ANALOG SIGNAL PROCESSING

ECE 210 & 211

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Objectives

- Average and rms signal power
- Parseval's theorem
- Total harmonic distortion
- Fourier Transforms
- Properties of Fourier transform
- Circuit interpretation for Fourier transforms
- Signal Energy and Spectrum
- Signal Bandwidth

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- **Average and rms signal power**
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Average Signal Power

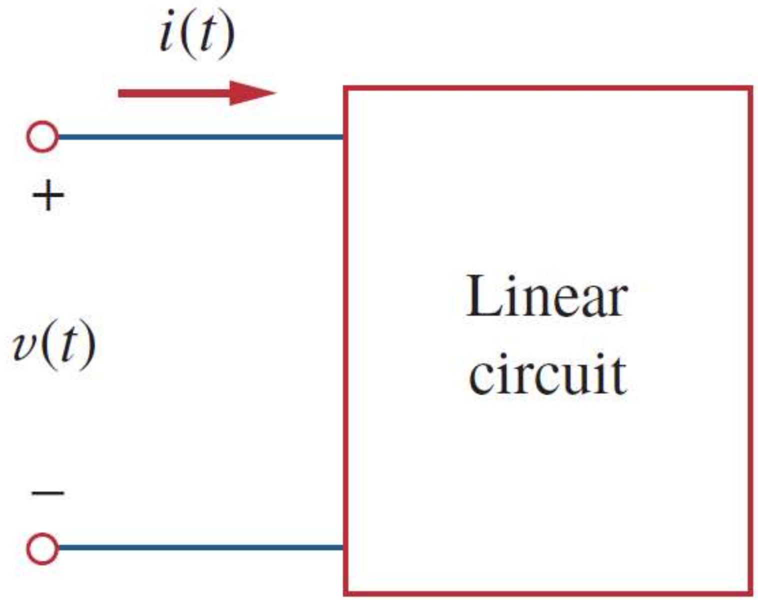
- To find the average power absorbed by a circuit due to **a periodic excitation**, we write the voltage and current in amplitude-phase form

$$v(t) = V_{\mathrm{dc}} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n)$$

$$i(t) = I_{\mathrm{dc}} + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m)$$

Average Signal Power

- **Recall that for a passive sign convention, we will use following direction vs sign notation as given below**



- **The average power is,**

$$P = \frac{1}{T}\int_0^T vi\,dt$$

Average Signal Power

Substituting the values of *v* and *i* gives 4 terms as,

$$P = \underbrace{\frac{1}{T}\int_0^T V_{dc} I_{dc}\, dt}_{1} + \underbrace{\sum_{m=1}^{\infty} \frac{I_m V_{dc}}{T} \int_0^T \cos(m\omega_0 t - \phi_m)\, dt}_{2}$$

$$+ \underbrace{\sum_{n=1}^{\infty} \frac{V_n I_{dc}}{T} \int_0^T \cos(n\omega_0 t - \theta_n)\, dt}_{3}$$

$$+ \underbrace{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{V_n I_m}{T} \int_0^T \cos(n\omega_0 t - \theta_n)\cos(m\omega_0 t - \phi_m)\, dt}_{4}$$

Average Signal Power

- **The 2nd and 3rd term will be *vanished* due to integration over period of cosine**

- **The 4th term will also be zero when m ≠ n**

- **Evaluating 1st term gives,**

$$\boxed{P = V_{\mathrm{dc}} I_{\mathrm{dc}} + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)}$$

- **The total average power is the sum of the average powers in each harmonically related voltage and current**

Average Signal Power

The rms value of a periodic function *f(t)* is given by,

$$F_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt} \qquad \textbf{Eq. 1}$$

Consider *f(t)* as,

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

substituting the value of *f(t)* into Eq. 1 gives,

Average Signal Power

$$F_{\text{rms}}^2 = \frac{1}{T}\int_0^T \left[a_0^2 + 2\sum_{n=1}^{\infty} a_0 A_n \cos(n\omega_0 t + \phi_n) \right.$$

$$\left. + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \cos(n\omega_0 t + \phi_n)\cos(m\omega_0 t + \phi_m) \right] dt$$

$$= \frac{1}{T}\int_0^T a_0^2\, dt + 2\sum_{n=1}^{\infty} a_0 A_n \frac{1}{T}\int_0^T \cos(n\omega_0 t + \phi_n)\, dt$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \frac{1}{T}\int_0^T \cos(n\omega_0 t + \phi_n)\cos(m\omega_0 t + \phi_m)\, dt$$

Average Signal Power

Using the same reasoning used earlier for 2nd -4th terms and simplification gives,

$$F_{\mathrm{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2 \Rightarrow F_{\mathrm{rms}} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

in terms of Fourier coefficients, it can be written as,

$$F_{\mathrm{rms}} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)}$$

Average Signal Power

If *f(t)* is the current through a resistor *R*, then the power dissipated in the resistor is

$$P = RF_{\text{rms}}^2$$

Or if *f(t)* is the voltage across a resistor *R*, the power dissipated in the resistor is

$$P = \frac{F_{\text{rms}}^2}{R}$$

The power dissipated by the *1*- Ω resistance

$$P_{1\Omega} = F_{\text{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

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- **Parseval's theorem**
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- Fourier Transforms
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Average Power – Parseval's Theorem

- **The average power in a periodic signal is the sum of the average power in its dc component and the average powers in its harmonics**

$$P = F_{\mathrm{rms}}^2 = \boxed{a_0^2} + \boxed{\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

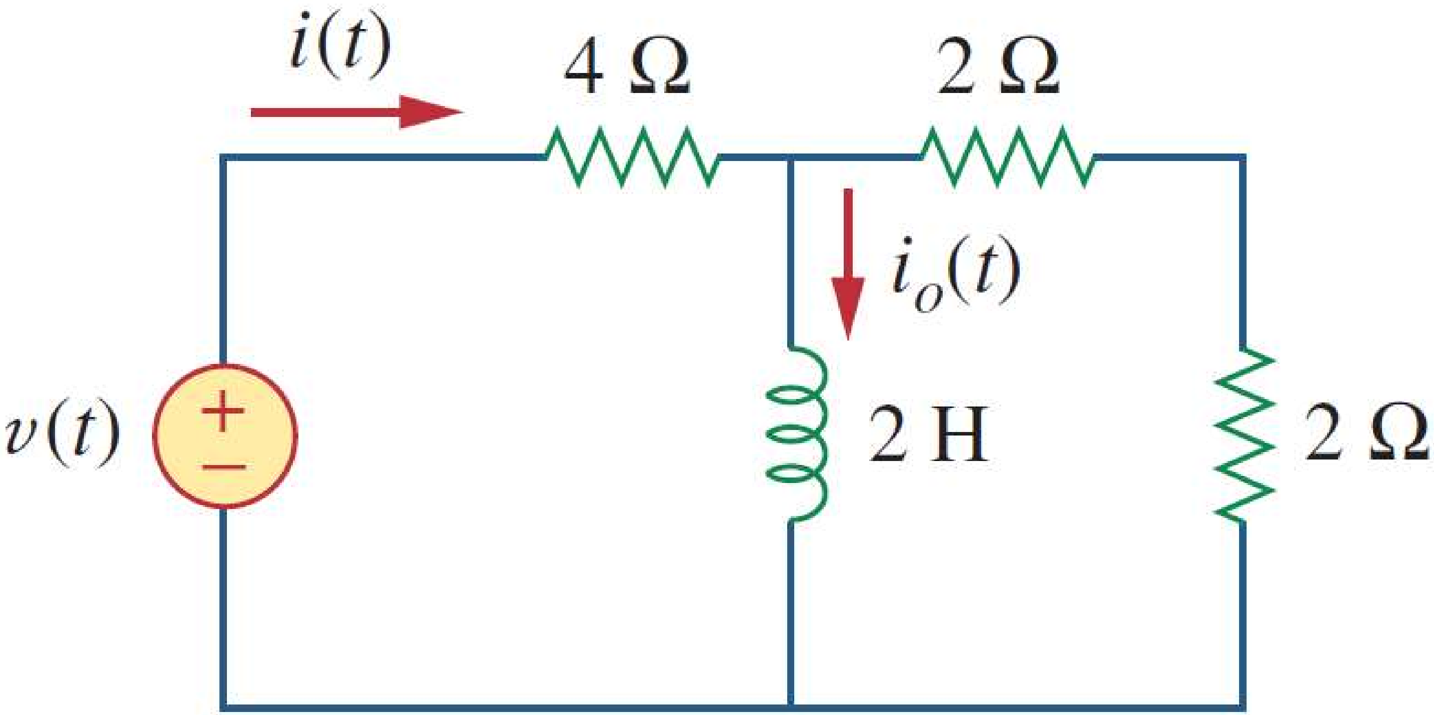
Power in the dc component

AC power in the *n*-th harmonic

Average Signal Power – Example 1

Question: **Find the response *$i_o(t)$* if the input *$v(t)$* is given by *Fourier series expansion* as,**

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nt - n \sin nt)$$



Average Signal Power – Example 1

Solution: **We can express the input voltage as,**

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + \tan^{-1} n)$$

$$= 1 - 1.414\cos(t + 45^\circ) + 0.8944\cos(2t + 63.45^\circ) - 0.6345\cos(3t + 71.56^\circ) - 0.4851\cos(4t + 78.7^\circ) + \cdots$$

We notice that $\omega_o = 1$, and $\omega_n = n\ rad/s$, the impedance at the source is,

$$Z = 4 + j\omega_n 2 \parallel 4 = \frac{8 + j\omega_n 8}{2 + j\omega_n}$$

Average Signal Power – Example 1

The input current is,

$$I = \frac{V}{Z} = \frac{2 + j\omega_n}{8 + j\omega_n 8} V$$

where V is the phasor form of the source voltage *v*(*t*)

By using current division,

$$I_o = \frac{4}{4 + j\omega_n 2} I = \frac{V}{4 + j\omega_n 4}$$

since, $\omega_n = n, I_o$ **can be expressed as,**

$$I_o = \frac{V}{4\sqrt{1 + n^2} \angle tan^{-1} n}$$

for dc component, $n = 0\ so, \omega_n = 0, So, V = 1, I_o = \frac{1}{4}$

Average Signal Power – Example 1

for *n*-th harmonic,

$$V = \frac{2(-1)^n}{\sqrt{1+n^2}} \angle tan^{-1} n$$

so that,

$$I_o = \frac{1}{4\sqrt{1+n^2} \angle tan^{-1} n} \frac{2(-1)^n}{\sqrt{1+n^2}} \angle tan^{-1} n$$

$$= \frac{(-1)^n}{2(1+n^2)}$$

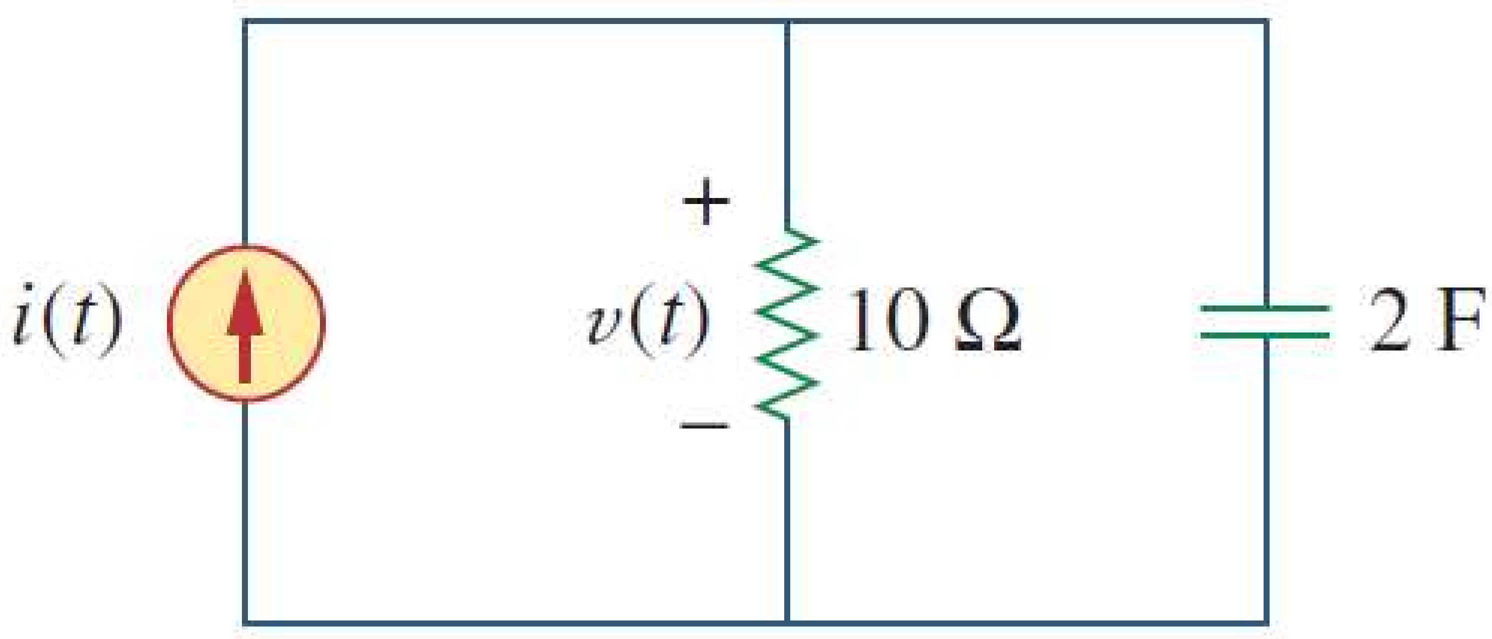
in the time domain,

$$i_o(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2(1+n^2)} cos\, nt\ A$$

Average Signal Power – Example 2

Question: **Determine the average power supplied to the circuit if,**

$$i(t) = 2 + 10\cos(t + 10^o) + 6\cos(3t + 35^o)\ A$$



Average Signal Power – Example 2

Solution: **The input impedance of the network is,**

$$\mathbf{Z} = 10 \parallel \frac{1}{j2\omega} = \frac{10\left(\frac{1}{j2\omega}\right)}{10 + \frac{1}{j2\omega}} = \frac{10}{1 + j20\omega}$$

Hence,

$$\mathbf{V} = \mathbf{IZ} = \frac{10\mathbf{I}}{\sqrt{1 + 400\omega^2} \angle tan^{-1} 20\omega}$$

Average Signal Power – Example 2

For dc component, $\omega = 0$,

$$I = 2\,A \qquad \text{and} \qquad V = 10(2) = 20\,V$$

Expected, as the capacitor is an *open circuit* to DC and the entire *2-A* current flows through the resistor

For $\omega = 1\,rad/s$,

$$\mathbf{I} = 10\angle 10^o\,A, \qquad V = \frac{10(10\angle 10^o)}{\sqrt{1+400\omega^2}\angle \tan^{-1}20}$$

$$= 5\angle -77.1^o$$

Average Signal Power – Example 2

For $\omega = 3\, rad/s$,

$$\mathrm{I} = 6\angle 35^o\ A, \qquad V = \frac{10(6\angle 35^o)}{\sqrt{1 + 3600\omega^2}\angle tan^{-1}60}$$

$$= 1\angle -54^o$$

Converting into time domain,

$$v(t) = 20 + 5\cos(t - 77.1^o) + 1\,cos(3t - 54^o)\ \mathrm{V}$$

Average Signal Power – Example 2

We obtain the average power supplied to the circuit by applying Eq.

$$\boxed{P = V_{\mathrm{dc}} I_{\mathrm{dc}} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)}$$

To get the proper signs of θ_n and ϕ_n, we will compare with values of v and i,

$$P = 2(20) + \frac{1}{2}(5)(10)cos[77.1^o - (-10^o)]$$
$$+\frac{1}{2}(1)(6)cos[54.0^o - (-35^o)]$$
$$= 40 + 1.24 + 0.005 = 41.6\ \mathrm{W}$$

Average Signal Power – Example 3

Question: From Example 1, estimate the rms value of voltage given as,

$$v(t) = 1 - 1.414\cos(t + 45^o) + 0.89\, cos(2t + 63.45^o)$$
$$-0.634\, cos(3t + 71.55^o)$$
$$-0.48\, cos(4t + 78.7^o) + \cdots V$$

Average Signal Power – Example 3

Solution:

Using the following relation to estimate rms value of the voltage *v(t)*

$$V_{rms} = \sqrt{a_o^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

$$V_{rms} = \sqrt{1^2 + \frac{1}{2}[(-1.41)^2 + (0.89)^2 + (-0.63)^2 + (-0.48)^2 + \cdots]}$$

$$V_{rms} = 1.649 \text{ V}$$

Average Signal Power – Example 3

- This is only an estimate, as we have not taken enough terms of the series
- The actual function represented by the Fourier series is

$$v(t) = \frac{\pi e^{t}}{sinh\pi}, \qquad -\pi < t < \pi$$

with $v(t) = v(t + T)$,

The exact rms value of this is 1.776 V

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Simple Harmonic Distortions

Suppose a system with *y(t)* as response is written as,

$$y(t) = \boxed{Af(t)} + \boxed{Bf^2(t)}$$

Linear Response

Desired ✓

Nonlinear Response

Undesired ✗

How to estimate the effects of undesired part of response!

Simple Harmonic Distortions

We know that

- **A linear system will respond a cosine input with a cosine output at *same frequency***
- **But if output generates *higher order frequency***

$$f(t) = cos(\omega_o t)$$

as,

$$y(t) = Acos(\omega_o t) + Bcos^2(\omega_o t)$$

$$y(t) = \frac{B}{2} + Acos(\omega_o t) + \frac{B}{2}cos(2\omega_o t)$$

Simple Harmonic Distortion

$$y(t) = \boxed{\frac{B}{2}} + \boxed{A\cos(\omega_o t) + \frac{B}{2}\cos(2\omega_o t)}$$

- **Clearly, you can see undesired harmonic part (DC term *plus* harmonic) is added into purely cosine input**
- **According to *Parseval's theorem*, average power is,**

$$P = F_{\text{rms}}^2 = \boxed{a_0^2} + \boxed{\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

DC Component

AC power in the *n*-th harmonic

Simple Harmonic Distortion

$$P_y = \boxed{\frac{B^2}{4}} + \boxed{\frac{A^2}{2} + \frac{B^2}{8}}$$

DC Component

Average power of fundamental and second harmonic

- **One important consideration is *simple or second harmonic distortion***
- **It can be achieved by taking ratio of average power in second harmonic and to the average power in fundamental,**

$$S.H.D = \frac{B^2}{8} : \frac{A^2}{2} = \frac{B^2}{4A^2}$$

Total Harmonic Distortion

- More generally, a non-linear system may respond to a pure cosine input using Fourier series as,

$$y(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t + \theta_n)$$

- Contains fundamental *plus* infinite higher order harmonics

- It's sensible to consider most affective harmonics in the calculation of overall distortion

Total Harmonic Distortion

- So, we take ratio of **second and higher order harmonics** to the **fundamental harmonic** is called ***total harmonic distortion***

$$T.H.D = \frac{\sum_{n=2}^{\infty}\frac{1}{2}c_n^2}{\frac{c_1^2}{2}} = \frac{\sum_{n=2}^{\infty}c_n^2}{c_1^2}$$

Second and higher order harmonics $= \sum_{n=2}^{\infty}\frac{1}{2}c_n^2$

From the signal,

$$y(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t + \theta_n)$$

Total Harmonic Distortion

Second and higher order harmonics = *Average Signal power* – (average power of first and fundamental harmonics)

$$\sum_{n=2}^{\infty} c_n^2 = 2\left(P_y - \left(\frac{c_o^2}{4} + \frac{1}{2}c_1^2\right)\right)$$

Putting the value,

$$T.H.D = \frac{P_y - \left(\frac{c_o^2}{4} + \frac{1}{2}c_1^2\right)}{\frac{1}{2}c_1^2}$$

Where P_y is the average signal power of *y(t)*

Total Harmonic Distortion – Example 4

Question: **A system is supposed to deliver ,**

$$y(t) = \cos(\omega_o t) \ @ \ \frac{\omega_o}{2\pi} = 60Hz$$

But, it actually delivers,

$$y(t) = \cos(\omega_o t) + \frac{1}{9}\cos(3\omega_o t) + \frac{1}{25}\cos(5\omega_o t)$$

What is the Total Harmonic Distortion (T.H.D) in the system?

Total Harmonic Distortion – Example 4

Solution:

- **It can be seen that taking *ratio* of second and third harmonic to the fundamental harmonic gives the *T.H.D*,**
- **As per Parseval's theorem,**

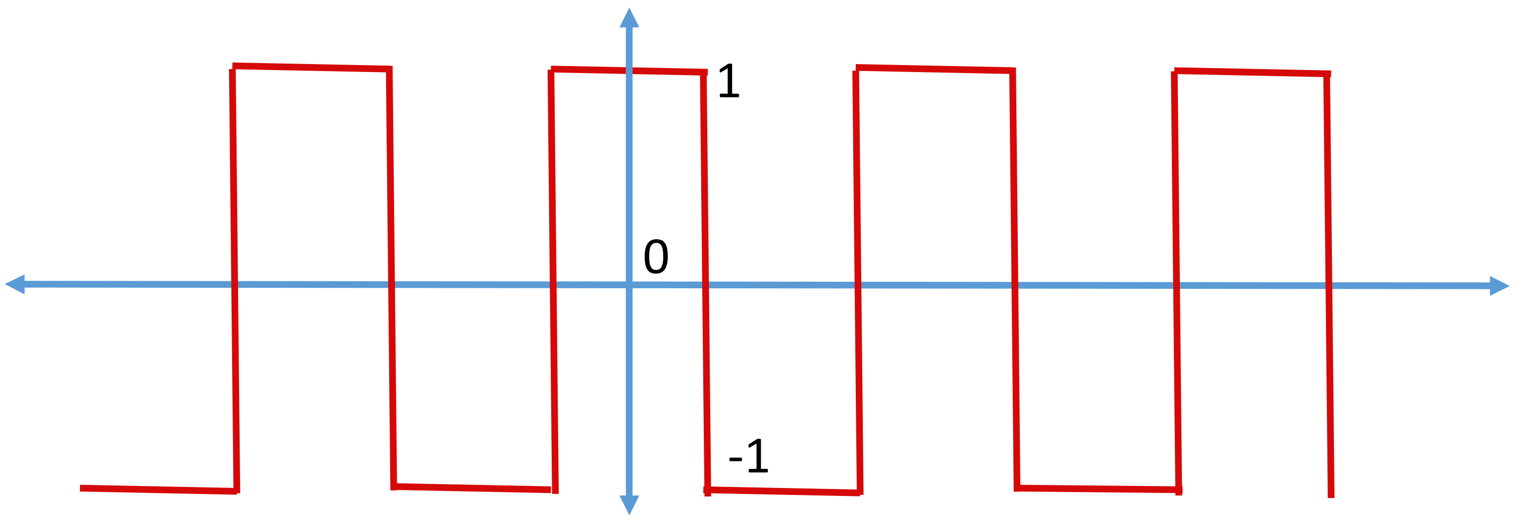
$$\boxed{T.H.D = \frac{\left(\frac{1}{9}\right)^2 + \left(\frac{1}{25}\right)^2}{1^2} \approx 1.40\%}$$

Total Harmonic Distortion – Example 5

Question: **The Fourier of zero-mean-square wave is given by,**

$$y(t) = \frac{4}{\pi}\left[\cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}cos(5t) - \frac{1}{7}cos(7t) + \cdots\right]$$

Assume that *y(t)* is proportional to *cos(t)*, find total harmonic distortion of the signal?



Total Harmonic Distortion – Example 5

Solution: **Since the signal as zero mean, so,**

$$c_o^2 = 0, \quad \text{also,} \quad \frac{c_1^2}{2} = \frac{8}{\pi^2}$$

Takin the average power of signal over entire period gives,

$$P_y = \frac{1}{2\pi}\int_0^{2\pi} 1^2 dt = 1$$

Using T.H.D relation,

$$T.H.D = \frac{1 - \frac{8}{\pi^2}}{\frac{8}{\pi^2}} \approx 23.4\%$$

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Fourier Transforms

- A non-sinusoidal periodic function can be represented by a Fourier series, provided that it satisfies the Dirichlet conditions

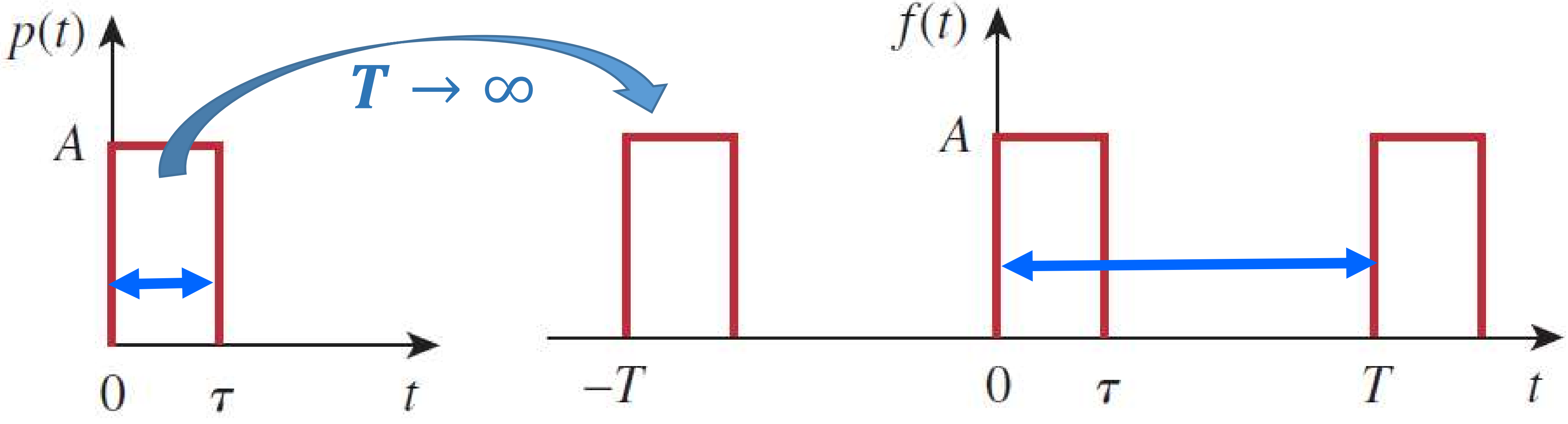
- Non-Periodic function is defined as a signal having non-repeating pattern in range $[-\infty , \infty]$

- What if the signal is non-periodic?

- Is Fourier series exist for non-periodic signals

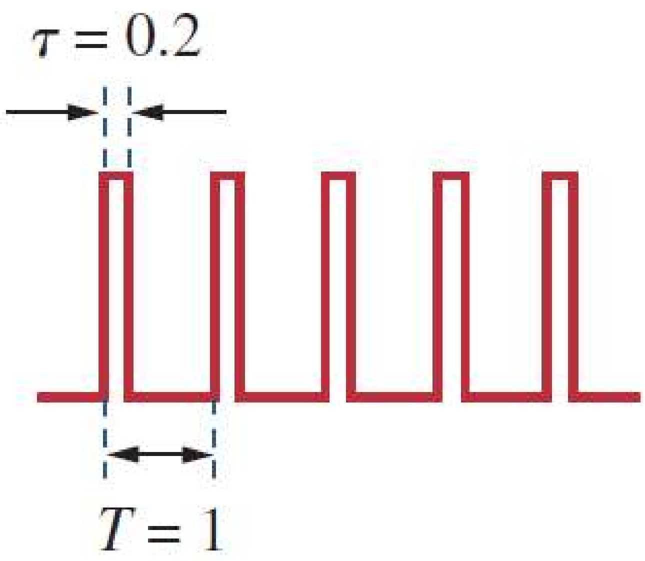
Fourier Transforms

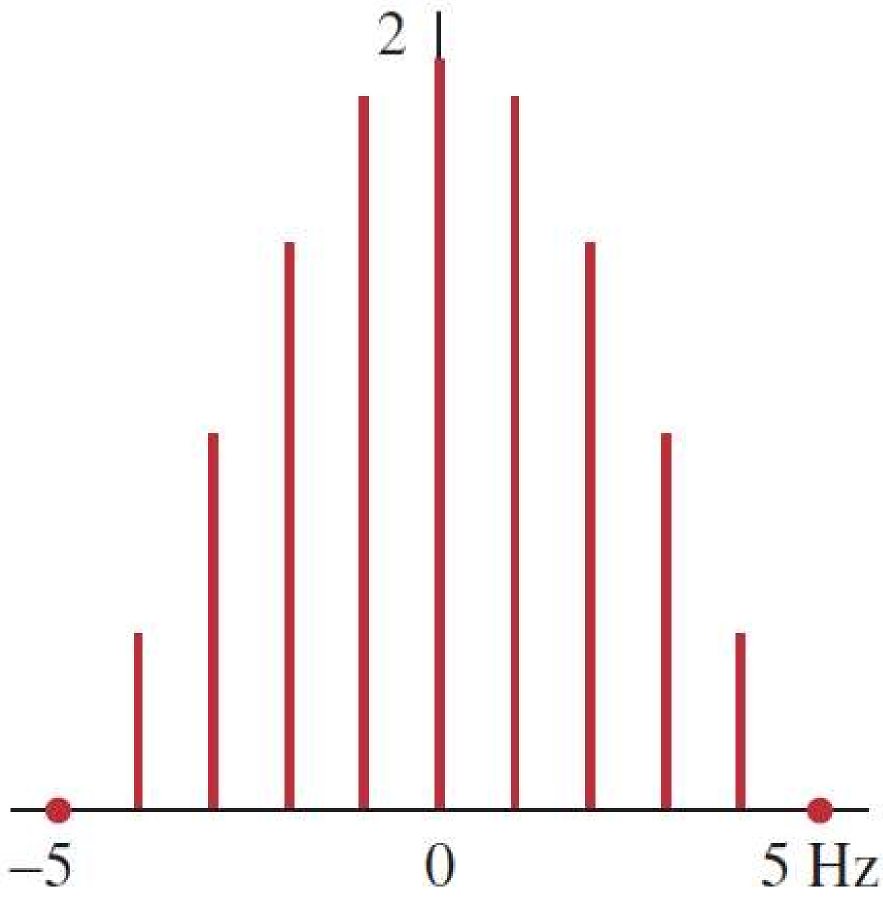
- **The answer is – Yes!**
- **Suppose a non-periodic function whose period extends up to ∞**



Fourier Transforms

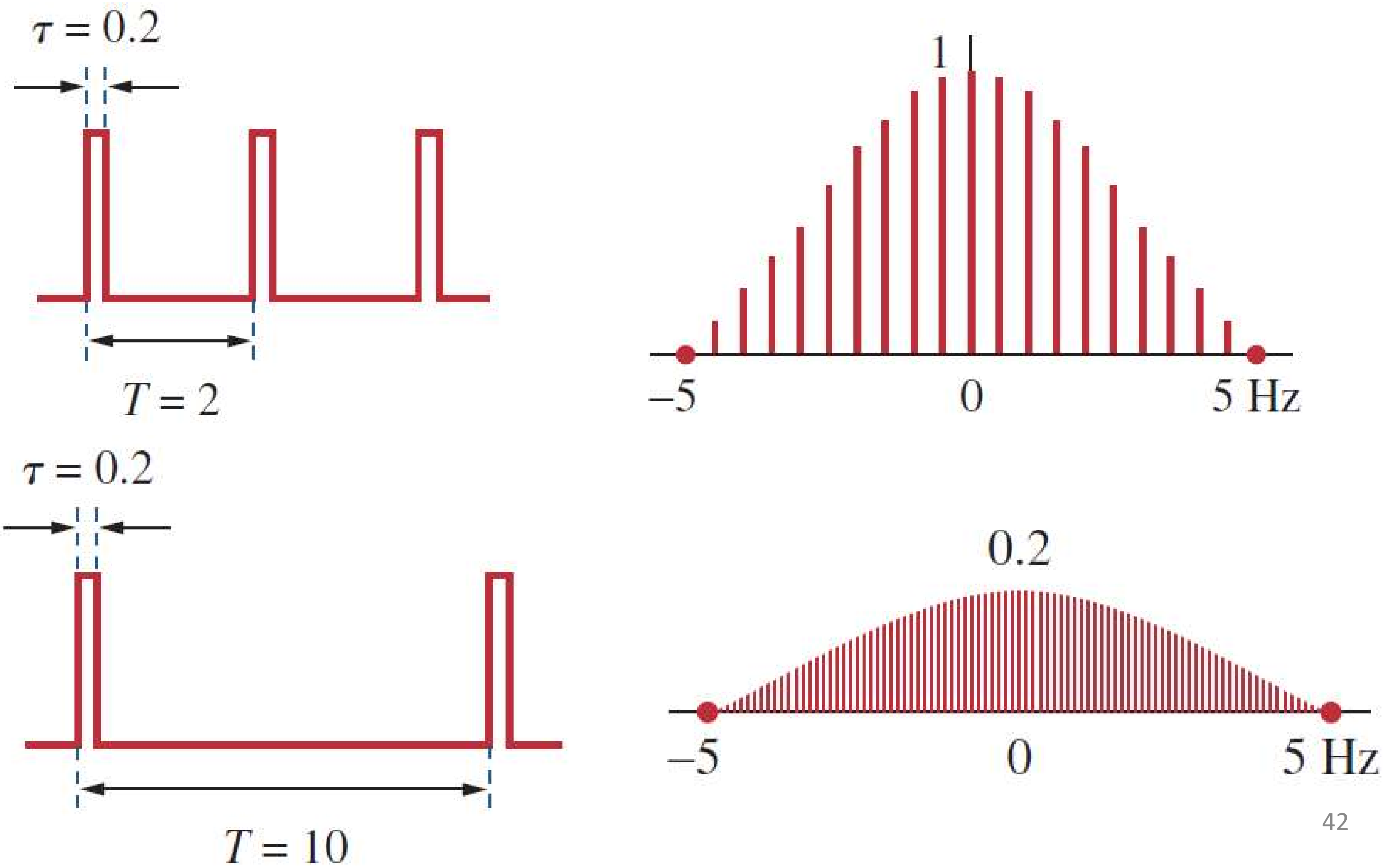
- **Observe the spectrum of signal when $T = 1$, $A = 10$ and $\tau = 0.2s$**





Fourier Transforms

- **Observe the spectrum of signal when $T = 2\ \&\ 10, A = 10\ and\ \tau = 0.2s$, respectively,**



Fourier Transforms

- The general shape remains same

- The frequency at which the envelope first becomes zero remains the same

- The amplitude of the spectrum and the spacing between adjacent components both decrease, while the number of harmonics increases

- Over a range of frequencies, the sum of the amplitudes of the harmonics remains almost constant

Fourier Transforms

To further understand the connection between *non-periodic function* and it's *periodic counterpart*, let's consider exponential form Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad \textbf{where,} \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

The fundamental frequency will be, $\omega_0 = \dfrac{2\pi}{T}$

and the spacing between adjacent harmonics is,

$$\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T}$$

Fourier Transforms

Substituting the value of C_n into *f(t)* and solving summation gives,

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt \right] \Delta\omega e^{jn\omega_0 t}$$

as T → ∞

$$\sum_{n=-\infty}^{\infty} \Rightarrow \int_{-\infty}^{\infty}$$

$$\Delta\omega \Rightarrow d\omega$$

$$n\omega_0 \Rightarrow \omega$$

Fourier Transforms

Putting the extensions, the equation becomes,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right]e^{j\omega t}\,d\omega$$

The term in the brackets is known as the *Fourier transform* of *f* (*t*) and is represented by *F*(*ω*)

$$\boxed{F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt}$$

$\mathcal{F}$ is the Fourier operator

Fourier Transforms

- The *Fourier transform* (FT) is an integral transformation of $f(t)$ from the time domain to the frequency domain

- The Fourier transform can also be represented in terms of $F(\omega)$, called *Inverse Fourier transform* (IFT)

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

Transform Pair: $f(t) \quad \Leftrightarrow \quad F(\omega)$

Fourier Transforms

- The *Fourier transform* $F(\omega)$ exist when the Fourier Integral has convergence

- A sufficient but not necessary condition that *f(t)* has a Fourier transform is that it be completely integrable in the sense that

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

- We will use s instead of $j\omega$ to avoid complexity and replace it after algebraic evaluation

Fourier Transforms – Example 6

Question: **Find the Fourier Transform of the following function**

$$(a) = \delta(t - t_0)$$
$$(b) = e^{j\omega_o t}$$
$$(c) = cos(\omega_o t)$$

Fourier Transforms – Example 6

Solution: (a) For the Fourier Transform of the impulse function, we will apply sifting property,

$$F(\omega) = \mathcal{F}[\delta(t - t_0)] = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\, dt = e^{-j\omega t_0}$$

For the special case when $t = t_o$

$$\mathcal{F}[\delta(t)] = 1$$

- **This shows that the *magnitude* of the spectrum of the impulse function is constant**
- **All frequencies are equally represented in the impulse function**

Fourier Transforms – Example 6

Solution: (b)

Let, $F(\omega) = \delta(\omega - \omega_0)$

then we can write *f(t)*

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega$$

Using sifting property of impulse function

$$f(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

As *f(t)* and $F(\omega)$ are Fourier transform pair so as $e^{j\omega_o t}$ and $2\pi\delta(t - t_0)$

$$\boxed{\mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0)}$$

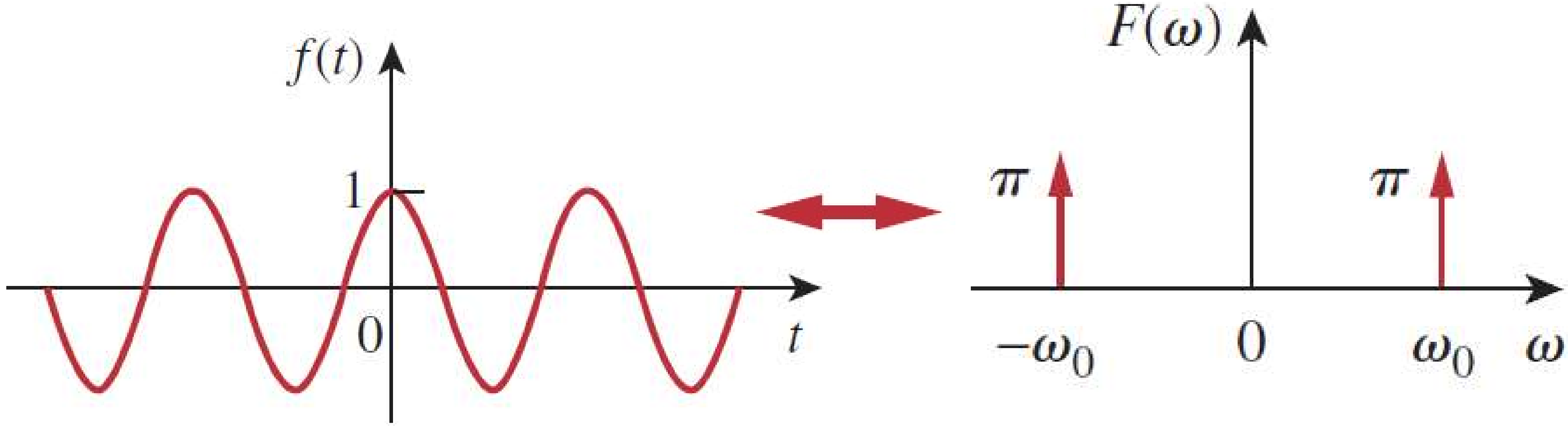
Fourier Transforms – Example 6

Solution: (c) By using results of (a) and (b), we can write,

$$\mathcal{F}[\cos \omega_0 t] = \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

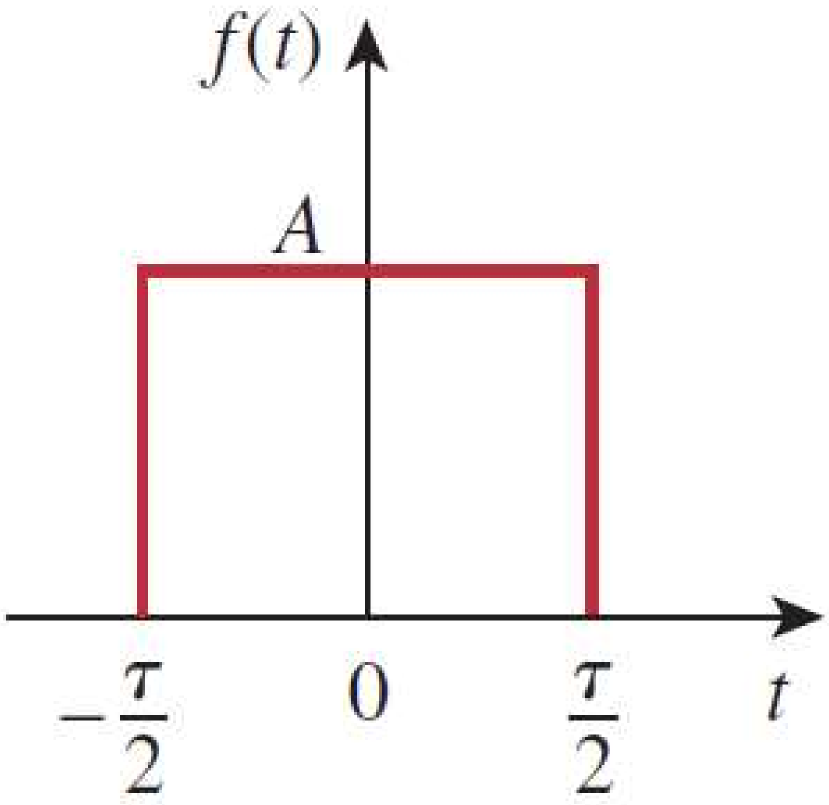
$$= \frac{1}{2}\mathcal{F}[e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[e^{-j\omega_0 t}]$$

$$= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transforms – Example 7

Question: **Derive the Fourier transform of a single rectangular pulse (gate function) of width τ and height A?**



Fourier Transforms – Example 7

Solution:

$$F(\omega) = \int_{-\tau/2}^{\tau/2} Ae^{-j\omega t}\,dt = -\frac{A}{j\omega}e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{\omega}\left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right)$$

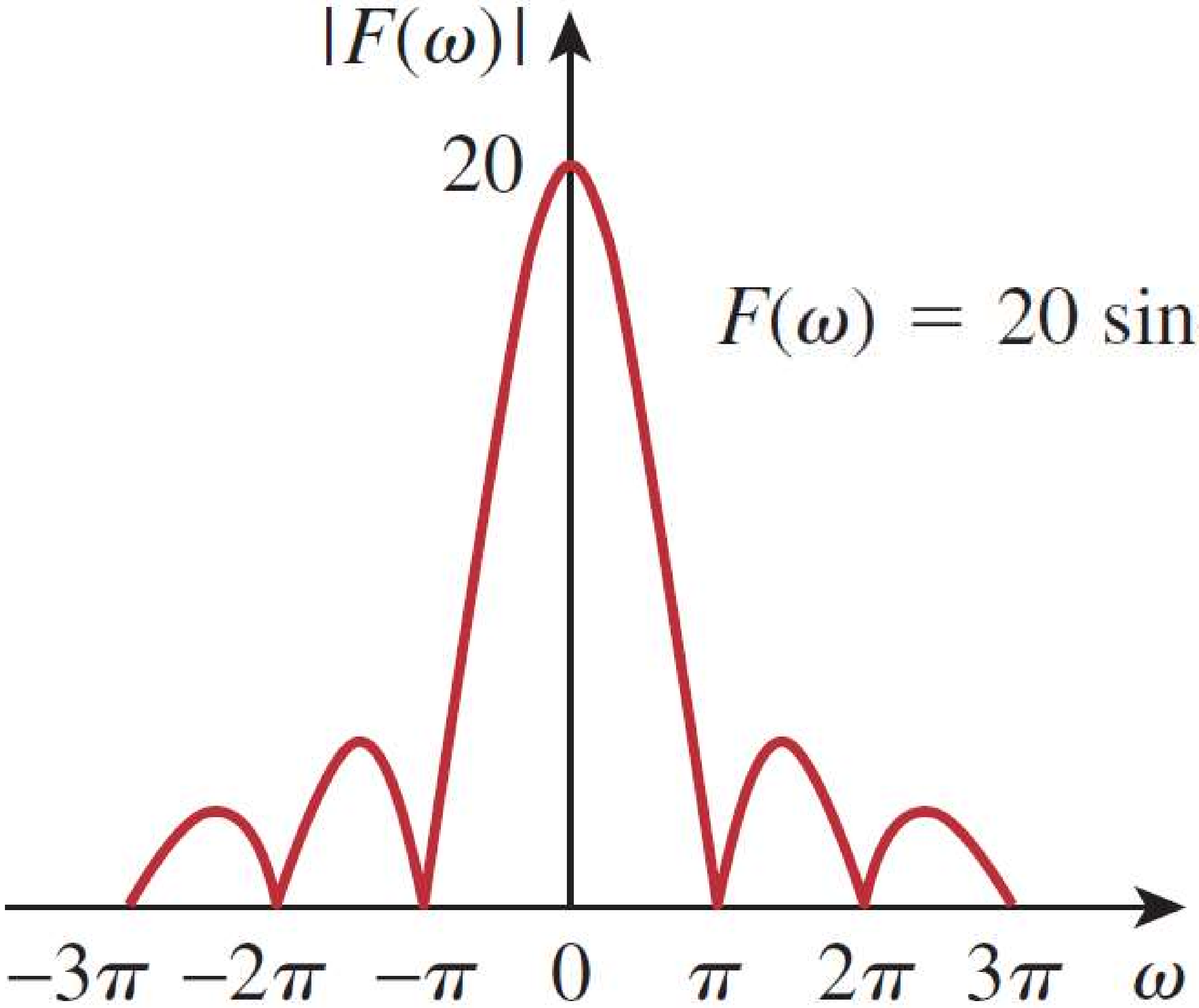
$$= A\tau\frac{\sin\omega\tau/2}{\omega\tau/2} = A\tau\,\mathrm{sinc}\,\frac{\omega\tau}{2}$$

If we take $A = 10$, and $\tau = 2\ s$, then the solution will be a sinc function

$$F(\omega) = 20\,\mathrm{sinc}\,\omega$$

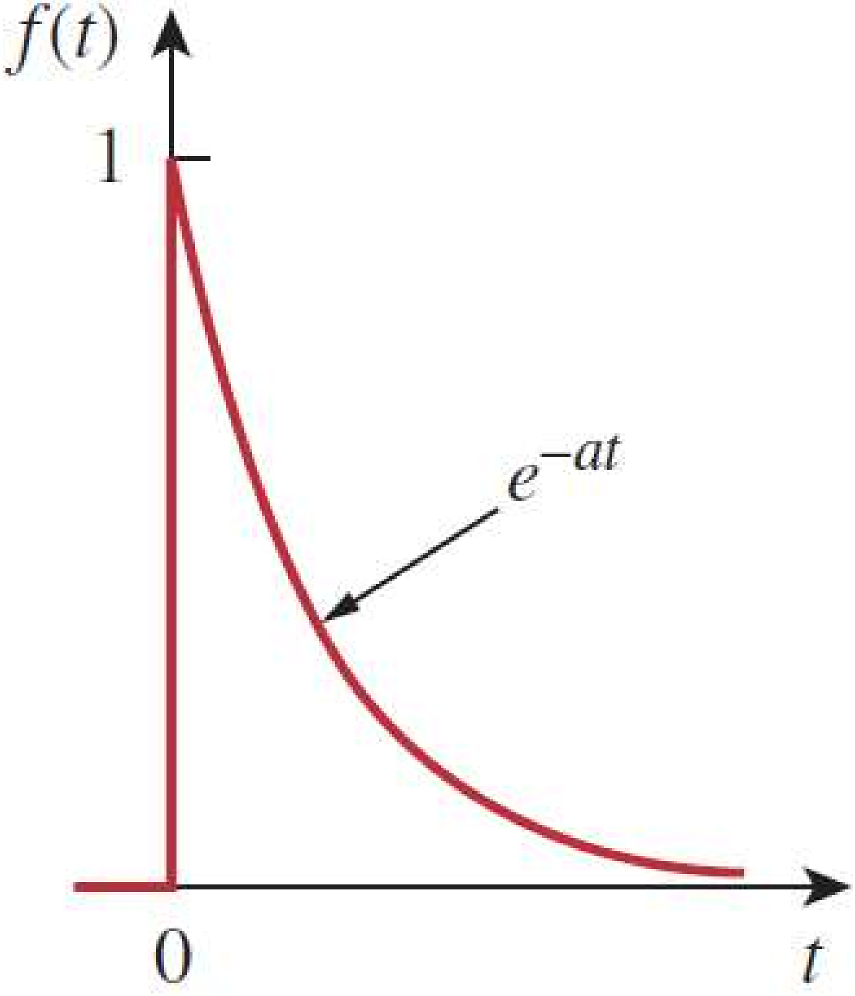
Fourier Transforms – Example 7

Amplitude spectrum



Fourier Transforms – Example 7

Question: **Derive the Fourier transform of a "switched ON" exponential function shown below?**



Fourier Transforms – Example 7

Solution:

$$f(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$$

Hence,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$

Objectives

- Average and rms signal power
- Parseval's theorem
- Total harmonic distortion
- Fourier Transforms
- **Properties of Fourier transform**
- Circuit interpretation for Fourier transforms
- Signal Energy and Spectrum
- Signal Bandwidth

Properties of Fourier Transforms

Linearity:

$$\boxed{\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(\omega) + a_2 F_2(\omega)}$$

a_1 and a_2 are two arbitrary constants

Examp

$$F[\sin\omega_0 t] = \frac{1}{2j}[\mathcal{F}(e^{j\omega_0 t}) - \mathcal{F}(e^{-j\omega_0 t})]$$

$$= \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$= j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

Properties of Fourier Transforms

Time scaling: **If $F(\omega) = \mathcal{F}\big(f(t)\big)$, then**

$$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)}$$

a **is arbitrary constants**

Properties of Fourier Transforms

Time shifting: **If $F(\omega) = \mathcal{F}\big(f(t)\big)$, then**

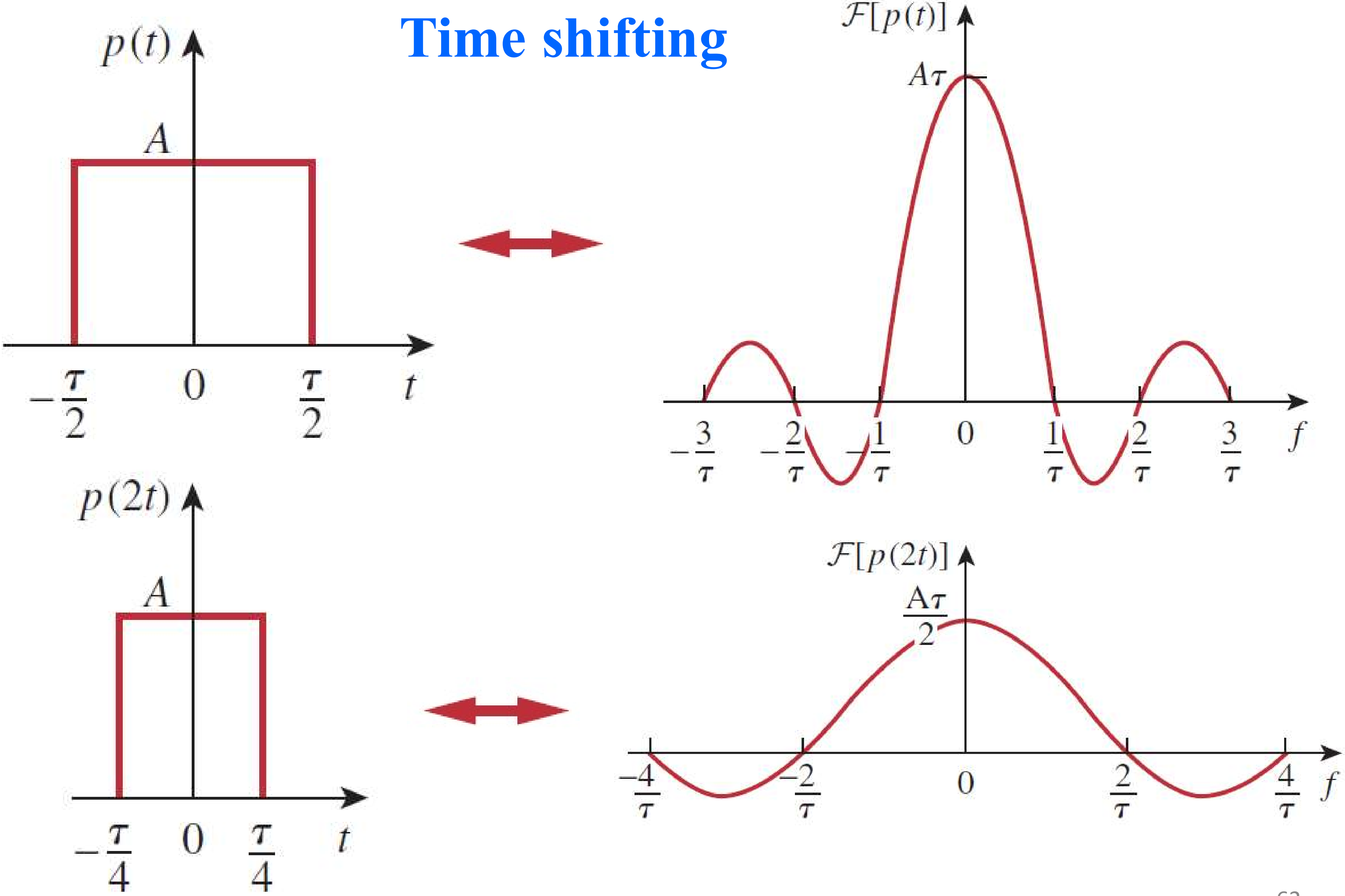
$$\mathcal{F}[f(t - t_0)] = e^{-j\omega t_0} F(\omega)$$

ω_o is arbitrary constants used for delay

- **A delay in the time domain corresponds to a phase shift in the frequency domain**

Properties of Fourier Transforms

Time shifting



Properties of Fourier Transforms

Frequency shifting (Amplitude modulation):

If $F(\omega) = \mathcal{F}\big(f(t)\big)$, then

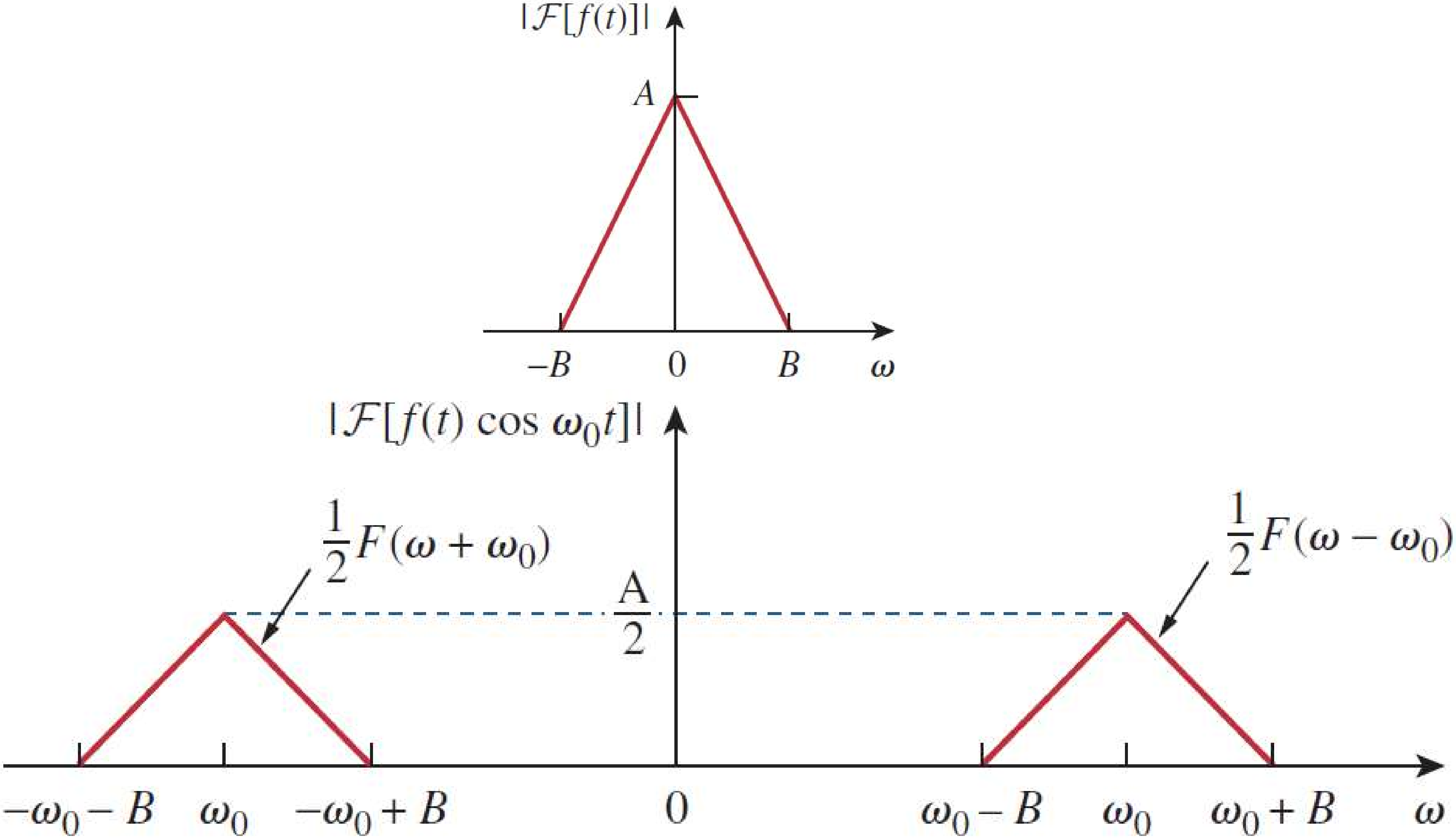
$$\boxed{\mathcal{F}\left[f(t)e^{j\omega_0 t}\right] = F(\omega - \omega_0)}$$

ω_o is arbitrary constants used for delay

- **A frequency shift in the frequency domain adds a phase shift to the time function**

Properties of Fourier Transforms

Frequency shifting (Amplitude modulation)



Properties of Fourier Transforms

Time differentiation:

If $\boldsymbol{F(\omega) = \mathcal{F}\big(f(t)\big)}$, then

$$\boxed{\mathcal{F}[f'(t)] = j\omega F(\omega)}$$

- **The transform of the derivative of *f (t)* is obtained by multiplying the transform of *f(t)* by $j\omega$**
- **For higher order differentiation,**

$$\boxed{\mathcal{F}\left[f^{(n)}(t)\right] = (j\omega)^n F(\omega)}$$

Properties of Fourier Transforms

Time integration: **If $\boldsymbol{F(\omega) = \mathcal{F}\big(f(t)\big)}$, then**

$$\mathcal{F}\left[\int_{-\infty}^{t} f(t)\, dt\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$$

- **The transform of the integral of *f(t)* is obtained by dividing the transform of *f(t)* by and *adding* the result to the impulse term that reflects the dc component *F(0)***

Properties of Fourier Transforms

Time Reversal: **If $\boldsymbol{F(\omega)} = \boldsymbol{\mathcal{F}\big(f(t)\big)}$, then**

$$\boxed{\mathcal{F}[f(-t)] = F(-\omega) = F^*(\omega)}$$

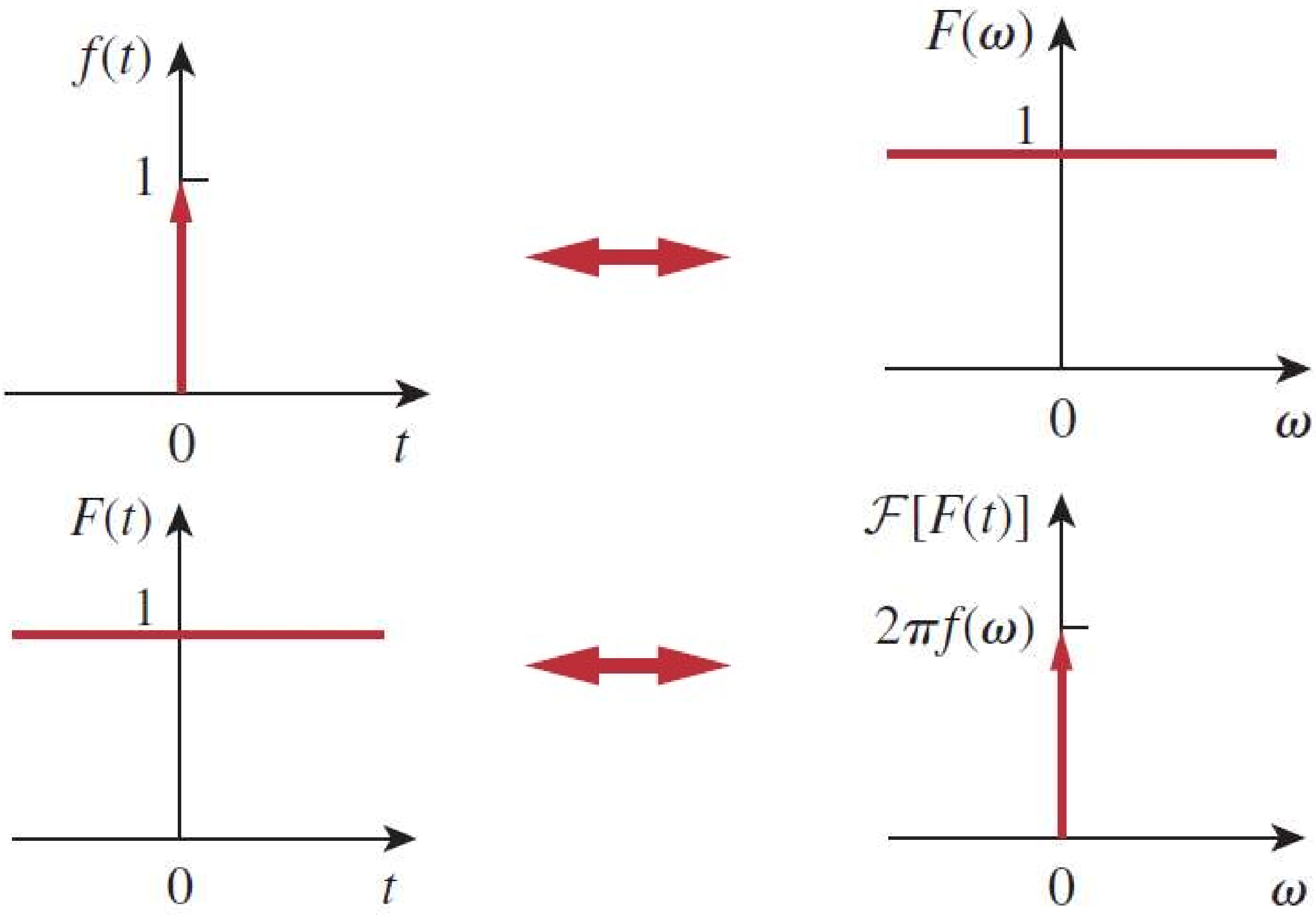
$*$ **is complex conjugate**

- **Reversing $f(t)$ about the time axis reverses $\mathrm{F}(\boldsymbol{\omega})$ about the frequency axis**

Properties of Fourier Transforms

Duality: **If $F(\omega) = \mathcal{F}\big(f(t)\big)$, then**

$$\boxed{\mathcal{F}[f(t)] = F(\omega) \qquad \Rightarrow \qquad \mathcal{F}[F(t)] = 2\pi f(-\omega)}$$



Properties of Fourier Transforms

Convolution:

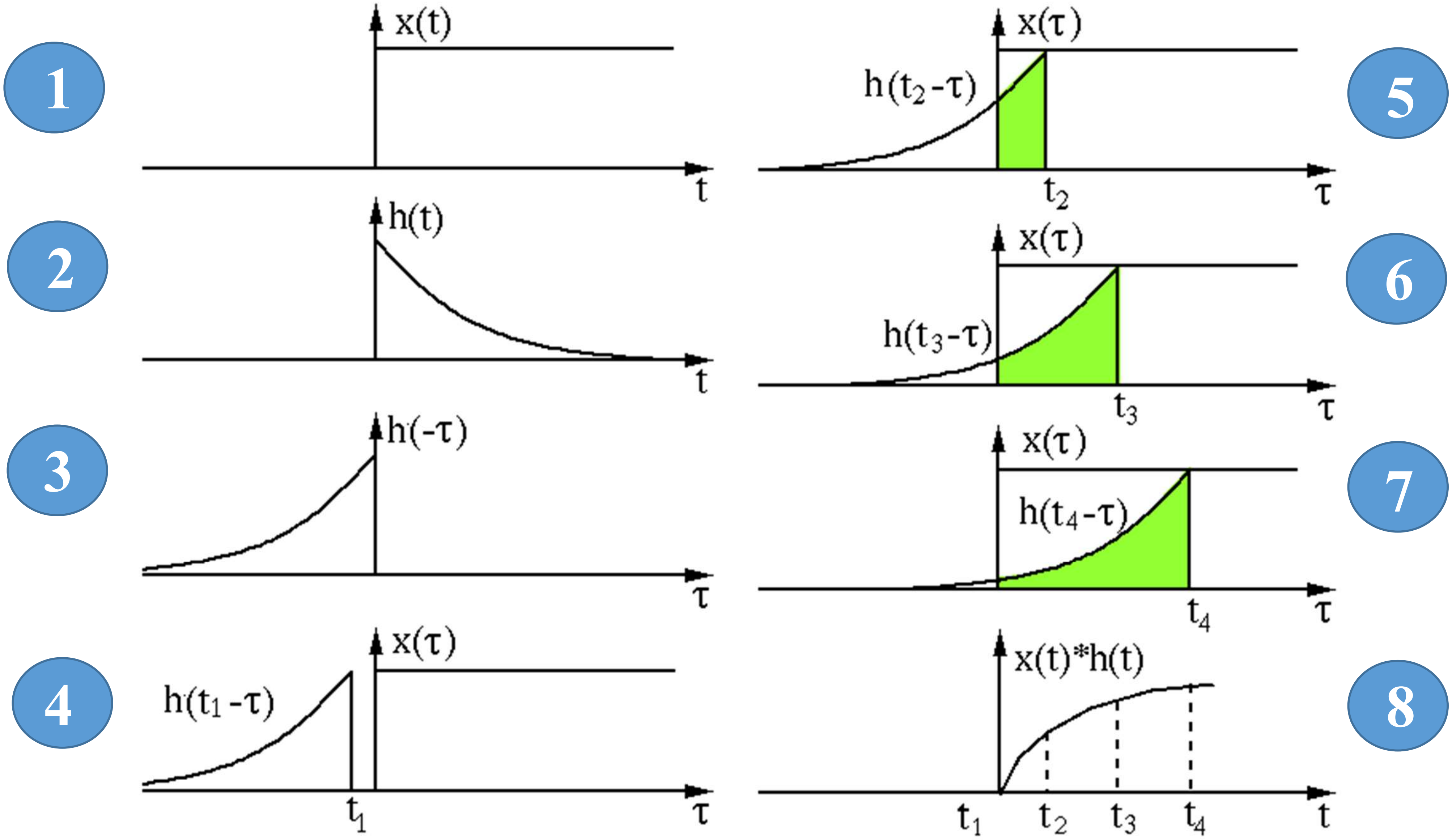
If *x(t)* is the input excitation to a circuit with an impulse function of *h(t)*, then the output response *y(t)* is given by the convolution integral

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda$$

if *X(ω)*, *H(ω)* and *Y(ω)* are Fourier transform of *x(t)*, *h(t)* and *y(t)*, respectively, then

$$\boxed{Y(\omega) = \mathcal{F}[h(t) * x(t)] = H(\omega)X(\omega)}$$

Convolution –Graphical illustration



Fourier Transforms – Summary

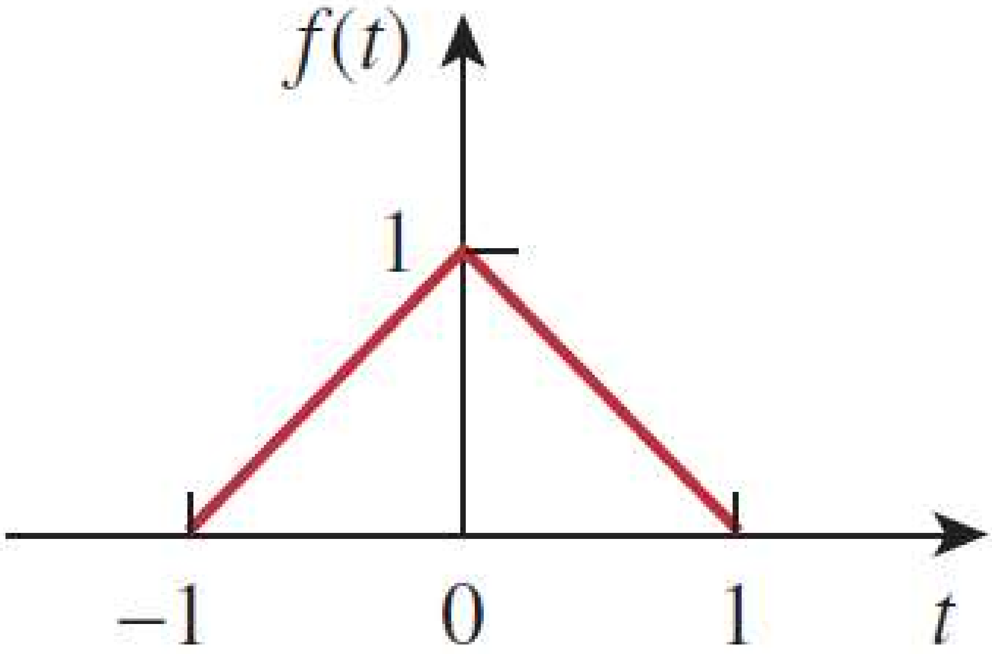
Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t - a)$	$e^{-j\omega a} F(\omega)$
Frequency shift	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Modulation	$\cos(\omega_0 t) f(t)$	$\frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$
Time differentiation	$\frac{df}{dt}$	$j\omega F(\omega)$
	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$

Fourier Transforms – Summary

Property	$f(t)$	$F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \dfrac{d^n}{d\omega^n} F(\omega)$
Reversal	$f(-t)$	$F(-\omega)$ or $F^*(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in t	$f_1(t) * f_2(t)$	$F_1(\omega) F_2(\omega)$
Convolution in ω	$f_1(t) f_2(t)$	$\dfrac{1}{2\pi} F_1(\omega) * F_2(\omega)$

Fourier Transforms – Example 8

Question: **Find the Fourier transform of the following signal?**

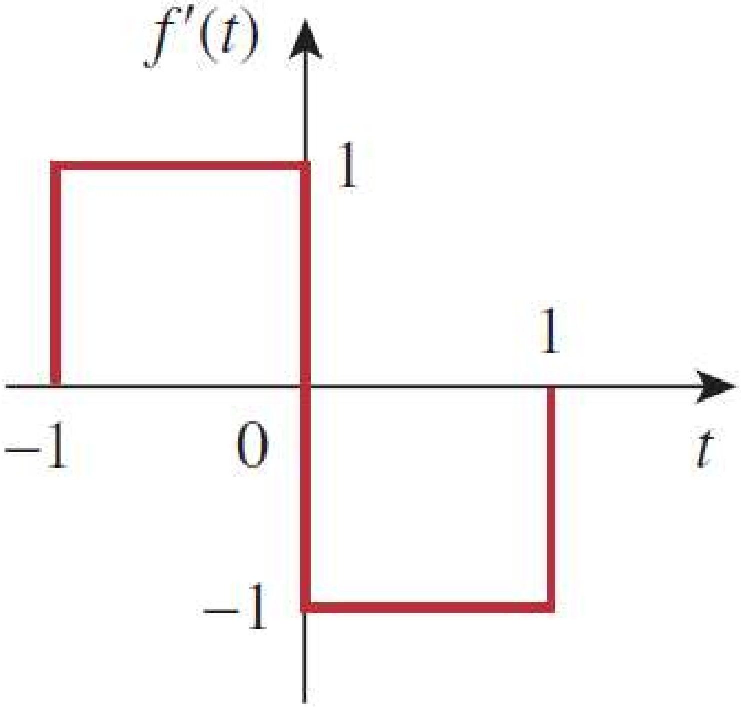


Fourier Transforms – Example 8

Solution: Applying derivative property will lead to the easy path towards solution, The signal can be written as,

$$f(t) = \begin{cases} 1 + t, & -1 < t < 0 \\ 1 - t, & 0 < t < 1 \end{cases}$$

It's first derivative will result into,

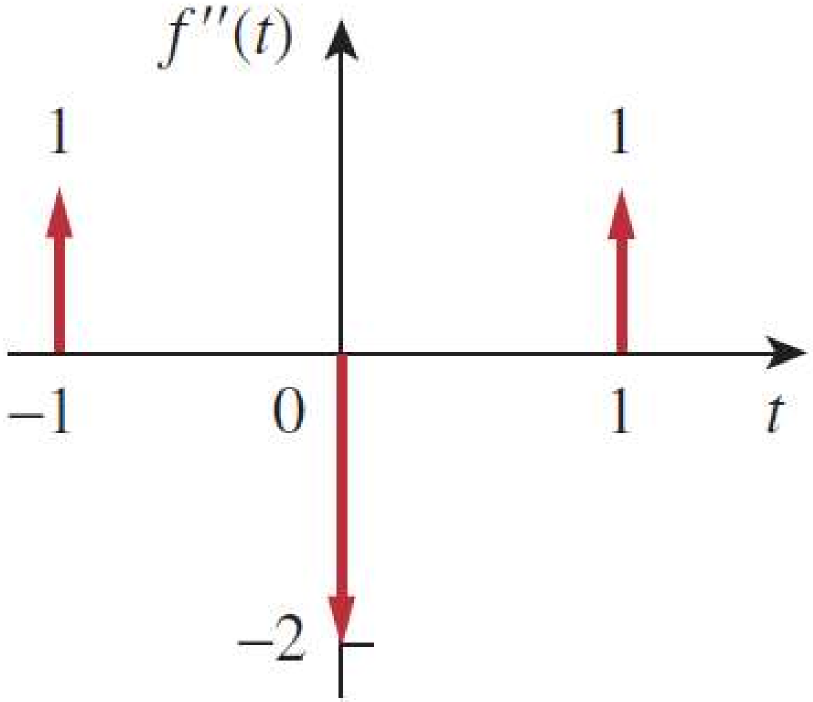


$$f'(t) = \begin{cases} 1, & -1 < t < 0 \\ -1, & 0 < t < 1 \end{cases}$$

Fourier Transforms – Example 8

It's second derivative

$$f''(t) = \delta(t+1) - 2\delta(t) + \delta(t-1)$$



taking Fourier transform of both sides,

$$(j\omega)^2 F(\omega) = e^{j\omega} - 2 + e^{-j\omega} = -2 + 2\cos\omega$$

$$\boxed{F(\omega) = \frac{2(1-\cos\omega)}{\omega^2}}$$

Fourier Transforms – Example 9

Question: Obtain the inverse Fourier transform of following function?

$$\text{(a)}\ F(\omega) = \frac{10j\omega + 4}{(j\omega)^2 + 6j\omega + 8}$$

$$\text{(b)}\ G(\omega) = \frac{\omega^2 + 21}{\omega^2 + 9}$$

Fourier Transforms – Example 9

Solution: (a)

To avoid algebraic complexity, we use *s* instead of *jω* for a moment and replace later on, Using partial fraction expansion

$$F(s) = \frac{10s + 4}{s^2 + 6s + 8}$$

$$= \frac{10s + 4}{(s + 4)(s + 2)}$$

$$= \frac{A}{s + 4} + \frac{B}{s + 2}$$

Fourier Transforms – Example 9

$$A = (s + 4)F(s)\big|_{s=-4} = \left.\frac{10s + 4}{(s + 2)}\right|_{s=-4} = \frac{-36}{-2} = 18$$

$$B = (s + 2)F(s)\big|_{s=-2} = \left.\frac{10s + 4}{(s + 4)}\right|_{s=-2} = \frac{-16}{2} = -8$$

Substituting the values of A , B and replacing s with $j\omega$

$$F(j\omega) = \frac{18}{j\omega + 4} + \frac{-8}{j\omega + 2}$$

using transform pair,

$$\boxed{f(t) = (18e^{-4t} - 8e^{-2t})u(t)}$$

Fourier Transforms – Example 9

Solution: (b)

Applying simplification for $G(\omega)$,

$$G(\omega) = \frac{\omega^2 + 21}{\omega^2 + 9} = 1 + \frac{12}{\omega^2 + 9}$$

using transform pair,

$$\boxed{g(t) = \delta(t) + 2e^{-3|t|}}$$

Objectives

- Average and rms signal power
- Parseval's theorem
- Total harmonic distortion
- Fourier Transforms
- Properties of Fourier transform
- **Circuit interpretation for Fourier transforms**
- Signal Energy and Spectrum
- Signal Bandwidth

Circuit solution with Fourier Transform

- The Fourier transform generalizes the phasor technique to non-periodic functions

- We apply Fourier transforms to circuits with non-sinusoidal excitations in exactly the same way we apply phasor techniques to circuits with sinusoidal excitations

- Ohm's law is still valid

$$V(\omega) = Z(\omega)I(\omega)$$

- Fourier transform cannot solve initial conditions

Circuit solution with Fourier Transform

- **We transform the functions for the circuit elements into the *frequency domain* and take the *Fourier transforms of the excitations*, then we can apply**
  - ✓ **Voltage/current division**
  - ✓ **Source transformation,**
  - ✓ **Mesh analysis**
  - ✓ **Node analysis**
  - ✓ **Thevenin's theorem**
  - ✓ **Norton's theorem**

- **We take the inverse Fourier transform to obtain the response in the time domain**

Circuit solution with Fourier Transform

RLC **components Transforms**

$$
\boxed{\begin{array}{ccc} R & \Rightarrow & R \\ L & \Rightarrow & j\omega L \\ C & \Rightarrow & \dfrac{1}{j\omega C} \end{array}}
$$

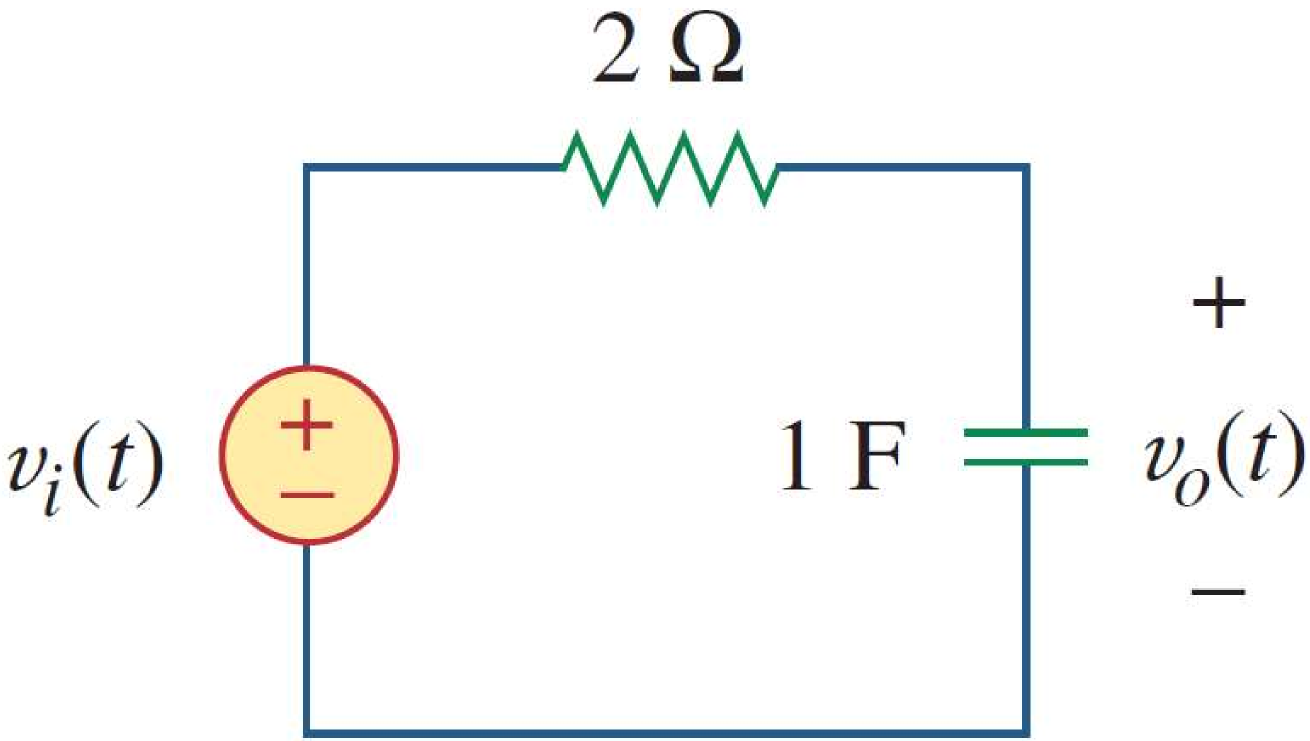
Transfer function

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \qquad\qquad Y(\omega) = H(\omega)X(\omega)$$

Fourier Transform – Example 10

Question: **Find $v_o(t)$ if the $v_i(t) = 2e^{-3t}u(t)$?**



Fourier Transform – Example 10

Solution: **The Fourier transform of the input voltage $v_i(t) = 2e^{-3t}u(t)$ is,**

$$V_i(\omega) = \frac{2}{3 + j\omega}$$

and the transfer function by using voltage division is,

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega}{2 + 1/j\omega} = \frac{1}{1 + j2\omega}$$

$$V_o(\omega) = V_i(\omega)H(\omega) = \frac{2}{(3 + j\omega)(1 + j2\omega)}$$

Fourier Transform – Example 10

$$V_o(\omega) = \frac{1}{(3 + j\omega)(0.5 + j\omega)}$$

By applying partial fraction,

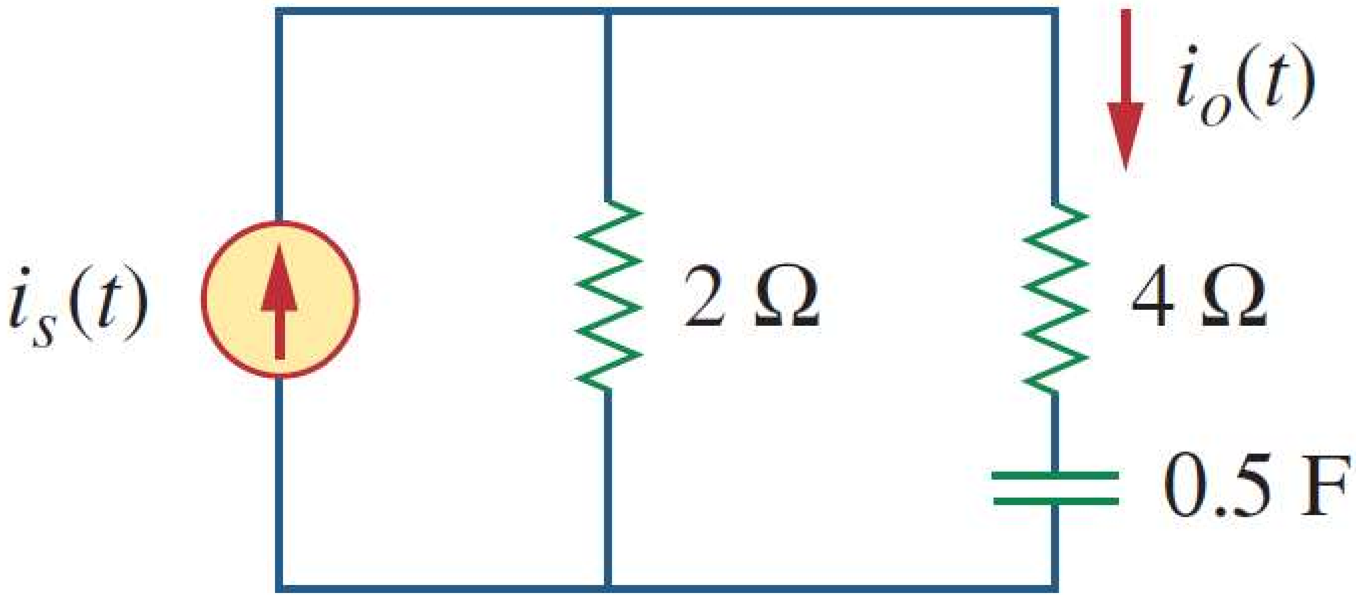
$$V_o(\omega) = \frac{-0.4}{3 + j\omega} + \frac{0.4}{0.5 + j\omega}$$

taking the inverse Fourier transform gives,

$$\boxed{v_o(t) = 0.4(e^{-0.5t} - e^{-3t})u(t)}$$

Fourier Transform – Example 11

Question: **Find $i_o(t)$ if the $i_s(t) = 10\, sin\,(2t)\, A$?**



Fourier Transform – Example 11

Solution: **Using current division rule gives,**

$$H(\omega) = \frac{I_o(\omega)}{I_s(\omega)} = \frac{2}{2 + 4 + 2/j\omega} = \frac{j\omega}{1 + j\omega 3}$$

If $\boldsymbol{i_s(t) = 10\,sin\,(2t)}$ then,

$$I_s(\omega) = j\pi 10[\delta(\omega + 2) - \delta(\omega - 2)]$$

$$I_o(\omega) = H(\omega)I_s(\omega) = \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3}$$

Fourier Transform – Example 11

Using inverse Fourier transform pairs,

$$i_o(t) = \mathcal{F}^{-1}[I_o(\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3} e^{j\omega t}\, d\omega$$

Using sifting property of impulse function,

$$\delta(\omega - \omega_0) f(\omega) = f(\omega_0)$$

or,

$$\int_{-\infty}^{\infty} \delta(\omega - \omega_0) f(\omega)\, d\omega = f(\omega_0)$$

Fourier Transform – Example 11

We obtain *$i_o(t)$* as,

$$i_o(t) = \frac{10\pi}{2\pi}\left[\frac{2}{1+j6}e^{j2t} - \frac{-2}{1-j6}e^{-j2t}\right]$$

$$= 10\left[\frac{e^{j2t}}{6.082e^{j80.54^\circ}} + \frac{e^{-j2t}}{6.082\, e^{-j80.54^\circ}}\right]$$

$$= 1.644\left[e^{j(2t-80.54^\circ)} + e^{-j(2t-80.54^\circ)}\right]$$

$$\boxed{= 3.288\cos(2t - 80.54^\circ)\ \text{A}}$$

Objectives

- Average and rms signal power
- Parseval's theorem
- Total harmonic distortion
- Fourier Transforms
- Properties of Fourier transform
- Circuit interpretation for Fourier transforms
- **Signal Energy and Spectrum**
- Signal Bandwidth

Signal Energy

- The concept of average power cannot be applied on *aperiodic signals*

- The period information required for average power is missing

- The Energy of signal is thus comprising range $[-\infty, \infty]$, given by

$$\boxed{W = \int_{-\infty}^{\infty} |f(t)|^2 dt}$$

$f(t)$ is an aperiodic signal

Signal Energy

- We cannot take Fourier series coefficients due to infinite limits

- It is practical to take FT (Fourier Transform) of the signal

- The Parseval's theorem –Now, Rayleigh's theorem for aperiodic signal will be,

$$\boxed{W = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega}$$

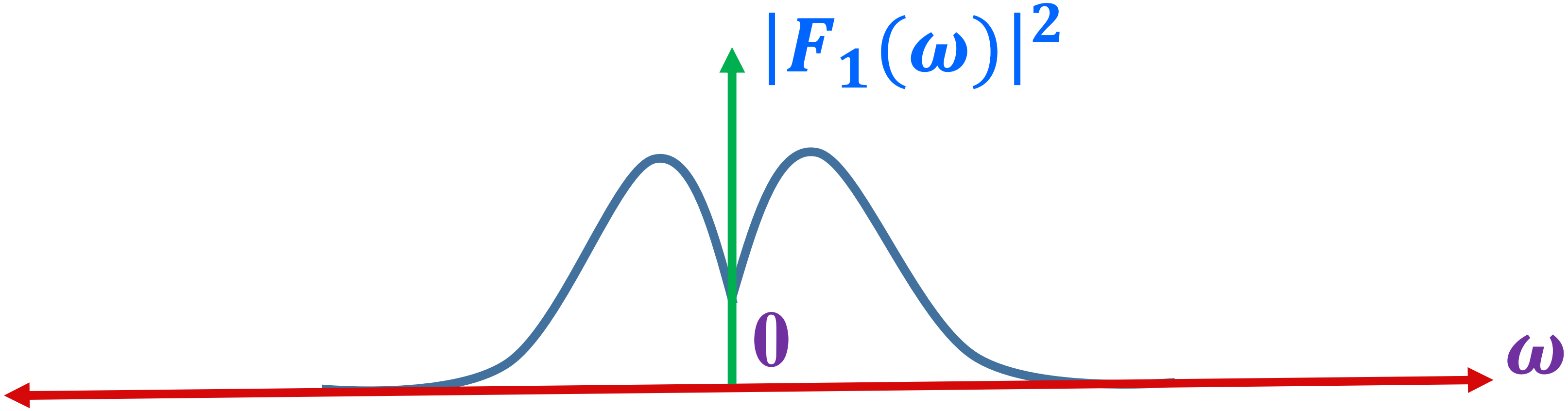
Signal Energy

- Parseval's theorem applies on all finite energy signals
- Any signal having finite period is called energy signal
- Note that;

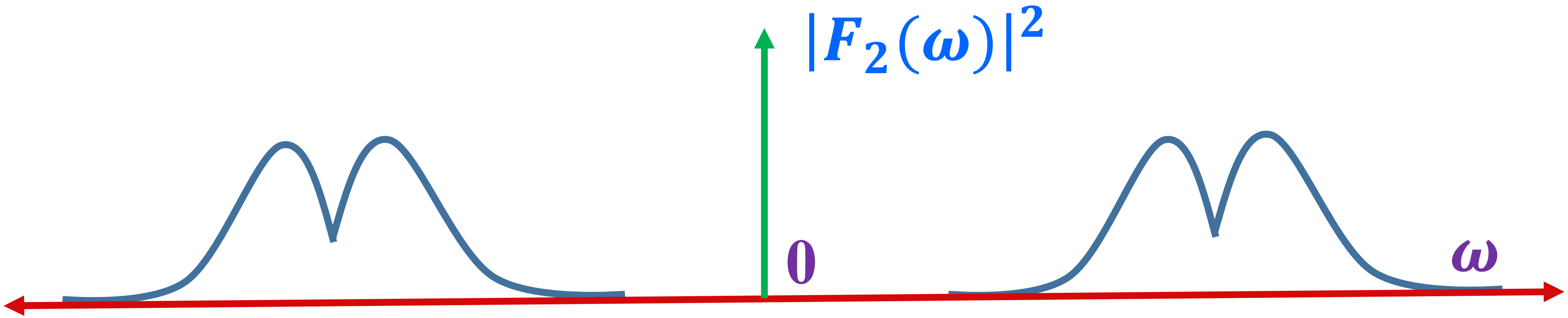
Periodic Signal ➡ $|F_n|^2$ ➡ Power Spectrum

Aperiodic Signal ➡ $|F(\omega)|^2$ ➡ Energy Spectrum

Signal Spectrum – Example Signals

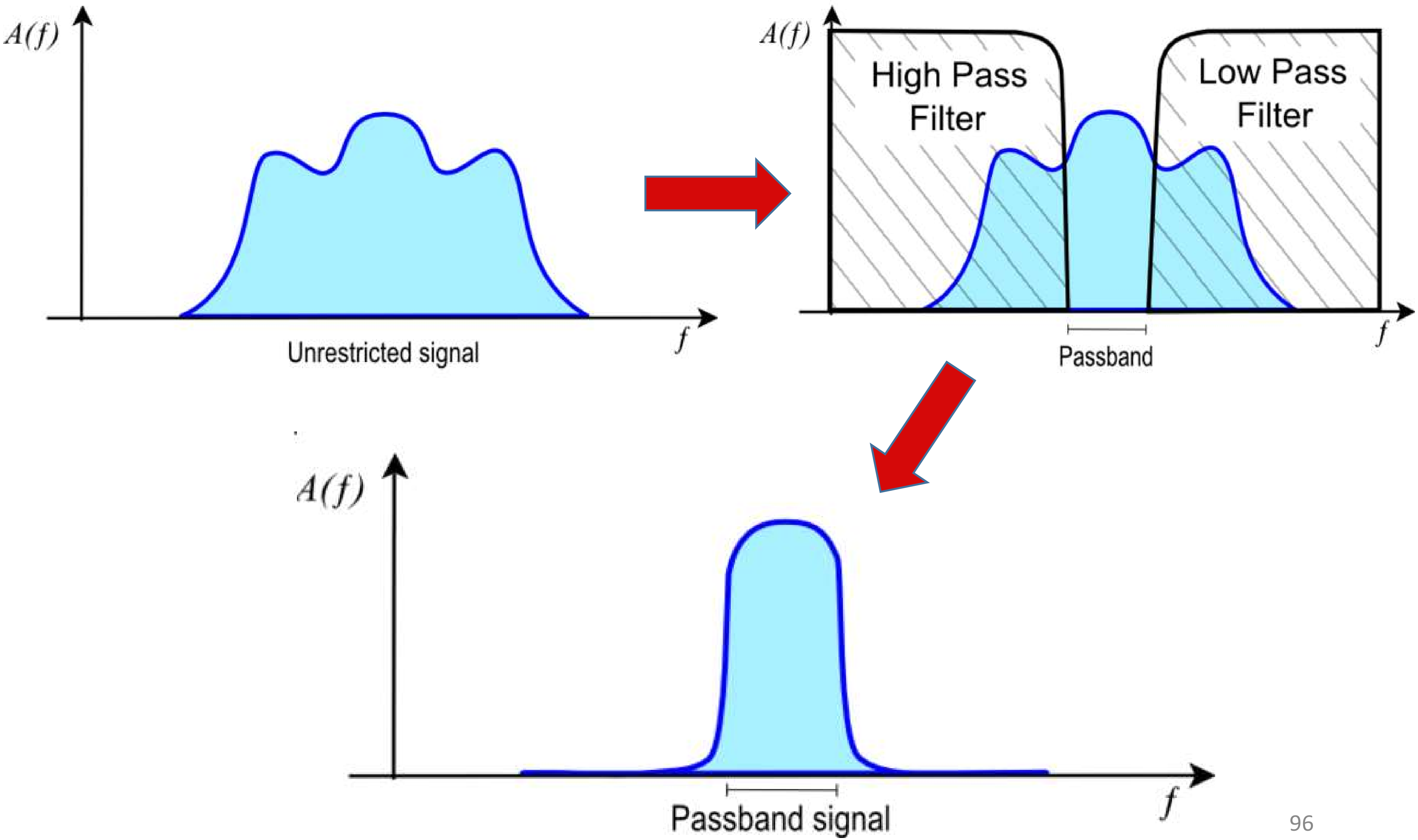


Energy Spectrum of a low pass signal



Energy Spectrum of a band pass signal

Signal Energy – PB Example



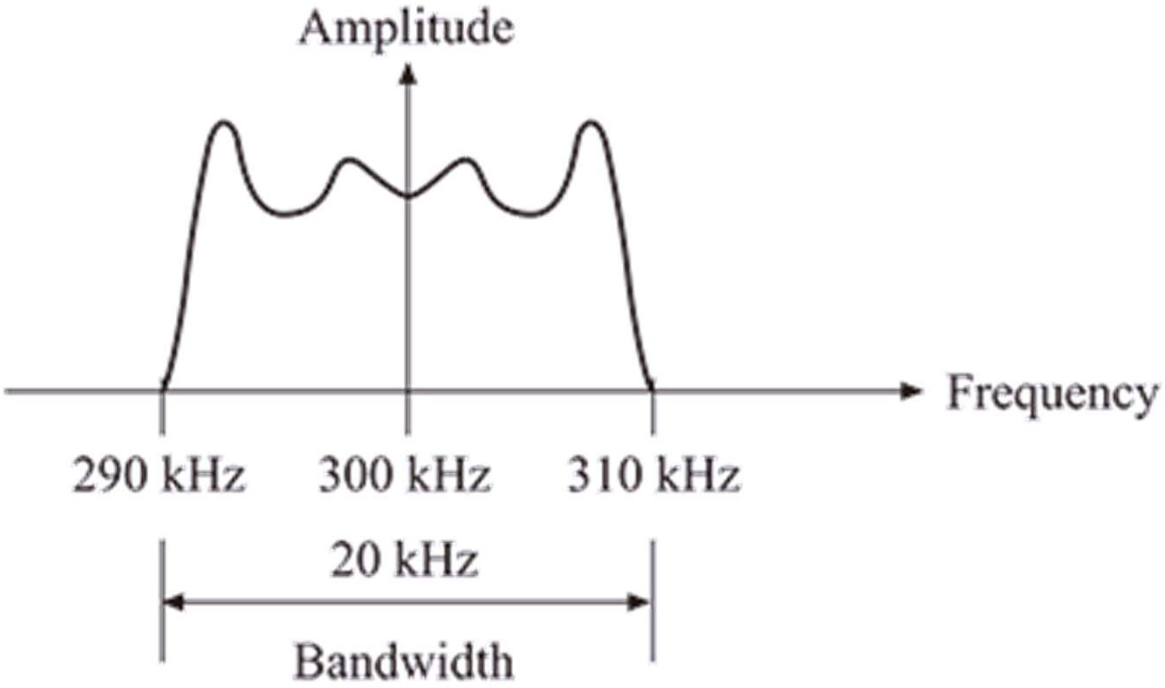
Objectives

- Average and rms signal power
- Parseval's theorem
- Total harmonic distortion
- Fourier Transforms
- Properties of Fourier transform
- Circuit interpretation for Fourier transforms
- Signal Energy and Spectrum
- **Signal Bandwidth**

Signal Bandwidth

- Bandwidth is the span of frequencies within the spectrum that is occupied by a given signal

- For example: For BW=20 KHz



Signal Bandwidth –LP Bandwidth

- **Bandwidth for low pass signal corresponds to a positive frequency as**

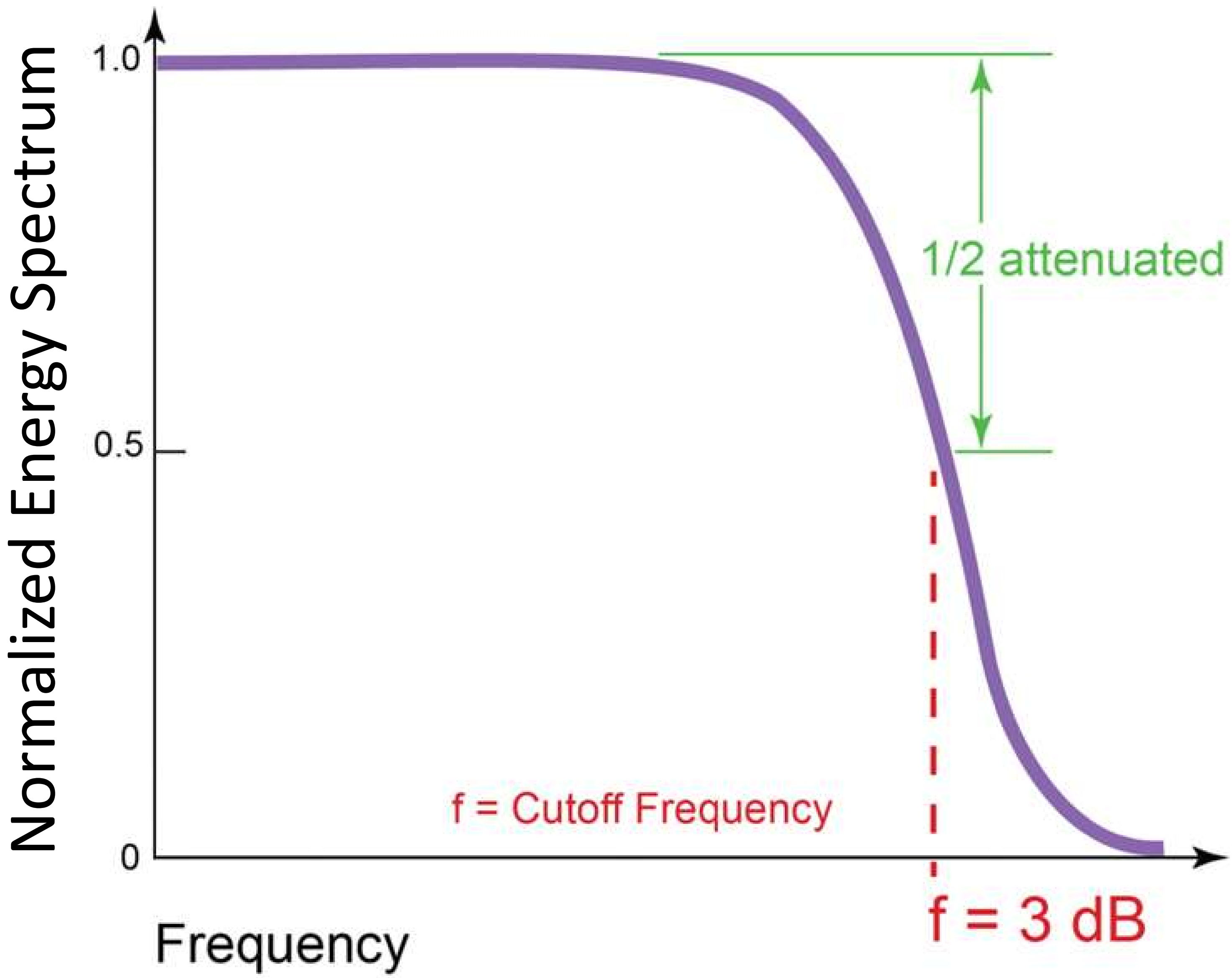
$$\omega = \Omega = 2\pi B$$

- **Beyond this, the energy spectrum $|F(\omega)|^2$ is too small to be ignored**
- **Mostly, we consider *3-dB bandwidth***

$$\boxed{3\text{-}dB\ bandwidth = \frac{|F(\Omega)|^2}{|F(0)|^2} = \frac{1}{2}}$$

- **The energy spectrum falls to one half of the spectral value $|F(0)|^2$ at DC level**

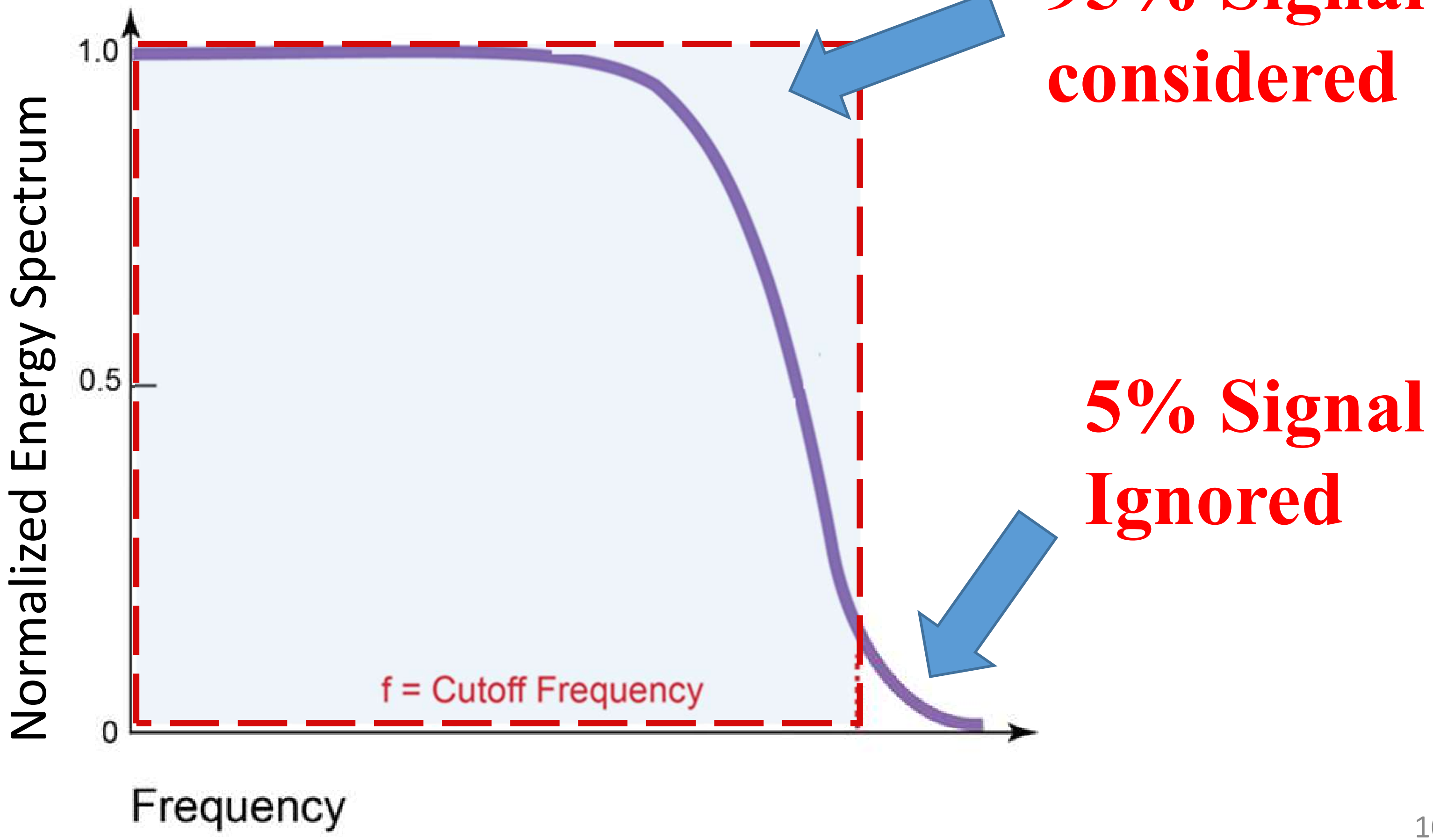
Signal Bandwidth –LP Bandwidth



Signal Bandwidth –LP Bandwidth

- **Another definition for $\Omega = 2\pi B$ requires that,**

$$\frac{1}{2\pi}\int_{-\Omega}^{\Omega}|F(\omega)|^2 d\omega = rW$$



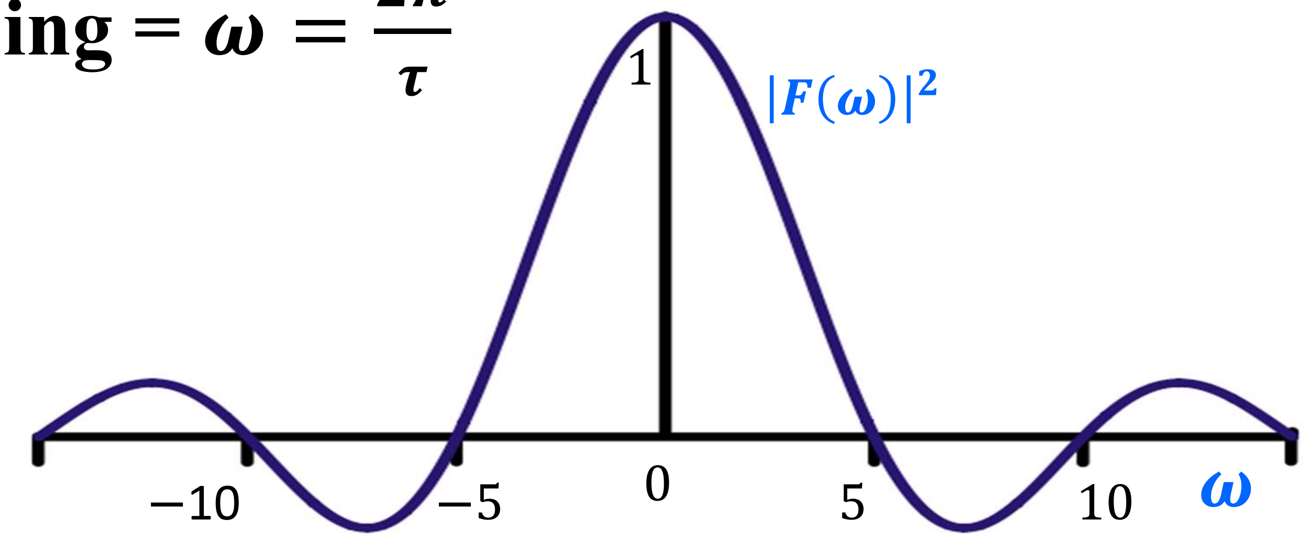
Signal Bandwidth – Example 8

Question: **The Fourier transform of the signal,**

$$f(t) = rect\left(\frac{t}{\tau}\right) \quad \text{is} \quad F(\omega) = \tau . sinc\left(\frac{\omega\tau}{2}\right)$$

So, the corresponding energy signal, $|F(\omega)|^2 = \tau^2 sinc^2\left(\frac{\omega\tau}{2}\right)$

First zero-crossing $= \omega = \dfrac{2\pi}{\tau}$



Show that choosing $\Omega = 2\pi B$ **bandwidth covers 90% of the signal?**

Signal Bandwidth – Example 8

Solution: **Given that,**

$$W = \int_{-\infty}^{\infty} \left| rect\left(\frac{t}{\tau}\right) \right|^2 dt = \int_{\frac{-t}{\tau}}^{\frac{t}{\tau}} dt = \tau$$

As, $\Omega = \frac{2\pi}{\tau}$ **is** $r\%$ **bandwidth of the signal, where** r **satisfies,**

$$\frac{1}{2\pi} \int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \tau^2 sinc^2\left(\frac{\omega\tau}{2}\right) d\omega = r\tau$$

Signal Bandwidth – Example 8

After change of variable as

$$x \equiv \frac{\omega\tau}{2}$$

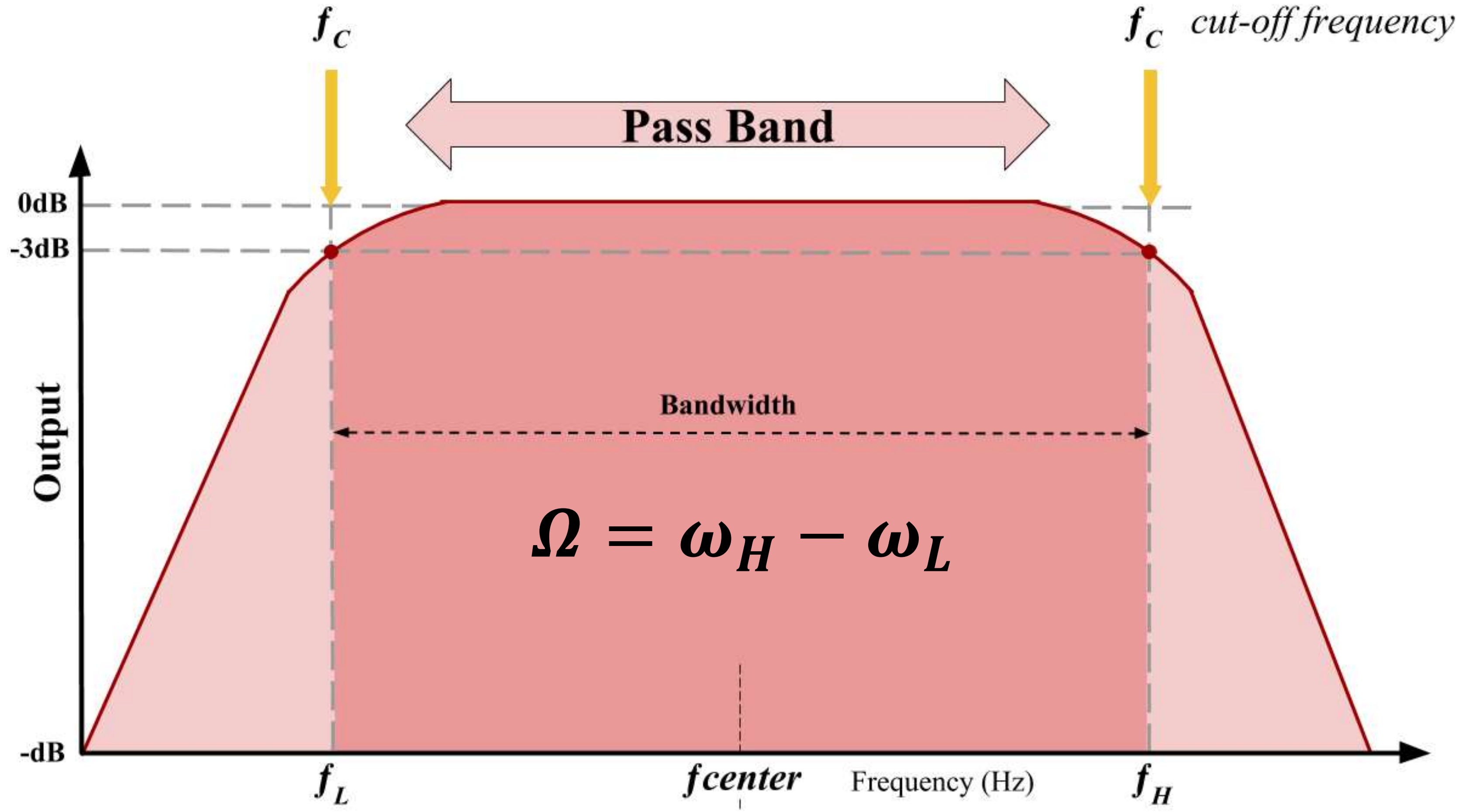
and using even integration,

$$r = \frac{2}{\pi}\int_{x=0}^{\pi} sinc^2(x)dx \approx 0.903$$

- **It can be seen that $\Omega = \frac{2\pi}{\tau}$ corresponding to the frequency of first null in the energy spectrum of the signal $rect\left(\frac{t}{\tau}\right)$,is the 90.3% bandwidth of the signal**

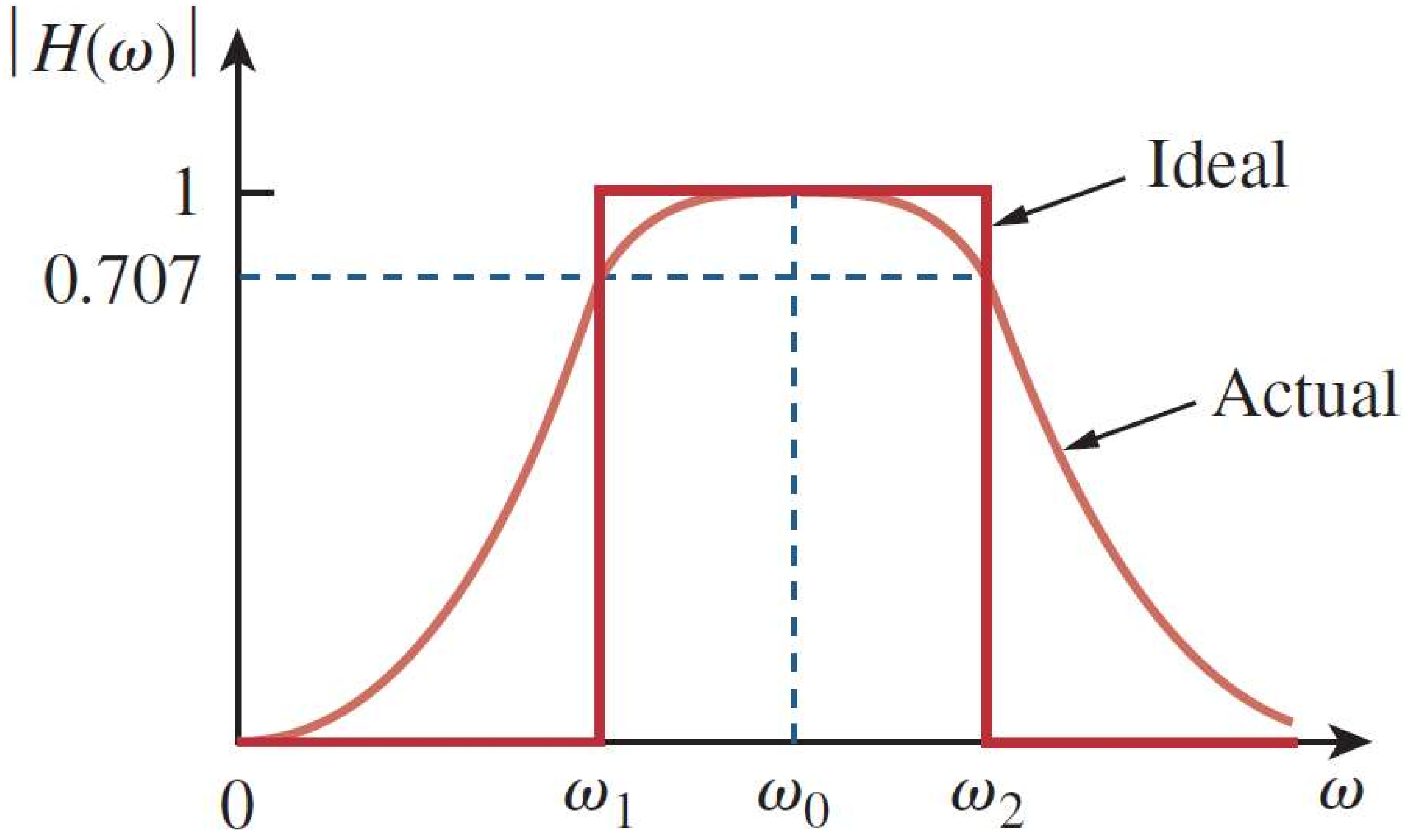
Signal Bandwidth –PB Bandwidth

- The difference between higher frequency spectrum and lower frequency spectrum is passband bandwidth



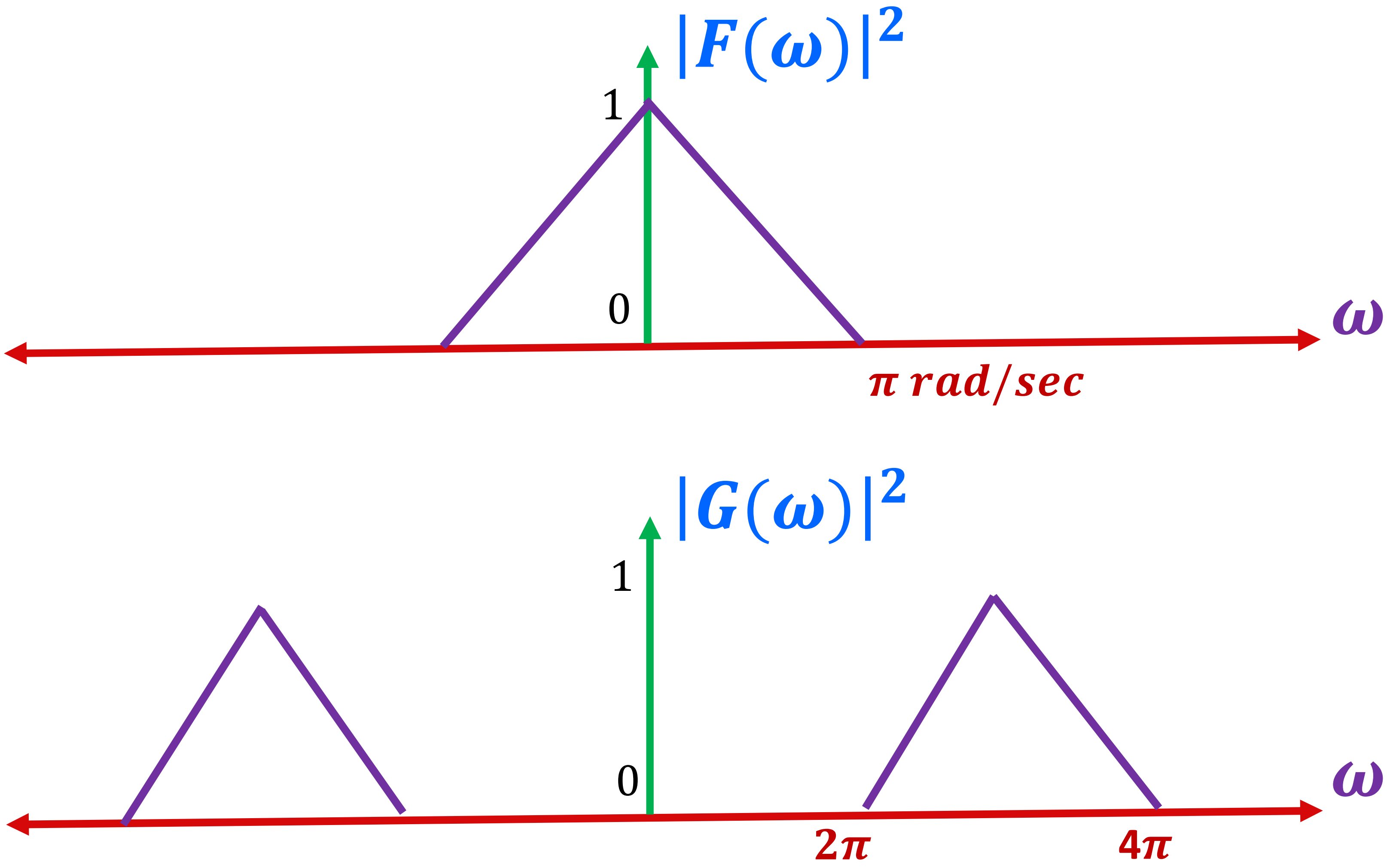
Signal Bandwidth –PB Bandwidth

- **A** **bandpass filter** **is designed to pass all frequencies within a band of frequencies** $\boldsymbol{\omega_1 < \omega < \omega_2}$



PB Bandwidth – Example 12

Question: **Find the 95% of the bandwidth of *f(t)* and *g(t)* for energy spectra shown below?**



PB Bandwidth – Example 12

Solution: **According to Parseval's theorem,**

- **Energy *W* of the signal *f(t)* is simply area under the curve *scaled by* $\frac{1}{2\pi}$**

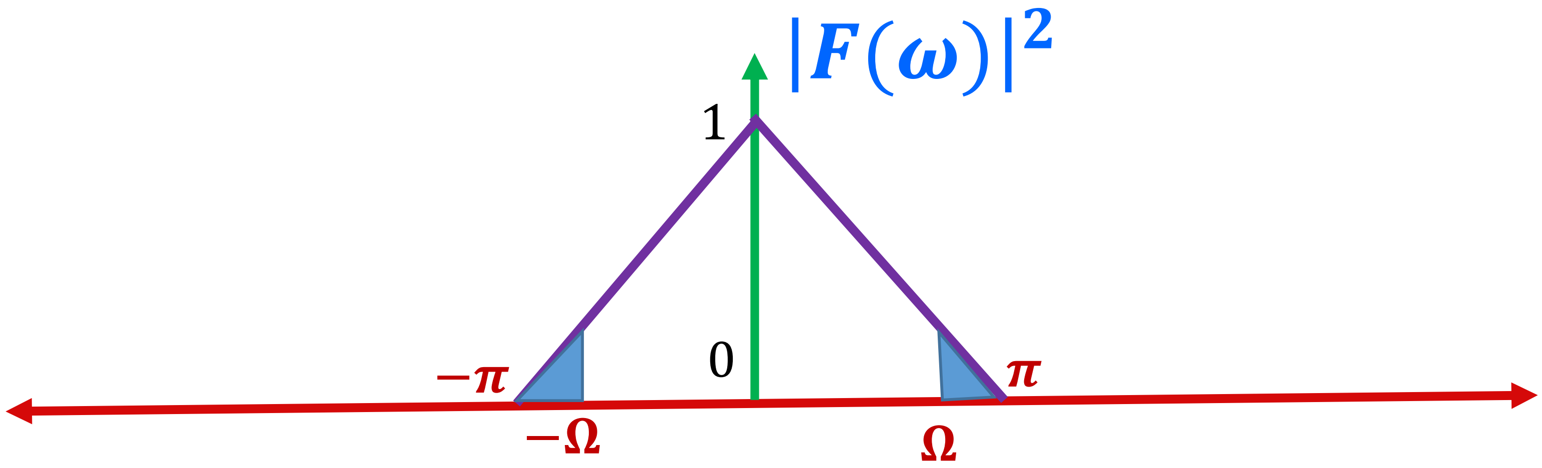
- **Using the formulae,**

$$W = \frac{1}{2\pi}\frac{2\pi \times 1}{2} = \frac{1}{2}$$

- **To determine 95% bandwidth, we compute the signal energy *outside* signal bandwidth (i.e. $|\omega| > \Omega$)**

- **Set this equal to 5% of *W (0.05W = 0.025)***

PB Bandwidth – Example 12



- This energy outside $|\omega| > \Omega$ equals $\frac{1}{2\pi}$ times the combined areas of right and left tips of $|F(\omega)|^2$

$$\frac{1}{2\pi}(\pi - \Omega)\frac{\pi - \Omega}{\pi}$$

- Setting this quantity equals to 0.025 yields,

PB Bandwidth – Example 12

$$(\pi - \Omega)^2 = 0.05\pi^2$$

$$\Omega = \pi(1 - \sqrt{0.05}) \approx 0.77\pi \ (rad/sec)$$

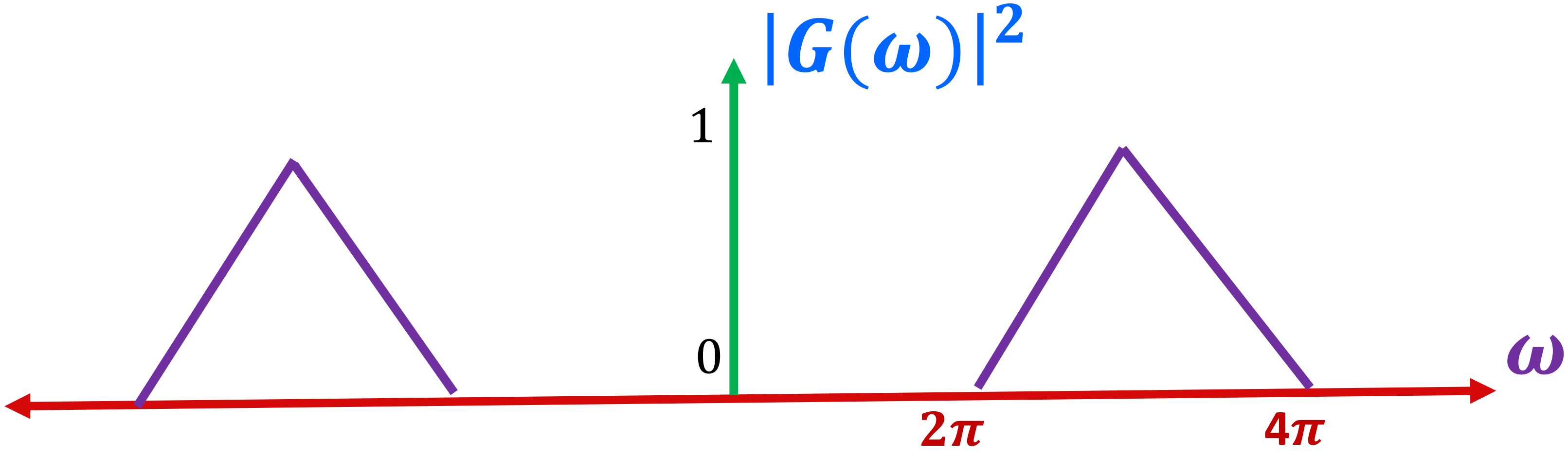
- Comparing the two signals energy spectra, it reveals that $|G(\omega)|^2$ is doubles as $|F(\omega)|^2$

- Simply, doubles the bandwidth will work for $|G(\omega)|^2$ 95% bandwidth acquisition

$$\Omega \approx 1.55\pi \ (rad/sec)$$

PB Bandwidth – Example 13

Question: What is 100% bandwidth of *g(t)?*



Solution: It can be seen that,

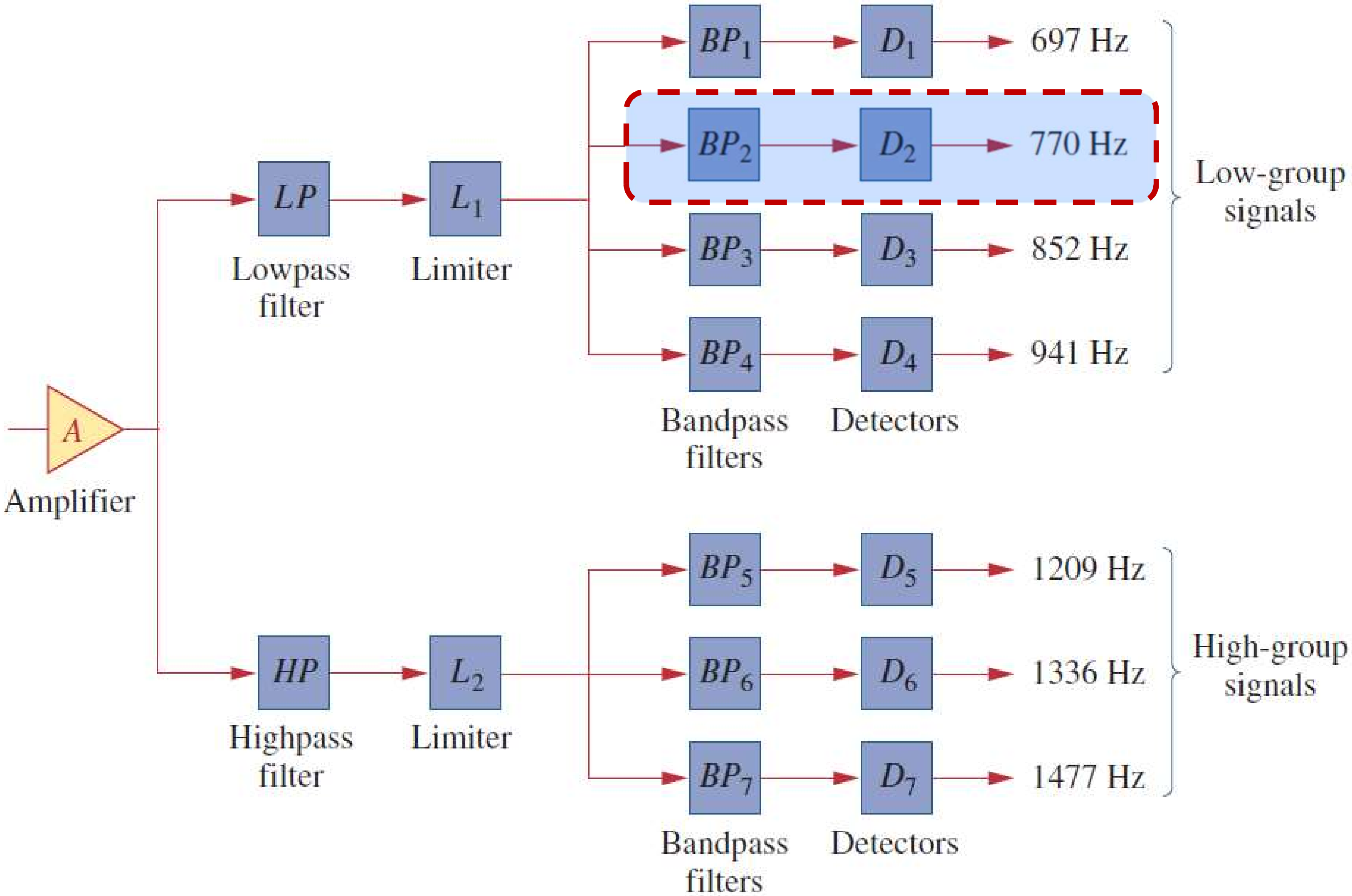
$$\omega_u = 4\pi \;\&\; \omega_l = 2\pi$$

$$\Omega = \omega_u - \omega_l = 4\pi - 2\pi = 2\pi\, rad/sec$$

or $B = 1\ Hz$ gives the 100% bandwidth

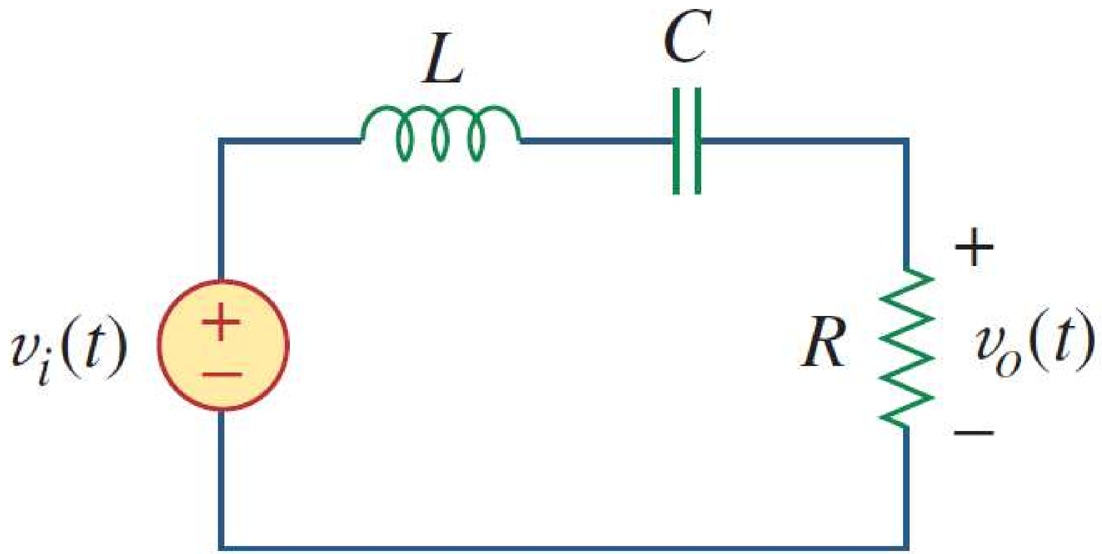
PB Bandwidth – Example 14

Question: **Using the standard 600 Ohms resistor used in telephone circuit, design *RLC* circuit for BP_2?**



PB Bandwidth – Example 14

Solution: Bandpass filter circuit (*R=600* Ω)



Since BP_2 passes frequencies 697 Hz to 852 Hz and is centered at $f_o = 770\ Hz$, The bandwidth is,

$$B = 2\pi(f_2 - f_1) = 2\pi(852 - 697) = 973.89 \text{ rad/s}$$

PB Bandwidth – Example 14

For inductance evaluation, we use relation,

$$L = \frac{R}{B} = \frac{600}{973.89} = 0.616\ \mathrm{H}$$

at resonant frequency, the circuit responds as pure non-resistive so,

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{4\pi^2 f_0^2 L}$$

$$= \frac{1}{4\pi^2 \times 770^2 \times 0.616} = 69.36\ \mathrm{nF}$$

Summary

- The total average power is the sum of the average powers in each harmonically related voltage and current

- Parseval's theorem states that the average power in a periodic signal is the sum of the average power in its dc component and the average powers in its harmonics

- Ratio of second and higher order harmonics to the fundamental harmonic is called total harmonic distortion

Summary

- The *Fourier transform* (FT) is an integral transformation of $f(t)$ from the time domain to the frequency domain

- A sufficient but not necessary condition that *f(t)* has a Fourier transform is that it be completely integrable

- By using different properties and transform pairs, many non-periodic inputs can be evaluated

- Circuits are solved using FT having basic theorems but initial conditions are not satisfied with FT

Summary

- **An aperiodic signal can be expressed in signal energy spectrum instead of power by taking FT over range $[-\infty, \infty]$**
- **For the case, Parseval's theorem is translated into Rayleigh theorem for infinite energy signals**
- **The bandwidth refers to the span of upper frequency and lower frequency spectrum**
- **The energy spectrum falls to one half of the spectral value at DC level**

Further reading

1. Ch. 6 (page 212-218), Ch. 7 (page 223-246), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.
2. Ch. 14 (page 650-670), Ch. 17 (page 782-785) and Ch. 18 (page 814-836), C. K. Alexander and M. Sadiku, ***Fundamentals of Electric Circuits***, 5th ed., McGraw-Hill, 2013.
3. Ch. 15 (page 755-765, 773-783), J. D. Irwin, and R. M. Nelms. ***Basic Engineering Circuit Analysis,*** 10th ed. Wiley, 2011.

Preview:

1. Ch. 7 (page 248-258), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.

Homework 9

Deadline: 10:00 PM, 27th April, 2022

Thank you!