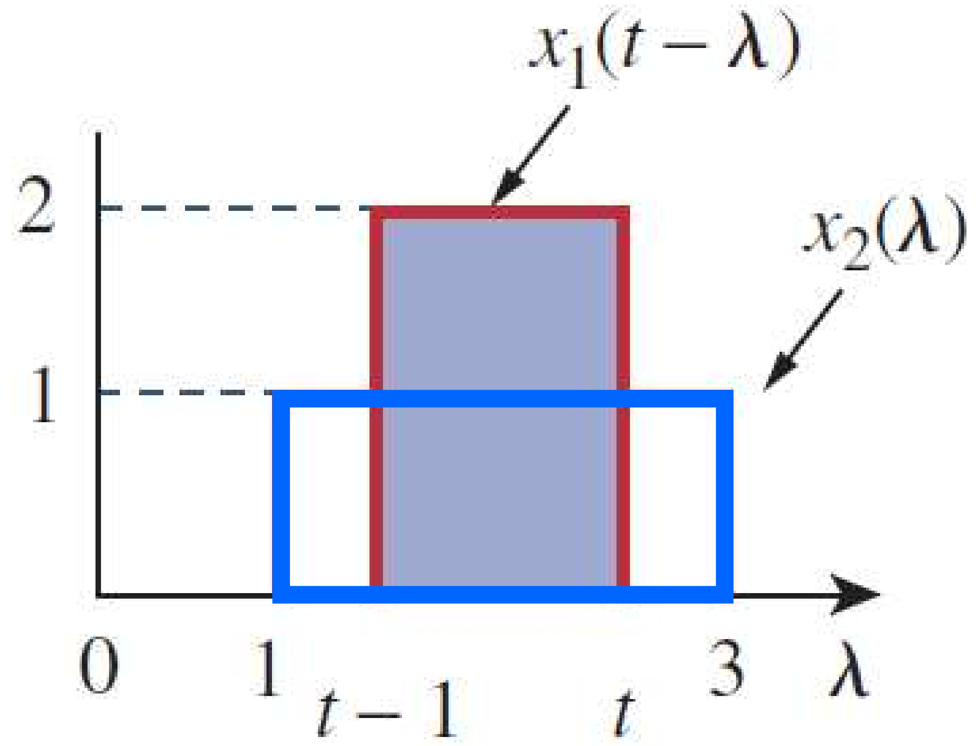
$$y(t) = \int_1^t (2)(1)\, d\lambda = 2\lambda \Big|_1^t = 2(t-1), \qquad 1 < t < 2$$



Graphical convolution – Example 1

- **The two signals completely overlap between $(t-1)$ and t for $2 < t < 3$**

$$y(t) = \int_{t-1}^{t} (2)(1)\, d\lambda = 2\lambda \Big|_{t-1}^{t} = 2, \qquad 2 < t < 3$$

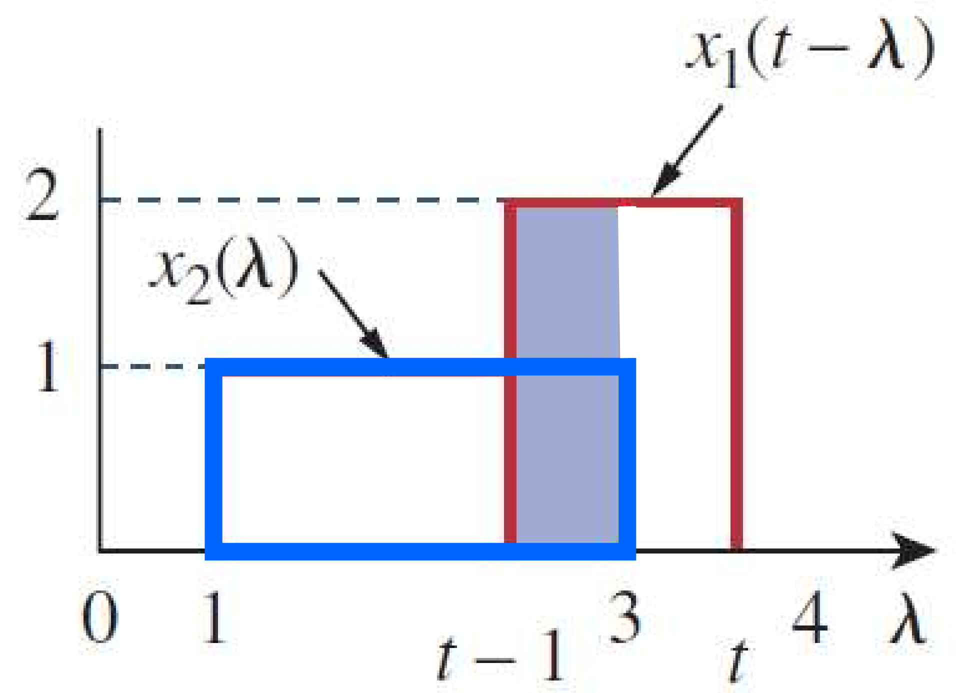


Graphical convolution –Example 1

- **The two signals overlap between $(t-1)$ and 3, for $3 < t < 4$**

$$y(t) = \int_{t-1}^{3} (2)(1)\, d\lambda = 2\lambda \Big|_{t-1}^{3}$$

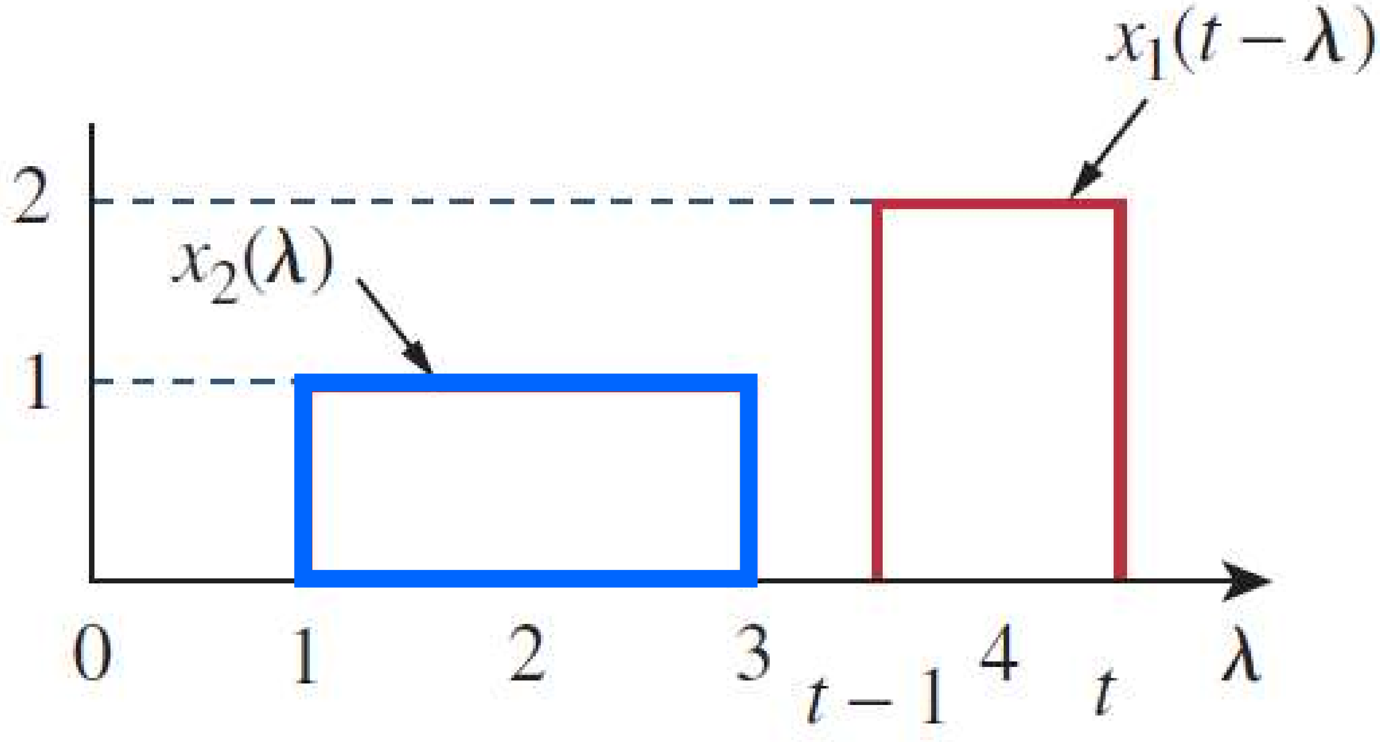
$$= 2(3 - t + 1) = 8 - 2t,$$



Graphical convolution – Example 1

➢ **The two signals do not overlap for $t > 4$,**

$$y(t) = 0, \qquad t > 4$$

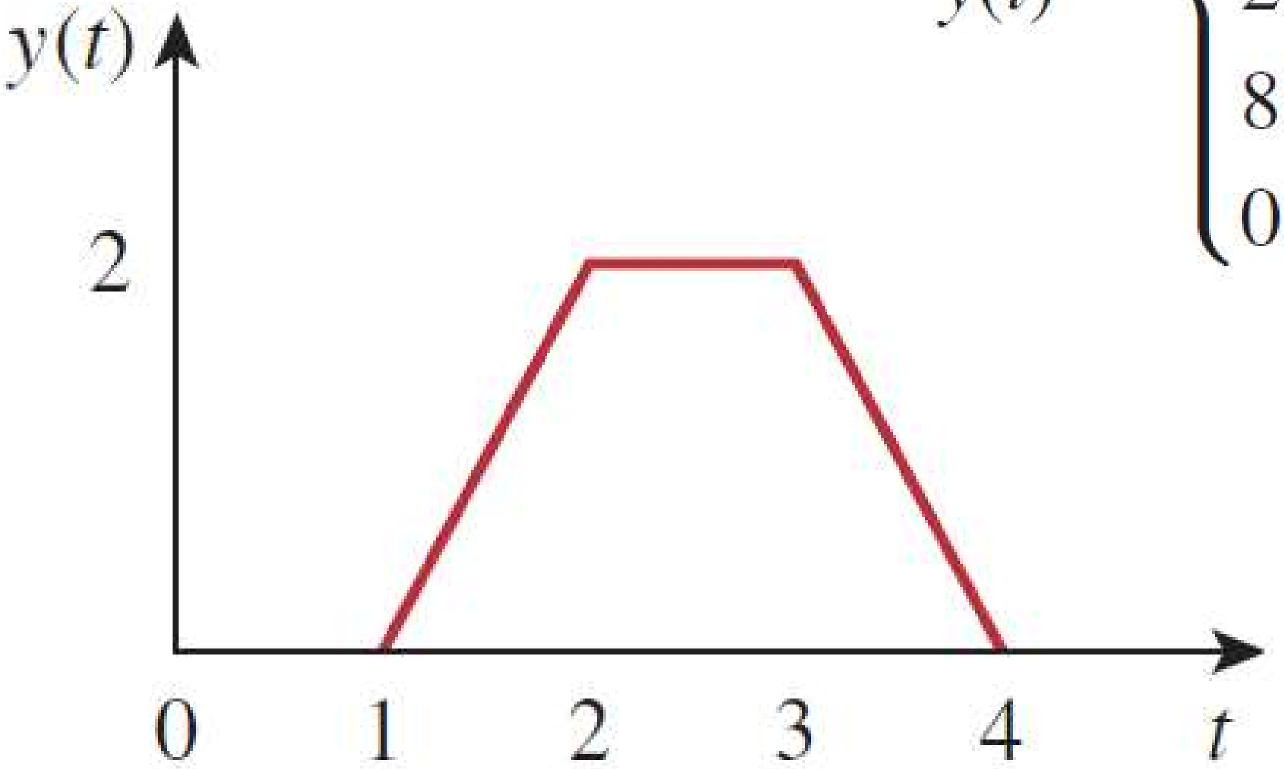


Graphical convolution – Example 1

***Step* 4:**

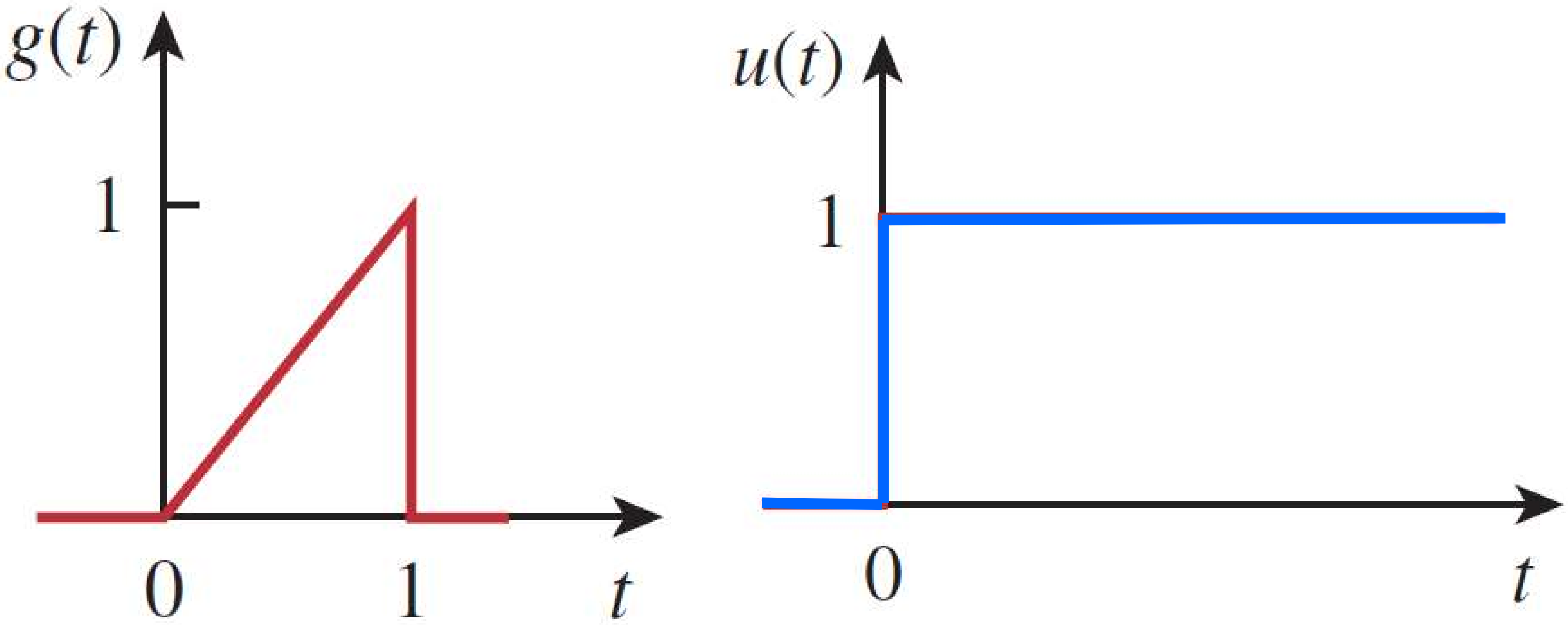
- **Combining all the results show the convolved signals as,**

$$y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 2t - 2, & 1 \le t \le 2 \\ 2, & 2 \le t \le 3 \\ 8 - 2t, & 3 \le t \le 4 \\ 0, & t \ge 4 \end{cases}$$



Graphical convolution – Example 2

Question: **Graphically convolve $g(t)$ and $u(t)$ as show in figure?**



Graphical convolution – Example 2

Solution: Let $y(t) = g(t) * u(t)$, we can find convolution in two ways

Method 1: Suppose we fold $g(t)$ and shift it to any t, since $g(t) = t \quad for\ 0 < t < 1$, originally,

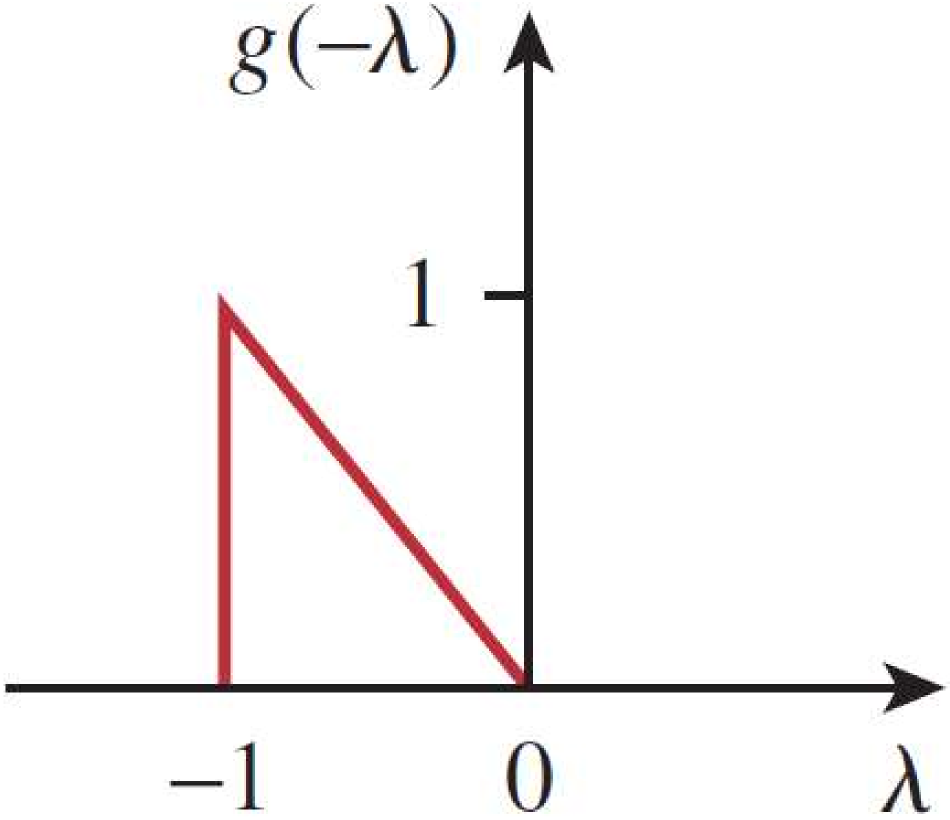
$$g(t-\lambda) = t - \lambda, 0 < t - \lambda < 1 \quad \text{or} \quad t - 1 < \lambda < t.$$

- There is no overlap between these two functions when $t < 0$, so that $y(0) = 0$ in this case

Graphical convolution – Example 2

- $For\ t < 0$, there is no overlap area

$$y(0) = 0$$

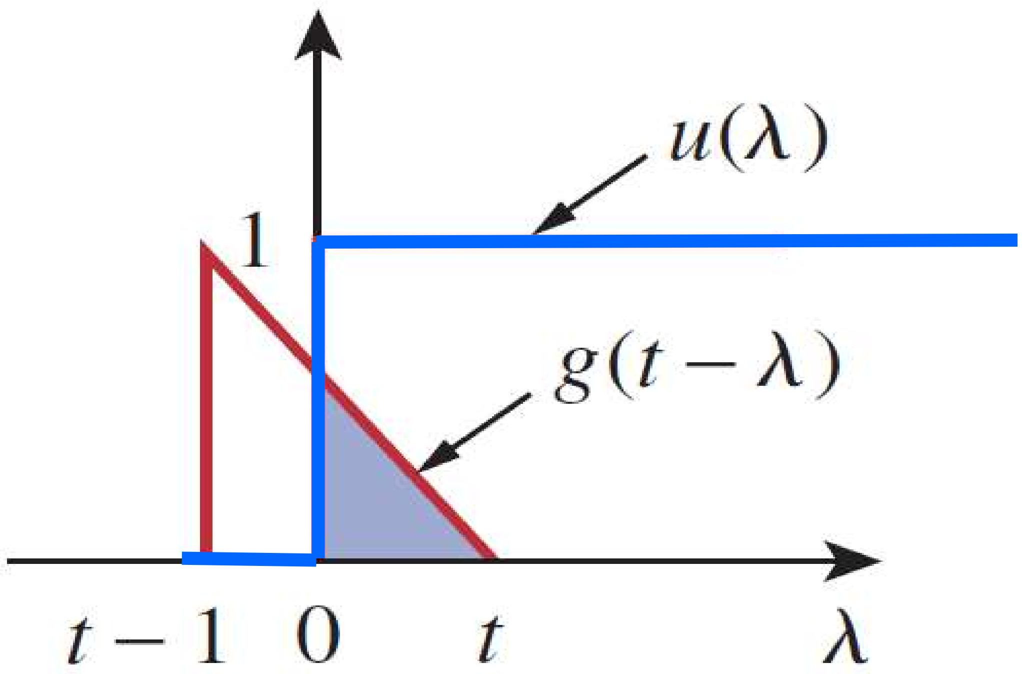


Graphical convolution – Example 2

- $\boldsymbol{For\ 0 < t < 1},$

$$y(t) = \int_0^t (1)(t-\lambda)\,d\lambda = \left(t\lambda - \frac{1}{2}\lambda^2\right)\Big|_0^t$$

$$= t^2 - \frac{t^2}{2} = \frac{t^2}{2}, \qquad 0 \le t \le 1$$

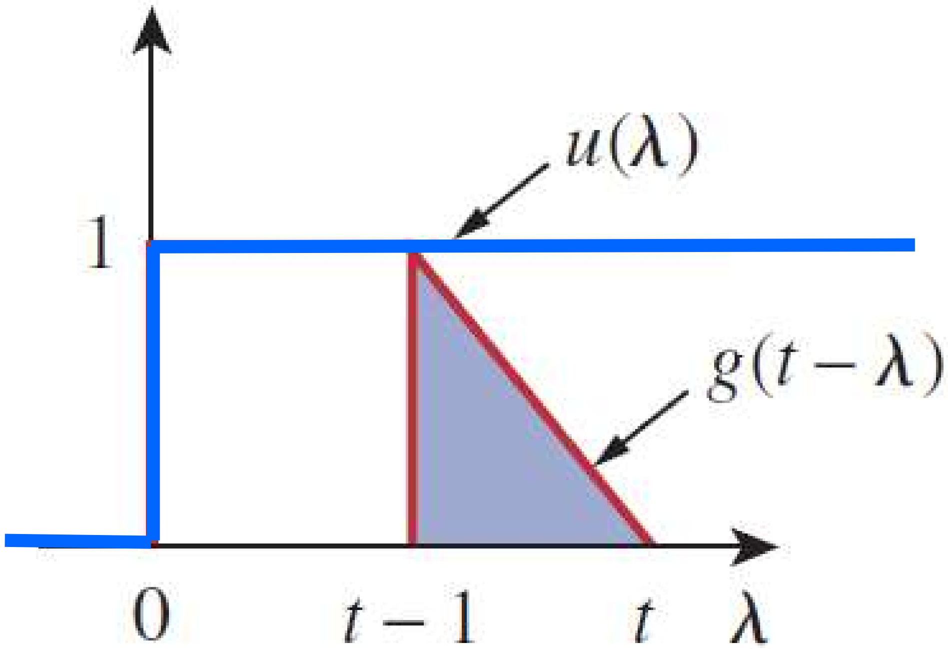


Graphical convolution – Example 2

- $\boldsymbol{For\ t > 1,}$

$$y(t) = \int_{t-1}^{t} (1)(t-\lambda)\,d\lambda$$

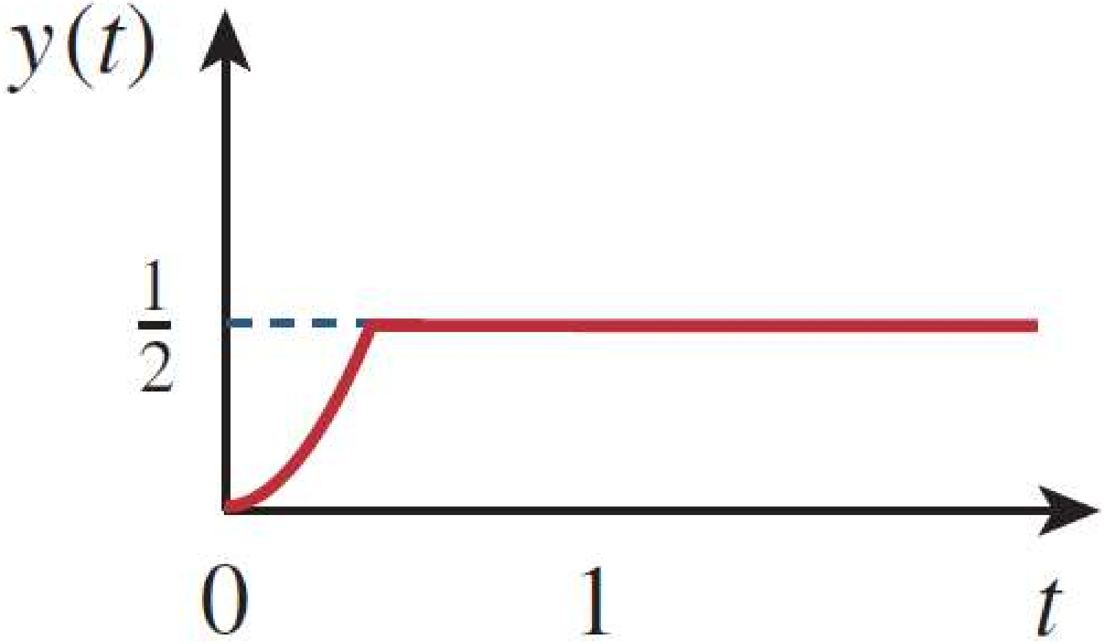
$$= \left(t\lambda - \frac{1}{2}\lambda^2\right)\Bigg|_{t-1}^{t} = \frac{1}{2}, \qquad t \geq 1$$



Graphical convolution – Example 2

- **Combining all the integrations for specified period,**

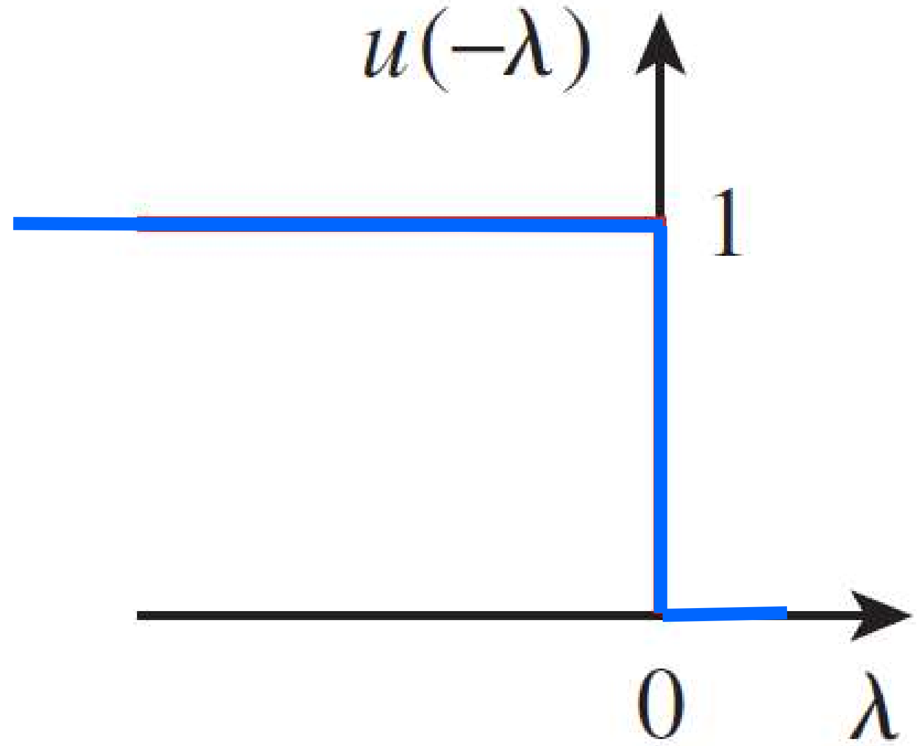
$$y(t) = \begin{cases} \frac{1}{2}t^2, & 0 \le t \le 1 \\ \frac{1}{2}, & t \ge 1 \end{cases}$$



Graphical convolution – Example 2

Method 2: Instead of folding $g(t)$, we fold u(t) and shift it to any *t*, since $u(t) = 1,\ for\ t > 0$, and

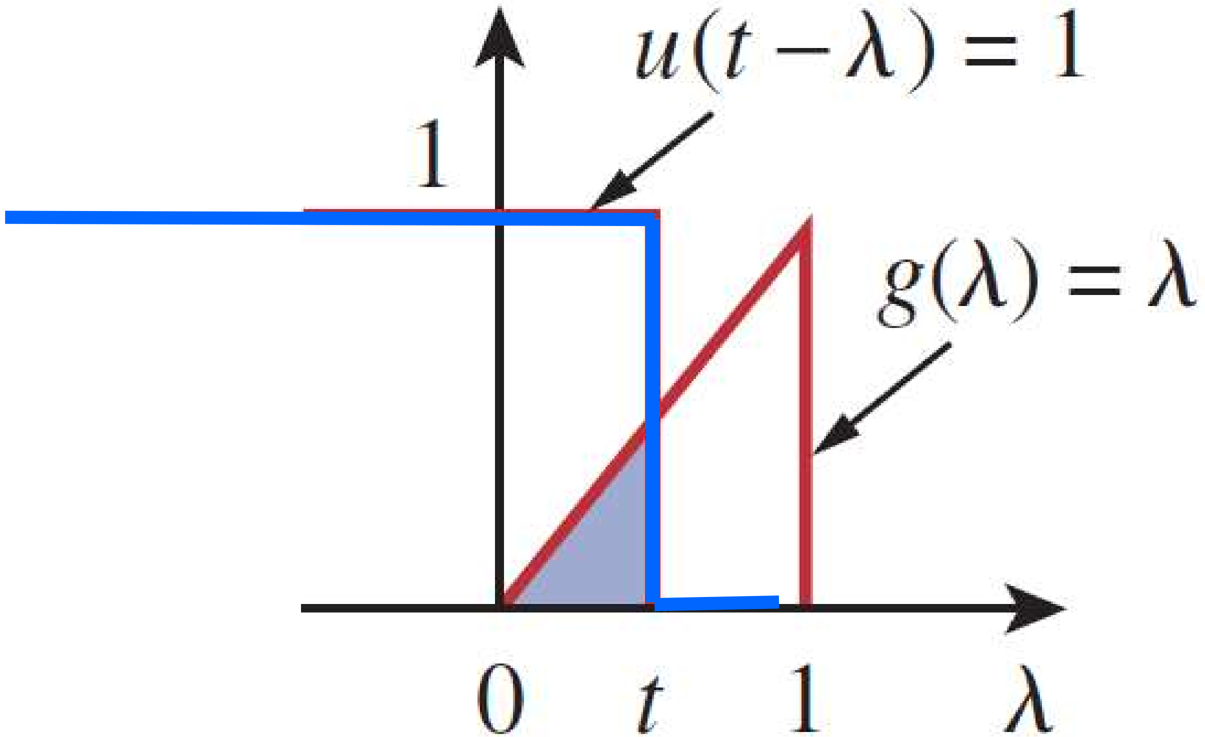
$$u(t - \lambda) = 1 \text{ for } t - \lambda > 0 \text{ or } \lambda < t$$



Graphical convolution – Example 2

➢ ***For*** $0 < t < 1$, **The function overlaps between 0 and *t***

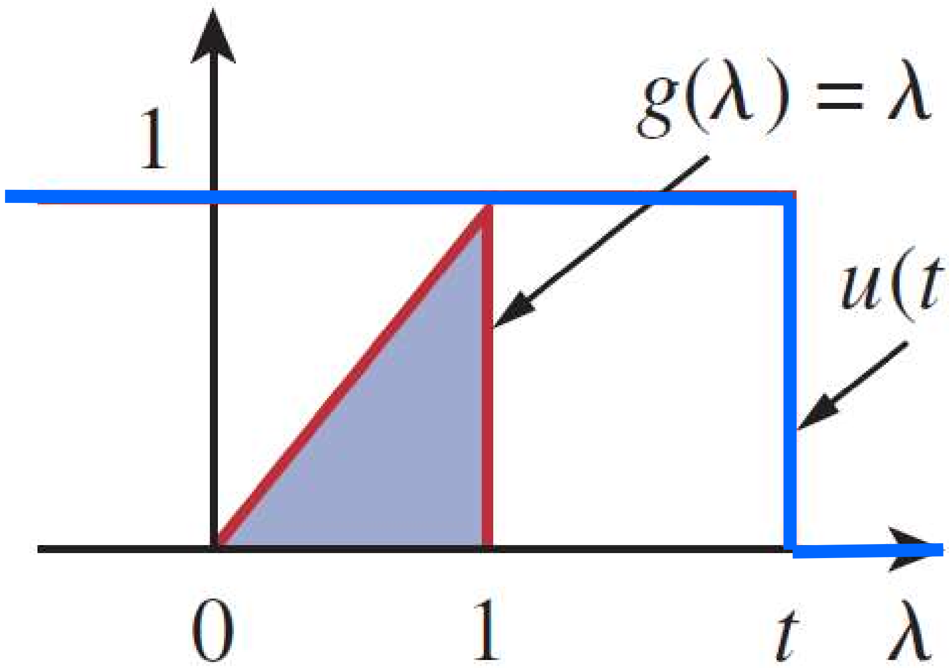
$$y(t) = \int_0^t (1)\lambda\, d\lambda = \frac{1}{2}\lambda^2 \Big|_0^t = \frac{t^2}{2}, \qquad 0 \le t \le 1$$



Graphical convolution – Example 2

- ***For*** $t > 1$, The function overlaps between 0 and *1*

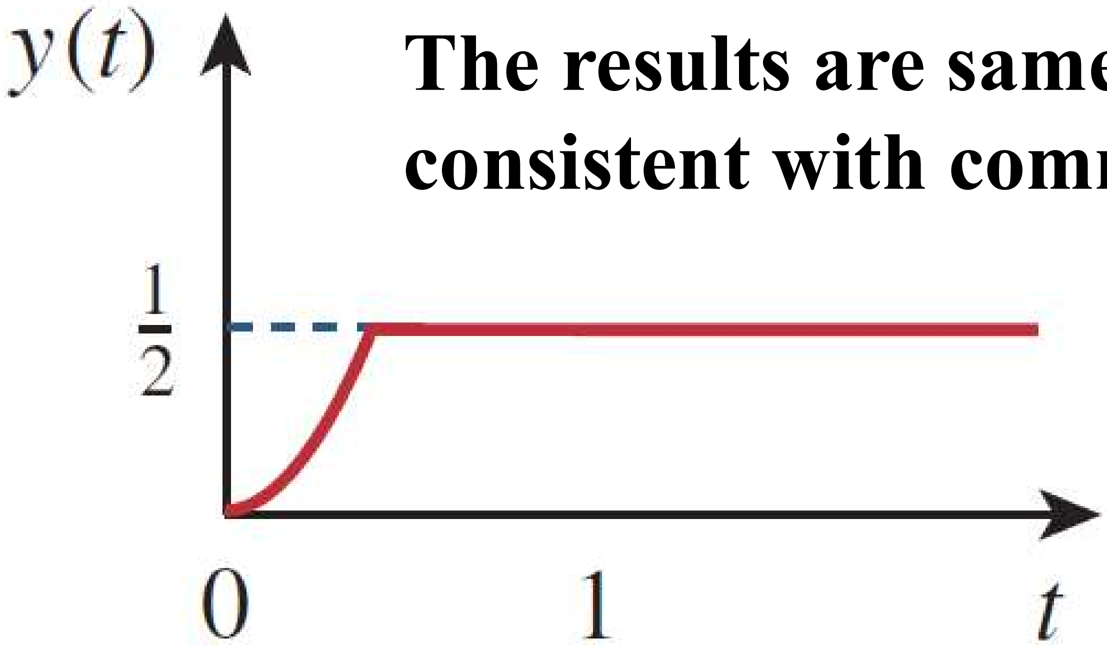
$$y(t) = \int_0^1 (1)\lambda \, d\lambda = \frac{1}{2}\lambda^2 \Big|_0^1 = \frac{1}{2}, \qquad t \geq 1$$



Graphical convolution – Example 2

- **Combining all the integrations for specified period,**

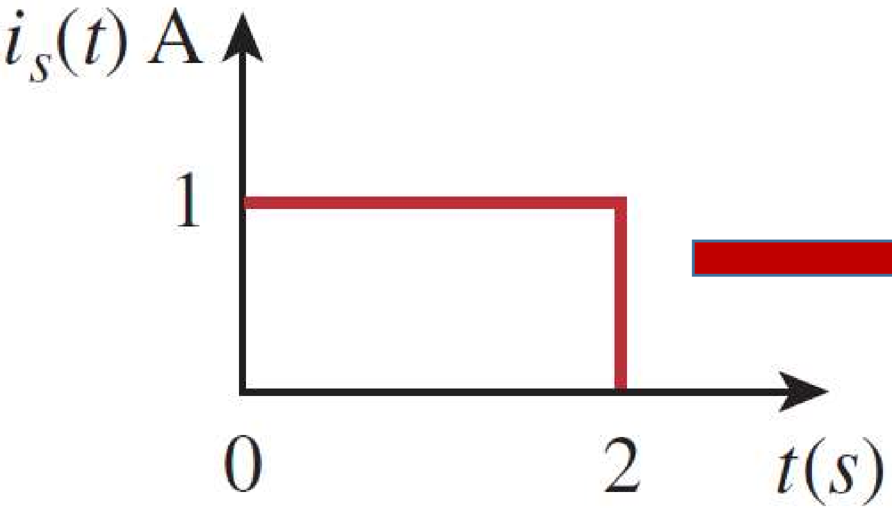
$$y(t) = \begin{cases} \frac{1}{2}t^2, & 0 \le t \le 1 \\ \frac{1}{2}, & t \ge 1 \end{cases}$$

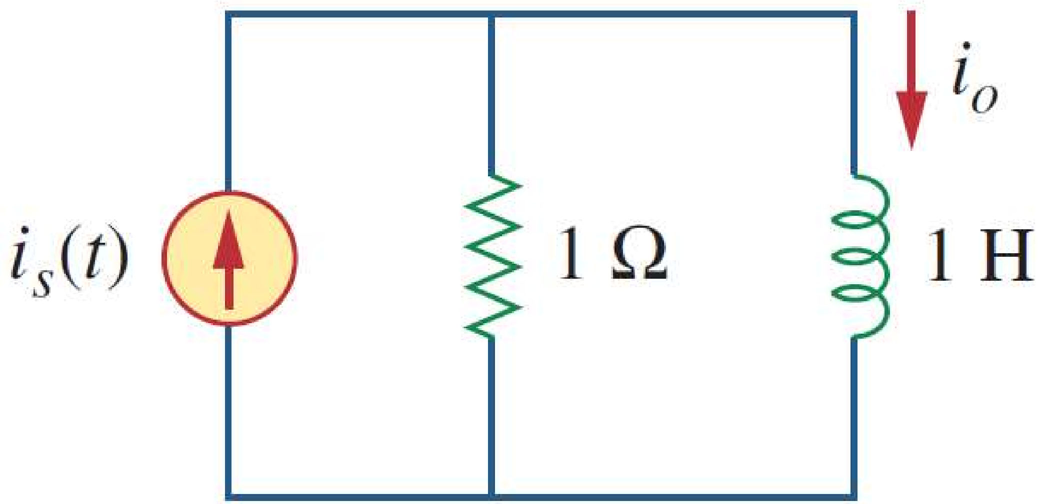


The results are same in both cases and consistent with commutation property

Graphical convolution – Example 3

Question: Use convolution and *s-plane*, to obtain $i_o(t)$ due to the excitation $i_s(t)$ in the circuit





Graphical convolution – Example 3

Solution: **We first need impulse response $h(t)$ of the circuit,**

- **Applying current division rule,**

$$I_o = \frac{1}{s+1} I_s$$

Hence,

$$H(s) = \frac{I_o}{I_s} = \frac{1}{s+1}$$

taking the inverse Laplace gives,

$$h(t) = e^{-t}u(t)$$

Graphical convolution – Example 3

To use the convolution integral directly, recall that the response is given in the *s*-domain as

$$I_o(s) = H(s)I_s(s)$$

with the given $i_s(t)$,

$$i_s(t) = u(t) - u(t-2)$$

so that,

$$i_o(t) = h(t) * i_s(t) = \int_0^t i_s(\lambda)h(t-\lambda)\,d\lambda$$

$$= \int_0^t [u(\lambda) - u(\lambda - 2)]e^{-(t-\lambda)}\,d\lambda$$

Graphical convolution – Example 3

- **The best way to handle the integral is to do the two parts separately, for $0 < t < 2$,**

$$i_o'(t) = \int_0^t (1)e^{-(t-\lambda)}\,d\lambda = e^{-t}\int_0^t (1)e^{\lambda}\,d\lambda$$

$$= e^{-t}(e^t - 1) = 1 - e^{-t}, \qquad 0 < t < 2$$

for $t > 2$,

$$i_o''(t) = \int_2^t (1)e^{-(t-\lambda)}\,d\lambda = e^{-t}\int_2^t e^{\lambda}\,d\lambda$$

$$= e^{-t}(e^t - e^2) = 1 - e^2 e^{-t}, \qquad t > 2$$

Graphical convolution – Example 3

Combining both the results,

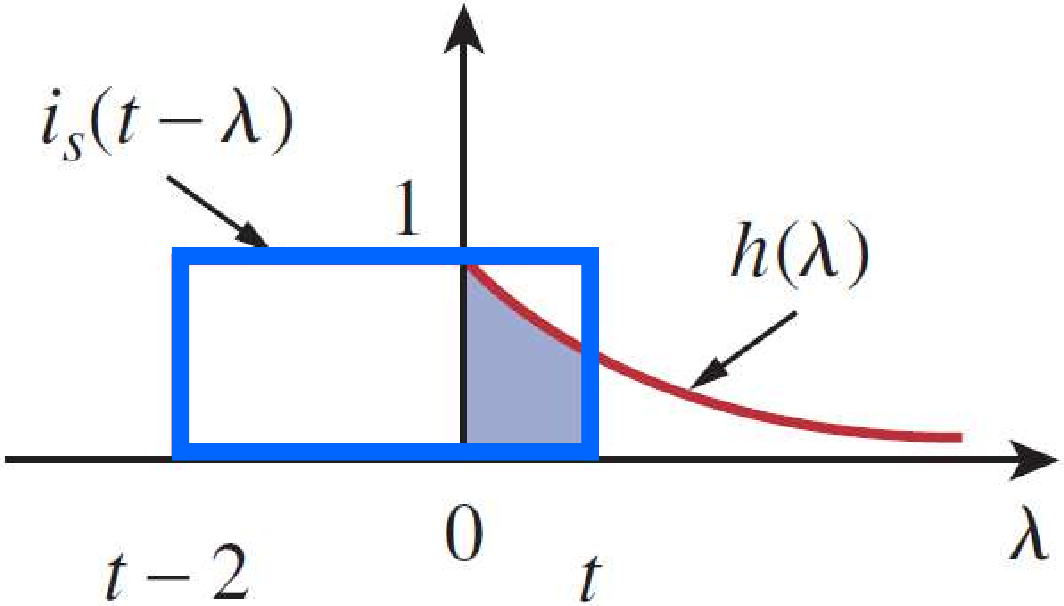
$$
\begin{aligned}
i_o(t) &= i_o'(t) - i_o''(t) \\
&= (1 - e^{-t})[u(t-2) - u(t)] \\
&\quad - (1 - e^2 e^{-t})u(t-2)
\end{aligned}
$$

$$
= \begin{cases} 1 - e^{-t}\ \text{A}, & 0 < t < 2 \\ (e^2 - 1)e^{-t}\ \text{A}, & t > 2 \end{cases}
$$

Graphical convolution – Example 3

- To use the graphical convolution, we may fold $\boldsymbol{i_s(t)}$ and shift by $\boldsymbol{t}$, the overlap between $\boldsymbol{i_s(t-\lambda)}$ and $\boldsymbol{h(\lambda)}$ is from 0 to $\boldsymbol{t}$,

$$i_o(t) = \int_0^t (1)e^{-\lambda}\, d\lambda = -e^{-\lambda}\Big|_0^t = (\mathbf{1} - \boldsymbol{e}^{-t})\,\mathrm{A}$$



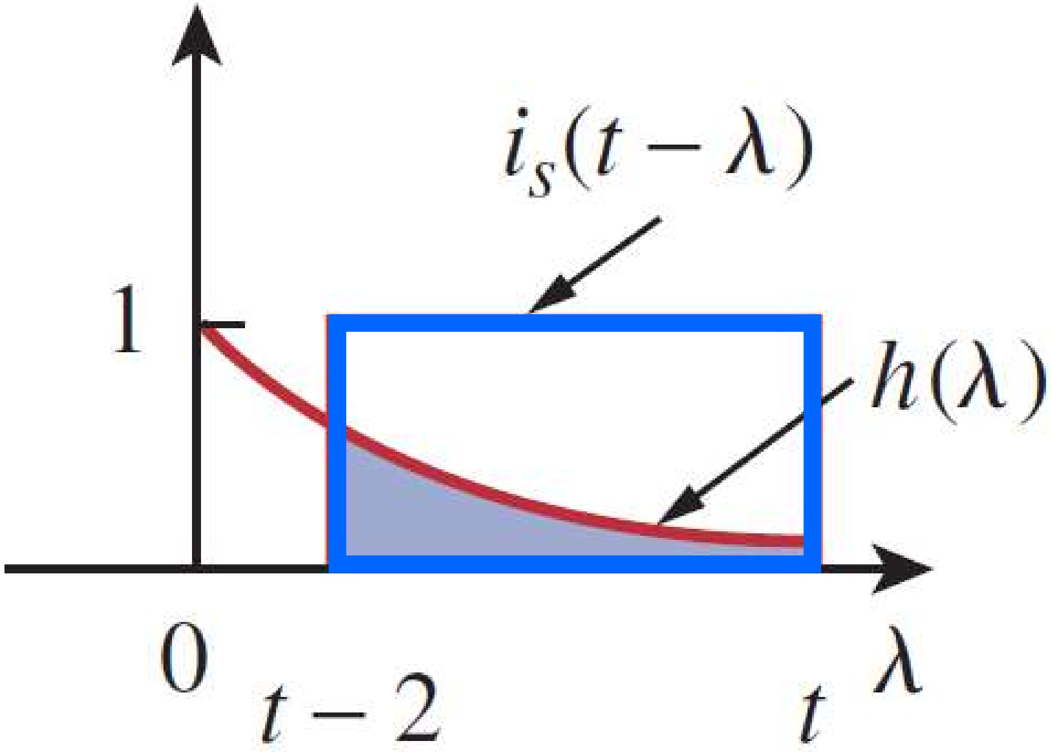
$$0 \le t \le 2$$

Graphical convolution – Example 3

➢ **For $t > 2$, the functions overlap between $(t - 2)$ and t,**

$$i_o(t) = \int_{t-2}^{t} (1)e^{-\lambda}\, d\lambda$$

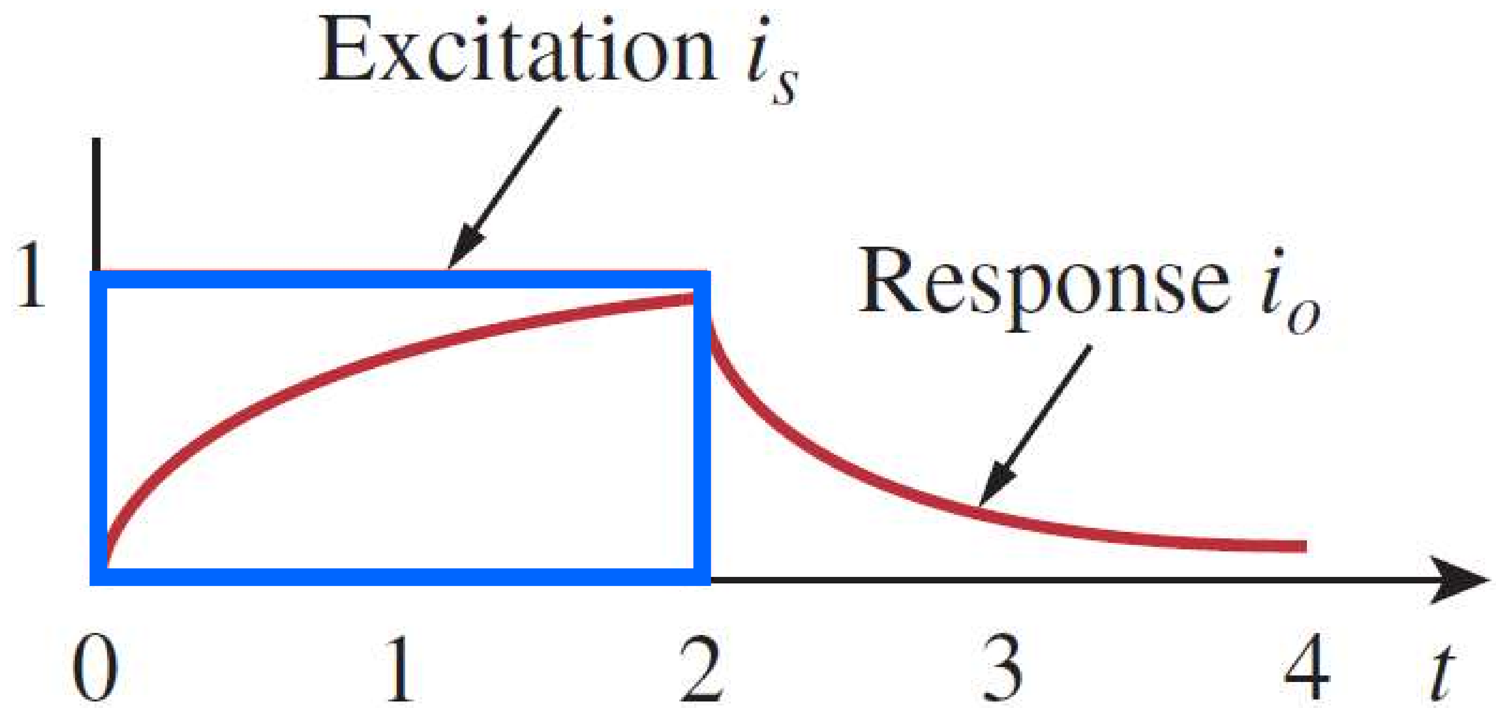
$$= (e^2 - 1)e^{-t}\ \mathrm{A}$$



Graphical convolution – Example 3

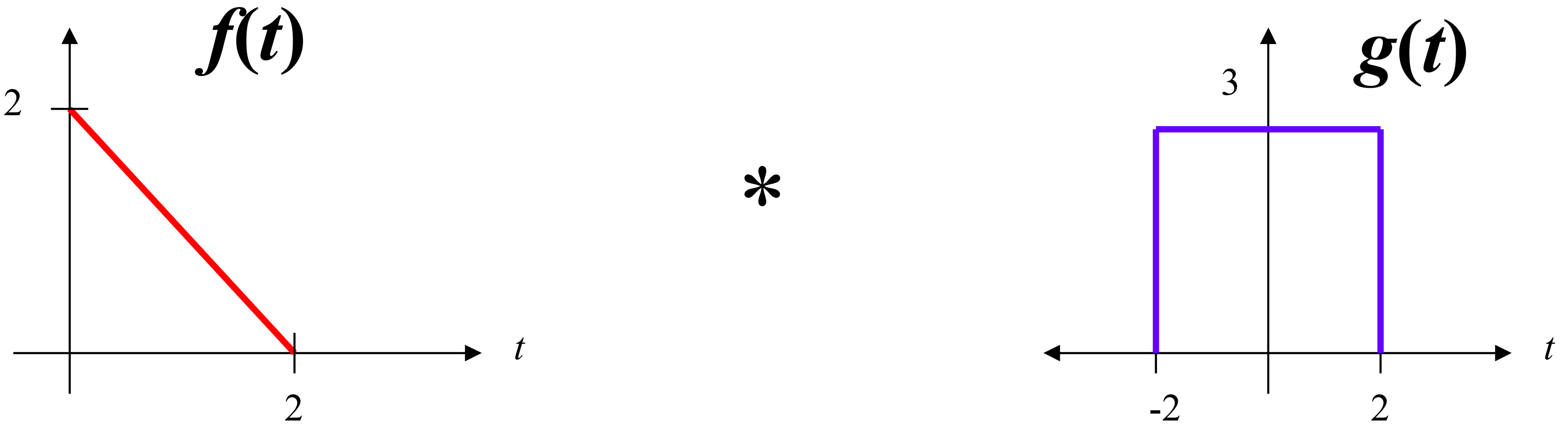
Combining both the results,

$$i_o(t) = \begin{cases} \mathbf{1 - e^{-t}\ A}, & 0 \le t \le 2 \\ \mathbf{(e^2 - 1)e^{-t}\ A}, & t \ge 2 \end{cases}$$

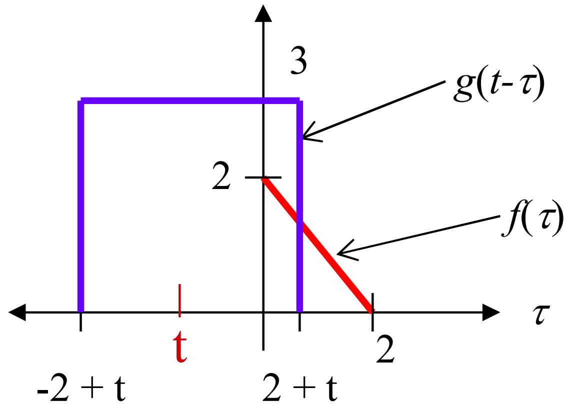


Graphical convolution – Example 4

Convolve the following two functions:



- **Replace t with τ in $f(t)$ and $g(t)$**
- **Choose to flip and slide $g(\tau)$ since it is simpler and symmetric**
- **Functions overlap like this:**

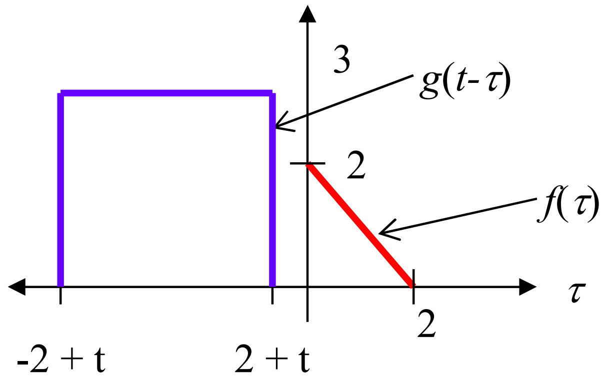


Graphical convolution – Example 4

- Convolution can be divided into 5 parts

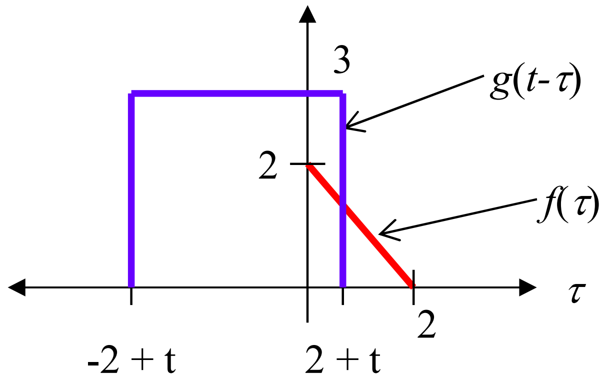
I. $t < -2$

- Two functions do not overlap
- Area under the product of the functions is zero



II. $-2 \le t < 0$

- Part of $g(t)$ overlaps part of $f(t)$
- Area under the product of the functions is



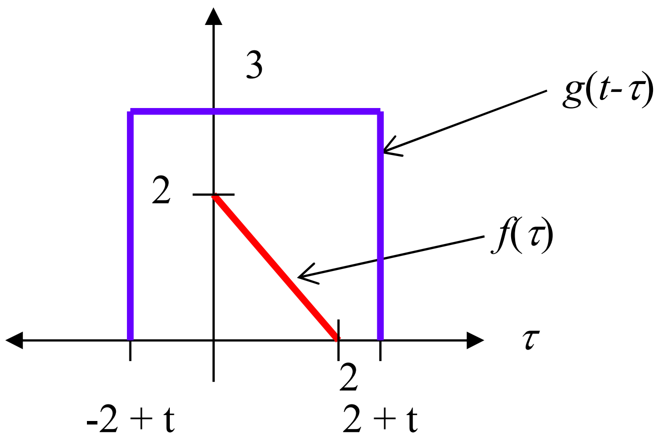
$$\int_0^{2+t} 3(-\tau+2)\,d\tau = 3\left(-\frac{\tau^2}{2}+2\tau\right)\Bigg|_0^{2+t} = -\frac{3(2+t)^2}{2}+6(2+t) = -\frac{3t^2}{2}+6$$

Graphical convolution – Example 4

III. $0 \le t < 2$

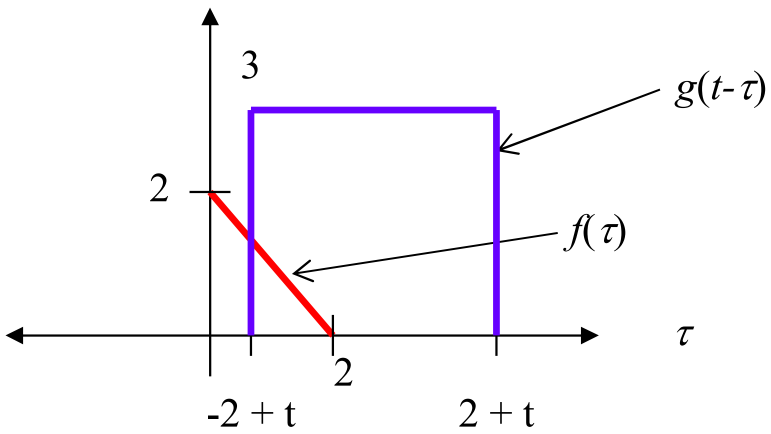
- Here, $g(t)$ completely overlaps $f(t)$
- Area under the product is just

$$\int_0^2 3(-\tau + 2)\,d\tau = 3\left(-\frac{\tau^2}{2} + 2\tau\right)\Bigg|_0^2 = 6$$



IV. $2 \le t < 4$

- Part of $g(t)$ and $f(t)$ overlap
- Calculated similarly to $-2 \le t < 0$



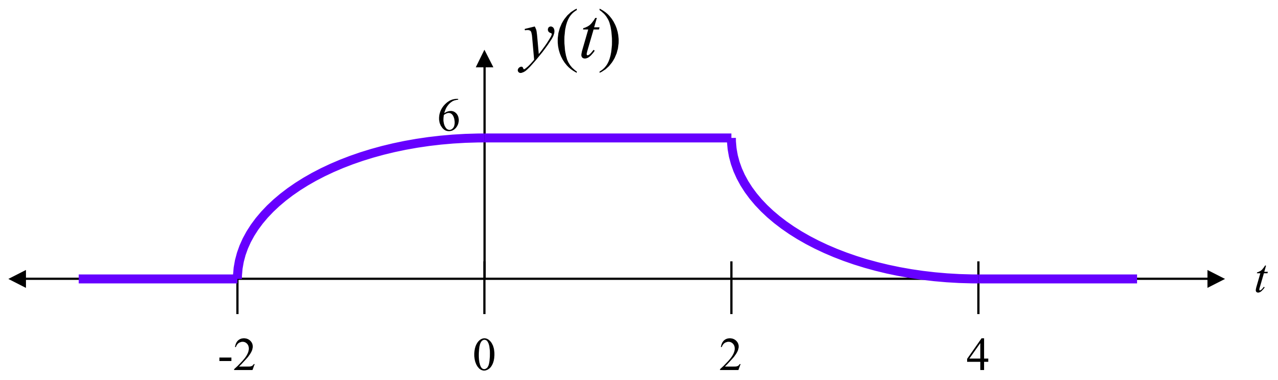
V. $t \ge 4$

- $g(t)$ and $f(t)$ do not overlap
- Area under their product is zero

Graphical convolution – Example 4

Result of convolution (5 intervals of interest):

$$y(t) = f(t) * g(t) = \begin{cases} 0 & \text{for } t < -2 \\ -\dfrac{3}{2}t^2 + 6 & \text{for } -2 \le t < 0 \\ 6 & \text{for } 0 \le t < 2 \\ \dfrac{3}{2}t^2 - 12t + 24 & \text{for } 2 \le t < 4 \\ 0 & \text{for } t \ge 4 \end{cases}$$



Objectives

- Graphical Convolution
- Examples of graphical convolution
- Impulse response
- Properties of impulse response

Impulse $\delta(t)$

- **We know that convolution is similar to multiplication**

- **In multiplication, there is a multiplicative identity "*one*" exist such that $\alpha * 1 = \alpha$**

- **We may be interested in a *signal* instead of a number having analogous behavior satisfying,**

$$p(t) * f(t) = f(t)$$

- **There is no signal that can exactly fulfill above stated condition, rather, we may use a good approximation**

Impulse $\delta(t)$

- **Consider the rectangular pulse signal given by,**

$$p_\epsilon(t) = \frac{1}{\epsilon} rect\left(\frac{t}{\epsilon}\right)$$

This pulse has unit area and very small ϵ, so that it is very narrow and tall

- **Writing the limit when $\epsilon \to 0$, it can be shown that,**

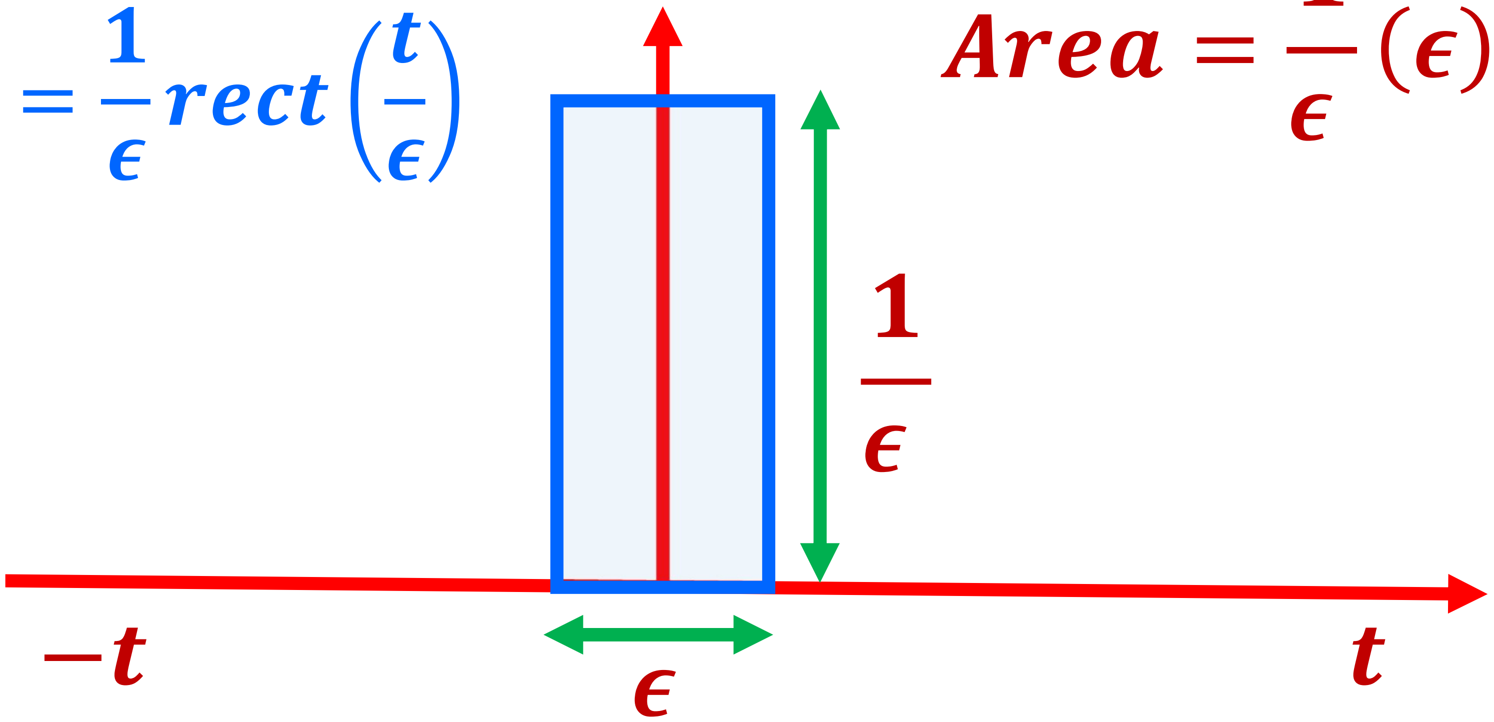
$$\lim_{\epsilon \to 0} \{p_\epsilon(t) * f(t)\} = f(t)$$

we can draw this waveform taking some small value of ϵ: *small enough to be drawn*

Impulse $\delta(t)$

$$p_\epsilon(t) = \frac{1}{\epsilon} rect\left(\frac{t}{\epsilon}\right)$$

$$Area = \frac{1}{\epsilon}(\epsilon) = 1$$



Identity Pulse

You can observe that if we decrease the value of ϵ, the waveform get taller i.e. approaching to ∞ when $\epsilon \to 0$

Impulse $\delta(t)$

If we replace the limiting function with a special signal -$\delta(t)$ such that

$$\boxed{\lim_{\epsilon\to 0} \{ p_\epsilon(t)} * f(t)\} = f(t)$$

$$\delta(t) * f(t) = f(t)$$

- **The properties* of impulse signal are related to response when one of the two signals is always impulse function, making it identity as**

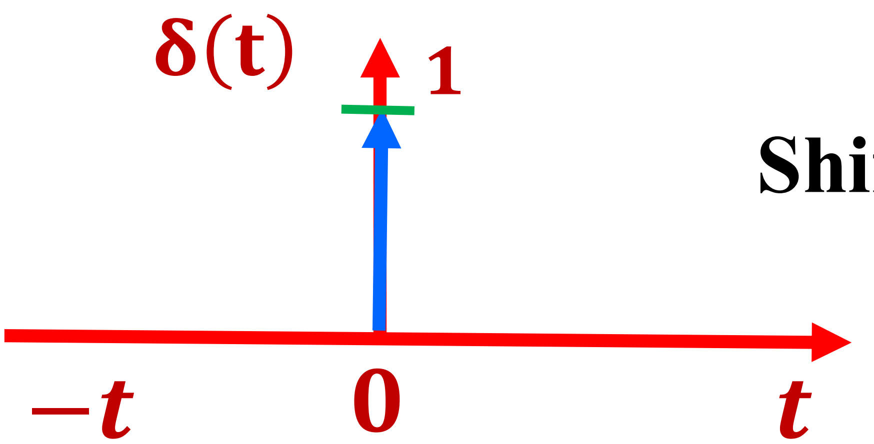
$$\delta(t) \leftrightarrow 1$$

***The properties of $\delta(t)$ are listed in the table 9.3 (Text book)**

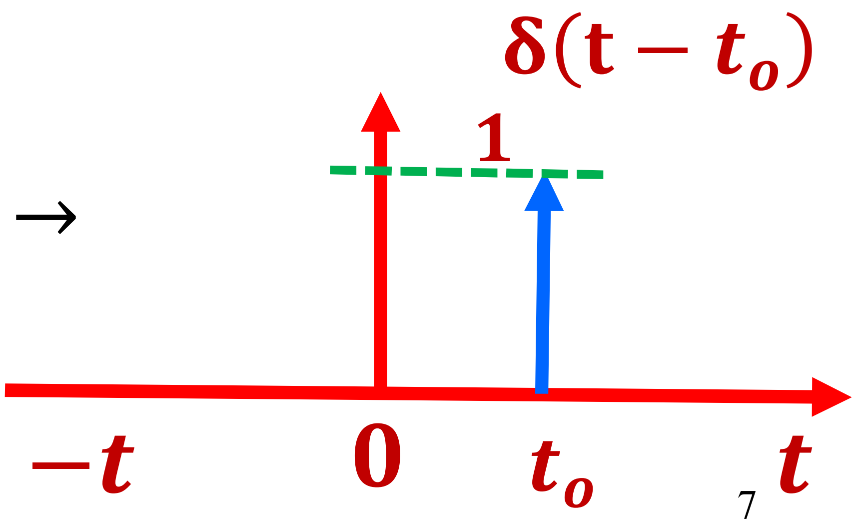
Impulse $\delta(t)$

- This property of Fourier transform of impulse is result of applying Fourier time-convolution property to identity $\delta(t) * f(t) = f(t)$
- Given $\delta(t) \leftrightarrow 1$, the IFT of 1 must be,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\, d\omega$$



Shifting →



Impulse $\delta(t)$

- **One important property of impulse called *sifting property* can be written as,**

$$\int_{-\infty}^{\infty} \delta(t - t_o) f(t) dt = f(t_o)$$

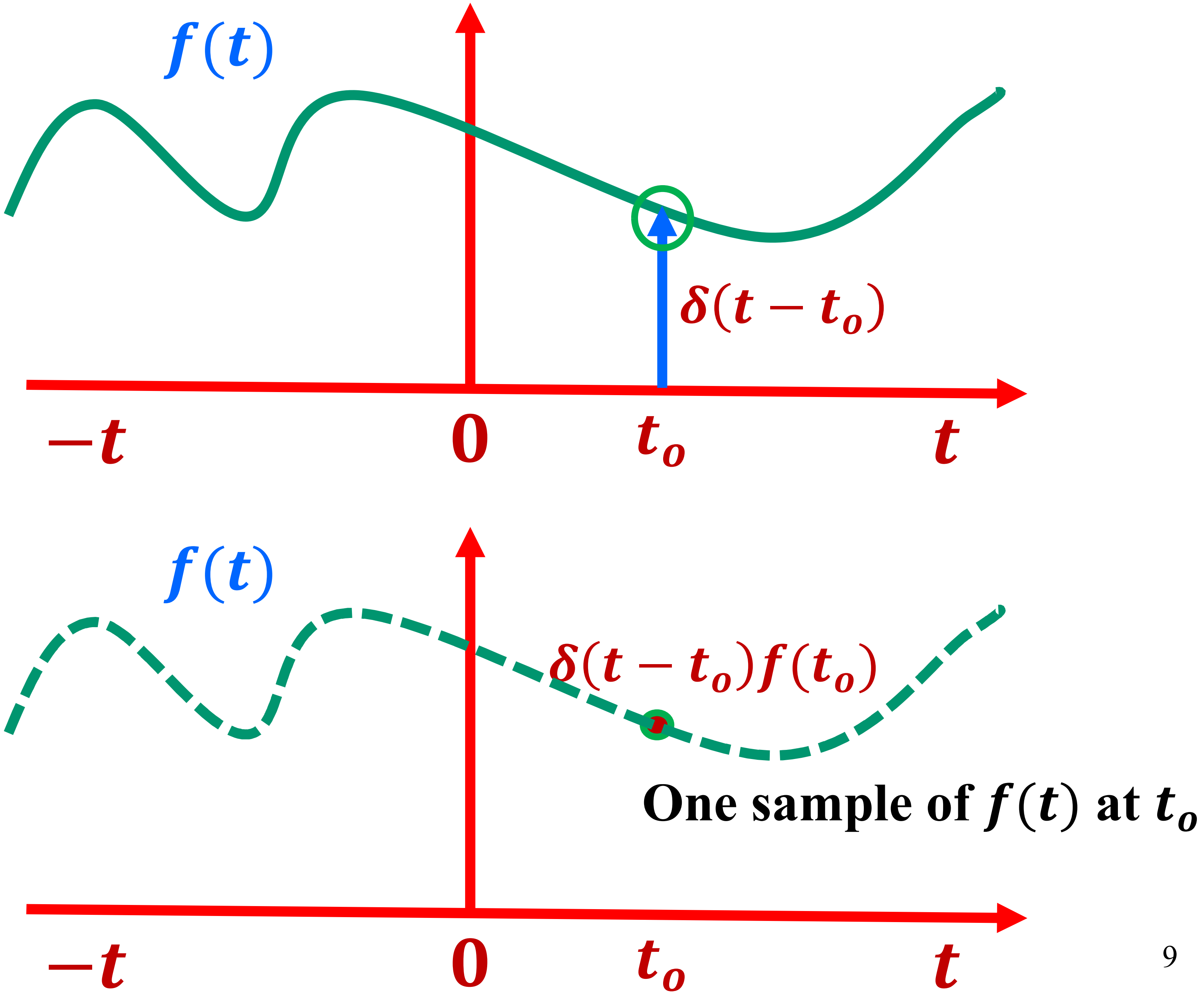
results into *sampling property,* if

$$\lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{\infty} p_\epsilon(t - t_o) f(t) dt \right\} = f(t_o)$$

or,

$$\delta(t - t_o) f(t) = \delta(t - t_o) f(t_o)$$

Impulse $\delta(t)$ – Sampling



Impulse $\delta(t)$ – Example 1

Question: **Using Fourier transform property of impulse, $\delta(t) \leftrightarrow 1$, determine**

$$c(t) = a(t) * b(t)$$

if

$$a(t) = u(t)$$

and

$$B(\omega) = 1 - \frac{1}{1 + j\omega}$$

Impulse $\delta(t)$ – Example 1

Solution: Using the fact that $\delta(t) \leftrightarrow 1$, and

$$e^{-t}u(t) \leftrightarrow \frac{1}{1+j\omega}$$

we have,

$$b(t) = \delta(t) - e^{-t}u(t)$$

Thus,

$$c(t) = a(t) * b(t) = u(t) * \delta(t) - u(t) * e^{-t}u(t)$$

$$c(t) = u(t) - u(t)(1 - e^{-t})$$

$$c(t) = e^{-t}u(t)$$

Impulse $\delta(t)$ – Example 2

Question: Given that $f(t) = \delta(t - t_o)$, determine the energy W_f of the signal $f(t)$?

Solution: Since $\delta(t - t_o) \leftrightarrow e^{-j\omega t_o}$, therefore the energy spectrum of the signal is $|F(\omega)|^2 = 1$, Using Rayleigh theorem,

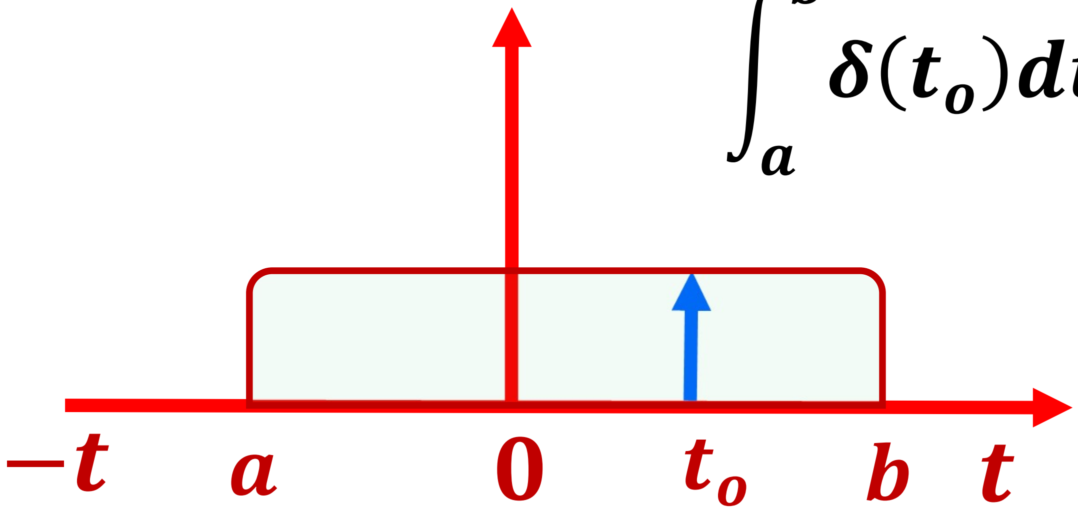
$$W_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \infty$$

Since it is infinite signal, so not an energy signal (i.e. cannot be generated in lab)

Impulse $\delta(t)$ – Limits

- **It is not necessary to take $[-\infty, \infty]$ as limits of integration to calculate impulse response for each time**

- **If you know the location of impulse, then it is sufficient to include a *finite integration window* to see impulse response**



$$\int_{a}^{b} \delta(t_o)\,dt = 1$$

$$\int_{-\infty}^{a} \delta(t_o)\,dt = 0$$

$$\int_{b}^{\infty} \delta(t_o)\,dt = 0$$

Impulse $\delta(t)$ – Example 3

Question: Find the derivative of function?

$$y(t) = t^2 u(t)$$

Solution: Due to unit step function, $\frac{dy}{dt} = 0$ for $t < 0$, and $\frac{dy}{dt} = 2t$ for $t > 0$, combining both results,

$$\frac{dy}{dt} = 2t\, u(t)$$

Alternatively, using the *product rule of differentiation* and properties of impulse, we obtain

$$\frac{dy}{dt} = \frac{d}{dt}\left(t^2 u(t)\right) = 2t\, u(t) + t^2 \frac{du}{dt}$$

$$= 2t\, u(t) + t^2 \delta(t) = 2t\, u(t) + 0^2 \delta(t) = 2t\, u(t)$$

Impulse $\delta(t)$ – Example 3

Question: Find the derivative of function?

$$z(t) = e^{2t}u(t)$$

Solution: Due to unit step function, $\frac{dz}{dt} = 0$ **for** $t < 0$**, and** $\frac{dz}{dt} = 2e^{2t}$ **for** $t > 0$ **, combining both results,**

$$\frac{dz}{dt} = 2e^{2t}u(t)$$

is wrong due to discontinuity of $z(t)$ **at** $t = 0$**, where its derivative is undefined**

- **As a result,** $\frac{dz}{dt}$ **is not the function** $2e^{2t}u(t)$

Impulse $\delta(t)$ – Example 3

- The integral of $2e^{2t}u(t)$ from $[-\infty, t]$, over τ, does not lead to $z(t) = e^{2t}u(t)$ as it should if $2e^{2t}u(t)$ were the correct derivative

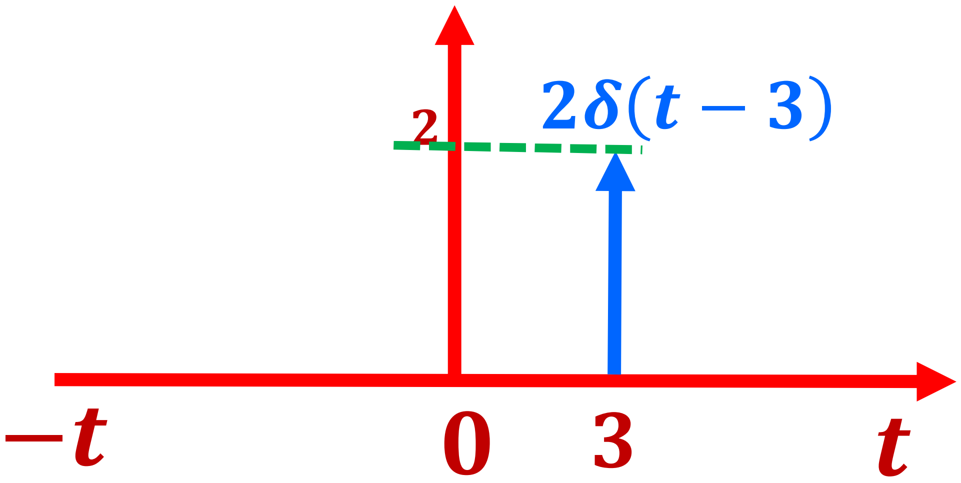
- However, we have to use the *product rule of differentiation* and properties of impulse,

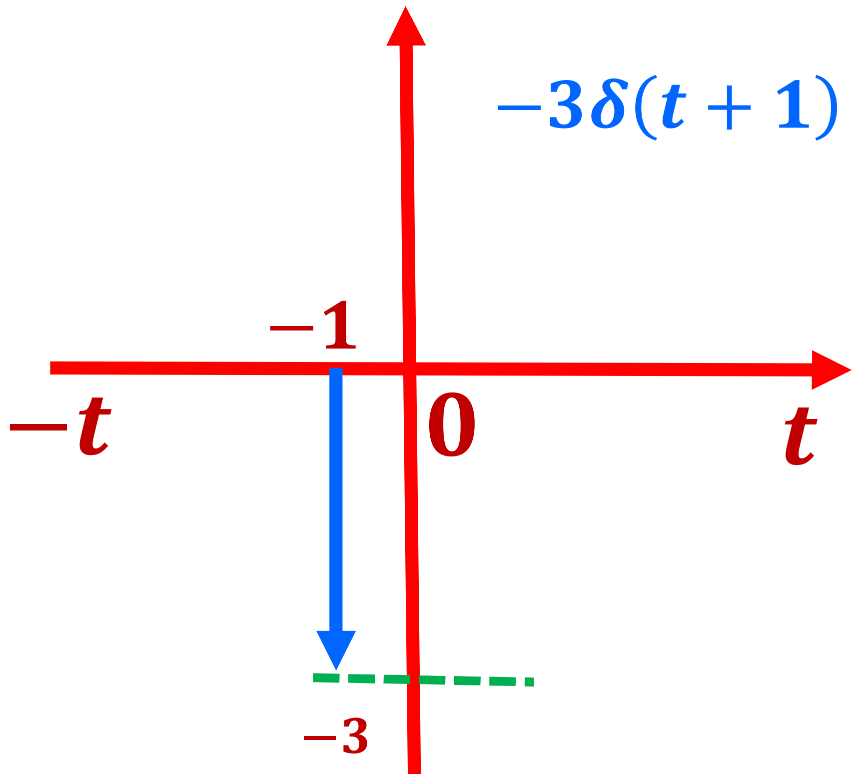
$$\frac{dz}{dt} = \frac{d}{dt}\left(e^{2t}u(t)\right) = 2e^{2t}\,u(t) + e^{2t}\frac{du}{dt}$$

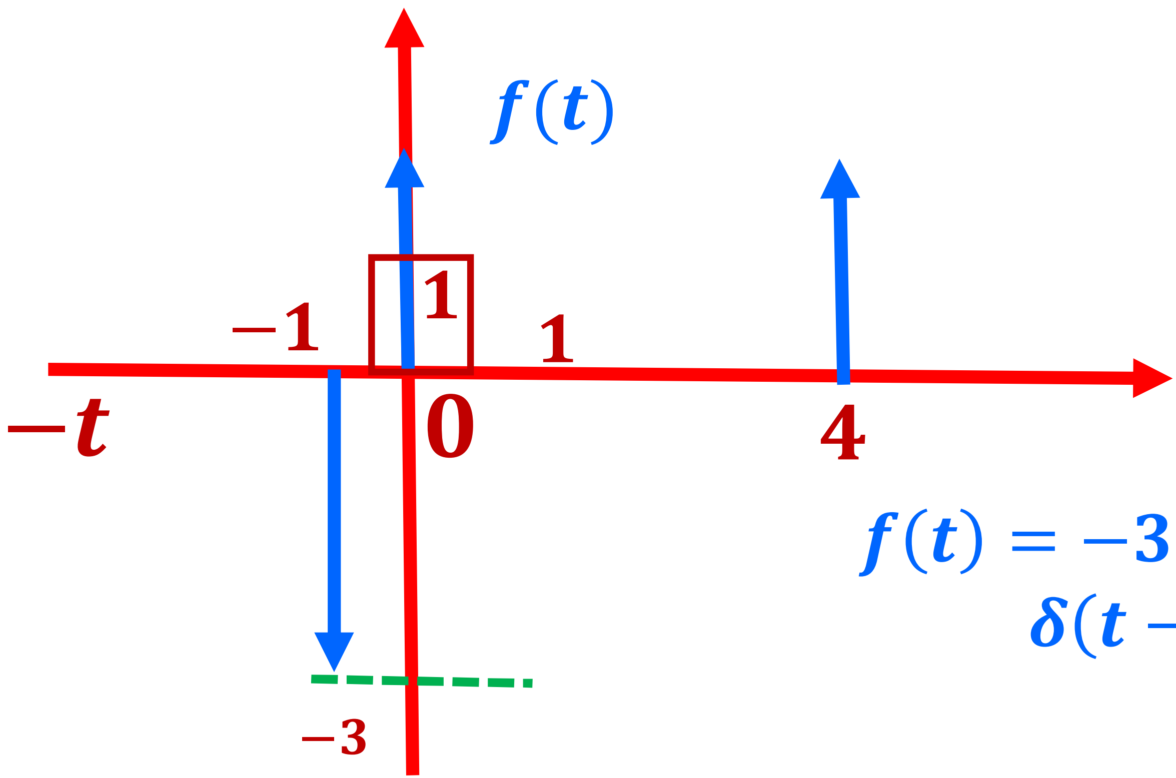
$$= 2e^{2t}\,u(t) + e^{2t}\delta(t) = 2e^{2t}\,u(t) + e^{0}\delta(t)$$

$= 2e^{2t}\,u(t) + \delta(t)$ is the right answer

Impulse $\delta(t)$ – Graphical representation







$$f(t) = -3\delta(t+1) + 2\delta(t) + \delta(t-4) + \mathrm{rect(t)}$$

Impulse $\delta(t)$ – Example 4

Question: **Find the Fourier transform of $rect(t)$,using Fourier time derivative property and the fact that,**

$$\delta(t - t_o) \leftrightarrow e^{-j\omega t_0}$$

Solution: **We know that,**

$$rect(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$

$$\frac{d}{dt}(rect(t)) = \frac{d}{dt}\left[u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)\right]$$

Impulse $\delta(t)$ – Example 4

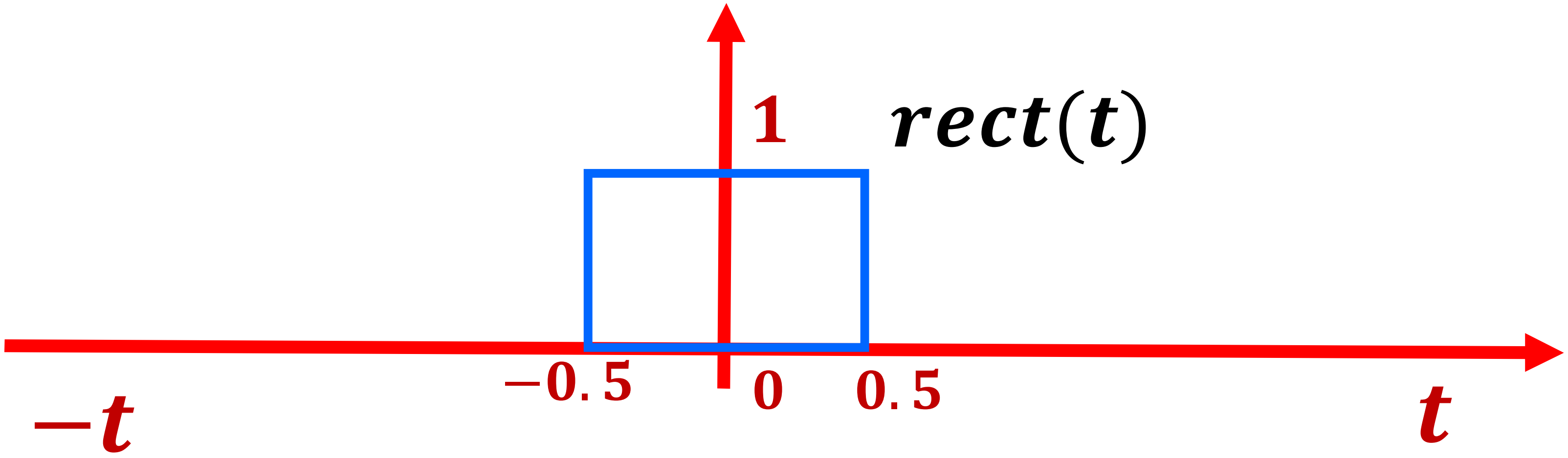
$$= \delta\left(t + \frac{1}{2}\right) - \delta\left(t - \frac{1}{2}\right) \leftrightarrow e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}$$

Since, by Fourier time derivative property , the Fourier transform of $\frac{d}{dt} rect(t)$ is $j\omega$ times the Fourier transform of $rect(t)$

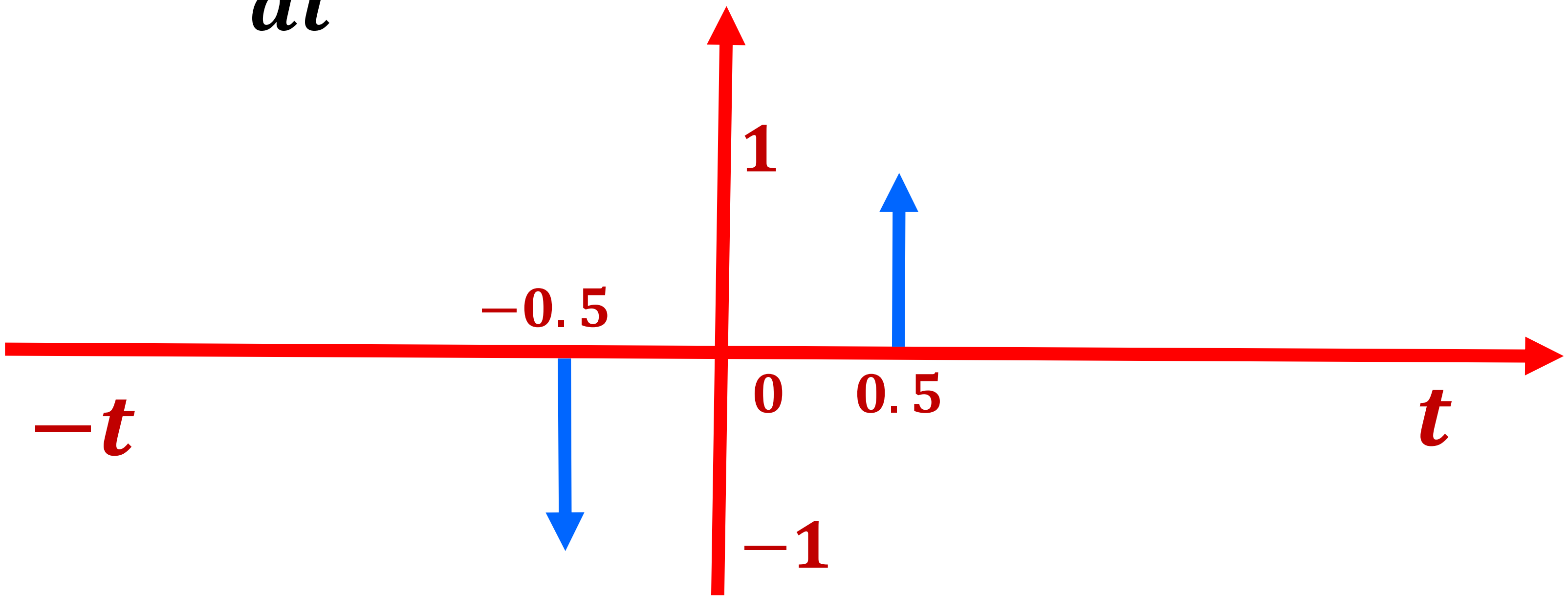
$$rect(t) \leftrightarrow \frac{1}{j\omega}\left(e^{0.5j\omega} - e^{-0.5j\omega}\right) = \frac{e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}}{j2.\frac{\omega}{2}}$$

$$= \frac{sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} = sinc\left(\frac{\omega}{2}\right)$$

Impulse $\delta(t)$ – Example 4



$$\frac{d}{dt}rect(t) = \delta(t+0.5) - \delta(t-0.5)$$



Objectives

- Graphical Convolution
- Examples of graphical convolution
- Impulse response
- **Properties of impulse response**

Impulse response $\delta(t)$ – Example 5

Question: **Suppose that a high pass filter with frequency response,**

$$H(\omega) = \frac{j\omega}{1 + j\omega}$$

has input,

$$f(t) = rect(t)$$

Determine the zero state response $y(t)$, using the system impulse response $h(t)$ and convolution?

Impulse response $\delta(t)$ – Example 5

Solution: There is no direct transform pair for $\frac{j\omega}{1+j\omega}$,

However,

$$\frac{j\omega}{1+j\omega} = \frac{j\omega+1-1}{1+j\omega} = 1 - \frac{1}{1+j\omega}$$

therefore, using the matching pairs from table*, we conclude that,

$$h(t) = \delta(t) - e^{-t}u(t)$$

using $y(t) = h(t) * f(t)$, and the convolution property of impulse response, we obtain

$$y(t) = \left(\delta(t) - e^{-t}u(t)\right) * rect(t)$$

$$y(t) = rect(t) - e^{-t}u(t) * rect(t)$$

Summary

- **The convolution of two signals consists of following steps: *time-reversing of signal , shifting it, multiplying it point by point with the second signal*, and *integrating the product***

- **Convolution simplifies the solution *once* you know the unit impulse response for any input function**

- **Graphical convolution further simplifies the solution by using integrating discrete overlapped areas**

Summary

- The unit impulse is a signal having unit area defined at any time *t*

- Multiplication of any signal with impulse results into same signal defined at discrete location of impulse

- We can find zero state response of an LTI system using impulse response

Further reading

1. Ch. 9 (page 289-314), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.
2. Ch.15 (page 698-720), Charles K. Alexander & Sadiku, ***Fundamentals of electric circuits***, 5th ed., McGraw-Hill, 2013.
3. Ch. 15 (page 782-785), J. D. Irwin, and R. M. Nelms. ***Basic Engineering Circuit Analysis,*** 10th ed. Wiley, 2011.

Preview:

1. Ch. 9 (page 314-325), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.

Homework 10

Deadline: 10:00 PM, 4th May, 2022

Thank you!