ANALOG SIGNAL PROCESSING

ECE 210 & 211

2022.4.12

Prof. Yang Xu （徐杨）

yangxu-isee@zju.edu.cn

Lab &Teaching Assistants:

Yue Dai (yuedai@zju.edu.cn)
Baoyu Wang (by.wang@zju.edu.cn)
Weiming Ma (22141072@ zju.edu.cn)
Jiangming Lin (3170104620@zju.edu.cn)
Yongliang Xie (22141005@zju.edu.cn)
Shuang Li (l211493@zju.edu.cn)

ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

Objectives

- Bode plot (continued)
- LTI system response to co-sinusoids input
- LTI system response to multifrequency co-sinusoids input
- Fourier coefficients of periodic signals
- Simplification by Symmetrical considerations
- Circuit interpretation for Fourier series

Objectives

- **Bode plot (continued)**
- **LTI system response to co-sinusoids input**
- **LTI system response to multifrequency co-sinusoids input**
- **Fourier coefficients of periodic signals**
- **Simplification by Symmetrical considerations**
- **Circuit interpretation for Fourier series**

Bode Plot

Normally, a transfer function looks like (in general form)

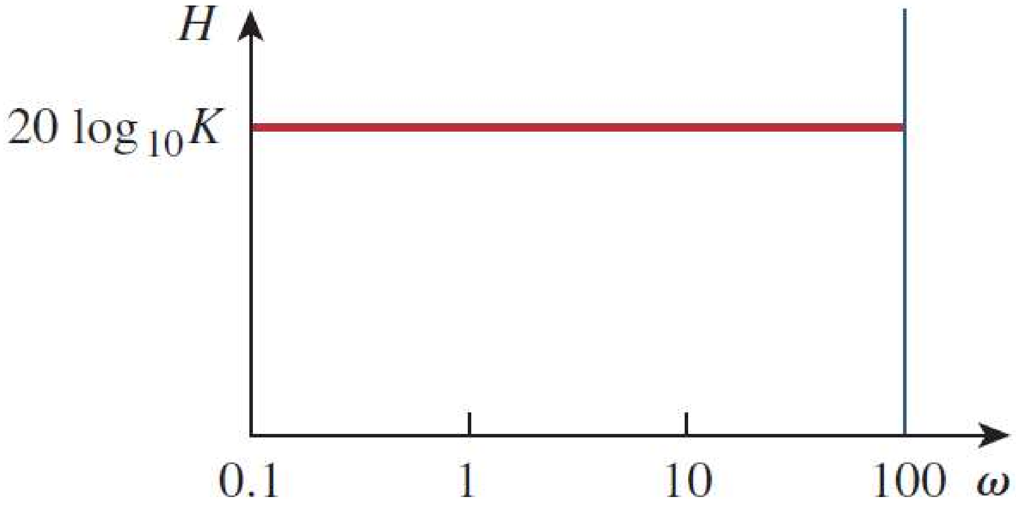
$$H(\omega)=\frac{K(j\omega)^{\pm 1}\left(1+\frac{j\omega}{z_1}\right)\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]\cdots}$$

The initial terms has 7 parts of equation

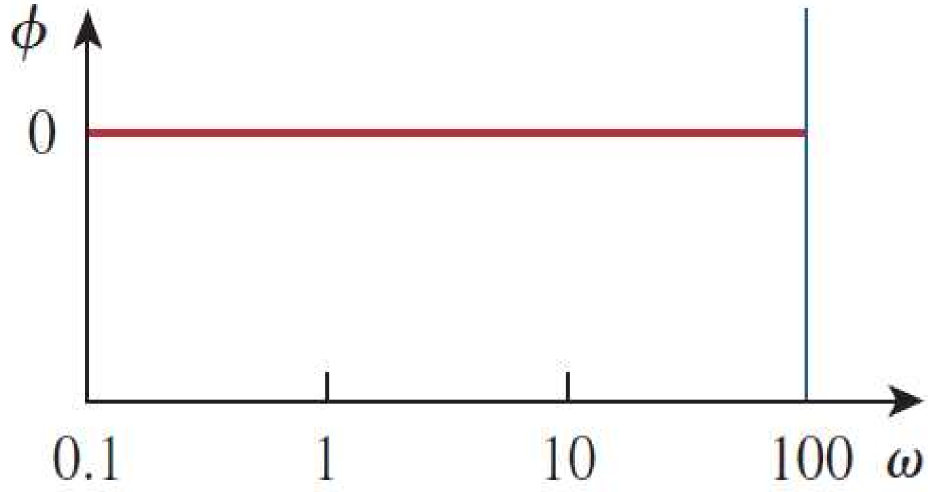
$$\begin{cases}(j\omega)^{+1}=Zero\ at\ origin\\(j\omega)^{-1}=Pole\ at\ origin\end{cases}$$

Bode Plot – Gain K

$$H(\omega) = \frac{\boxed{K}(j\omega)^{\pm 1}\left(1+\frac{j\omega}{z_1}\right)\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]\cdots}$$



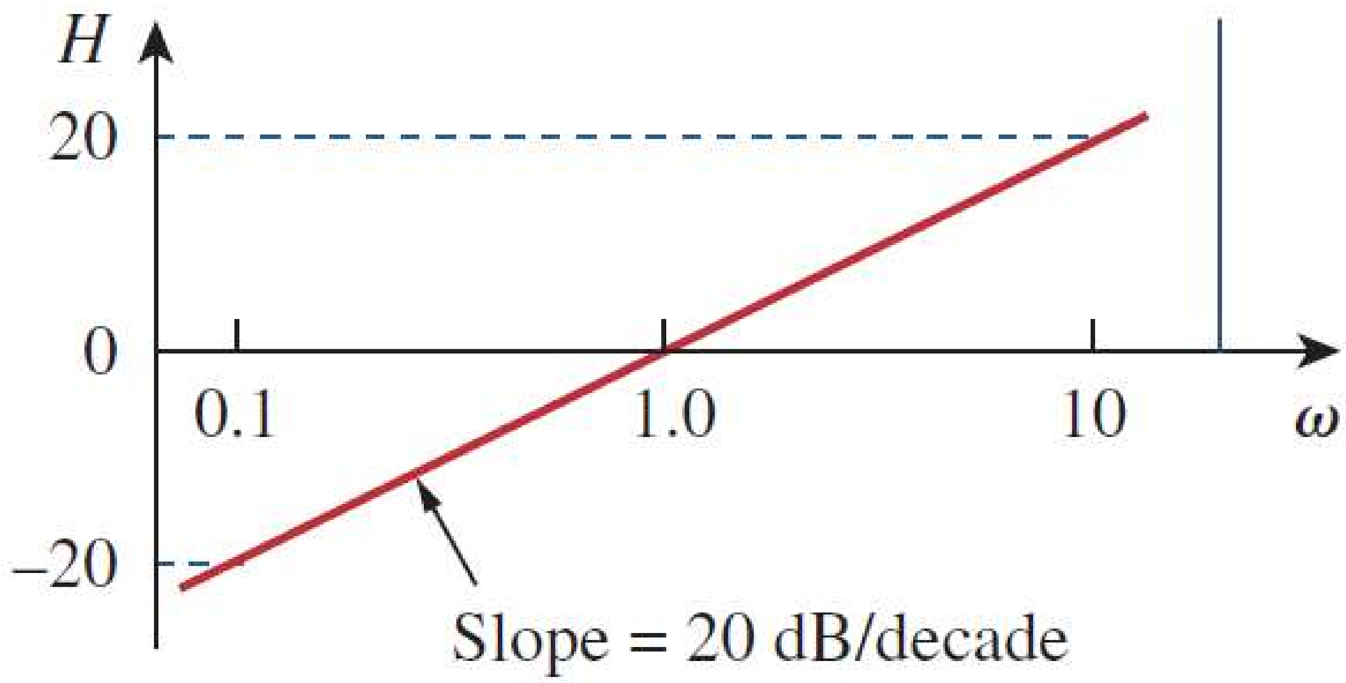
Magnitude



Phase

Bode Plot -Pole/zero at the origin

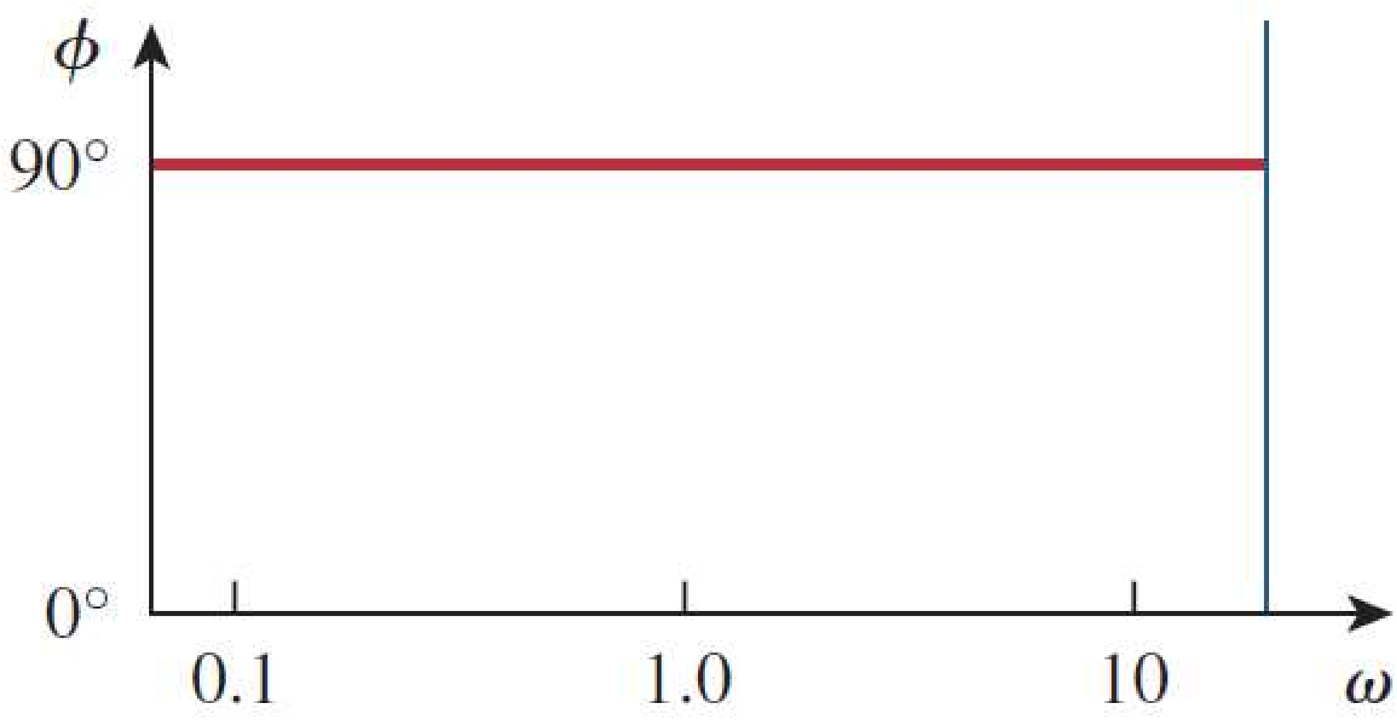
$$H(\omega)=\frac{K\boxed{(j\omega)^{\pm 1}}\left(1+\frac{j\omega}{z_1}\right)\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\dots}{\left(1+\frac{j\omega}{p_1}\right)\left[1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]\dots}$$



Magnitude

Bode Plot -Pole/zero at the origin

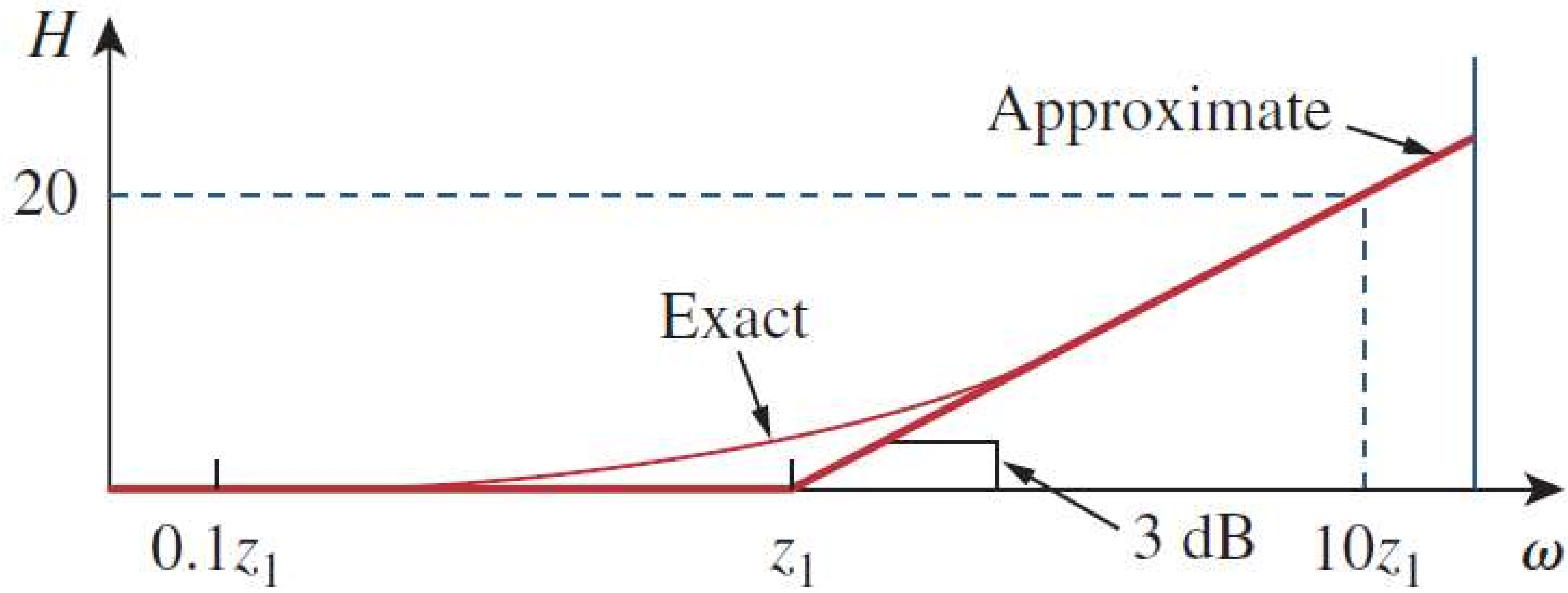
$$H(\omega) = \frac{K\boxed{(j\omega)^{\pm 1}}\left(1+\frac{j\omega}{z_1}\right)\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]\cdots}$$



Phase

Bode Plot – Simple pole/zero

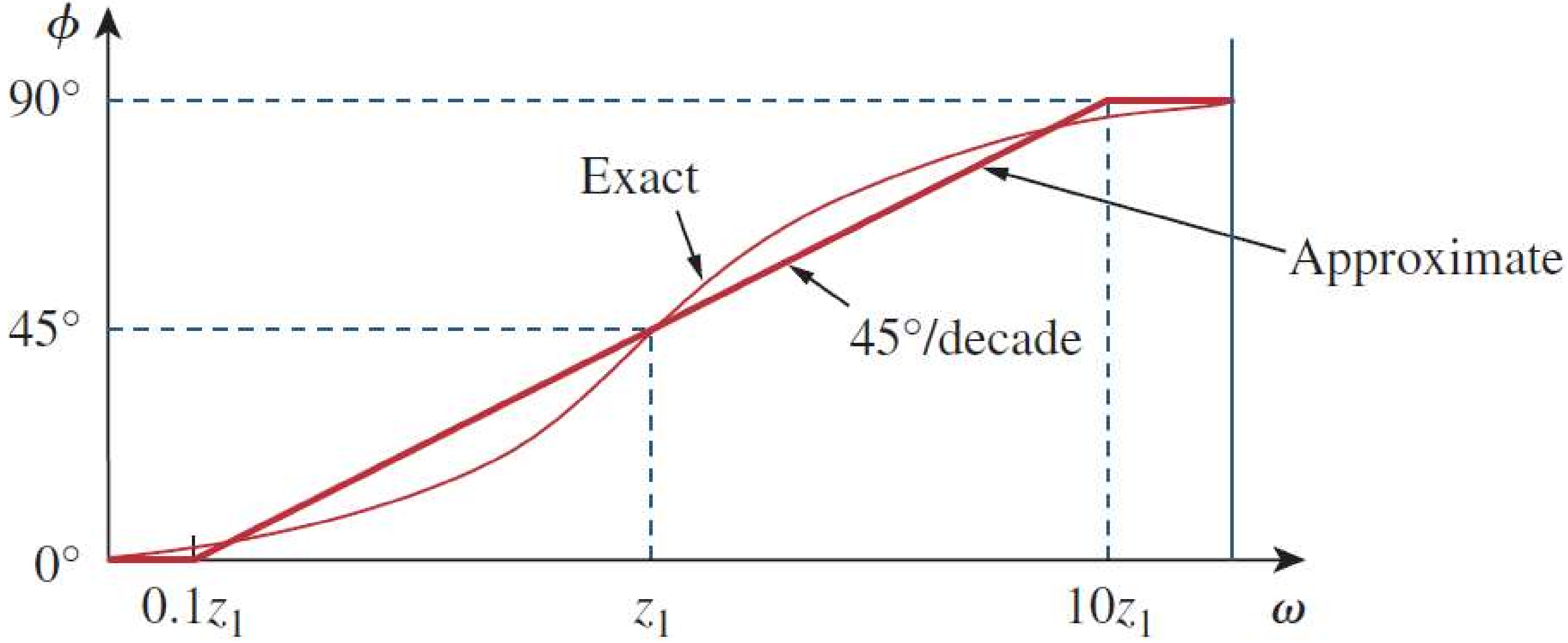
$$H(\omega) = \frac{K(j\omega)^{\pm 1}\boxed{\left(1+\frac{j\omega}{z_1}\right)}\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]\cdots}$$



Magnitude

Bode Plot – Simple pole/zero

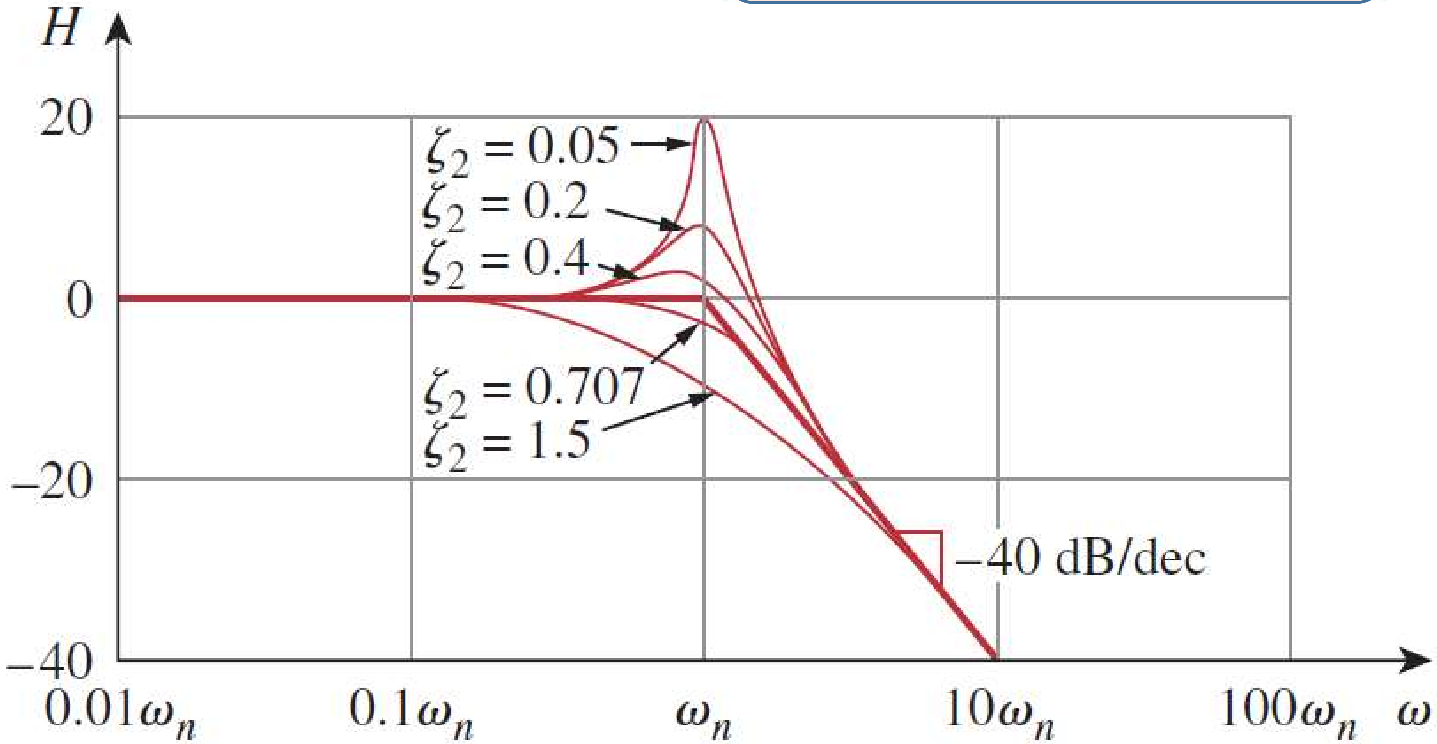
$$H(\omega) = \frac{K(j\omega)^{\pm 1}\boxed{\left(1+\frac{j\omega}{z_1}\right)}\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right]\cdots}$$



Phase

Bode Plot – Quadratic pole/zero

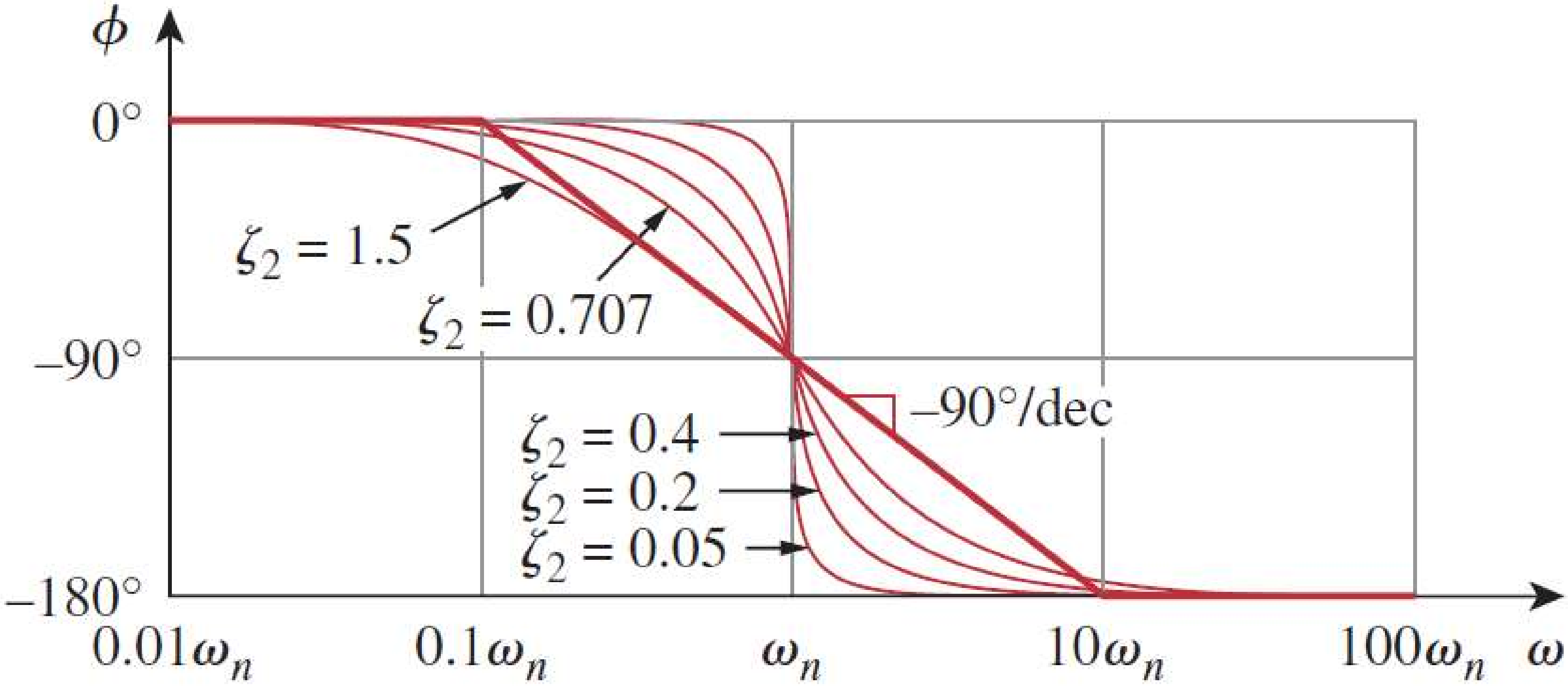
$$H(\omega)=\frac{K(j\omega)^{\pm 1}\left(1+\frac{j\omega}{z_1}\right)\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[\boxed{1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}\right]\cdots}$$



Magnitude

Bode Plot – Quadratic pole/zero

$$H(\omega)=\frac{K(j\omega)^{\pm 1}\left(1+\frac{j\omega}{z_1}\right)\left[1+j2\xi_1\frac{\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right]\cdots}{\left(1+\frac{j\omega}{p_1}\right)\left[\boxed{1+j2\xi_2\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}\right]\cdots}$$



Phase

Bode Plot – Example 1

Question: **Construct the Bode plots for the transfer function?**

$$H(\omega) = \frac{200j\omega}{(j\omega + 2)(j\omega + 10)}$$

Bode Plot – Example 1

Solution:

We first put $H(\omega)$ in the standard form by dividing out the poles and zeros. Thus,

$$H(\omega) = \frac{10j\omega}{\left(1 + j\frac{\omega}{2}\right)\left(1 + j\frac{\omega}{10}\right)}$$

$$= \frac{10|j\omega|}{\left|1 + j\frac{\omega}{2}\right|\left|1 + j\frac{\omega}{10}\right|}\angle(90^o - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10})$$

Bode Plot – Example 1

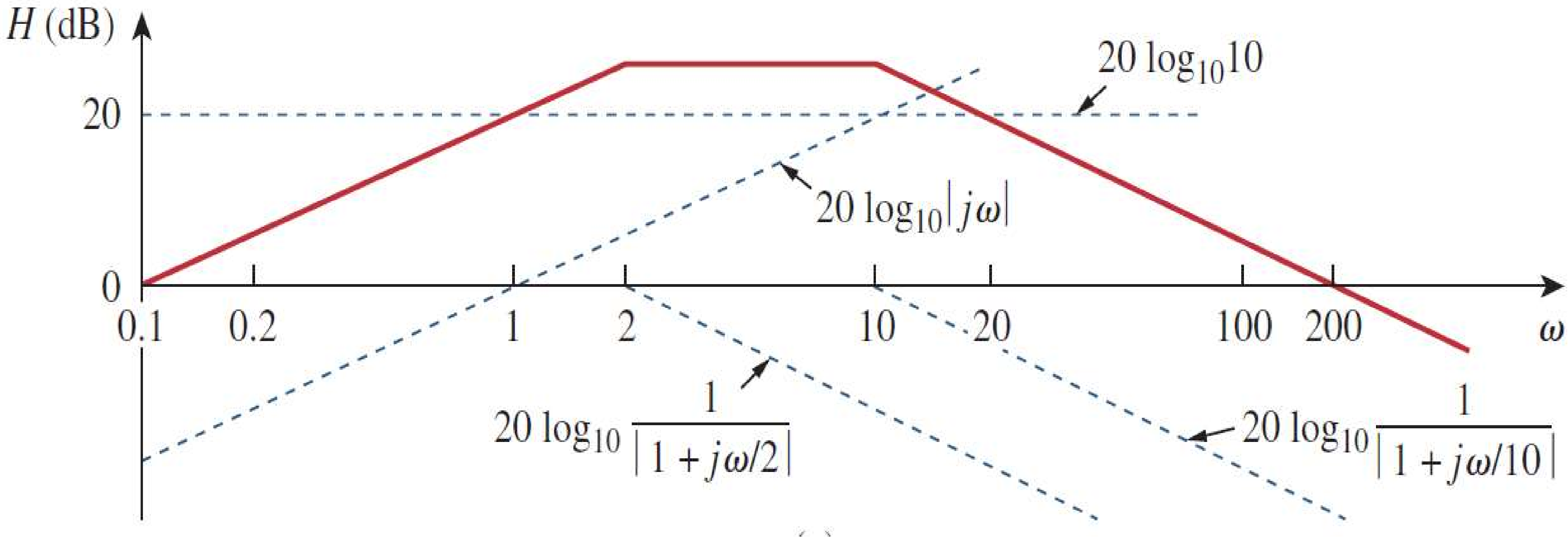
Hence , the magnitude and phase will be,

$$H_{dB} = 20log_{10}10 + 20log_{10}|j\omega| - 20log_{10}\left|1+\frac{j\omega}{2}\right| - 20log_{10}\left|1+\frac{j\omega}{10}\right|$$

$$\phi = 90^{o} - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10}$$

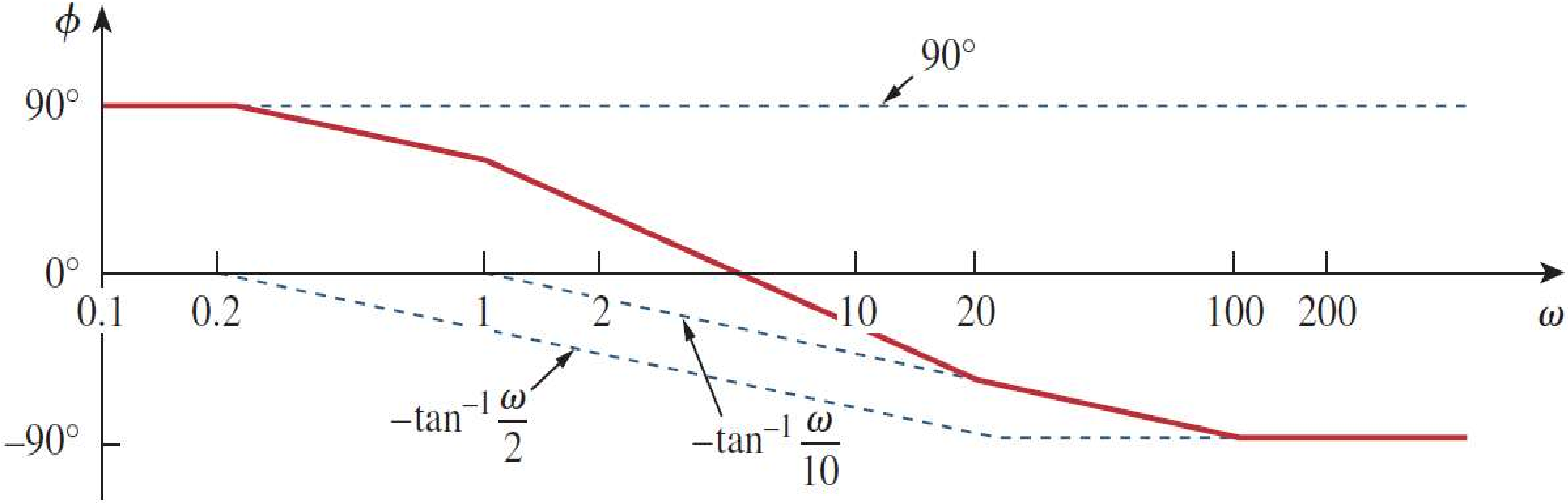
Bode Plot – Example 1

$$H_{dB} = 20log_{10}10 + 20log_{10}|j\omega| - 20log_{10}\left|1+\frac{j\omega}{2}\right| - 20log_{10}\left|1+\frac{j\omega}{10}\right|$$



Bode Plot – Example 1

$$\phi = 90^o - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10}$$



Bode Plot – Example 2

Question: Construct the Bode plots for the transfer function?

$$H(\omega) = \frac{j\omega + 10}{j\omega(j\omega + 5)^2}$$

Bode Plot – Example 2

Solution: **Putting $H(\omega)$ in the standard form,**

$$H(\omega) = \frac{0.4\left(1 + j\frac{\omega}{10}\right)}{j\omega\left(1 + j\frac{\omega}{5}\right)^2}$$

Hence , the magnitude and phase will be,

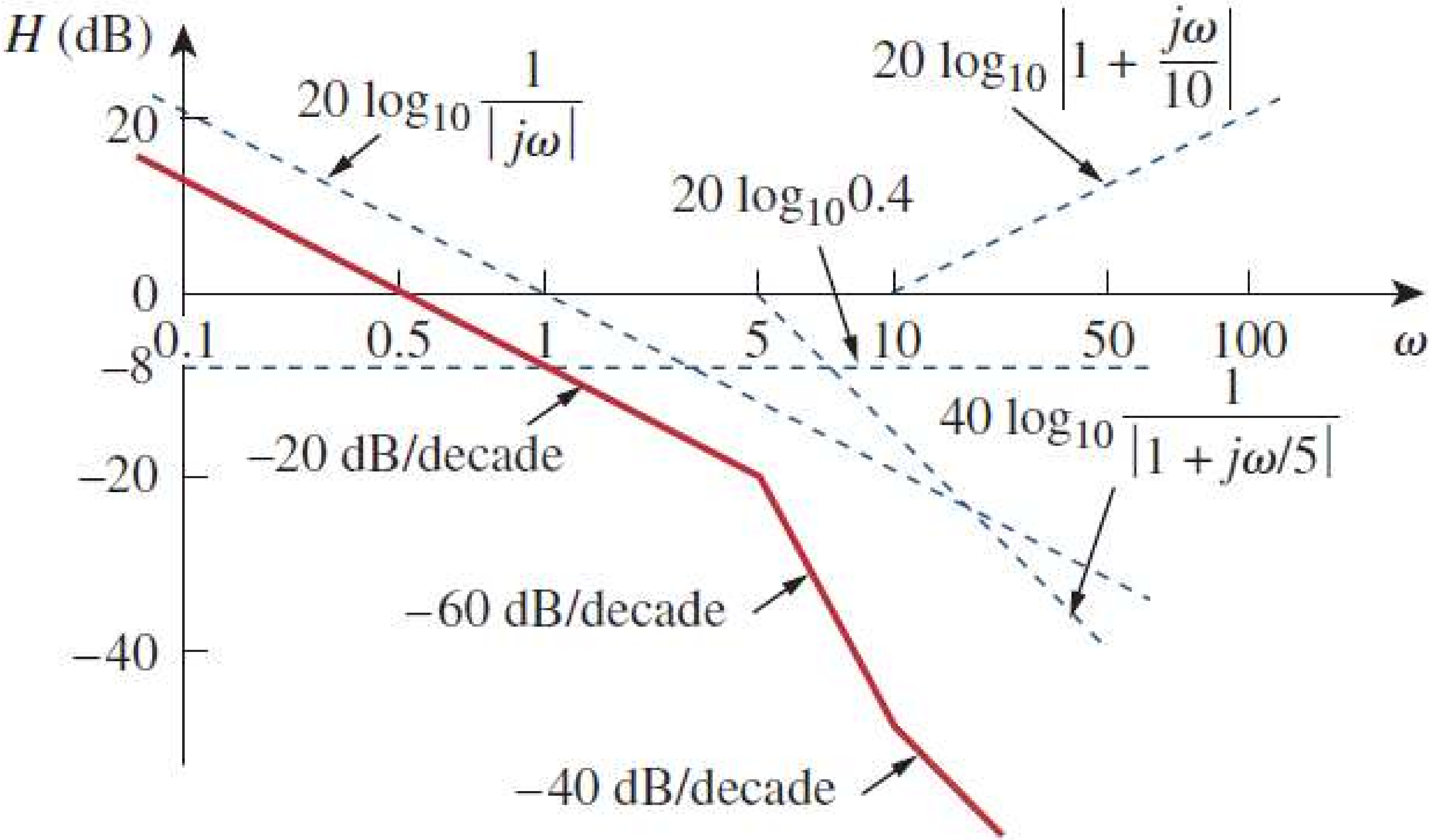
$$H_{dB} = 20log_{10}0.4 + 20log_{10}\left|1 + \frac{j\omega}{10}\right|$$

$$-20log_{10}|j\omega| - 40log_{10}\left|1 + \frac{j\omega}{5}\right|$$

$$\phi = 0^o + tan^{-1}\frac{\omega}{10} - 90^o - 2tan^{-1}\frac{\omega}{5}$$

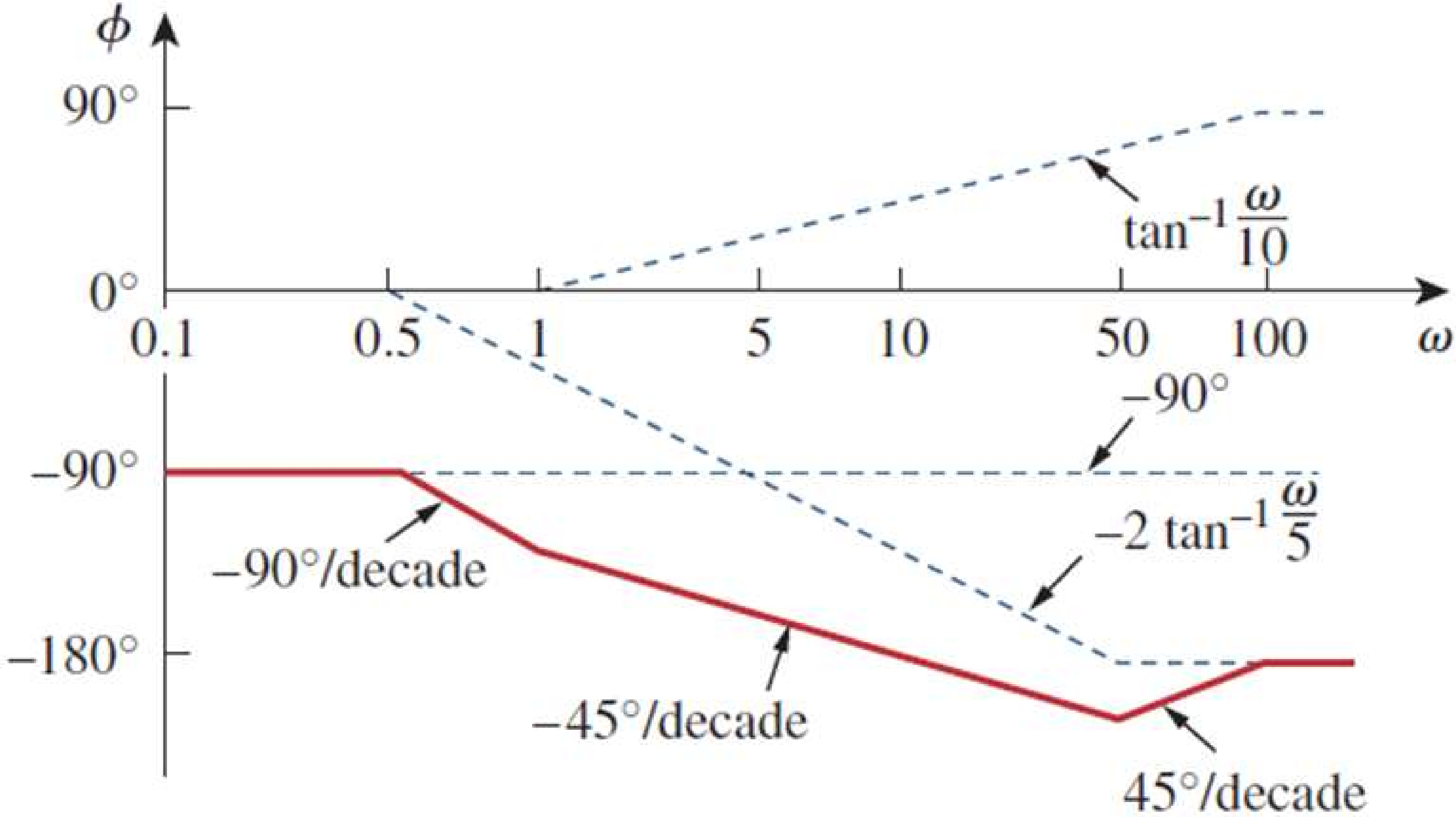
Bode Plot – Example 2

$$H_{dB} = 20log_{10}0.4 + 20log_{10}\left|1+\frac{j\omega}{10}\right| - 20log_{10}|j\omega| - 40log_{10}\left|1+\frac{j\omega}{5}\right|$$



Bode Plot – Example 2

$$\phi = 0^o + tan^{-1}\frac{\omega}{10} - 90^o - 2tan^{-1}\frac{\omega}{5}$$



Objectives

- Bode plot (continued)
- **LTI system response to co-sinusoids input**
- LTI system response to multifrequency co-sinusoids input
- Fourier coefficients of periodic signals
- Simplification by Symmetrical considerations
- Circuit interpretation for Fourier series

LTI system response to co – sinusoids

For the steady – state systems,

$$\cos(\omega t) \longrightarrow \boxed{\text{LTI}} \longrightarrow |\mathrm{H}(\omega)|\cos(\omega t + \angle H(\omega))$$

$$\sin(\omega t) \longrightarrow \boxed{\text{LTI}} \longrightarrow |\mathrm{H}(\omega)|\sin(\omega t + \angle H(\omega))$$

and for linearity,

Scaled input $\longrightarrow \boxed{\text{Linear}} \longrightarrow$ **Scaled output**

time-delayed input $\longrightarrow \boxed{\text{Linear}} \longrightarrow$ **time-delayed output**

LTI system response to co – sinusoids

From this we can conclude that,

$$|F|\cos(\omega t+\theta) \longrightarrow \boxed{\text{LTI}} \longrightarrow |H(\omega)||F|cos(\omega t+\theta+\angle H(\omega))$$

and

$$|F|sin(\omega t+\theta) \longrightarrow \boxed{\text{LTI}} \longrightarrow |H(\omega)||F|sin(\omega t+\theta+\angle H(\omega))$$

LTI system response to co – sinusoids

It is interesting to note that,

- **The LTI systems converts their co-sinusoidal inputs of frequency ω into co-sinusoidal outputs having the same frequency and following amplitude and phase:**

  - **Output amplitude = Input amplitude *multiplied* by $|H(\omega)|$**

  - **Output phase = Input phase *plus* $\angle H(\omega)$**

Co – sinusoids input: Example 3

Question: **Determine the steady-state system responses $y_1(t)$ and $y_2(t)$ for given driving functions $f_1(t)$ and $f_2(t)$ for transfer function $H(\omega)$?**

$$H(\omega) = \frac{1}{1 + j\omega} = \frac{1}{\sqrt{1 + \omega^2}} \angle - tan^{-1}(\omega)$$

$$f_1(t) = 1\cos(0.5t)\,V$$

$$f_2(t) = 1\cos(2t)\,V$$

Co – sinusoids input: Example 3

Solution:

We apply the input-output relation previously shown,

$$y_1(t) = \underbrace{\boxed{|H(0.5)|}}_{\text{Magnitude}} 1\cos\left(0.5t + \underbrace{\boxed{\angle H(0.5)}}_{\text{Phase}}\right)\ V$$

$$H(0.5) = \frac{1}{\sqrt{1+0.5^2}} = 0.894$$

$$\angle H(0.5) = -tan^{-1}(0.5) = -26.56^o$$

$$y_1(t) = 0.894\cos(0.5t - 26.56^o)\ V$$

Co – sinusoids input: Example 3

Similarly, for *$f_1(t)$*,

$$y_2(t) = \underbrace{\boxed{|H(2)|}}_{\text{Magnitude}} 1\cos\left(2t + \underbrace{\boxed{\angle H(2)}}_{\text{Phase}}\right)\ V$$

$$H(2) = \frac{1}{\sqrt{1+2^2}} = 0.447$$

$$\angle H(2) = -tan^{-1}(2) = -63.46^o$$

$$y_2(t) = 0.447\cos(2t - 63.46^o)\ V$$

Co – sinusoids input: Example 4

Question: **An LTI system has system response as low pass filter denoted as *H(ω)* converts input**

$$f(t) = 2\sin(12t)$$

into a steady-state output

$$y(t) = \sqrt{2}\sin(12t + \theta)$$ **for some real valued θ**

Determine *H(12)* and also compare the *average power* that the *f(t)* and *y(t)* would deliver to 1 Ω resistor?

Co – sinusoids input: Example 4

Solution: First we write,

$$|H(12)| = \frac{Y}{F} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

From the statement of the question ,

$$H(12) = \frac{1}{\sqrt{2}} e^{j\theta}$$

The average power per ohm for input *f(t)*,

$$P_f = \frac{1}{2}|F|^2 = 2$$

While for output,

$$P_y = \frac{1}{2}|Y|^2 = 1$$

Co – sinusoids input: Example 4

Comparing both input and output ,

$$\frac{P_y}{P_f} = \frac{1}{2}$$

which makes $\omega = 12 \frac{rad}{sec}$, the half power frequency of the filter $H(\omega)$

Co – sinusoids input: Example 5

Question: What is the steady-state response of the system

$$H(\omega) = \frac{2 + j\omega}{4 + j\omega}$$ to a DC input $f(t) = 5$?

Solution: Since,

$$H(0) = \frac{2 + j0}{4 + j0} = 0.5$$

The steady-state response will be,

$$y(t) = H(0)5 = 2.5$$

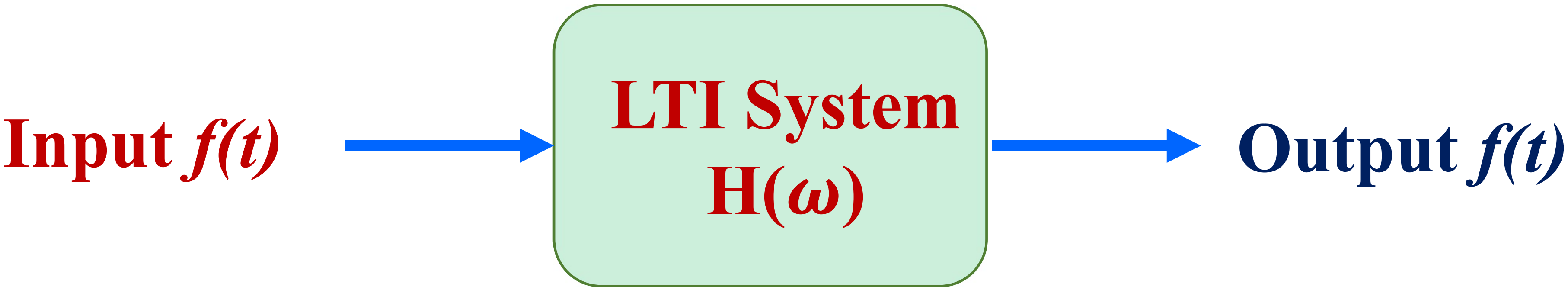
Objectives

- Bode plot (continued)
- LTI system response to co-sinusoids input
- LTI system response to multifrequency co-sinusoids input
- Fourier coefficients of periodic signals
- Simplification by Symmetrical considerations
- Circuit interpretation for Fourier series

LTI Sys. response: Multi-frequency Input

We will apply following methods to evaluate LTI response to multifrequency input

- **Principle of Superposition**
- **Input-output relation for co-sinusoids**



$$f(t) = \sum_{n=1}^{N} |F_n| \cos(\omega_n t + \theta_n)$$

$$y(t) = \sum_{n=1}^{N} |H(\omega_n)||F_n| \cos(\omega_n t + \theta_n + \chi(\omega_n))$$

Multi-frequency Input – Example 6

Question: **The input of a low pass filter**

$$H(\omega) = \frac{1}{1 + j\omega}$$

is $f(t) = 1cos(0.5t) + 1cos(\pi t)$

Determine the system output *y(t)*?

Multi-frequency Input – Example 6

Solution: **According to relation shown for multifrequency,**

$$y(t) = y_1(t) + y_2(t)$$

$$y(t) = |H(0.5)|1\cos\big(0.5t + \angle H(0.5)\big) + |H(\pi)|1\cos\big(\pi t + \angle H(\pi)\big)$$

For $y_1(t)$

$$H(0.5) = \frac{1}{\sqrt{1 + 0.5^2}} = 0.894$$

$$\angle H(0.5) = -tan^{-1}(0.5) = -26.56^o$$

$$y_1(t) = 0.894\cos(0.5t - 26.56^o)\ V$$

Multi-frequency Input – Example 6

For $y_2(t)$

$$H(\pi) = \frac{1}{\sqrt{1+\pi^2}} = 0.303$$

$$\angle H(\pi) = -tan^{-1}(\pi) = -72.34^o$$

$$y_2(t) = 0.303\cos(\pi t - 72.34^o)\ V$$

$$y(t) = y_1(t) + y_2(t)$$

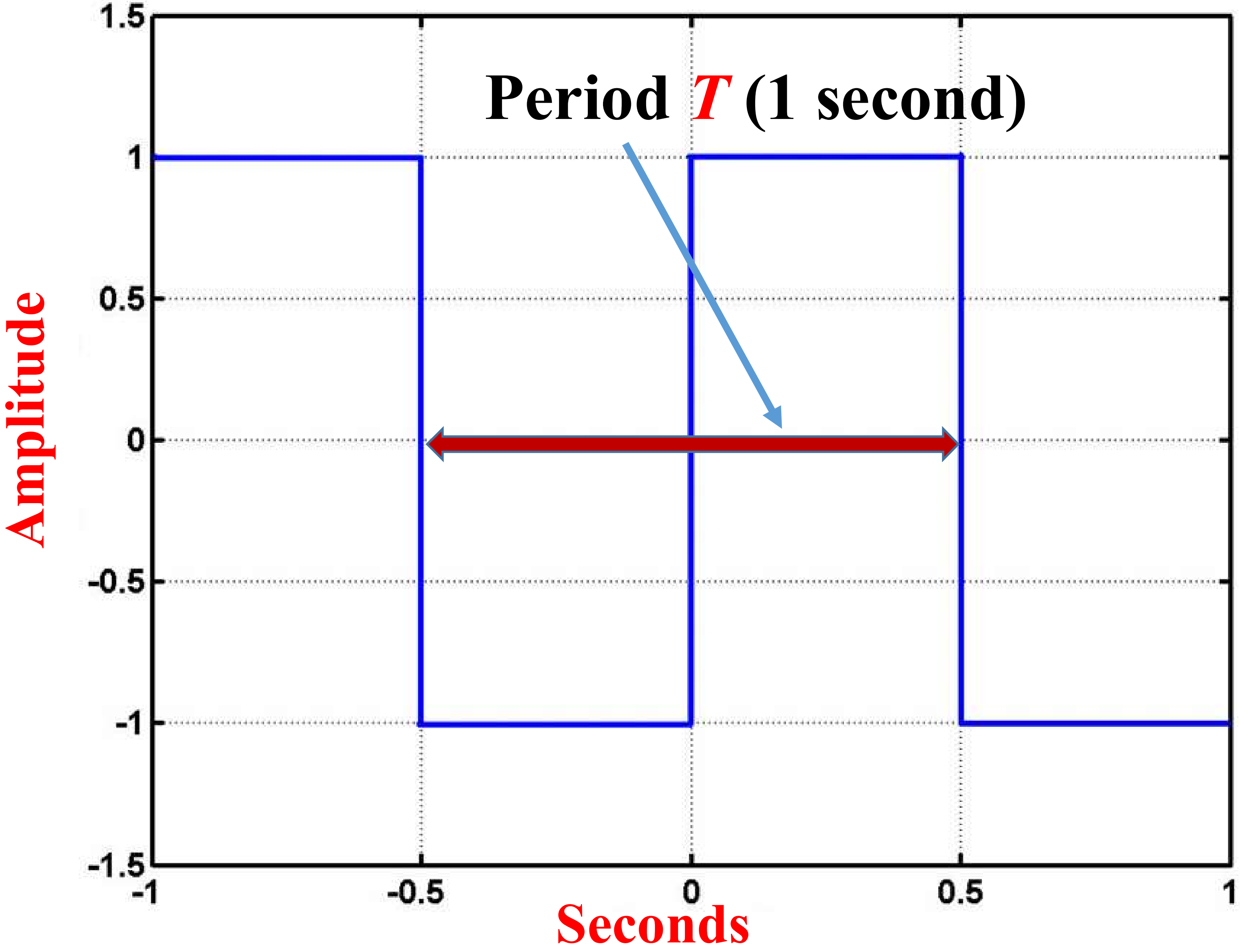
$$y(t) = 0.894\cos(0.5t - 26.56^o) + 0.303\cos(\pi t - 72.34^o)\ \mathrm{V}$$

Objectives

- Bode plot (continued)
- LTI system response to co-sinusoids input
- LTI system response to multifrequency co-sinusoids input
- **Fourier coefficients of periodic signals**
- Simplification by Symmetrical considerations
- Circuit interpretation for Fourier series

Fourier Series and coefficients

Periodic Signal $f(t) = f(t + nT)$



Fourier Series and coefficients

Fourier stated that :

- **A non-sinusoidal periodic function can be expressed as an *infinite sum* of sinusoidal functions**

Periodic Signal: $f(t) = f(t + nT)$

Where n is any integer and T is the period of signal

- **Any practical periodic function of frequency ω_o can be expressed as an *infinite sum of sine or cosine functions* that are integral multiples of ω_o**

Fourier Series and coefficients

Periodic Signal is expressed as per *Fourier's theorem* as,

$$f(t) = a_0 + a_1 \cos\omega_0 t + b_1 \sin\omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

which can be reduced to,

$$f(t) = \underbrace{\boxed{a_0}}_{\text{dc}} + \underbrace{\boxed{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}}_{\text{ac}}$$

$\omega_o = \frac{2\pi}{T}$ **is the fundamental frequency (rad/s)**

Fourier Series and coefficients

Decomposition

- $cosn\omega_o t$ and $\sin n\omega_o t$ are n-th harmonics of $f(t)$
- Even harmonic for even n and odd for odd n
- a_n and b_n are called *Fourier coefficients*
- a_o is the dc component of average value of $f(t)$
- a_n and b_n are the amplitudes of the sinusoids in the ac component

Fourier Series and coefficients

- **It resolves *f* (*t*) into a dc component and an ac component comprising an infinite series of harmonic sinusoids**

Dirichlet conditions to satisfy Fourier series

1. ***f(t)* should be single valued everywhere**
2. ***f(t)* has a finite number of finite discontinuities in any one period**
3. ***f(t)* has a finite number of maxima and minima in any one period**
4. **The integral $\int_{t_o}^{t_o+T} |f(t)|dt < \infty$ for any t_o**

Fourier Series and coefficients

DC component (constant)

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

AC component Harmonics

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

Fourier Series and coefficients

Another form of expressing Fourier series is in terms of amplitude and phase components,

$$f(t) = a_0 + \sum_{n=1}^{\infty} \boxed{A_n} \cos(n\omega_0 t + \boxed{\phi_n})$$

$$\boxed{A_n = \sqrt{a_n^2 + b_n^2},} \qquad \boxed{\phi_n = -\tan^{-1}\frac{b_n}{a_n}}$$

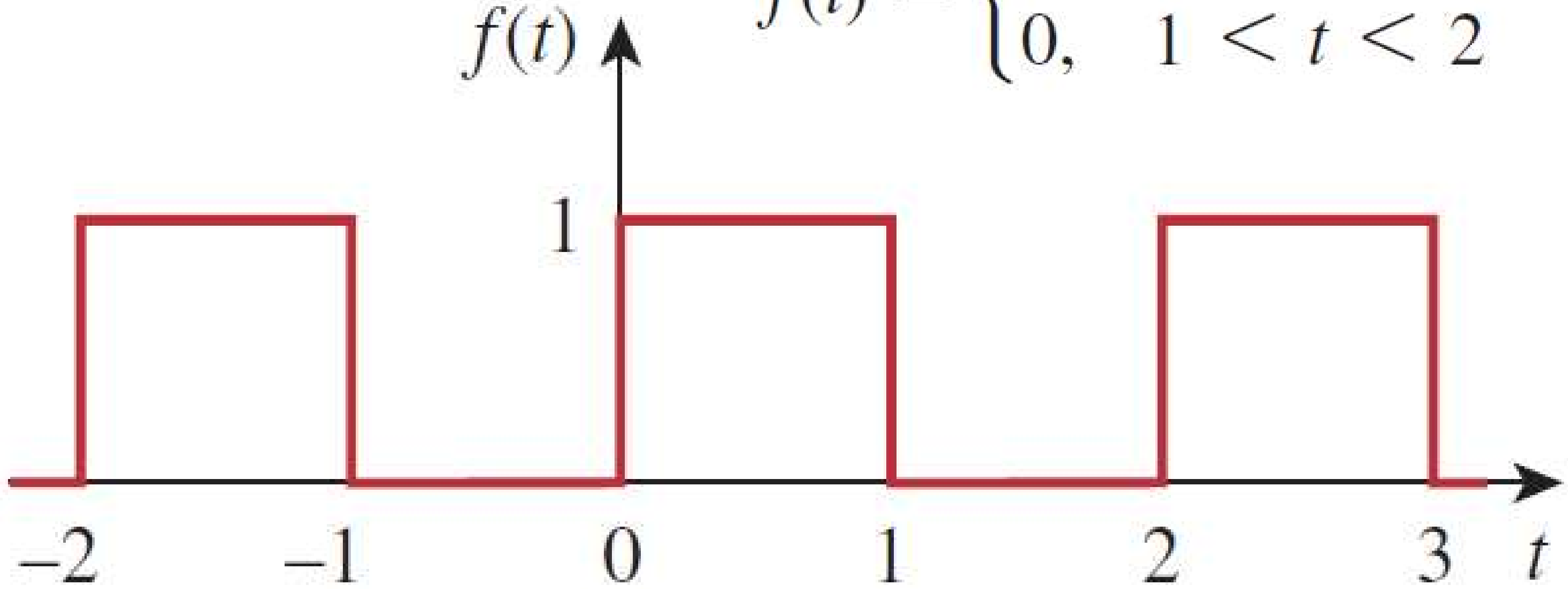
Or, $A_n \underline{/\phi_n} = a_n - jb_n$

- **The frequency spectrum of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency**

Fourier Series – Example 7

Question: **Determine the Fourier series of the waveform shown . Obtain the amplitude and phase spectra?**

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$



Fourier Series – Example 7

Solution: **From the waveform,**

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \; as \; T = 2 \; s$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}t\,\Big|_0^1 = \frac{1}{2}$$

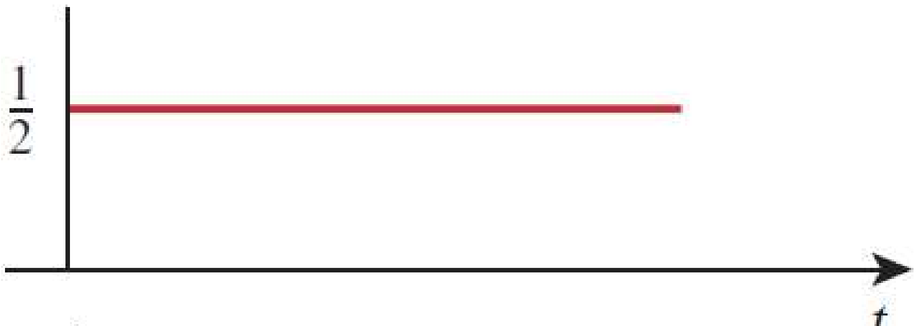
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

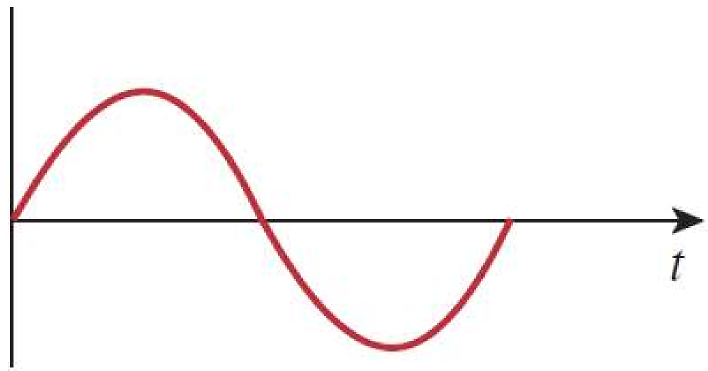
Fourier Series – Example 7

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \cdots$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k - 1$$



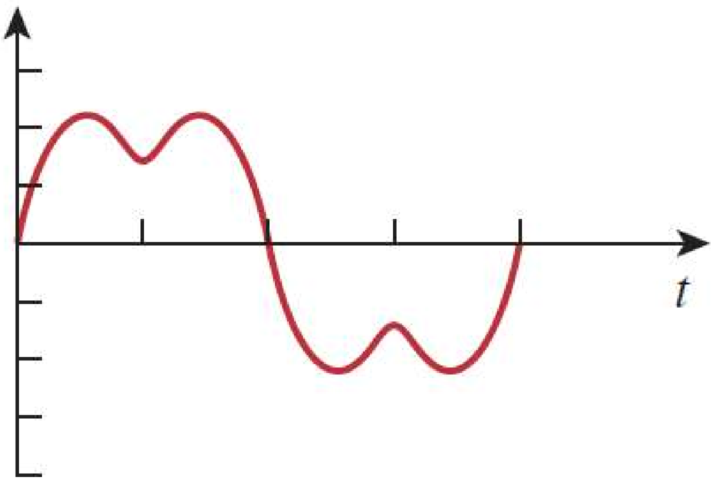
dc component



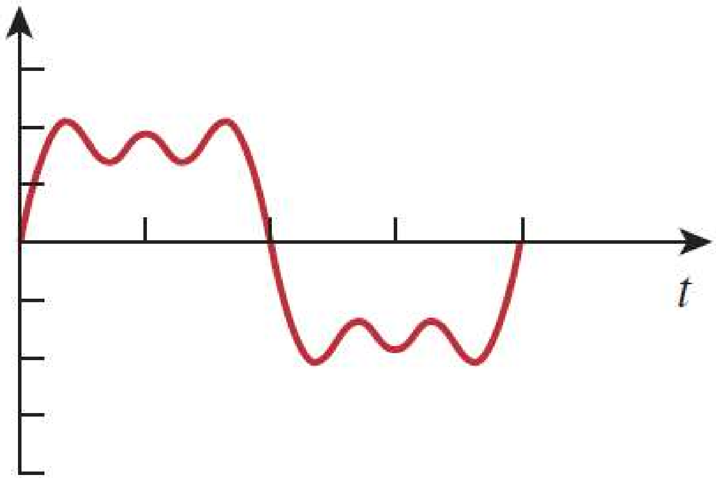
Fundamental ac component

Fourier Series – Example 7

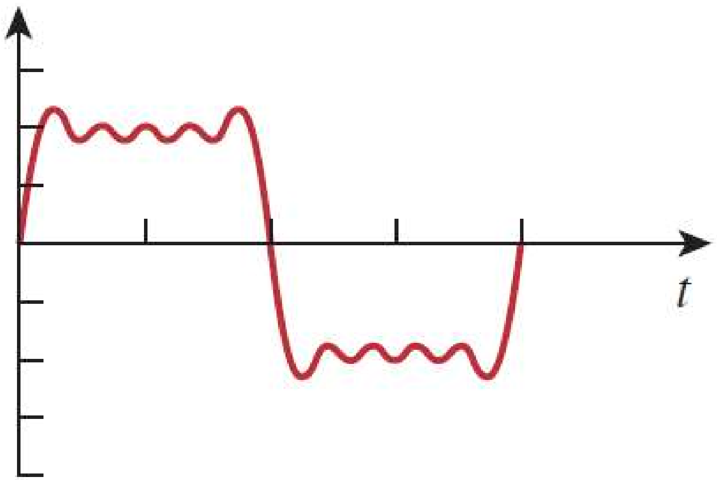
ac components harmonics



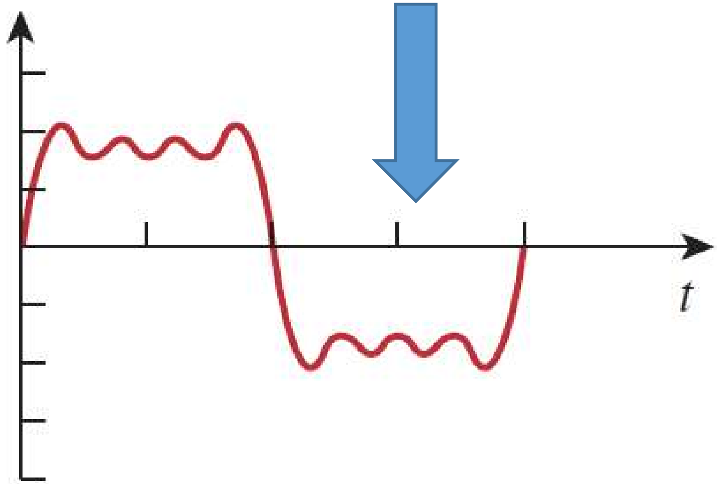
Sum of first two ac components



Sum of first three ac components



Sum of first five ac components



Sum of first four ac components

Fourier Series – Example 7

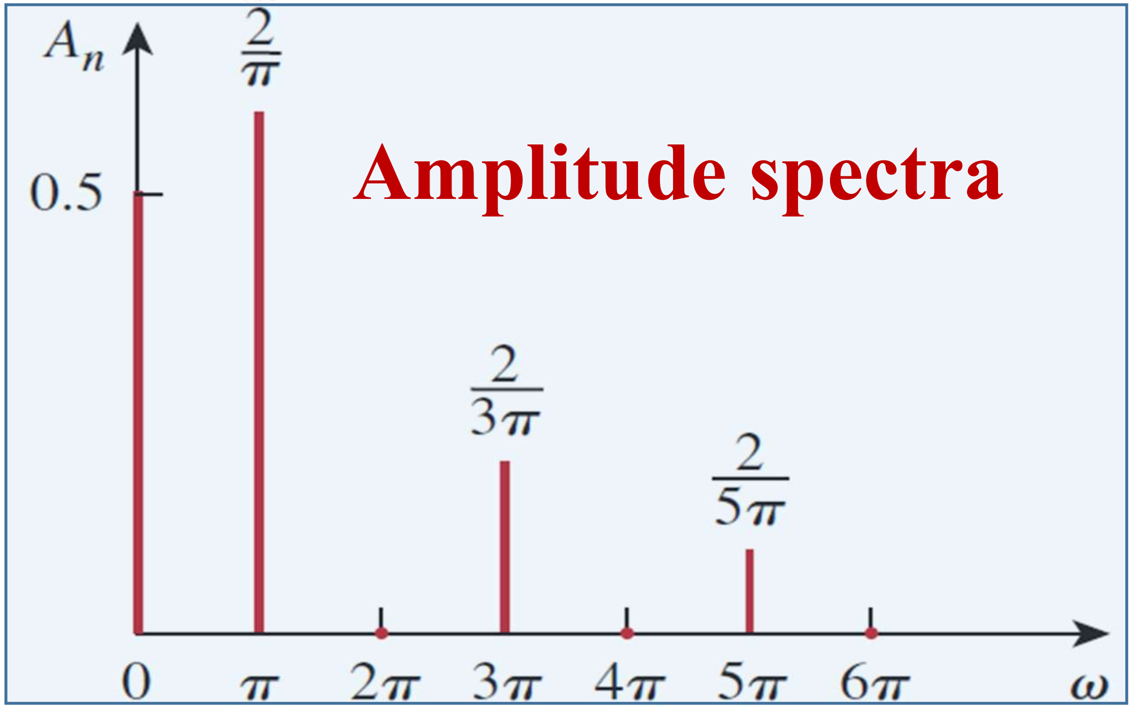
Another approach

Amplitude and phase spectra for the signal

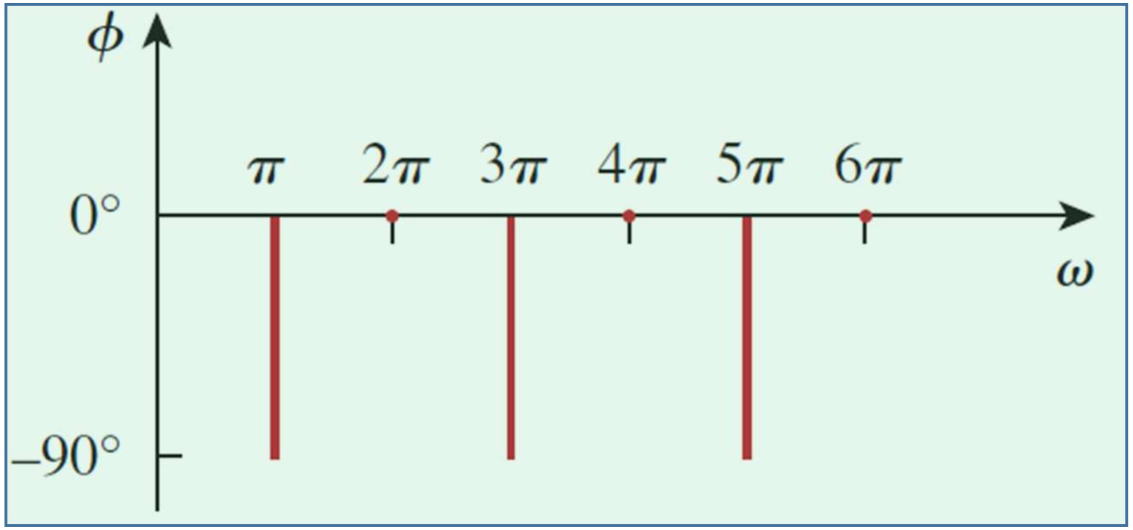
$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$\phi_n = -\tan^{-1}\frac{b_n}{a_n} = \begin{cases} -90^\circ, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Fourier Series – Example 7



Phase spectra



Objectives

- Bode plot (continued)
- LTI system response to co-sinusoids input
- LTI system response to multifrequency co-sinusoids input
- Fourier coefficients of periodic signals
- Simplification by Symmetrical considerations
- Circuit interpretation for Fourier series

Symmetry considerations

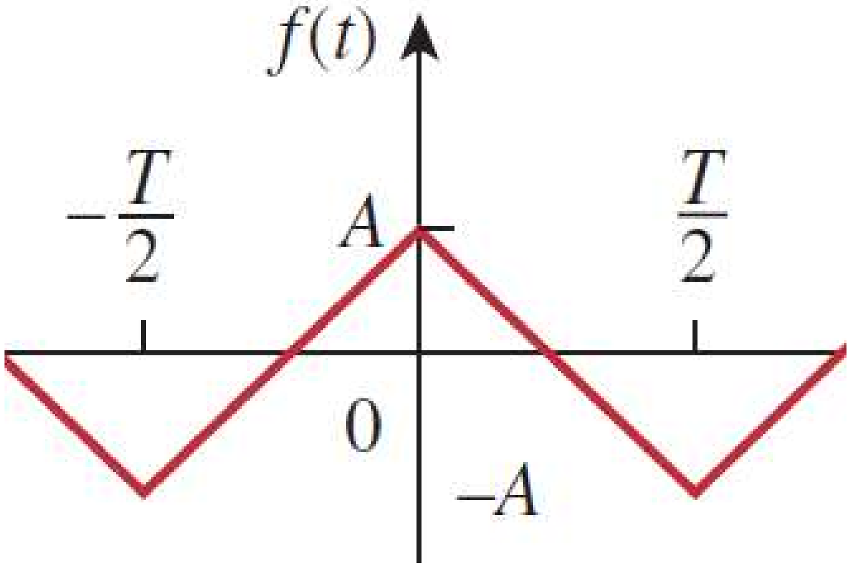
- **To avoid tedious calculations of integrals, symmetry is considered as it exists for sinusoidal and co – sinusoidal**

  - **Even Symmetry**
  - **Odd Symmetry**
  - **Half-wave symmetry**

Even Symmetry

- **A function *f(t)* is *even* if its plot is symmetrical about the vertical axis; that is,**

$$\boxed{f(t) = f(-t)}$$



$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

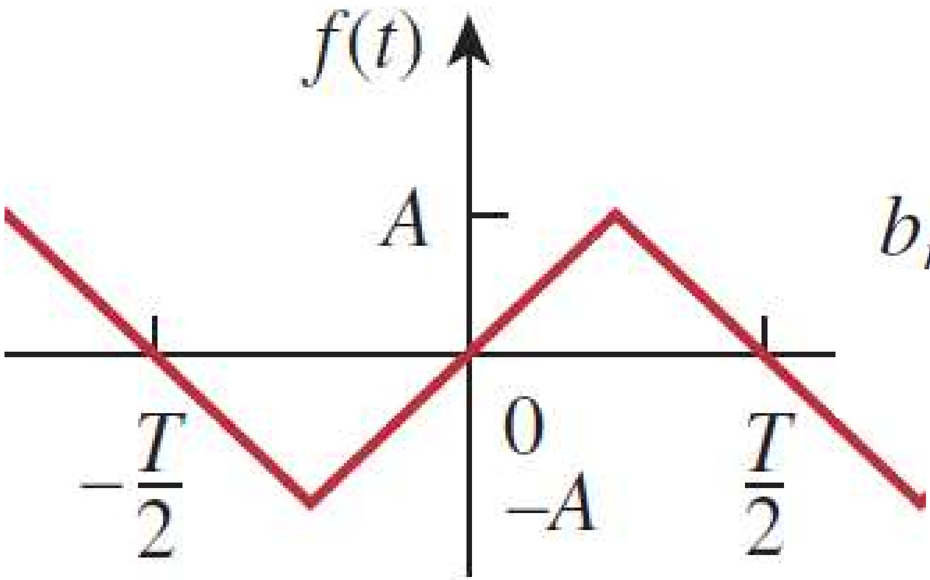
$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = 0$$

Odd Symmetry

- **A function *f(t)* is *odd* if its plot is asymmetrical about the vertical axis; that is,**

$$\boxed{f(-t) = -f(t)}$$



$$a_0 = 0, \qquad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$$

Half – wave Symmetry

A function is half-wave (odd) symmetric if

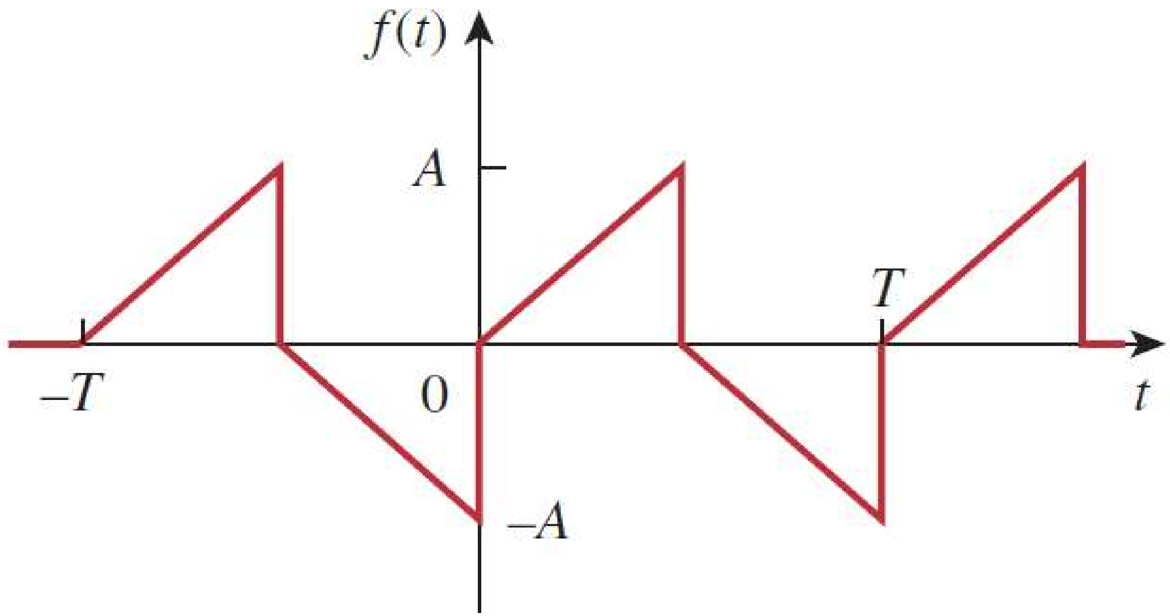
$$\boxed{f\left(t-\frac{T}{2}\right)=-f(t)}$$

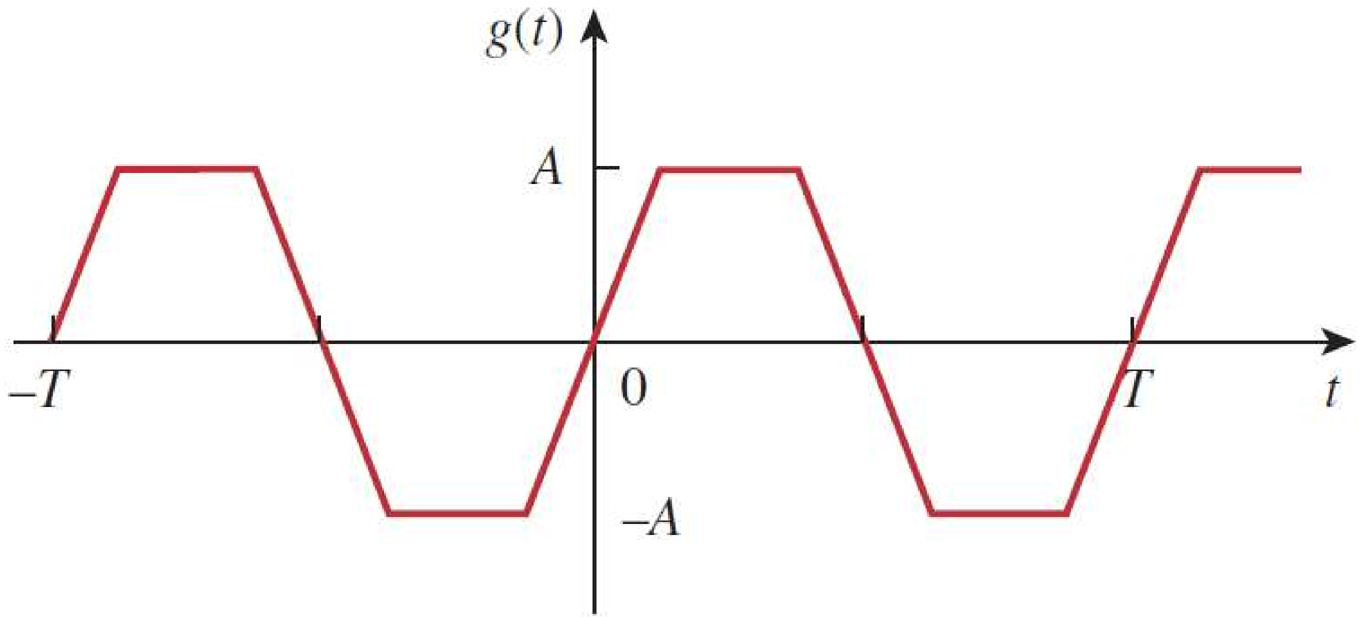
$$a_0 = 0$$

$$a_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\,dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

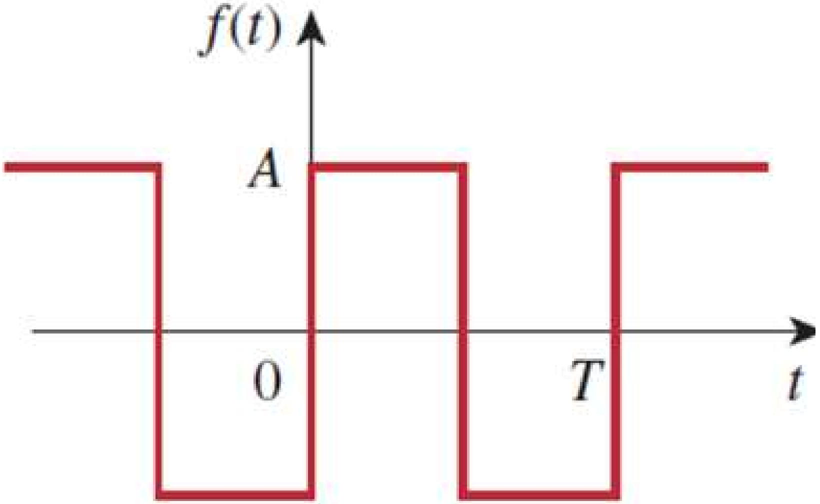
$$b_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin n\omega_0 t\,dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

Half – wave Symmetry

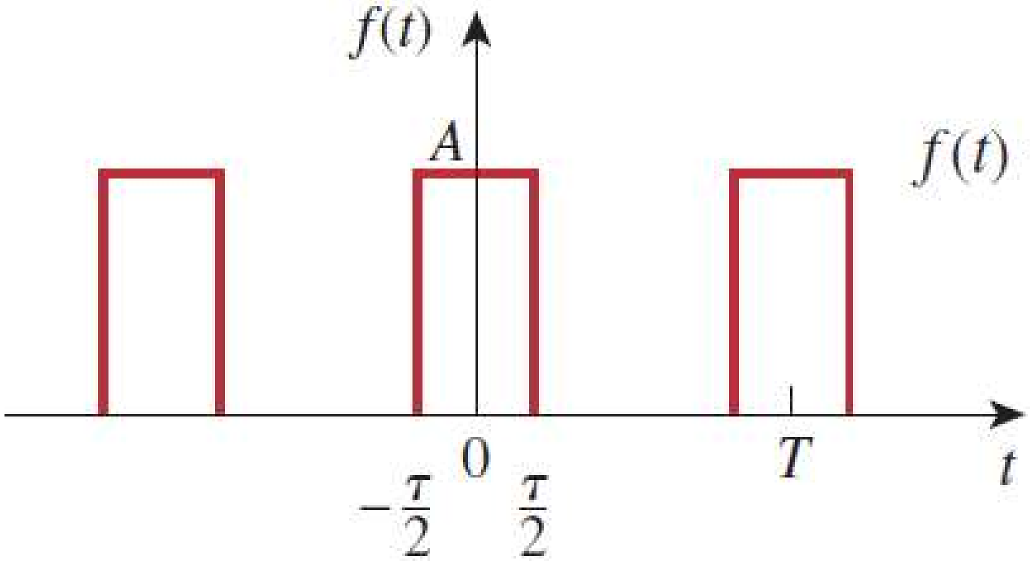




Fourier series of typical signals

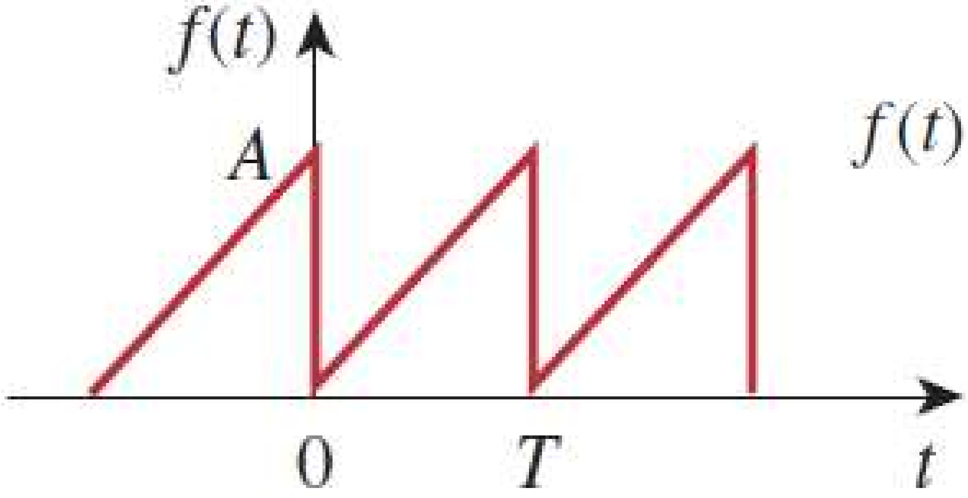


$$f(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)\omega_0 t$$

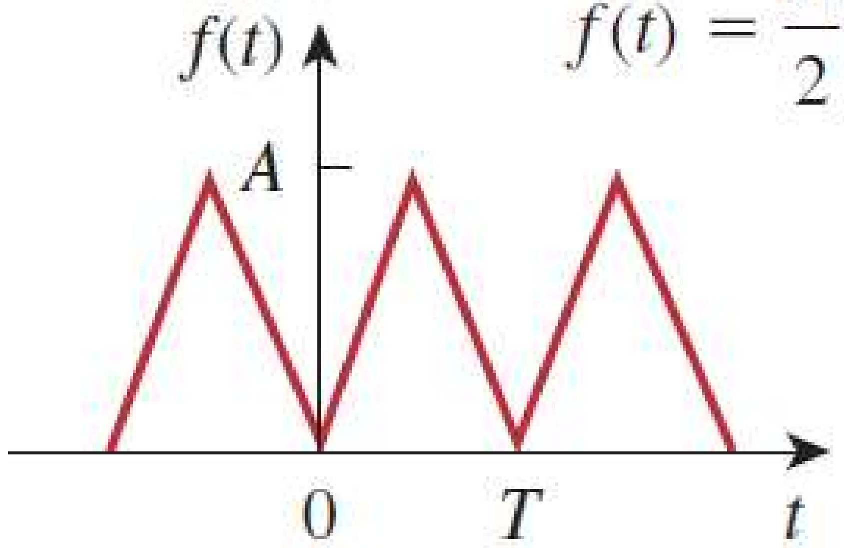


$$f(t) = \frac{A\tau}{T} + \frac{2A}{T} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi\tau}{T} \cos n\omega_0 t$$

Fourier series of typical signals



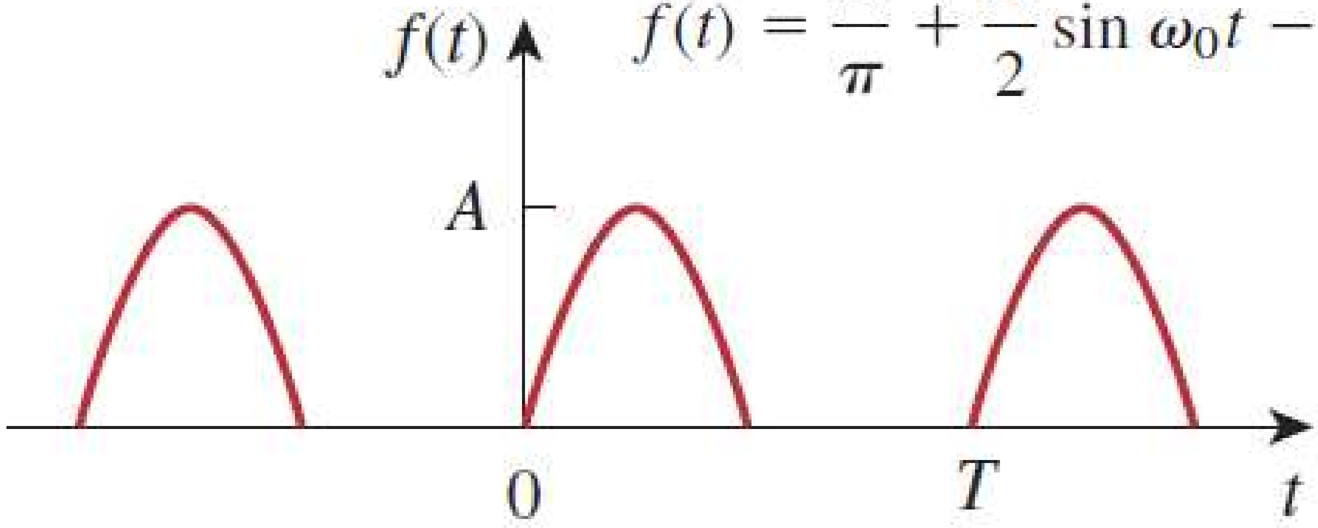
$$f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega_0 t}{n}$$



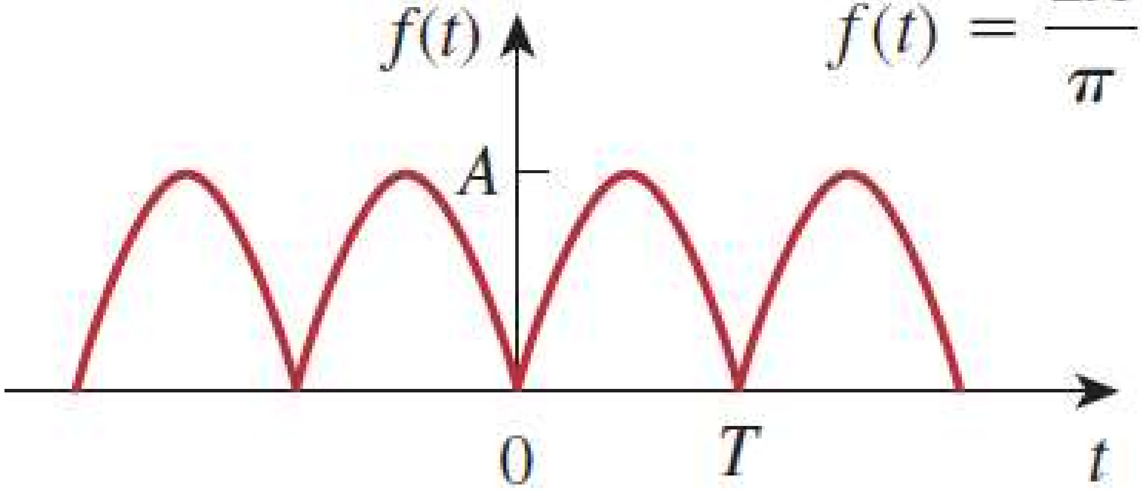
$$f(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\omega_0 t$$

Fourier series of typical signals

$$f(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega_0 t - \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2n\omega_0 t$$

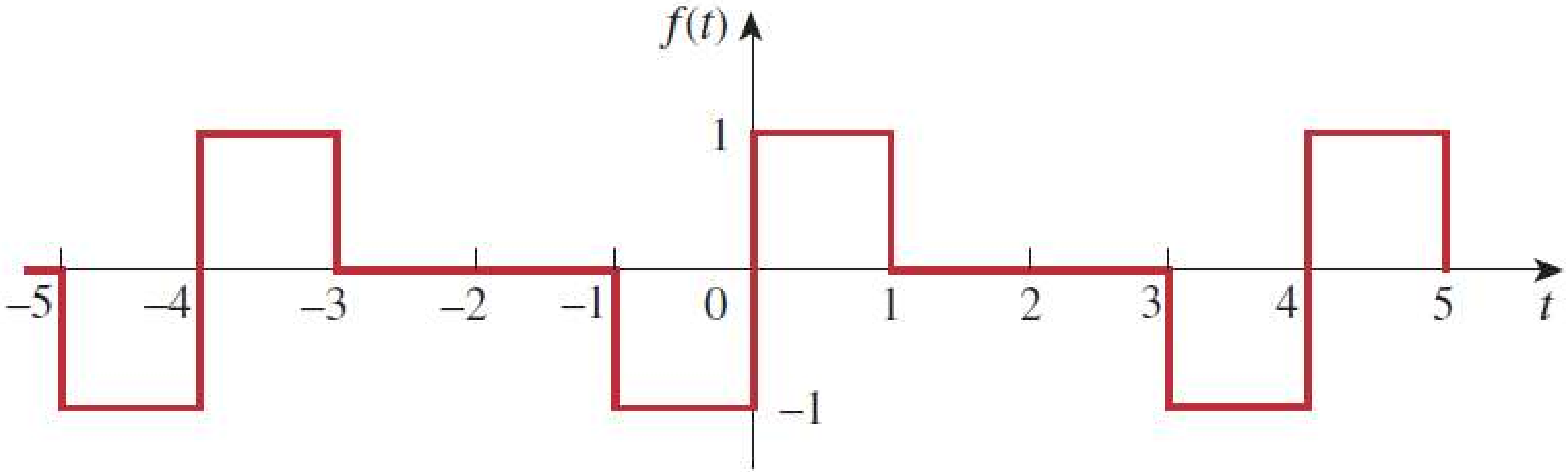


$$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos n\omega_0 t$$



Symmetrical Signals – Example 8

Question: **Find Fourier series expansion for the given *f(t)*?**



Symmetrical Signals – Example 8

Solution: The function *f(t)* is an odd function, So,

$$a_o = a_n = 0$$

The period is T = 4, hence, $\omega_o = {}^{\pi}/_{2}$ so that,

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^1 = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$\boxed{f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}t}$$

Objectives

- Bode plot (continued)
- LTI system response to co-sinusoids input
- LTI system response to multifrequency co-sinusoids input
- Fourier coefficients of periodic signals
- Simplification by Symmetrical considerations
- Circuit interpretation for Fourier series

Circuit application of Fourier series

Steps for Applying Fourier Series

1. Express the excitation as a Fourier series

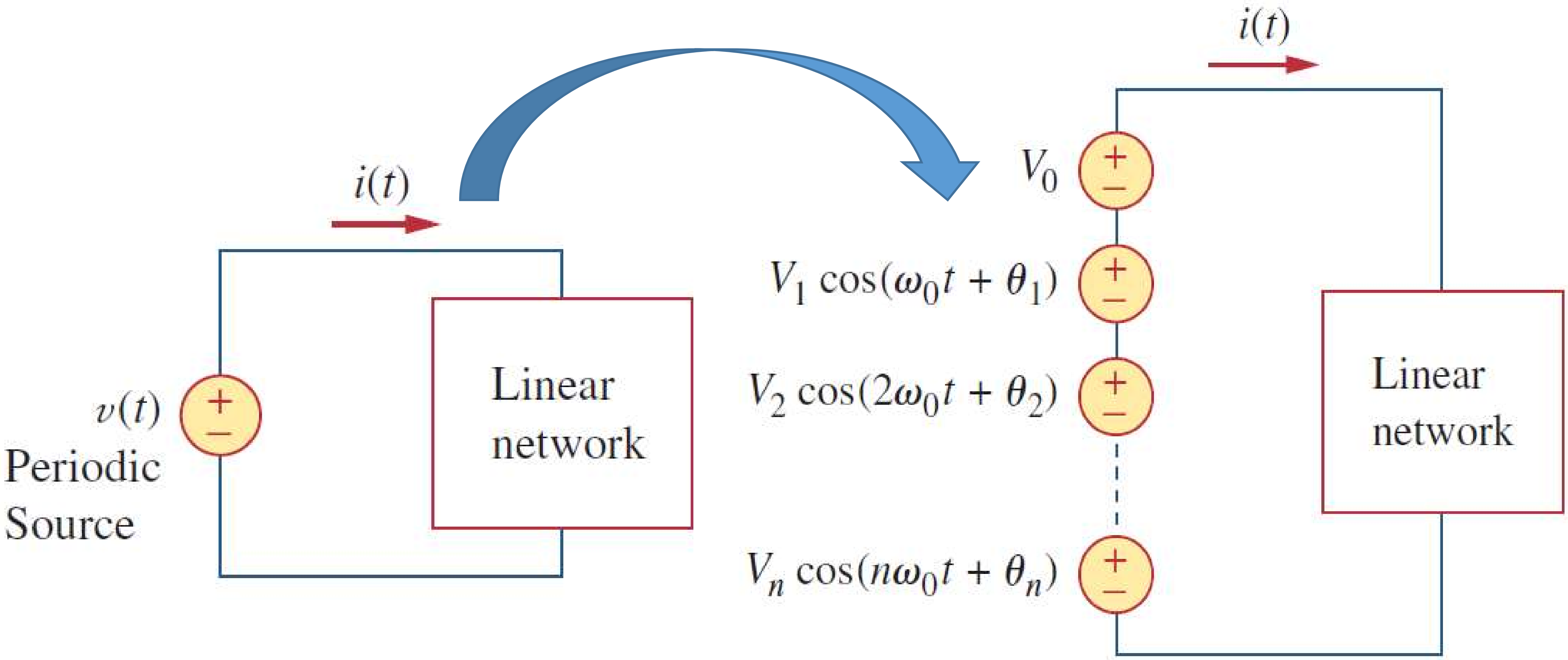
2. Transform the circuit from the time domain to the frequency domain

3. Find the response of the dc and ac components in the Fourier series

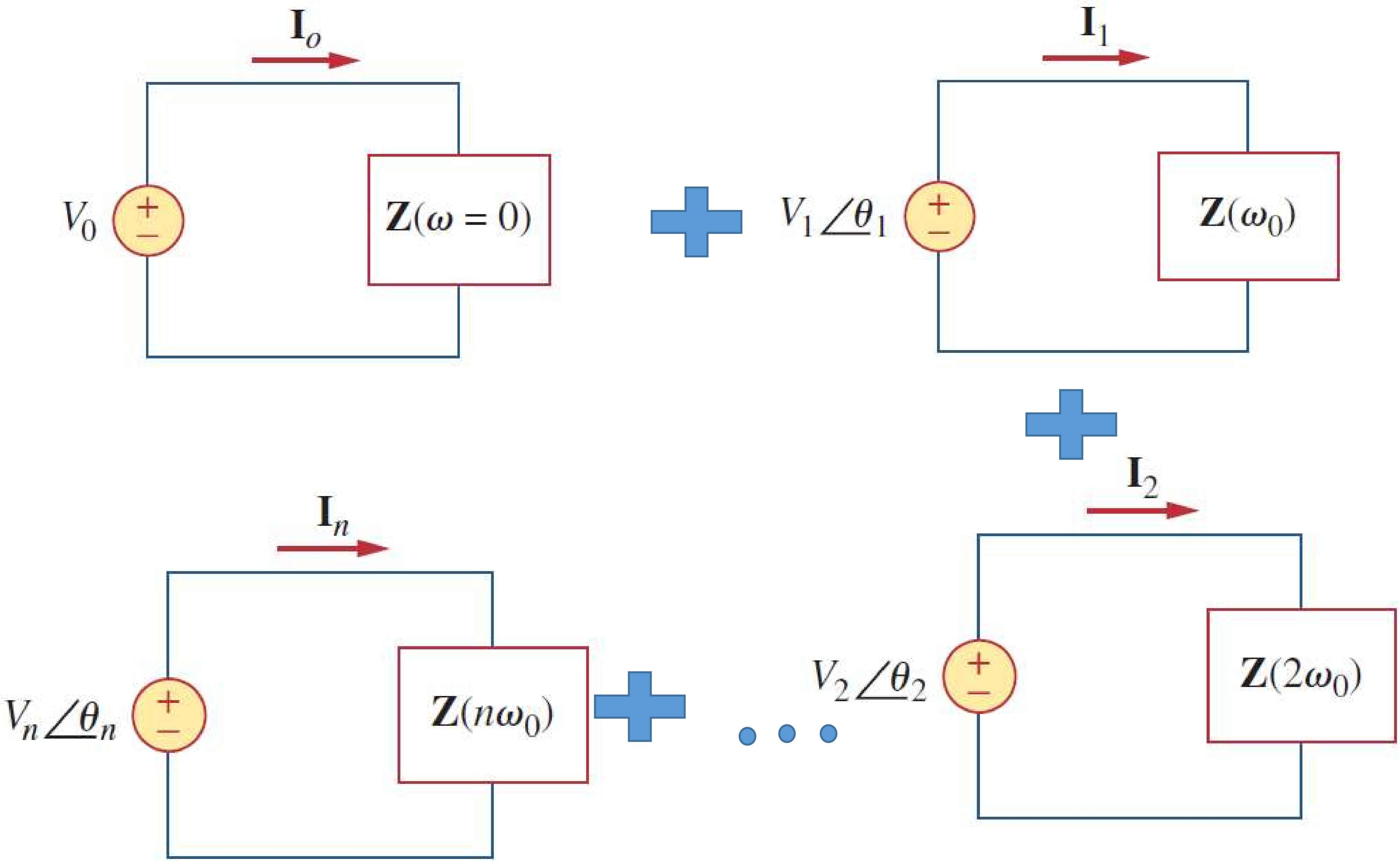
4. Add the individual dc and ac responses using the superposition principle

Circuit application of Fourier series

Superposition of all phasors of periodic sources



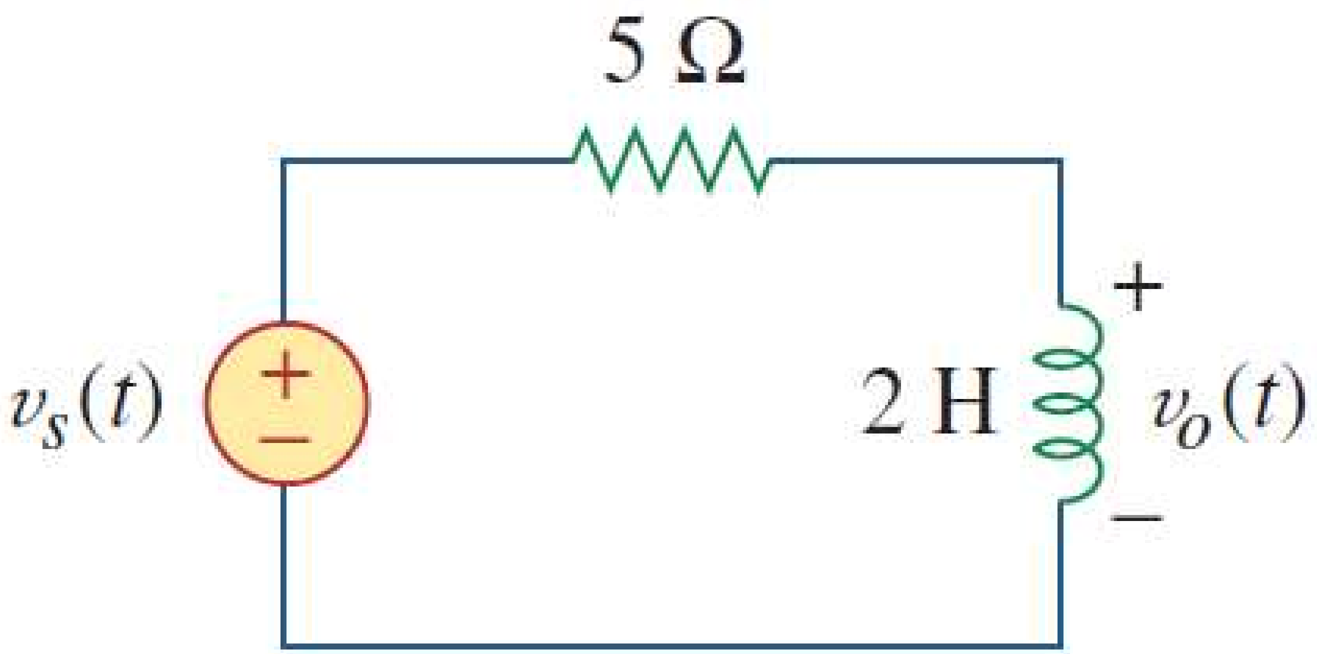
Circuit application of Fourier series



Circuit application – Example 9

Question: **Find $v_o(t)$ in the circuit by using value of $v_s(t)$ given below?**

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k - 1$$



Circuit application – Example 9

Solution: By using voltage division rule,

$$\mathbf{V}_o = \frac{j\omega_n L}{R + j\omega_n L}\mathbf{V}_s = \frac{j2n\pi}{5 + j2n\pi}\mathbf{V}_s$$

For DC component , $\omega_o = 0$ or $n = 0$,

$$V_s = \frac{1}{2}, \quad \text{so,} \quad V_o = 0$$

This is expected, since the inductor is a short circuit to dc. For the n-th harmonic,

$$\mathbf{V}_s = \frac{2}{n\pi}\angle -90^\circ$$

Circuit application – Example 9

and the corresponding response is,

$$\mathbf{V}_o = \frac{2n\pi \angle 90^\circ}{\sqrt{25 + 4n^2\pi^2} \angle \tan^{-1} 2n\pi/5} \left(\frac{2}{n\pi} \angle -90^\circ \right)$$

$$= \frac{4 \angle -\tan^{-1} 2n\pi/5}{\sqrt{25 + 4n^2\pi^2}}$$

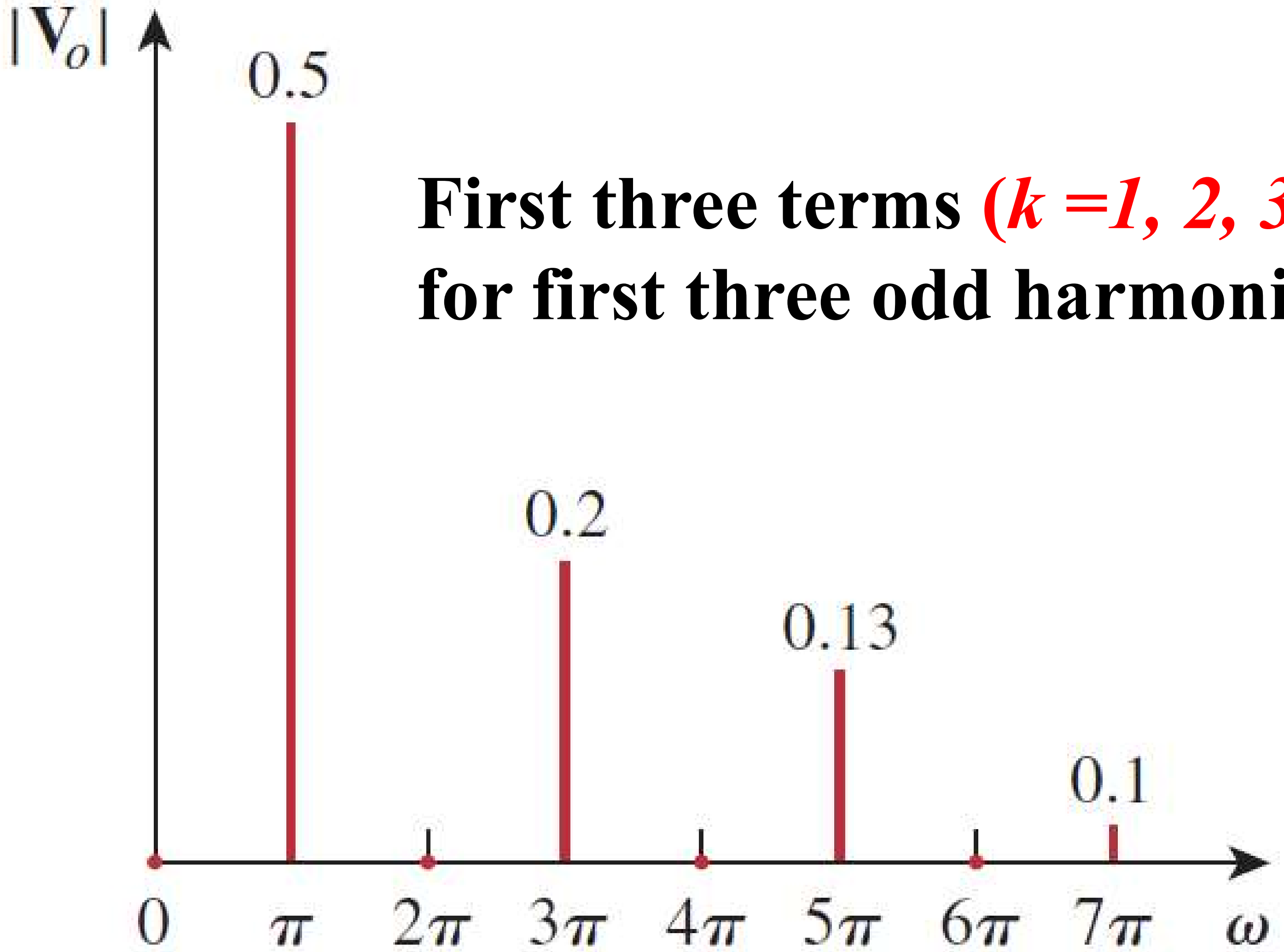
converting into time domain,

$$v_o(t) = \sum_{k=1}^{\infty} \frac{4}{\sqrt{25 + 4n^2\pi^2}} \cos\left(n\pi t - \tan^{-1} \frac{2n\pi}{5} \right),$$

$$n = 2k - 1$$

Circuit application – Example 9

$$v_o(t) = 0.4981\cos(\pi t - 51.49°) + 0.2051\cos(3\pi t - 75.14°) + 0.1257\cos(5\pi t - 80.96°) + \cdots \text{ V}$$



First three terms *(k =1, 2, 3 or n = 1, 3, 5)* for first three odd harmonics

System response – Example 10

Question: **The input of a linear system,**

$$H(\omega) = \frac{2 + j\omega}{3 + j\omega}$$

is the periodic function,

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{n}{1 + n^2} e^{-jn4t}$$

Find the Fourier coefficients of Y_n of the periodic system output *y(t)*?

Circuit application –Example 10

Solution: **The input Fourier coefficients are,**

$$F_n = \frac{n}{1+n^2}$$

and the fundamental frequency is $\omega_o = 4\, rad/s$

$$H(n\omega_o) = H(n4) = \frac{2+jn4}{3+jn4}$$

and the Fourier coefficients of the system output Y_n

$$\boxed{Y_n = H(nw_o)F_n = \frac{2+jn4}{3+jn4} \cdot \frac{n}{1+n^2}}$$

Summary

- The Bode plot describes the magnitude and phase of transfer function in terms of function of frequency

- The LTI systems converts their co-sinusoidal inputs of frequency ω into co-sinusoidal outputs having the same frequency: amplitude multiplied and phase added for input output relation

- The linearity and superposition are key principles for applying multifrequency input co-sinusoids

Summary

- Any practical periodic function of frequency ω_o can be expressed as an infinite sum of sine or cosine functions that are integral multiples of ω_o
- The Dirichlet conditions are sufficient to satisfy Fourier series
- Using symmetrical simplification reduces tedious job of calculations in Fourier transform
- Superposition and input-output relation jointly solve Fourier series for periodic excitations

Further reading

1. Ch. 4 (page 166-181) and Ch. 6 (page 185-208), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.
2. Ch. 4 (page 613-619) and Ch. 17 (page 760-780), C. K. Alexander and M. Sadiku, ***Fundamentals of Electric Circuits***, 5th ed., McGraw-Hill, 2013.
3. Ch. 12 (page 597-607) and Ch. 15 (page 751-773), J. D. Irwin, and R. M. Nelms. ***Basic Engineering Circuit Analysis,*** 10th ed. Wiley, 2011.

Preview:

1. Ch. 7 (page226-247), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.

Homework 8

Deadline: 10:00 PM, 20th April, 2022

Thank you!