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**Zhejiang University – University of Illinois at Urbana-Champaign Institute**
**ECE-210 Analog Signal Processing Spring 2022**
**Homework #8: Submission Deadline 20th April (10:00 PM)**

1. The function $f(t)$ is periodic with period $T = 4s$. Between t=0 and 4s, the function is described by:

$$f(t) = \begin{cases} 2, & 0 < t < 1s \\ -1, & 1 < t < 3s \\ 1, & 3 < t < 4s \end{cases}$$

(a) Plot $f(t)$ between $t = -5s$ and $t = 7s$.
(b) Determine the exponential Fourier coefficients $F_n$ of $f(t)$ for $n = 0$, $n = \pm1$, and $n = \pm2$.
(c) Using the result of part(b), determine the compact-form Fourier coefficients $C_0$, $C_1$and $C_2$.

b)

$$F_n = \frac{1}{4}\left( \int_0^1 2\cdot e^{-j\frac{\pi}{2}nt} dt - \int_1^3 e^{-j\frac{\pi}{2}nt} dt + \int_3^4 e^{-j\frac{\pi}{2}nt} dt \right.$$

$$= \frac{1}{4}\left( 2\cdot \frac{2j}{\pi n} (e^{-j\frac{\pi}{2}n} - 1) - \frac{2j}{\pi n}(e^{-3j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}) + \frac{2j}{\pi n}(e^{-j2\pi n} - e^{-j\frac{3\pi n}{2}}) \right)$$

$$= \frac{j}{4\pi n}(e^{-j\frac{\pi}{2}n} - 1) +_{8} \frac{j}{\pi n}(e^{-j2n\pi} - 2e^{-3j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})$$

$$= \frac{j}{\pi n} e^{-j\frac{\pi}{2}n} - \frac{j}{2\pi n} - \frac{j}{\pi n} e^{-3j\frac{\pi}{2}n} + \frac{j}{2\pi n}\cdot e^{-j\frac{\pi}{2}n}$$

$$= \frac{j}{\pi n}(-2j\sin\frac{n\pi}{2}) - \frac{j}{2\pi n} + \frac{j}{2\pi n}(\cos\frac{n\pi}{2} - j\sin\frac{n\pi}{2})$$

$$= \frac{2}{\pi n}\sin\frac{n\pi}{2} + \frac{1}{2\pi n}\sin\frac{n\pi}{2} + \frac{j}{2\pi n}(\cos\frac{n\pi}{2} - 1)$$

$$= \frac{5}{2\pi n}\sin\frac{n\pi}{2} + \frac{j}{2\pi n}(\cos\frac{n\pi}{2} - 1)$$

for $n=0$ =

$$F_n = \frac{1}{4}\left( \int_0^1 2e^{-j\frac{\pi}{2}n\cdot t} dt + \int_1^3 -e^{-j\frac{\pi}{2}nt} dt + \int_3^4 e^{-jn\cdot\frac{\pi}{2}t} dt \right)$$

$$\Rightarrow \frac{1}{4}(2-2+1) = \frac{1}{4}$$

for $n=1$

$$\therefore F_1 = \frac{5}{2\pi}\cdot\sin\frac{\pi}{2} + \frac{j}{2\pi}(\cos\frac{\pi}{2} - 1)$$

$$= \frac{5}{2\pi} - \frac{j}{2\pi}$$

$$F_{-1} = \frac{5}{2\pi} + \frac{j}{2\pi}$$

$$F_2 = \frac{5}{4\pi}\cdot\sin\pi + \frac{j}{4\pi}(-1-1)$$

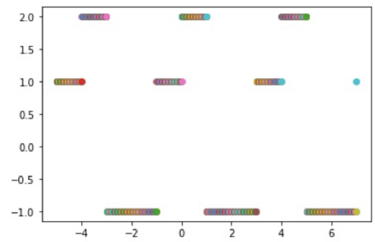
$$= \frac{-j}{2\pi}$$

$$F_{-2} = \frac{j}{2\pi}$$

a)

[17]:

```python
# Here is the code for ECE210 Homework#8;
# periodic function ploting;
def func(t):
    a = t % 4
    if (a >= 0 and a < 1):
        return 2
    elif (a >= 1) and (a < 3):
        return -1
    elif (a >= 3) and (a < 4):
        return 1
list1 = np.linspace(-5,7,1000)
for i in list1:
    plt.scatter(i,func(i))
```




c)

$$C_0 = \frac{1}{4}$$

$$C_1 = \frac{\sqrt{26}}{\pi}$$

$$C_2 = \frac{1}{\pi}$$

2. Consider an LTI system whose frequency response is

$$H(\omega) = \frac{\sin(4\omega)}{\omega}$$

If the input to this system is a periodic signal

$$f(t) = \begin{cases} +1, & 0 < t < 4s \\ -1, & 4 < t < 8s \end{cases}$$

with period $T = 8s$.
Determine the corresponding system output $y(t)$

① First calculate Fourier Series.

$$F_n = \frac{1}{T}\left(\int_0^4 e^{-j\omega t n}dt + \int_4^8 -e^{-j\omega t n}dt\right)$$

$$= \frac{1}{8}\left(\int_0^4 e^{-j\frac{\pi}{4}tn}dt - \int_4^8 e^{-j\frac{\pi}{4}tn}dt\right)$$

$$= \frac{1}{8}\cdot\frac{4}{-j\pi n}\left(e^{-j\frac{\pi}{4}t}\Big|_{t=0}^{t=4} - e^{-j\frac{\pi}{4}tn}\Big|_{t=4}^{t=8}\right)$$

$$= \frac{1}{-2j\pi n}\left[e^{-j\pi n} - e^{-j\cdot 0\cdot n} - (e^{-j\cdot 2\pi\cdot n} - e^{-j\pi n})\right]$$

$$= \frac{j}{2\pi n}\left(2e^{-j\pi\cdot n} - e^{-j2\pi n} - 1\right)$$

$$= \frac{j}{2\pi n}\left[2(\cos n\pi - j\sin n\pi) - \cos 2n\pi + j\sin 2n\pi - 1\right]$$

$$= \frac{j}{2\pi n}(2\cos n\pi - 2j\sin n\pi - 2)$$

$$= \frac{j}{\pi n}(\cos n\pi - j\sin n\pi - 1)$$

② define Response:

$$Y_n = H(n w_0)\cdot F_n$$

$$= \frac{4\sin n\pi}{n\pi}\cdot F_n \Rightarrow \text{As } 4\cdot\frac{\sin n\pi}{n\pi} = 0$$

$\therefore Y_n = 0$ $\therefore$ output is $y(t) = 0$

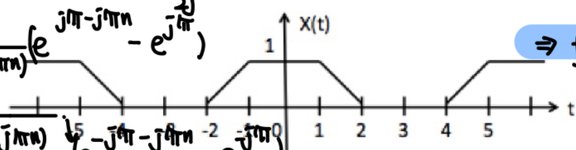
3. Determine the Fourier series representations for the following signals:
(a) A periodic signal $x(t)$ with period of $T = 2s$ and

$$x(t) = e^{-t}\ for -1 < t < 1s$$

(b) A periodic signal $x(t)$ with period $T = 4s$ and

$$x(t) = \begin{cases} \sin(\pi t), & 0 \le t \le 2s \\ 0, & 2 < t \le 4s \end{cases}$$

(c)




(a)

$$F_n = \frac{1}{T}\int_T e^{-t}\cdot e^{-jw_0 nt}dt$$

$$w_0 = \frac{2\pi}{T} = \pi$$

$$\therefore F_n = \frac{1}{2}\int_{-1}^{1} e^{-t}\cdot e^{-j\pi nt}dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+j\pi n)t}dt$$

$$= \frac{1}{-2(1+j\pi n)}e^{-(1+j\pi n)t}\Big|_{-1}^{1} = \frac{-1}{2(1+j\pi n)}\left(e^{1+j\pi n} - e^{-1-j\pi n}\right)$$

$$= \frac{e^{1+j\pi n} - e^{-1-j\pi n}}{2(1+j\pi n)}$$

$$\Rightarrow f(t) = \sum_{-\infty}^{\infty}\frac{e^{1+j\pi n} - e^{-1-j\pi}}{2(1+j\pi n)}\cdot e^{j\pi nt}$$

(b)

$$F_n = \frac{1}{T}\int_0^2 \sin(\pi t)\cdot e^{-j\cdot\frac{\pi}{2}nt}dt$$

$$= \frac{1}{4}\int_0^2 \frac{e^{j\pi t} - e^{-j\pi t}}{2j}\cdot e^{-j\frac{\pi}{2}n\cdot t}dt$$

$$= \frac{1}{4}\int_0^2 \frac{1}{2j}e^{(j\pi - j\frac{\pi}{2}\cdot n)t} - \frac{1}{2j}\cdot e^{-(j\pi + j\frac{\pi}{2}n)t}dt$$

$$= \frac{1}{4j(2j\pi - j\pi n)}\left(e^{j\pi - j\pi n} - e^{-j\pi}\right)$$

$$+ \frac{1}{4j(2j\pi + j\pi n)}\left(e^{-j\pi - j\pi n} - e^{-j\pi}\right)$$

as $e^{j\cdot 2\pi} = 1 \Rightarrow \left(\frac{1}{4j(2j\pi + j\pi n)} + \frac{1}{4j(2j\pi - j\pi n)}\right)\cdot\left(e^{j\pi - j\pi n} - e^{j\pi}\right)$

$$= \left(-\frac{1}{8\pi + 4\pi n} + \frac{1}{-8\pi + 4\pi n}\right)\left(\cos[\pi(1-n)] + j\sin[\pi(1-n)] - \cos\pi - j\sin\pi\right)$$

$$\left(\frac{8\pi + 4\pi n - 4\pi n + 8\pi}{16n^2\pi^2 - 64\pi^2}\right)\left[(-1)^{n-1} + 1\right] = \frac{16\pi\cdot[(-1)^n + 1]}{16n^2\pi^2 - 64\pi^2} = \frac{(-1)^{n-1} + 1}{n^2\pi - 4\pi}$$

$$\Rightarrow x(t) = \sum_{-\infty}^{\infty}\frac{(-1)^{n-1} + 1}{n^2\pi - 4\pi}e^{j\frac{\pi}{2}nt}$$

(c) $T = 6s$: $\quad \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$ rad/s

$$\Rightarrow F_n = \frac{1}{T}\int_T f(t)\, e^{-j\omega t \cdot n} dt$$

$$= \frac{1}{6}\left(\int_{-2}^{-1} (t+2)\cdot e^{-j\frac{\pi}{3}t\cdot n} dt \;①+ \int_{-1}^{1} e^{-j\frac{\pi}{3}t\cdot n} dt \;②+ \int_{1}^{2} (2-t)\, e^{-j\frac{\pi}{3}t n} dt \;③\right)$$

① $\frac{3j}{\pi n}\left[(t+2)e^{-j\frac{\pi}{3}t\cdot n} - \int e^{-j\frac{\pi}{3}t\cdot n}dt\right]\Big|_{-2}^{-1} = \frac{3j}{\pi n}\left[(\;1\;)e^{-j\frac{\pi}{3}t\cdot n} - \cancel{\frac{3j}{\pi\cdot n} e^{-j\frac{\pi}{3}t\cdot n}}\right]\Big|_{-2}^{-1} = \frac{3j}{\pi n}\left[\; e^{j\frac{\pi}{3}n} - \cancel{(t+2)e^{j\frac{\pi}{3}n}} - \frac{3j}{\pi n}\cdot e^{j\frac{\pi}{3}n} + \frac{3j}{\pi n}\cdot e^{2j\frac{\pi}{3}tn}\right]$

② $\frac{3j}{\pi n} e^{-j\frac{\pi}{3}nt}\Big|_{-1}^{1} = \frac{3j}{\pi n}\left(e^{-j\frac{\pi}{3}n} - e^{j\frac{\pi}{3}n}\right)$

③ $\frac{3j}{\pi n}\cdot\left[(2-t)\, e^{-j\frac{\pi}{3}n\cdot t} + \int e^{-j\frac{\pi}{3}n\cdot t}dt\right] = \frac{3j}{\pi n}\left[(2-t)e^{-j\frac{\pi}{3}n\cdot t} + \frac{3j}{\pi n}\cdot e^{-j\frac{\pi}{3}\cdot nt}\right]\Big|_1^2$

$\qquad = \frac{3j}{\pi n}\left(-e^{-j\frac{\pi}{3}n} + \frac{3j}{\pi n}\cdot e^{-j\frac{\pi}{3}2n} - \frac{3j}{\pi n}\cdot e^{-j\frac{\pi}{3}n}\right)$

$\Rightarrow F_n = \frac{3j}{\pi n}\left[\cancel{e^{j\frac{\pi}{3}n}} - \frac{3j}{\pi n}\cdot e^{j\frac{\pi}{3}n} + \frac{3j}{\pi n}\cdot e^{2j\frac{\pi}{3}n}\right] + \frac{3j}{\pi n}\left(\cancel{e^{-j\frac{\pi}{3}n}} - \cancel{e^{j\frac{\pi}{3}n}}\right) + \frac{3j}{\pi n}\left(\cancel{-e^{-j\frac{\pi}{3}n}} + \frac{3j}{\pi n}\cdot e^{-j\frac{\pi}{3}2n} - \frac{3j}{\pi n}\cdot e^{-j\frac{\pi}{3}n}\right)$

$= \frac{9}{\pi^2 n^2} e^{j\frac{\pi}{3}n} - \frac{9}{\pi^2 n^2} e^{2j\frac{\pi}{3}n} - \frac{9}{\pi^2 n^2} e^{-j\frac{2\pi}{3}n} + \frac{9}{\pi^2 n^2} e^{-j\frac{\pi}{3}n}$

$= \frac{9}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} + \cancel{j\sin\frac{n\pi}{3}} - \cos\frac{2n\pi}{3} - \cancel{j\sin\frac{2n\pi}{3}} - \cos\frac{2n\pi}{3} + \cancel{j\sin\frac{2n\pi}{3}} + \cos\frac{n\pi}{3} - \cancel{j\sin\frac{n\pi}{3}}\right)$

$= \frac{18}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)$

$\Rightarrow x(t) = \sum_{-\infty}^{\infty} \frac{18}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)\cdot e^{j\frac{\pi}{3}nt}$

4. Let

$$x(t) = \begin{cases} t, & 0 \le t \le 1s \\ 2-t, & 1 \le t \le 2s \end{cases}$$

be a periodic signal with fundamental period $T = 2s$ and exponntial Fourier coefficients $X_n$.
(a) Determine the value of $X_0$.
(b) Determine the Fourier series representation of $\frac{dx(t)}{dt}$
(c) Use the result of part (b) and the differential property of the Fourier series to help determine the Fourier series coefficients of $x(t)$.

(a) $X_0 = \frac{1}{T}\int_T x(t)\, e^{-j\pi n\cdot t} dt\Big|_{n=0} = \frac{1}{2}\left(\int_0^1 t\,dt + \int_1^2 2-t\,dt\right) = \frac{1}{2}\left(\frac{1}{2} + 2t - \frac{1}{2}t^2\Big|_1^2\right) = \frac{1}{4} + (2 - \frac{1}{2}\times 3) + 2$
$= \frac{1}{2}$

(b) for $\frac{dx(t)}{dt} = \begin{cases} 1 & t\in[0,1] \\ -1 & t\in[1,2] \end{cases}$

$\Rightarrow X_n = \frac{1}{T}\int_T x'(t)\, e^{-j\pi n\cdot t} dt$

$= \frac{1}{2}\left(\int_0^1 e^{-j\pi n t}dt - \int_1^2 e^{-j\pi n t}dt\right)$

$= \frac{1}{2}\left[\frac{j}{\pi n}\left(e^{-j\pi n} - 1\right) - \frac{j}{\pi n}\left(e^{-2j\pi n} - e^{j\pi n}\right)\right]$

$= \frac{j}{\pi n} e^{-j\pi n} - \frac{j}{2\pi n} - \frac{j}{2\pi n} e^{-2j\pi n}$

$\because e^{-j2\pi n} = 1$

$\therefore \frac{j}{\pi n}(-1)^n - \frac{j}{\pi n} = X_n \quad \Rightarrow \quad \frac{dx(t)}{dt} = \sum_{-\infty}^{\infty}\left(\frac{j}{\pi n}(-1)^n - \frac{j}{\pi n}\right)\cdot e^{j\pi n t}$

(c) $x(t) = \int \sum_{-\infty}^{\infty}\left(\frac{j}{\pi n}(-1)^n - \frac{j}{\pi n}\right)\cdot e^{j\pi n t}dt$

$= \sum_{-\infty}^{\infty}\left(\frac{(-1)^n}{j\pi n}\cdot\frac{j}{\pi n} - \frac{j}{\pi n}\cdot\frac{1}{j\pi n}\right) e^{j\pi n t}$

$= \sum_{-\infty}^{\infty}\left(\frac{(-1)^n}{\pi^2 n^2} - \frac{1}{\pi^2 n^2}\right) e^{j\pi n t}$

5. Let the signal $f(t) = \sin^4(t)$ be the input of an LTI system with frequency response $H(\omega) = 2\,e^{-j\omega\pi/2}$ for $\omega \,\epsilon\, [-2, 2]$ rad/s and zero elsewhere. Obtain the steady-state response $y(t)$ of the system to the input $f(t)$.

$$f(t) = \sin^4(t) = \left(\frac{1-\cos 2t}{2}\right)^2$$
$$= \frac{1}{4}(1 - 2\cos 2t + \cos^2 2t)$$
$$= \frac{1}{4}\left(1 - 2\cos 2t + \frac{1+\cos 4t}{2}\right)$$
$$= \frac{1}{4}\left(\frac{3}{2} - 2\cos 2t + \frac{1}{2}\cos 4t\right)$$
$$= \frac{1}{8}(3 - 4\cos 2t + \cos 4t)$$
$$= \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t$$

$\Rightarrow$ for term $\frac{3}{8}$

$$F_n = \frac{1}{T}\int_0^T \frac{3}{8}\cdot e^{-j\omega nt}\,dt = 0$$
$$= \frac{3}{8\cancel{T}}\cdot\frac{j\cancel{T}}{2\pi n}\,e^{-j\frac{2\pi}{T}nt}$$
$$\lim_{T\to\infty} F_n = \frac{3j}{16\pi n}\,e^{-j\cdot\frac{2\pi}{\infty}nt} = \frac{3j}{16\pi n}$$

For $\frac{3}{8}$ signal $H\left(\frac{2\pi}{\infty}\right) \Rightarrow 2$ $\Rightarrow$ output $= \frac{3}{4}$

For $-\frac{1}{2}\cos 2t$: $H(\omega) = H(\pi)$ $\because \omega > 2$ $\therefore = 0$

For $\cos 4t$ $H(\omega) = H\left(\frac{\pi}{2}\right)$ $\Rightarrow$ $Y_n = H\left(\frac{\pi}{2}\right)\cdot F_n = \frac{1}{8}e^{j\cdot 4t}\cdot 2\cdot e^{-j\frac{\pi^2}{4}} = \frac{1}{4}\cdot e^{j\left(4t-\frac{\pi^2}{4}\right)}$

$\Rightarrow$ output: $\frac{1}{4}\cos\left(4t - \frac{\pi^2}{4}\right) + \frac{3}{4}$ is output

For steady state sol.: $\frac{1}{4}\cos\left(4t - \frac{\pi^2}{4}\right) + \frac{3}{4}$