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ECE-210 Analog Signal Processing Spring 2022
Homework #13: Solution

1. Consider the following zero-state input-output relations for a variety of systems. In each case, determine whether the system is zero-state linear, time invariant, and causal.

   (a) $y(t) = f(t-1) + f(t+1)$.

   (b) $y(t) = 5f(t) * u(t)$.

   (c) $y(t) = \int_{-\infty}^{t-2} f(\tau)d\tau$.

   (d) $y(t) = \delta(t-4) * f(t) - \int_{-\infty}^{t+2} f(\tau)d\tau$.

   (e) $y(t) = f(t^2)$.

   Solution:

   (a) Proving linearity: Let the input be $f(t) = af_1(t) + bf_2(t)$. Then, following the input-output relation given, the output is

   $y(t) = f(t-1) + f(t+1) = af_1(t-1) + bf_2(t-1) + af_1(t+1) + bf_2(t+1)$

   $= a(f_1(t-1) + f_1(t+1)) + b(f_2(t-1) + f_2(t+1)) = ay_1(t) + by_2(t)$,

   where $y_1(t)$ and $y_2(t)$ are the outputs of inputs $f_1(t)$ and $f_2(t)$ respectively. Consequently, the system is linear.

   Proving if time-invariant: Let the input be $f_1(t) = f(t-t_0)$. Then the output is

   $y(t) = f_1(t-1) + f_1(t+1) = f(t-1-t_0) + f(t+1-t_0) = y(t-t_0)$.

   Therefore, the system is time-invariant.

   We recognize that the output y(t) depends on future values of the input f(t + 1). Hence, the system is noncausal.

   (b) Since the output is a convolution between the input and a system impulse response, the system is lineartime-invariant (LTI).

   Since the output $y(t)$ does not depend on future values of the input $f(t)$, the system is causal. (The system impulse response $h(t) = 5u(t)$ is right-sided.)

   (c) Since we can express the output as a convolution

   $y(t) = \int_{-\infty}^{t-2} f(\tau)d\tau = f(\tau) * u(t+2)$,

   the system is LTI.

   The output $y(t)$ does not depend on future values of the input $f(t)$, since $y(t) = \int_{-\infty}^{t-2} f(\tau)d\tau$ integrates $f(t)$ until $t-2$. So the system is causal.

   (d) Proving linearity: Let the input be $f(t) = af_1(t) + bf_2(t)$. Then, following the input-output relation given, the output is

   $y(t) = \delta(t-4) * (af_1(t) + bf_2(t)) - \int_{-\infty}^{t+2}(af_1(t) + bf_2(t))d\tau$

   $= a\delta(t-4) * f_1(t) + b\delta(t-4) * f_2(t)) - a\int_{-\infty}^{t+2} f_1(\tau)d\tau + b\int_{-\infty}^{t+2} f_2(\tau)d\tau = ay_1(t) + by_2(t)$,

   where $y_1(t)$ and $y_2(t)$ are the outputs of inputs $f_1(t)$ and $f_2(t)$ respectively. Consequently, the system is linear..

   Proving if time-invariant: Let the input be $f_1(t) = f(t-t_0)$. Then the output is

   $y_1(t) = \delta(t-4) * f_1(t) - \int_{-\infty}^{t+2} f_1(t)d\tau = \delta(t-4) * f(t-t_0) - \int_{-\infty}^{t+2} f(\tau - t_0)d\tau$

   $= [\delta(t-4) * f(t-t_0)]|_{t=t-t_0} - \int_{-\infty}^{t+2-t_0} f(\tau)d\tau = y(t-t_0)$

   Therefore , the system is time-invariant..

   The output $y(t)$ does not depend on future values of the input $f(t)$, since $y(t) = \int_{-\infty}^{t+2} f(\tau)d\tau$ integrates $f(t)$ until $t+2$. So the system is not causal.

(e) Proving linearity: Let the input be $af_1(t) + bf_2(t)$. Then the output is

$$\begin{aligned} y_{12}(t) &= af_1(\tau^2) + bf_2(\tau^2)d \\ &= ay_1(t) + by_2(t). \end{aligned}$$

Clearly the system is linear.
Proving if time-invariant: Let the input be $f_1(t) = f(t - t_0)$, then the output is

$$y_1(t) = f(\tau^2) = f((\tau - t_0)^2).$$

Clearly this is different than

$$y(t - t_0) = f(\tau^2 - t_0) \neq y_1(t).$$

Therefore, the system is time-varying.
Since the output $y(t)$ does depend on future values of the input $f(t)$, the system is not causal.

2. Find the impulse responses $h(t)$ of the LTI systems having the following unit-step responses.

(a) $g(t) = 3u(t - 3)$.
(b) $g(t) = t^2 u(t)$.
(c) $g(t) = (2 - e^{-t})u(t - 3)$.

Solution:

(a) If we know the unit-step response $g(t)$, then $h(t) = \frac{d}{dt} g(t)$. Therefore
$h(t) = \frac{d}{dt} g(t) = \frac{d}{dt}(3u(t - 3)) = 3\delta(t - 3)$.
(b) $h(t) = \frac{d}{dt} g(t) = \frac{d}{dt}(t^2 u(t)) = t^2 \delta(t) + u(t)2t = 2tu(t)$.
(c) $h(t) = \frac{d}{dt} g(t) = \frac{d}{dt}((2 - e^{-t})u(t - 3)) = (2 - e^{-t})\delta(t - 3) + u(t - 3)e^{-t}$.

3. If the unit-step response of an LTI system is $g(t) = 3\text{rect}\left(\frac{t-3}{2}\right)$, find the system zero-state responses to the following inputs.

(a) $f(t) = \text{rect}(t)$.
(b) $f(t) = e^{-2t}u(t)$.
(c) $f(t) = 2\delta(t)$.

Solution:

(a) First, we need to find the impulse response:
$h(t) = \frac{d}{dt} g(t) = \frac{d}{dt}\left(3u(t - 2) - 3u(t - 4)\right) = 3\delta(t - 2) - 3\delta(t - 4)$.
Then, $y_{ZS}(t) = f(t) * h(t) = \text{rect}(t) * (3\delta(t - 2) - 3\delta(t - 4)) = 3rect(t - 2) - 3rect(t - 4)$.
(b) The impulse response is the same as in part (a), therefore,
$y_{ZS}(t) = f(t) * h(t) = e^{-2t} * (3\delta(t - 2) - 3\delta(t - 4)) = 3e^{-2(t-2)}u(t - 2) - 3e^{-2(t-4)}u(t - 4)$.
(c) The impulse response is the same as in part (a), therefore,
$y_{ZS}(t) = f(t) * h(t) = 2\delta(t) * (3\delta(t - 2) - 3\delta(t - 4)) = 6\delta(t - 2) - 6\delta(t - 4)$.

4. For each one of the 3 signals $f(t)$ in parts (a), (b), (c), (d), do the following

i. Obtain its Laplace transform $\hat{F}(s)$.
ii. Indicate the poles of $\hat{F}(s)$.
iii. Indicate the ROC of $\hat{F}(s)$.

(a) $f(t) = u(t) - u(t - 6)$
(b) $f(t) = te^{2(t-1)}u(t)$
(c) $f(t) = (t - 1)\, e^{-4t} + \delta(t)$

(d) $f(t) = e^{2t}\cos(t)u(t+1)$.

Solution:

(a) $f(t) = u(t) - u(t-6)$

i. Using the Laplace transform definition, we have

$$\hat{F}(s) = \int_{0^-}^{\infty} [u(t) - u(t-6)]\, e^{-st} dt = \int_{0^-}^{6} e^{-st} dt = \frac{1-e^{-6s}}{s}.$$

ii. poles:
Testing if $\hat{F}(s) \to \pm\infty$ as $s \to 0$: $\lim\limits_{s\to 0} \hat{F}(s) = \lim\limits_{s\to 0} \frac{1-e^{-6s}}{s} = \frac{0}{0}$ (indeterminate). Applying l'Hospital rule we find out that $s = 0$ is not a pole, because $\lim\limits_{s\to 0} \hat{F}(s) \neq \pm\infty$:

$$\lim_{s\to 0} \frac{1-e^{-6s}}{s} = \lim_{s\to 0} \frac{\frac{d}{ds}\left(1-e^{-6s}\right)}{\frac{d}{ds}(s)} = \lim_{s\to 0} \frac{6e^{-6s}}{1} = 6 \neq \infty.$$

There is a set of poles as $\mathrm{Re}\{s\} \to -\infty$. Therefore, we say that there is a "hidden" pole at $s = -\infty + j\omega$. List of poles:

$$s_1 = \{-\infty + j\omega\}.$$

zeros:
$s = 0$ is not a zero, since $\lim\limits_{s\to 0} \hat{F}(s) = 6 \neq 0$.
Testing if $\hat{F}(s) \to 0$ as $s \to \infty$: $\lim\limits_{s\to\infty} \hat{F}(s) = \lim\limits_{s\to\infty} \frac{1-e^{-6s}}{s} = 0$. Therefore $s = \infty + j\omega$ is a zero. Also, $\hat{F}(s) = 0$, when $e^{-6s} = 1 \Leftrightarrow 6s = 2\pi jn$, for all integers $n \neq 0$. Therefore the complete list of zeros is

$$z_{1,2} = \{\infty + j\omega,\ \frac{j\pi n}{3} \text{ for all integers } n \neq 0\}$$

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iii. We recognize the ROC as the region to the right of the rightmost pole : $\sigma = \mathrm{Re}\{s\} > -\infty$.

(b) $f(t) = te^{2(t-1)}u(t) = e^{-2}te^{2t}u(t)$

i. Using Laplace transform tables we obtain

$$\hat{F}(s) = \frac{e^{-2}}{(s-2)^2},$$

ii. list of poles: $s_{1,2} = 2$ (double pole), list of zeros: $z_{1,2} = \pm\infty + j\omega$

iii. ROC: $\sigma = \mathrm{Re}\{s\} > 2$.
This means that this Laplace integral converges only for values of s such that $\mathrm{Re}\{s\} > 2$.

(c) $f(t) = (t-1)\, e^{-4t} + \delta(t) = te^{-4t} - e^{-4t} + \delta(t)$

i. Using Laplace transform tables we obtain

$$\hat{F}(s) = \frac{1}{(s+4)^2} - \frac{1}{s+4} + 1 = \frac{s^2 + 7s + 13}{(s+4)^2} = \frac{\left(s - \frac{-7+j\sqrt{55}}{2}\right)\left(s - \frac{-7-j\sqrt{55}}{2}\right)}{(s+4)^2}.$$

ii. list of poles: $s_{1,2} = \{-4\,(\text{double})\}$, list of zeros: $z_{1,2} = \frac{-7\pm j\sqrt{77}}{2}$

iii. ROC: $\sigma = \mathrm{Re}\{s\} > -4$.
This means that this Laplace integral converges only for values of s such that $\mathrm{Re}\{s\} > -4$.

(d) $f(t) = e^{2t}\cos(t)u(t+1)$

i. The Laplace transform starts at $t = 0$. Therefore it will be the same as calculating the L.T of $f(t) = e^{2t}\cos(t)u(t)$. Using Table 11.1 , we obtain

$$f(t) = e^{2t}\cos(t)u(t) \longleftrightarrow \hat{F}(s) = \frac{s-2}{(s-2)^2 + 1} = \frac{s-2}{(s-(2+j))\,(s-(2-j))}.$$

ii. list of poles: $s_{1,2} = \{2+j,\, 2-j\}$, list of zeros: $z_{1,2} = \{2,\, +\infty + j\omega\}$
iii. ROC: $\sigma = \text{Re}\{s\} > 2$.
This means that this Laplace integral converges only for values of s such that $\text{Re}\{s\} > 2$.

5. Determine whether the LTIC systems with the following transfer functions are BIBO stable and explain why or why not.

(a) $\hat{H}_2(s) = 2 + \frac{s}{(s+1)(s-2)}$
(b) $\hat{H}_3(s) = \frac{s^2+4s+6}{(s+1+j6)(s+1-j6)}$
(c) $\hat{H}_1(s) = \frac{s^3+1}{(s+2)(s+4)}$
(d) $\hat{H}_4(s) = \frac{1}{s^2+16}$
(e) $\hat{H}_5(s) = \frac{s-2}{s^2-4}$.

Solution:

(a) $\hat{H}_2(s)$ has a pole in the RHP at s = 2, so the system is not BIBO stable.
(b) $\hat{H}_3(s)$ has two conjugate poles at s =-1-j6, and s = -1 + j6, both in the LHP, so the system is BIBO stable.
(c) $\hat{H}_1(s)$ has two conjugate poles at s =-2, s = -4 and s = $+\infty$. Because the pole at infinity is not confined to the LHP, the system is not BIBO stable.
(d) $\hat{H}_4(s)$ has two conjugate poles on the imaginary axis at s = j4, and s = -j4. The system is marginally stable, but not BIBO stable.
(e) $\hat{H}_5(s)$ has one pole at s = -2, so the system is BIBO stable. The unstable pole is cancelled with the unstable zeros.

6. For each of the following Laplace transforms $\hat{F}(s)$, determine the inverse Laplace transform $f(t)$.

(a) $\hat{F}(s) = \frac{s+3}{(s+2)(s+4)}$
(b) $\hat{F}(s) = \frac{1}{s(s-5)^2}$
(c) $\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)}$

Solution:

(a) Expressing as a PFE,
$\hat{F}(s) = \frac{K_1}{(s+2)} + \frac{K_2}{(s+4)}$
Applying the cover-up method, we have
$K_1 = 0.5,\; K_2 = 0.5$, therefore,
$f(t) = \left(\frac{1}{2}e^{-4t} + \frac{1}{2}e^{-2t}\right)u(t)$.
(b) Expressing as a PFE,
$\hat{F}(s) = \frac{1}{s(s-5)^2} = \frac{K_1}{s} + \frac{K_2}{(s-5)^2} + \frac{K_3}{(s-5)}$
Applying the cover-up method, we have
$K_1 = \frac{1}{25},\; K_2 = \frac{1}{5}$, and $K_1 = -\frac{1}{25}$, therefore,
$f(t) = \left(\frac{1}{25} + \frac{1}{25}te^{5t} - \frac{1}{25}e^{5t}\right)u(t)$.
(c) We first simplify the expression by writing
$\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)} = \frac{(s+1)(s+2)-(s+1)}{(s+1)(s+2)} = 1 - \frac{1}{(s+2)}$
Consequently, $f(t) = \delta(t) - e^{-2t}u(t)$.