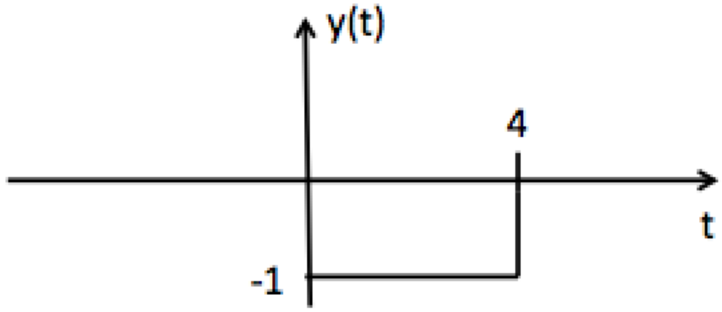
Zhejiang University – University of Illinois at Urbana-Champaign Institute

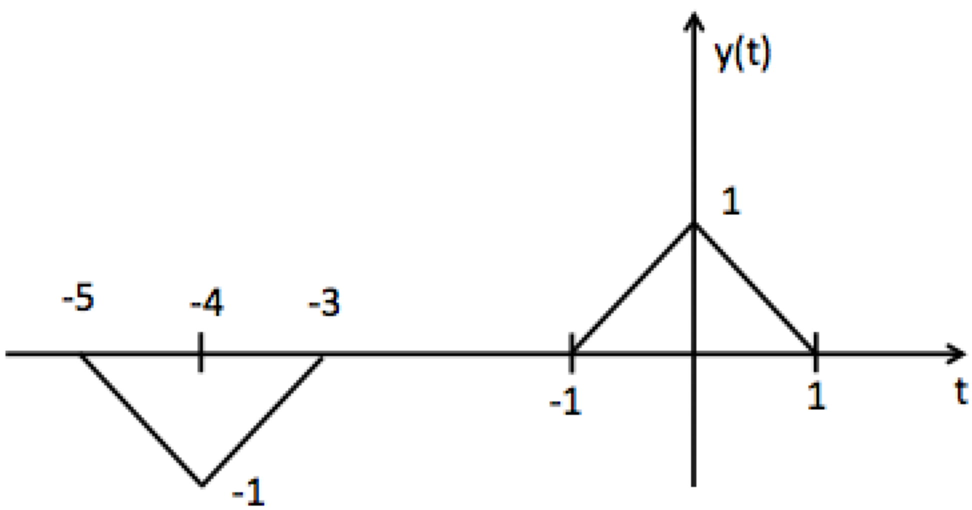
ECE-210 Analog Signal Processing Spring 2022

Homework #12: Solution

1. A system is described by an impulse response $h(t) = \delta(t-2) - \delta(t+2)$
   Sketch the system response $y(t) = h(t) * f(t)$ to the following inputs:
   (a) $f(t) = u(\frac{t-2}{2})$

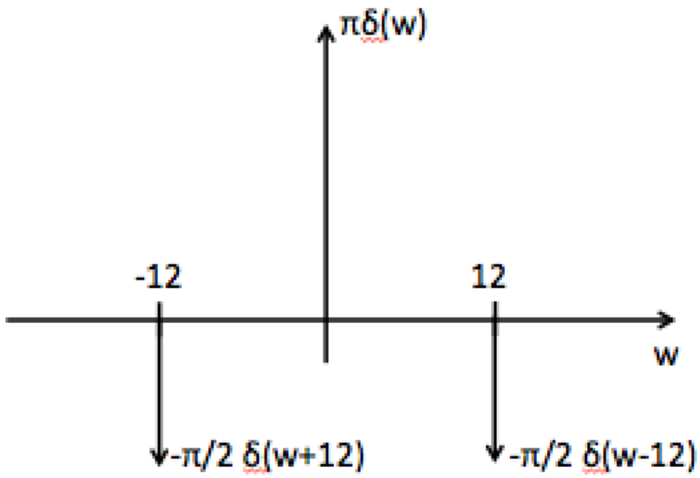


   (b) $f(t) = \triangle(\frac{t+2}{2})$



2. Determine the Fourier transform of the following signals —Simplify the results as much as possible.
   Sketch the result if it is real valued.
   Solution:
   (a) $f(t) = 4\cos(4t) + 3\sin(5t)$
   $f(t) = 4\cos(4t) + 3\sin(5t) \leftrightarrow 4\pi[\delta(w-4) + \delta(w+4)] + j3\pi[\delta(w+5) - \delta(w-5)]$
   This is not a real valued function

   (b) $x(t) = \sin^2(6t)$
   $x(t) = \sin^2(6t) = \frac{1}{2}(1 - \cos(12t)) \leftrightarrow \pi\delta(w) - \frac{\pi}{2}[\delta(w-12) + \delta(w+12)]$



(c) $y(t) = e^t u(-t) * \cos(2t)$
$y(t) \leftrightarrow \frac{1}{1-jw} \times \pi[\delta(w+2) + \delta(w-2)] = \frac{\pi}{1-2j}\delta(w-2) + \frac{\pi}{1+2j}\delta(w+2)$
This is not a real valued function

(d) $z(t) = [2 + 3\cos(2t)]\, e^{-t}\, u(t)$

$$\begin{aligned} y(t) &= [2 + 3\cos(2t)]e^{-t}u(t) \\ &= 2e^{-t}u(t) + 3\cos(2t)e^{-t}u(t) \\ &= \frac{2}{1+jw} + 3\frac{1+jw}{(1+jw)^2 + 4} \end{aligned}$$

This is not a real valued function

3. Determine the inverse Fourier transform of the following:
Solution:
(a) $F(w) = 3\pi[\delta(2w-2) - \delta(2w+2)] + 4\pi\delta(w)$
$F(w) = 3\pi[\delta(2w-2) - \delta(2w+2)] + 4\pi\delta(w)$
$F(w) = 3\pi[\delta(2(w-1)) - \delta(2(w+1))] + 4\pi\delta(w)$
$F(w) = \frac{3\pi}{2}[\delta(w-1) - \delta(w+1)] + 4\pi\delta(w)$
$f(t) = j\frac{3}{2}\sin(t) + 2$

(b) $A(w) = 2\pi\sin(5w)$
Using symmetry property
$a(t) = -\pi j[\delta(t+5) - \delta(t-5)]$

(c) $B(w) = \sum_{n=-\infty}^{\infty} 2\pi\frac{1}{1+n^2}\delta(2w-3n)$
Using frequency shift property for each individual delta function

$$\begin{aligned} f(t)e^{jw_0 t} &\leftrightarrow F(w-w_0) \\ B(w) &= \sum_{n=-\infty}^{\infty} 2\pi\frac{1}{1+n^2}\delta(2w-3n) \\ &\leftrightarrow \sum_{n=-\infty}^{\infty} \frac{1}{2}\cdot\frac{1}{1+n^2}e^{j\frac{3}{2}nt} \end{aligned}$$

(d) $C(w) = \frac{8}{jw-2} + 4\pi\delta(w)$
$c(t) = -8e^{2t}u(-t) + 2$

4. (a) Show that the following LTI systems with impulse responses:

$$h_1(t) = u(t)$$
$$h_2(t) = -2\delta(t) + 5e^{-2t}\, u(t)$$
$$h_3(t) = 2te^{-t}\, u(t)$$

All have the same response to $x(t) = \cos(t)$
Solution:

$$h_1(t) = u(t) \leftrightarrow H_1(w) = \pi\delta(w) + \frac{1}{jw}$$
$$h_2(t) = -2\delta(t) + 5e^{-2t}\, u(t) \leftrightarrow H_2(w) = -2 + \frac{5}{2+jw}$$
$$h_3(t) = 2te^{-t}\, u(t) \leftrightarrow H_3(w) = \frac{2}{(1+jw)^2}$$
$$x(t) = \cos(t) \leftrightarrow X(w) = \pi[\delta(w-1) + \delta(w+1)]$$

Clearly $X(w)H_1(w) = X(w)H_2(w) = X(w)H_3(w) = j\pi[\delta(w+1) - \delta(w-1)]$ since $H_1(w), H_2(w), H_3(w)$ evaluated at $w = \pm 1$ equal to $\mp\pi j$

(b) Find the impulse response of another LTI system with the same response to $x(t) = \cos(t)$
(This problem illustrates the fact that the response to $\cos(t)$ cannot be used to specify an LTI uniformly)

As long as the Fourier transform of the signal evaluated at $w = \pm 1$ equal to $\mp\pi j$, it will have the same response. One possible solution is $h_4(t) = u(t) - rect(\frac{t}{2\pi})$

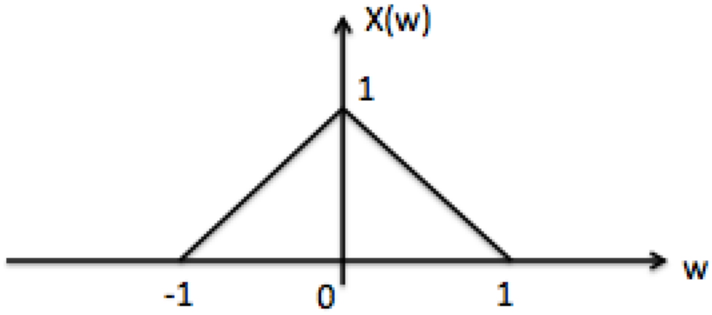
5. (a) Let $x(t)$ have the Fourier transform $\chi(w)$, and let $p(t)$ be periodic with fundamental frequency w_0 and Fourier series representation

$$p(t) = \sum_{n=-\infty}^{+\infty} P_n\, e^{jnw_0 t}$$

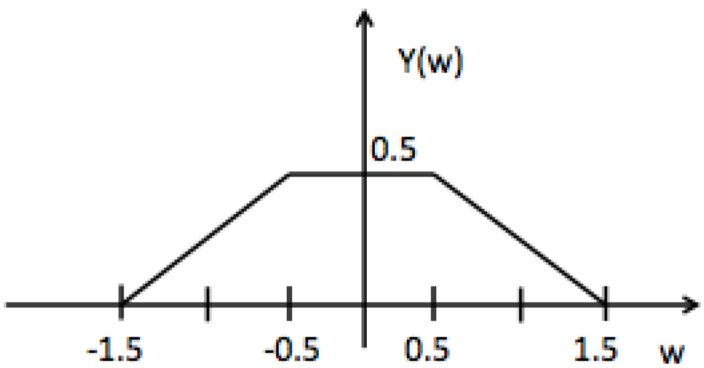
Determine an expression for the Fourier transform of $y(t) = x(t)\, p(t)$
Solution:

$$\begin{aligned} Y(w) &= \frac{1}{2\pi}\, X(w) * P(w) \\ &= \frac{1}{2\pi}\, X(w) * \sum_{n=-\infty}^{+\infty} P_n \cdot 2\pi\delta(w - nw_0) \\ &= \sum_{n=-\infty}^{+\infty} P_n \cdot X(w - nw_0) \\ &= \sum_{n=-\infty}^{+\infty} P_n \cdot \triangle(\frac{w - nw_0}{2}) \end{aligned}$$

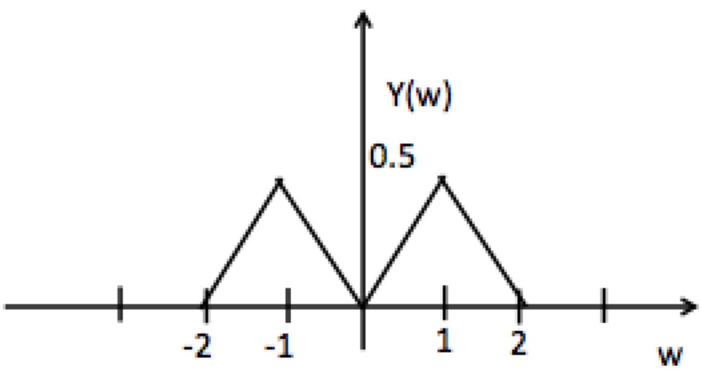
(b) Suppose that $\chi(w)$ is as depicted in the following figure:



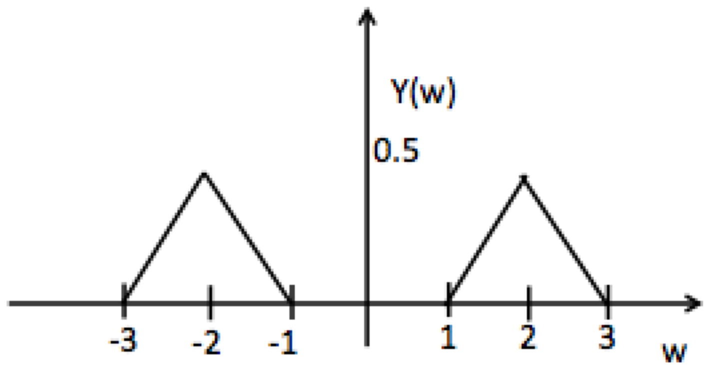
Sketch the spectrum of $y(t) = x(t)\,p(t)$ found in part(a) for each of the following choices of $p(t)$:
Solution:
(1) $p(t) = \cos(\frac{t}{2})$
$p(t) = \cos(\frac{t}{2}) = \frac{1}{2}e^{j\frac{1}{2}t} + \frac{1}{2}e^{-j\frac{1}{2}t}$
$Y(w) = \frac{1}{2}X(w - \frac{1}{2}) + \frac{1}{2}X(w + \frac{1}{2})$



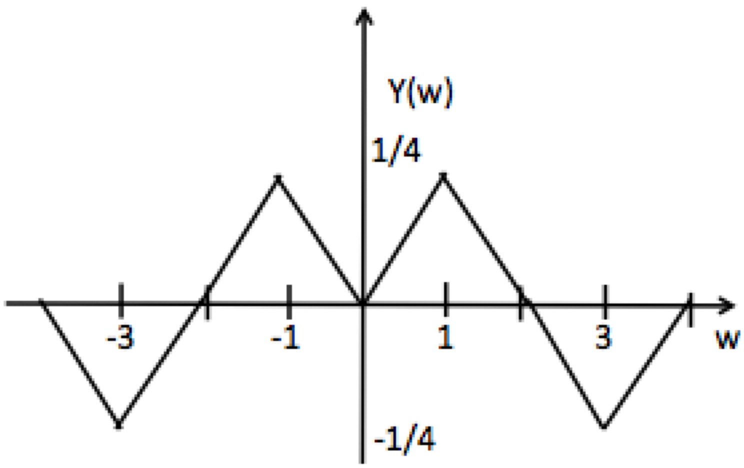
(2) $p(t) = \cos(t)$
$p(t) = \cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}$
$Y(w) = \frac{1}{2}X(w - 1) + \frac{1}{2}X(w + 1)$



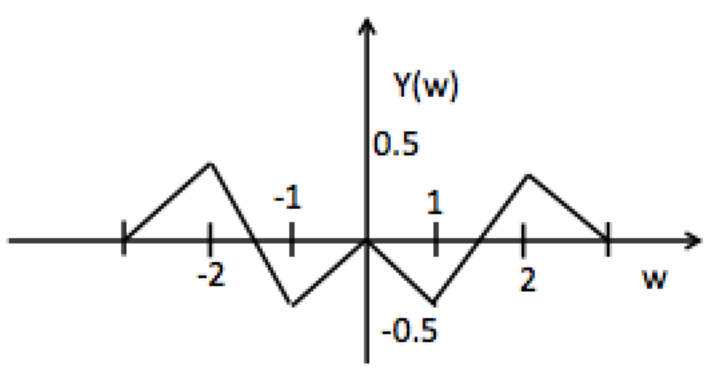
(3) $p(t) = \cos(2t)$
$p(t) = \cos(2t) = \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}$
$Y(w) = \frac{1}{2}X(w - 2) + \frac{1}{2}X(w + 2)$



(4) $p(t) = \sin(t)\,\sin(2t)$
$p(t) = \sin(t)\,\sin(2t) = \frac{1}{2}[\cos(t) - \cos(3t)]$
$Y(w) = \frac{1}{4}X(w-1) + \frac{1}{4}X(w+1) - \frac{1}{4}X(w-3) - \frac{1}{4}X(w+3)$



(5) $p(t) = \cos(2t) - \cos(t)$
$Y(w) = \frac{1}{2}X(w-2) + \frac{1}{2}X(w+2) - \frac{1}{2}X(w-1) - \frac{1}{2}X(w+1)$



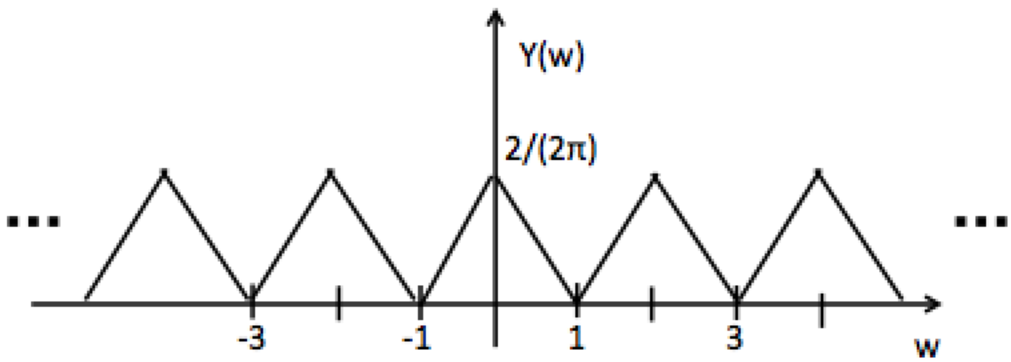
(6) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(w-n\frac{2\pi}{T})$$

$$P(w) = \frac{2\pi}{\pi} \sum_{n=-\infty}^{\infty} \delta(w-n\frac{2\pi}{\pi})$$

$$= 2 \sum_{n=-\infty}^{\infty} \delta(w-2n)$$

$$Y(w) = \frac{2}{2\pi} \sum_{n=-\infty}^{\infty} X(w-2n)$$

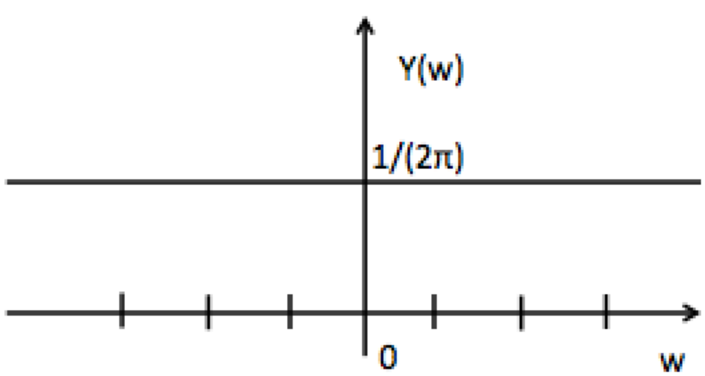


(7) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-2\pi n)$

$$P(w) = \frac{2\pi}{2\pi} \sum_{n=-\infty}^{\infty} \delta(w-n\frac{2\pi}{2\pi})$$

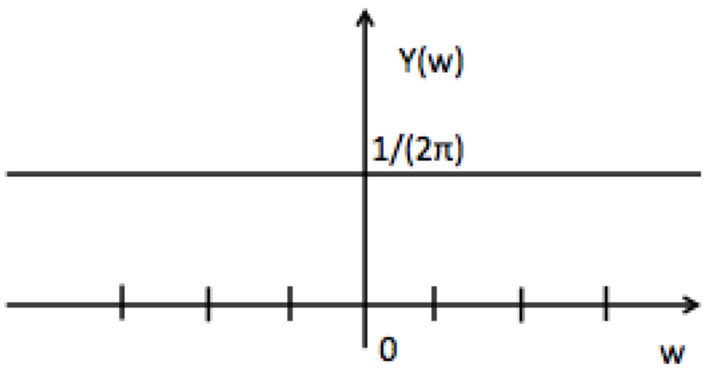
$$= \sum_{n=-\infty}^{\infty} \delta(w-n)$$

$$Y(w) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(w-n)$$



(8) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-4\pi n)$

$$P(w) = \frac{2\pi}{4\pi} \sum_{n=-\infty}^{\infty} \delta(w - n\frac{2\pi}{4\pi})$$
$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta(w - \frac{1}{2}n)$$
$$Y(w) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \delta(w - \frac{1}{2}n)$$



6. The input and the output of a causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{d\,y(t)}{dt} + 8\,y(t) = 2\,x(t)$$

Solution:
(a) Find the impulse response of this system.

$$((jw)^2Y(w) + 6jw + 8)Y(w) = 2X(w)$$
$$H(w) = \frac{2}{-w^2 + 6jw + 8}$$
$$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

(b) What is the response of this system if $x(t) = t\,e^{-2t}\,u(t)$?

$$Y(w) = \frac{1}{(2+jw)^2} \times \frac{2}{-w^2 + 6jw + 8}$$
$$= -\frac{1}{4(4+jw)} + \frac{1}{4(2+jw)} - \frac{1}{2(2+jw)^2} + \frac{1}{(2+jw)^3}$$
$$y(t) = -\frac{1}{4}e^{-4t}u(t) + \frac{1}{4}e^{-2t}u(t) - \frac{t}{2}e^{-2t}u(t) - \frac{t^2}{2}e^{-2t}u(t)$$

(c) Repeat part(a) for the causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{d\,y(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2\,x(t)$$

$$((jw)^2 + \sqrt{2}(jw) + 1)Y(w) = 2(jw)^2X(w) - 2X(w)$$
$$H(w) = \frac{2(jw)^2 - 2}{(jw)^2 + \sqrt{2}(jw) + 1}$$

Let S=jw

$$
\begin{aligned}
H(S) &= \frac{2S^2 - 2}{S^2 + \sqrt{2}S + 1} \\
&= 2(1 - \frac{2 + \sqrt{2}S}{S^2 + \sqrt{2}S + 1})
\end{aligned}
$$

Use the transform pair

$$
\begin{aligned}
e^{-at} sin(w_0 t)u(t) &\leftrightarrow \frac{w_0}{(a + jw)^2 + w_0^2} \\
e^{-at} cos(w_0 t)u(t) &\leftrightarrow \frac{a + jw}{(a + jw)^2 + w_0^2}
\end{aligned}
$$

$$
\begin{aligned}
H(S) &= 2(1 - \frac{2 + \sqrt{2}S}{S^2 + \sqrt{2}S + 1}) \\
&= 2(1 - \frac{2 + \sqrt{2}S}{(S + \frac{2}{\sqrt{2}})^2 + \frac{1}{2}}) \\
&= 2(1 - \frac{\sqrt{2}(S + \frac{\sqrt{2}}{2}) + 1}{(S + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}}) \\
&= 2(1 - \sqrt{2}\frac{(S + \frac{\sqrt{2}}{2})}{(S + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} + \frac{1}{(S + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}}) \\
&= 2(1 - \sqrt{2}\frac{(S + \frac{\sqrt{2}}{2})}{(S + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} + \sqrt{2}\frac{(\frac{\sqrt{2}}{2})}{(S + \frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2}) \\
h(t) &= 2\delta(t) - 2\sqrt{2}e^{-\frac{\sqrt{2}}{2}t}(\cos\frac{\sqrt{2}}{2}t + \sin\frac{\sqrt{2}}{2}t)u(t)
\end{aligned}
$$