ANALOG SIGNAL PROCESSING

ECE 210 & 211

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Objectives

- **Envelope detection**
- **Super heterodyne receiver with envelope detection**
- **Convolution**
- **The Fourier properties of convolution**

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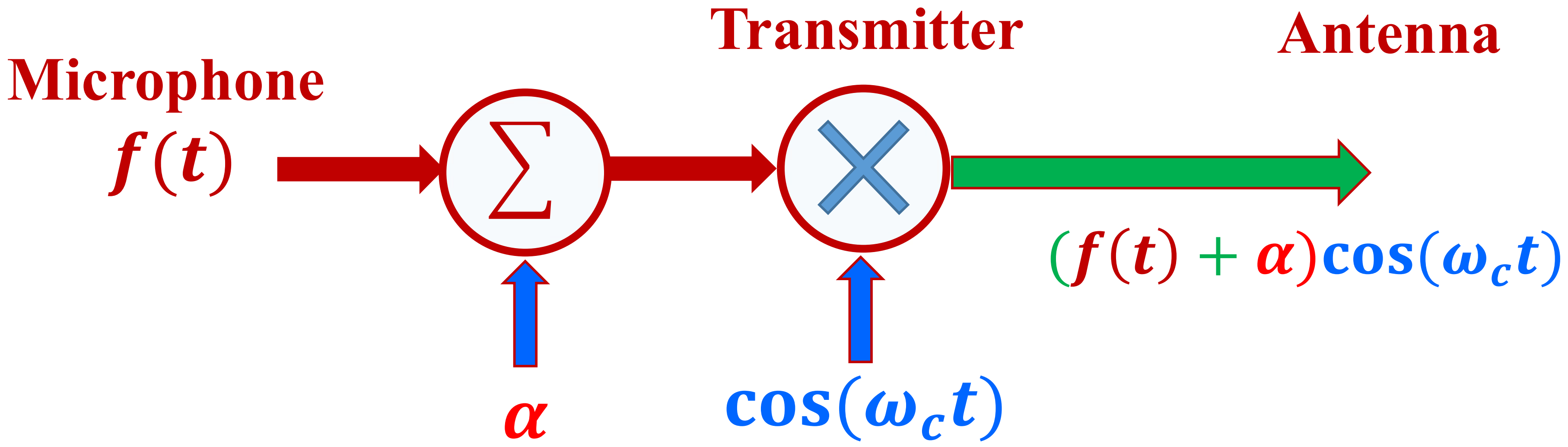
- Envelope detection
- Super heterodyne receiver with envelope detection
- Convolution
- The Fourier properties of convolution

Envelope Detection

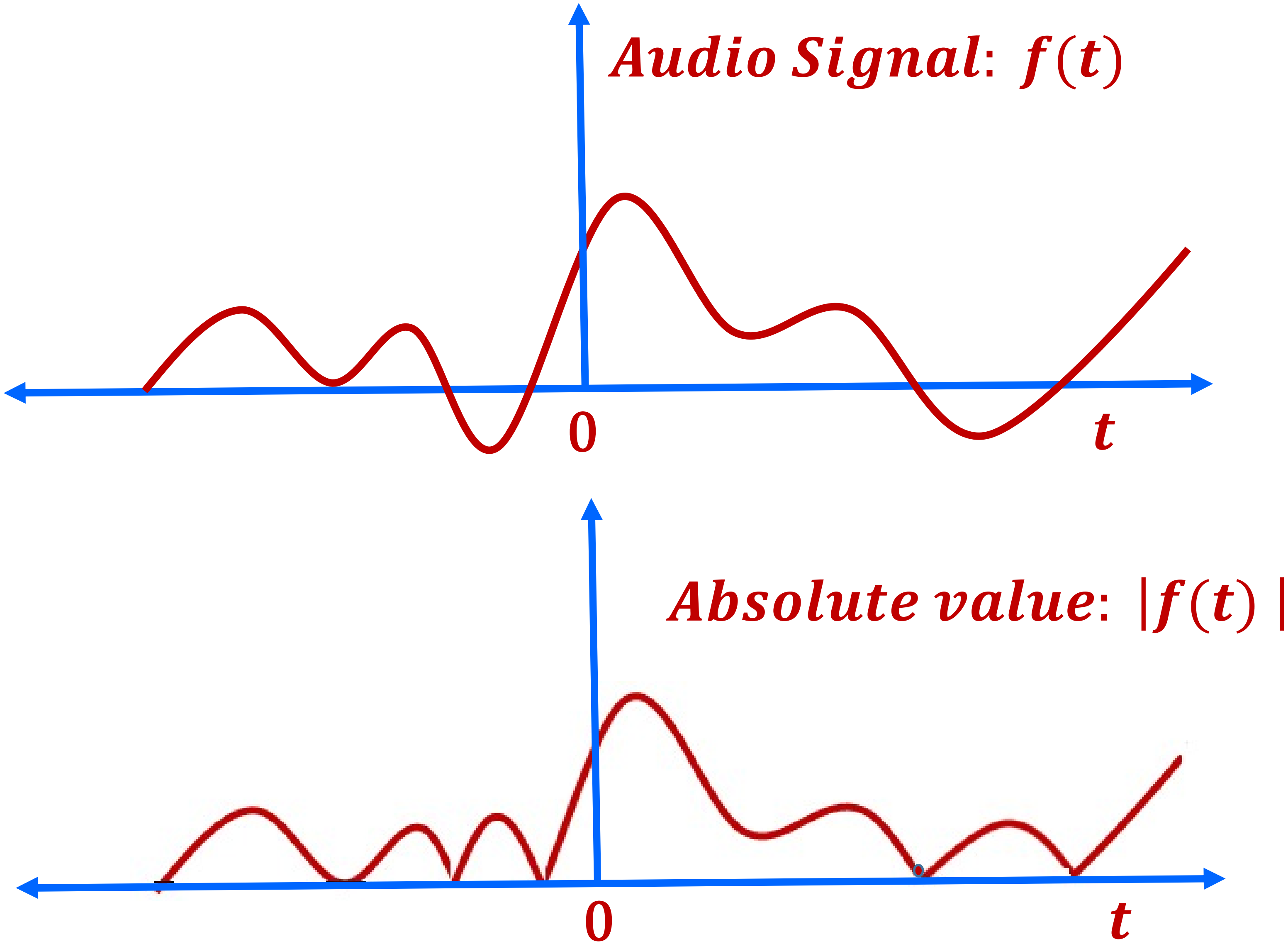
- Previously, we discussed about introducing some *DC offset* α in the voice signal $f(t)$,

- Summing function modulates the amplitude of the carrier signal, $cos(\omega_c t)$

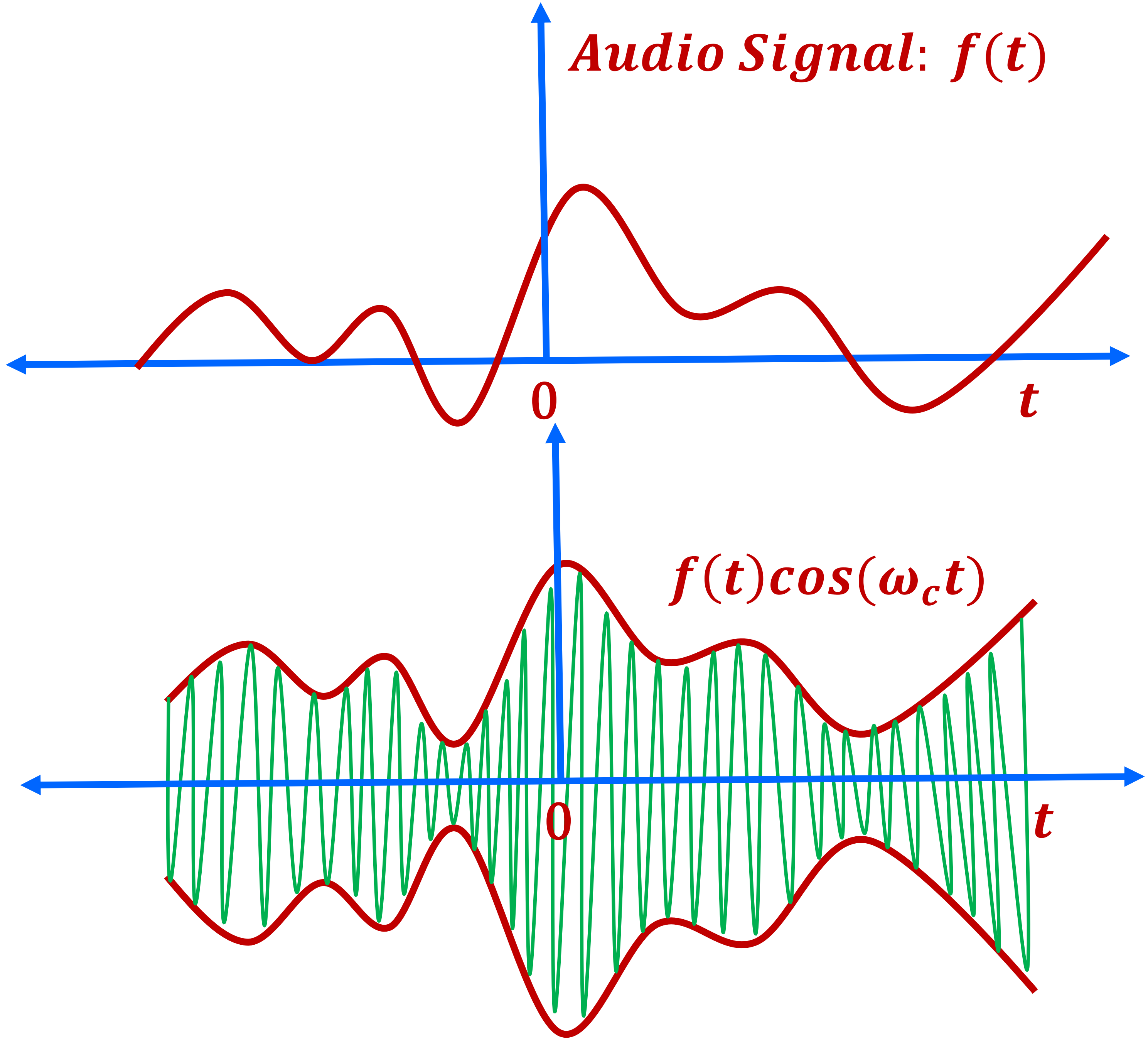
- The system should look like,



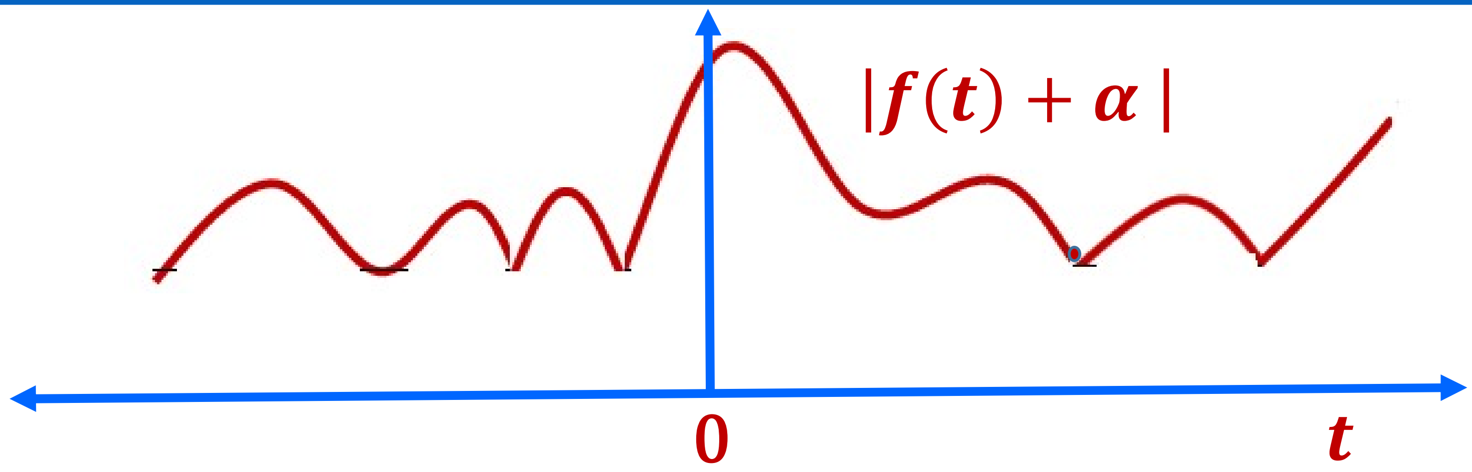
Envelope Detection – without DC offset

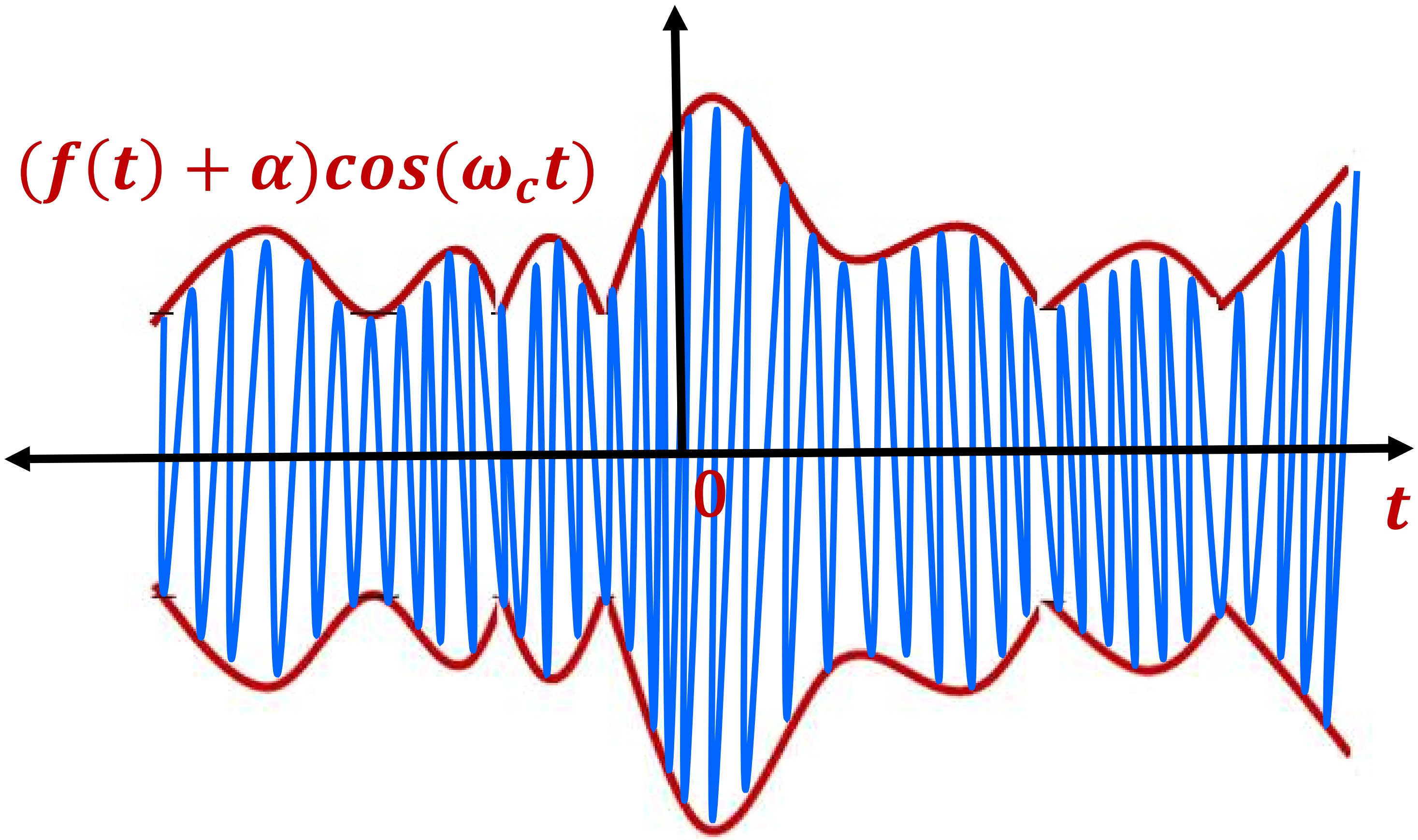


Envelope Detection – without DC offset



Envelope Detection – with DC offset





Envelope Detection

- Notice that the shape of $f(t)cos(\omega_c t)$ and $(f(t)+\alpha)cos(\omega_c t)$ are different due to "*rectification effect*"

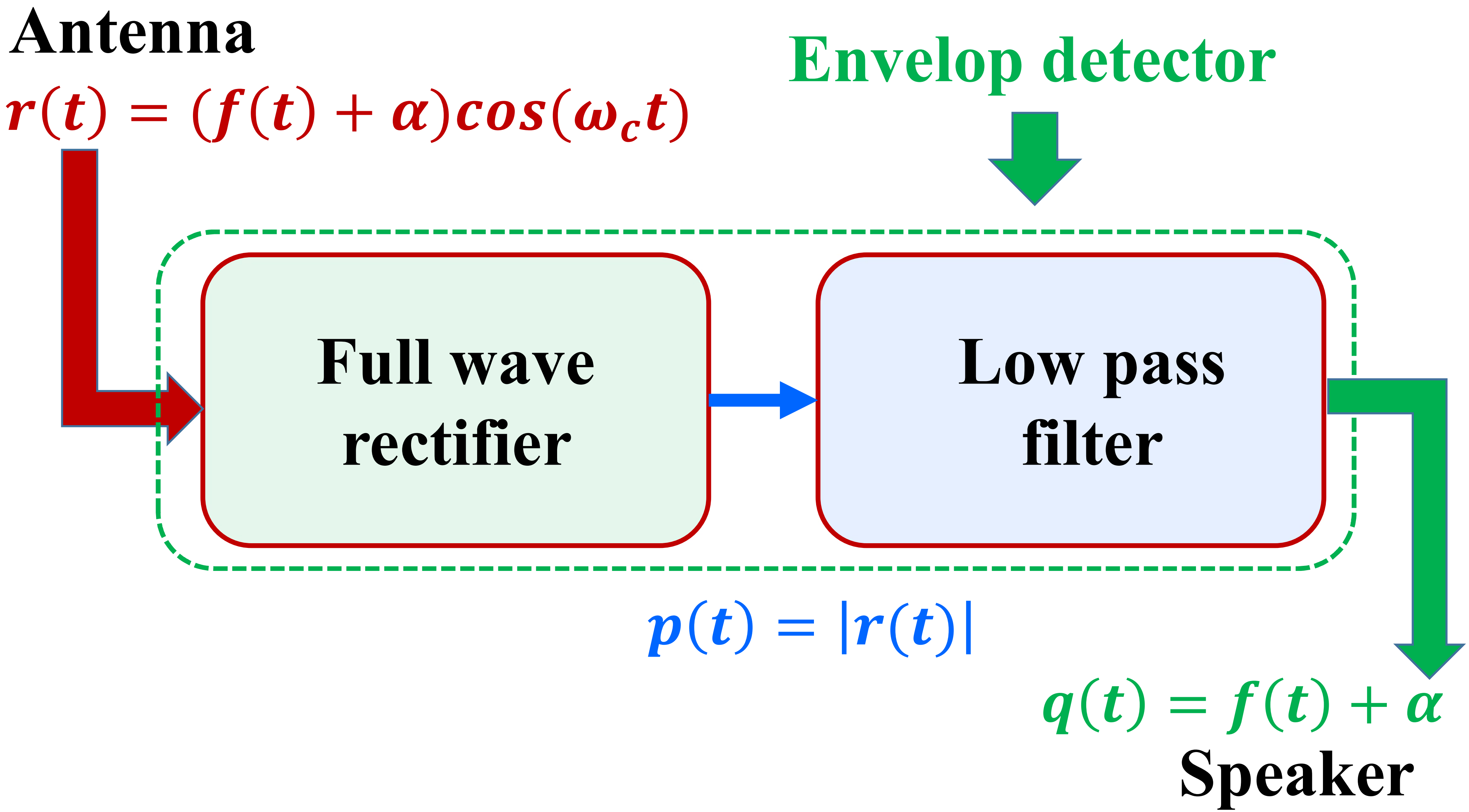
- Our intention is to design a detector that works when,

$$\alpha > max\,|f(t)|$$

- An ideal envelop detector should contain a full wave rectifier followed by a low pass filter

- Let's see the overall operation of an ideal envelop detector,

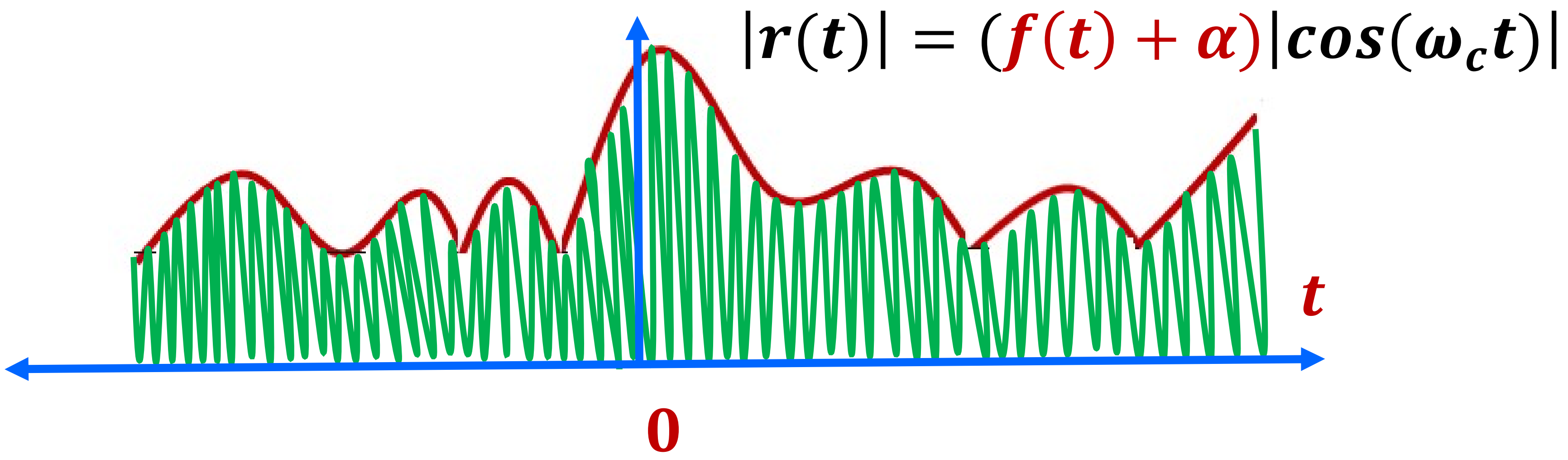
Envelope Detection



Envelope Detection – Rectification

- Let's assume that $\alpha > max\ |f(t)|$ so that $f(t) + \alpha > 0$ and $|f(t) + \alpha| = f(t) + \alpha$, then the rectifier output $p(t)$ will be,

$$p(t) = |r(t)| = |(f(t) + \alpha)cos(\omega_c t)\ |$$
$$= (f(t) + \alpha)|cos(\omega_c t)|$$



Envelope Detection – Low pass filter

- **Now, the signal has two components**
  - **High frequency cosine with ω_c as carrier frequency**
  - **Low frequency envelope signal having peaks containing $f(t)$**
- **We have to design an LPF to allow the shape of $f(t)$ and block all high frequency signals**
- **We can realize this by using Fourier series**

Envelope Detection – Low pass filter

High frequency cosine with ω_c as carrier frequency

- **The rectified cosine has period** $T = \frac{T_c}{2} = \frac{\frac{2\pi}{\omega_c}}{2} = \frac{\pi}{\omega_c}$

- **The fundamental frequency is** $\omega_o = 2\omega_c$

- **This can be expanded in Fourier series as,**

$$|\cos(\omega_c t)| = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos(n2\omega_c t)$$

Envelope Detection – Low pass filter

- **By choosing appropriate Fourier coefficient a_n, It was found that $a_o = \frac{4}{\pi}$** ***(Example 6.7-tex book)***
- **The rectifier output can be written as,**

$$p(t) = (f(t) + \alpha)|cos(\omega_c t)| = p_1(t) + p_2(t)$$

where,

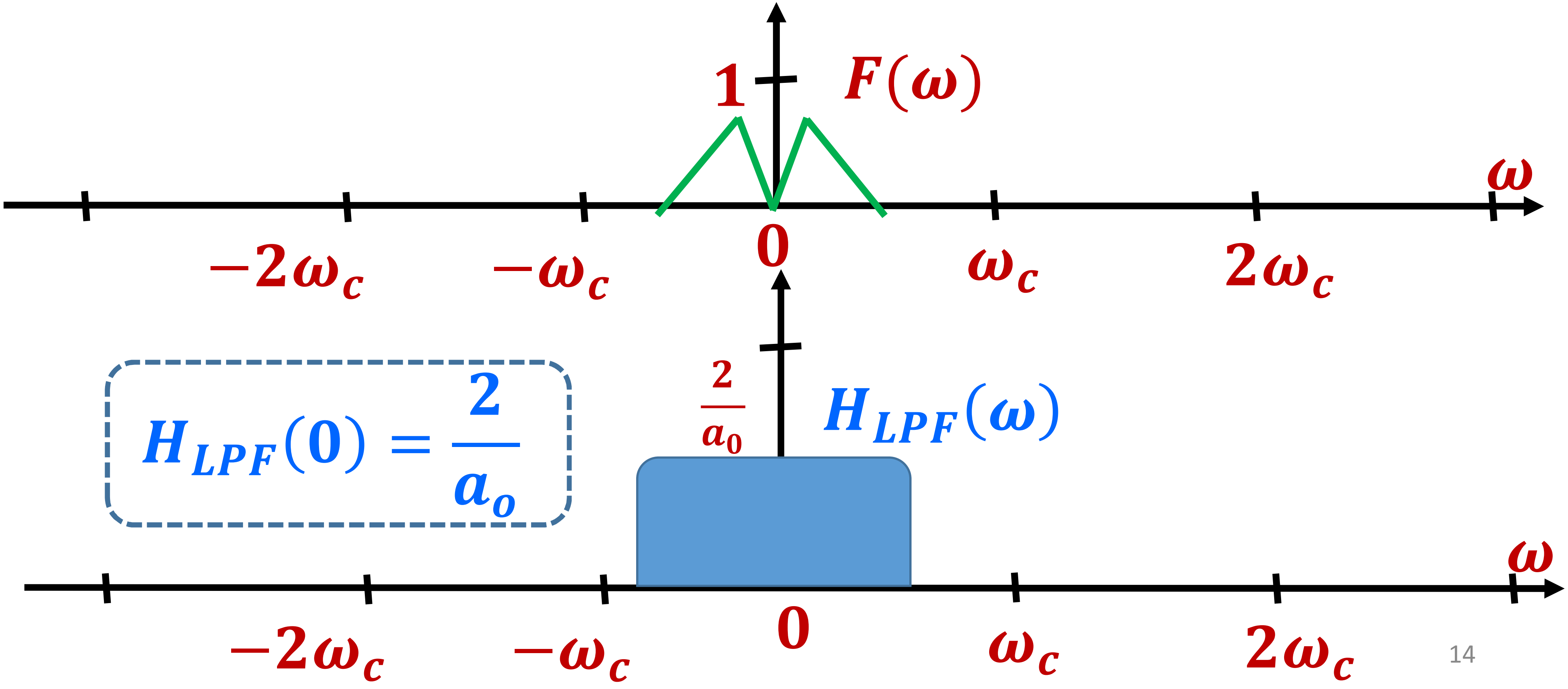
$$p_1(t) = \frac{a_o}{2} f(t) + \sum_{n=1}^{\infty} a_n f(t)\cos(n2\omega_c t)$$

$$p_2(t) = \frac{a_o}{2} \alpha + \sum_{n=1}^{\infty} a_n \alpha \cos(n2\omega_c t)$$

Envelope Detection – Low pass filter

- Now, the response of the filter $H_{LPF}(\omega)$ to the input $p_2(t)$ is $q_2(t) = \alpha$

$$H_{LPF}(n2\omega_c) = 0 \; for \; n > 1$$



$$H_{LPF}(0) = \frac{2}{a_o}$$

Envelope Detection – Low pass filter

- **To determine the filter response $q_1(t)$ to the input $p_1(t)$, we observe that,**

$$P_1(\omega) = \frac{a_o}{2} F(\omega) + \sum_{n=1}^{\infty} \frac{a_n}{2} \{F(\omega - n\omega_c t) + F(\omega + n\omega_c t)\}$$

- **Only the first term in the expression lies within the passband of the low pass filter**

$$Q_1(\omega) = P_1(\omega) H_{LP}(\omega) = F(\omega)$$

taking IFT,

$$q_1(t) = f(t)$$

Envelope Detection – Low pass filter

- Using the superposition , we can find the filter output for the input $p(t) = p_1(t) + p_2(t)$ as,

$$q(t) = q_1(t) + q_2(t) = f(t) + \alpha$$

which is the desired envelop of input

$$r(t) = (f(t) + \alpha)(cos(\omega_c t))$$

- In summary, an envelop detector detects the envelope of an AM signal corresponding to an audio signal that is offset to certain DC level

Envelope Detection – Linearity

- *Is envelope detector a linear or non-linear process?*
- **In general, it is non-linear as,**

$$\left(\boldsymbol{f_1(t) + f_2(t)cos(\omega_c t)}\right)$$

will be different from sum of envelopes $|\boldsymbol{f_1(t)}|$ and $|\boldsymbol{f_2(t)}|$ of the signals

$$\boldsymbol{f_1(t)cos(\omega_c t)} \qquad \& \qquad \boldsymbol{f_2(t)cos(\omega_c t)}$$

The envelopes of $\left(\boldsymbol{f_1(t) + f_2(t)cos(\omega_c t)}\right)$ is

$$|\boldsymbol{f_1(t) + f_2(t)}|$$

Envelope Detection – Linearity

but for the case,

$$|f_1(t) + f_2(t)| \neq |f_1(t)| + |f_2(t)|$$

unless, for each value of t, $f_1(t)$ and $f_2(t)$ have same algebraic sign

Envelope Detection – Example

Question: **Suppose that input signal to an envelop detector is,**

$$r(t) = (f(t) + \alpha)cos(\omega_c(t - t_o)) \quad \text{for positive } t_o$$

what will be the detector output while assuming $f(t) + \alpha > 0$?

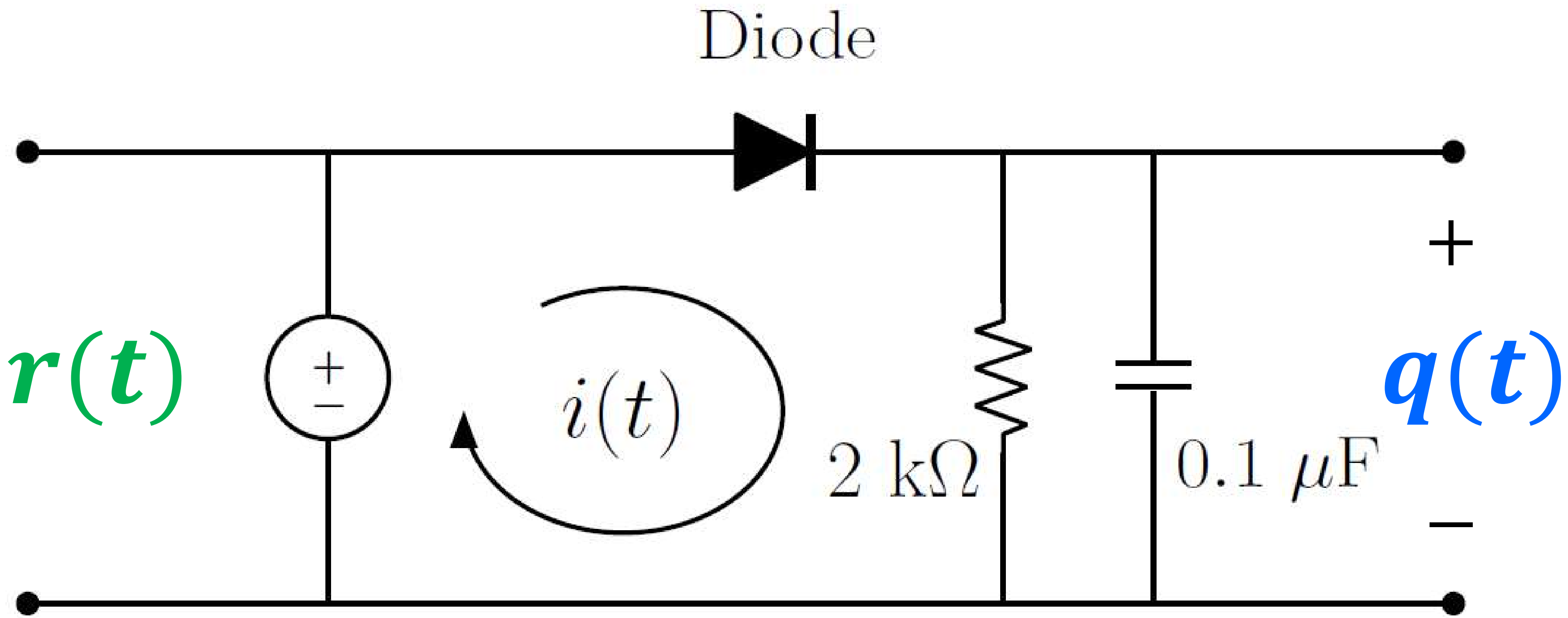
Solution: **The detector output will be**

$$f(t) + \alpha$$

- **Amplitude coefficients of Fourier series $|cos(\omega_c(t - t_o))|$ and $|cos(\omega_c(t)|$ are identical**
- **Envelope detection is insensitive to the phase shift of the carrier signal**

Envelope Detection – Practical circuit

- **When $r(t) > q(t)$, the diode conducts and charges capacitor up to voltage $q(t)$ that remains close to the envelope of $r(t)$**



Objectives

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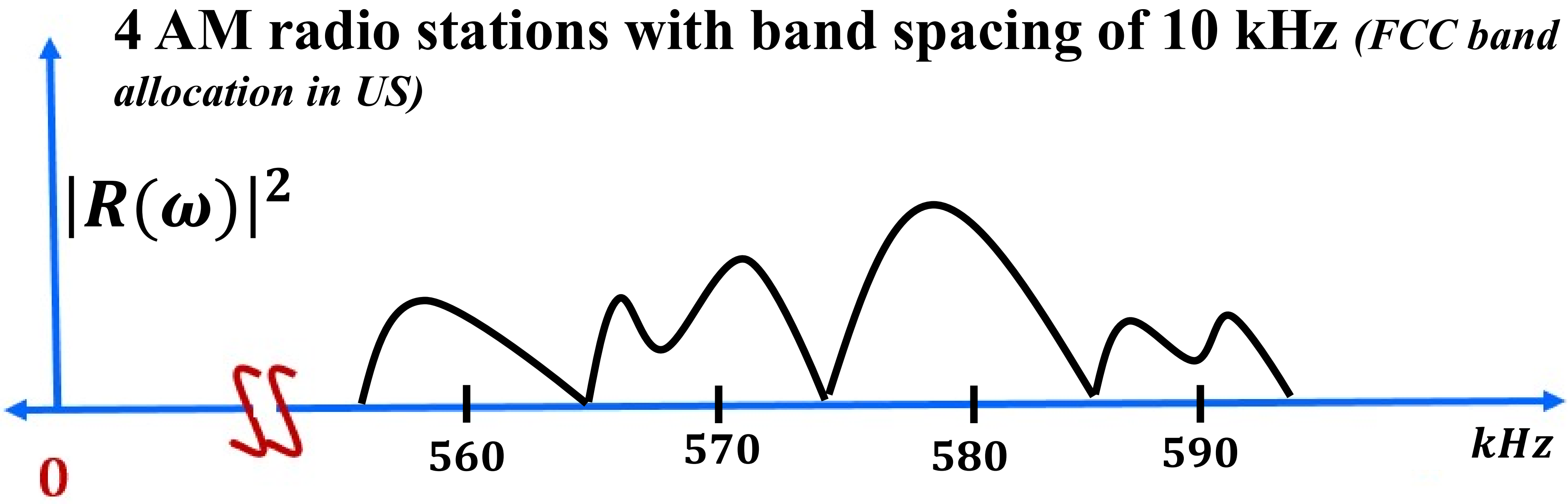
- **Super heterodyne receiver with envelope detection**
- Convolution
- The Fourier properties of convolution

Superheterodyne AM receiver with Env. Det.



- **The design of receiver with bandpass filter having tunable center frequency is complex as *bandwidth-to-center frequency ratio* is within 1%, for example**

$$BW\colon f_c = \frac{B}{f} = \frac{10kHz}{1MHz} = 1\%$$

Superheterodyne AM receiver with Env. Det.

- **The practical solution to this problem is to use a band pass filter with *fixed center frequency* that is *below* the AM band**

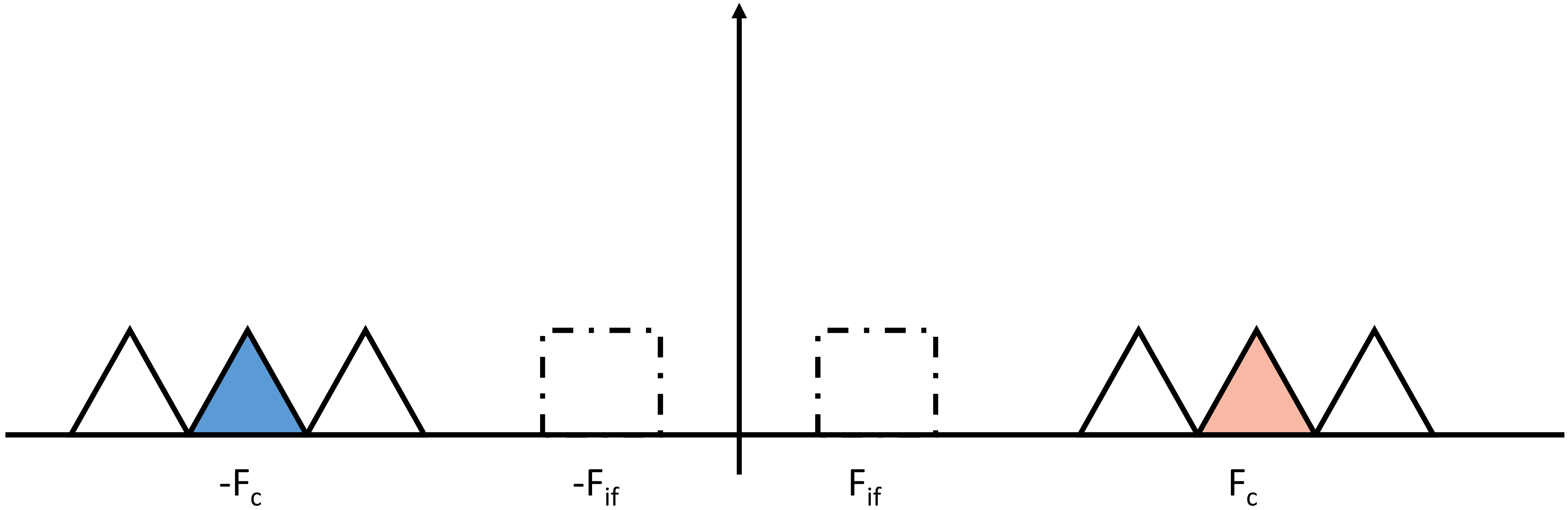
$$f = f_{IF} = \frac{\omega_{IF}}{2\pi} = 455\ kHz$$

- ***Shifting or heterodyning* the frequency band to the desired AM signal to the pass band of intermediate filter (*IF filter*)**

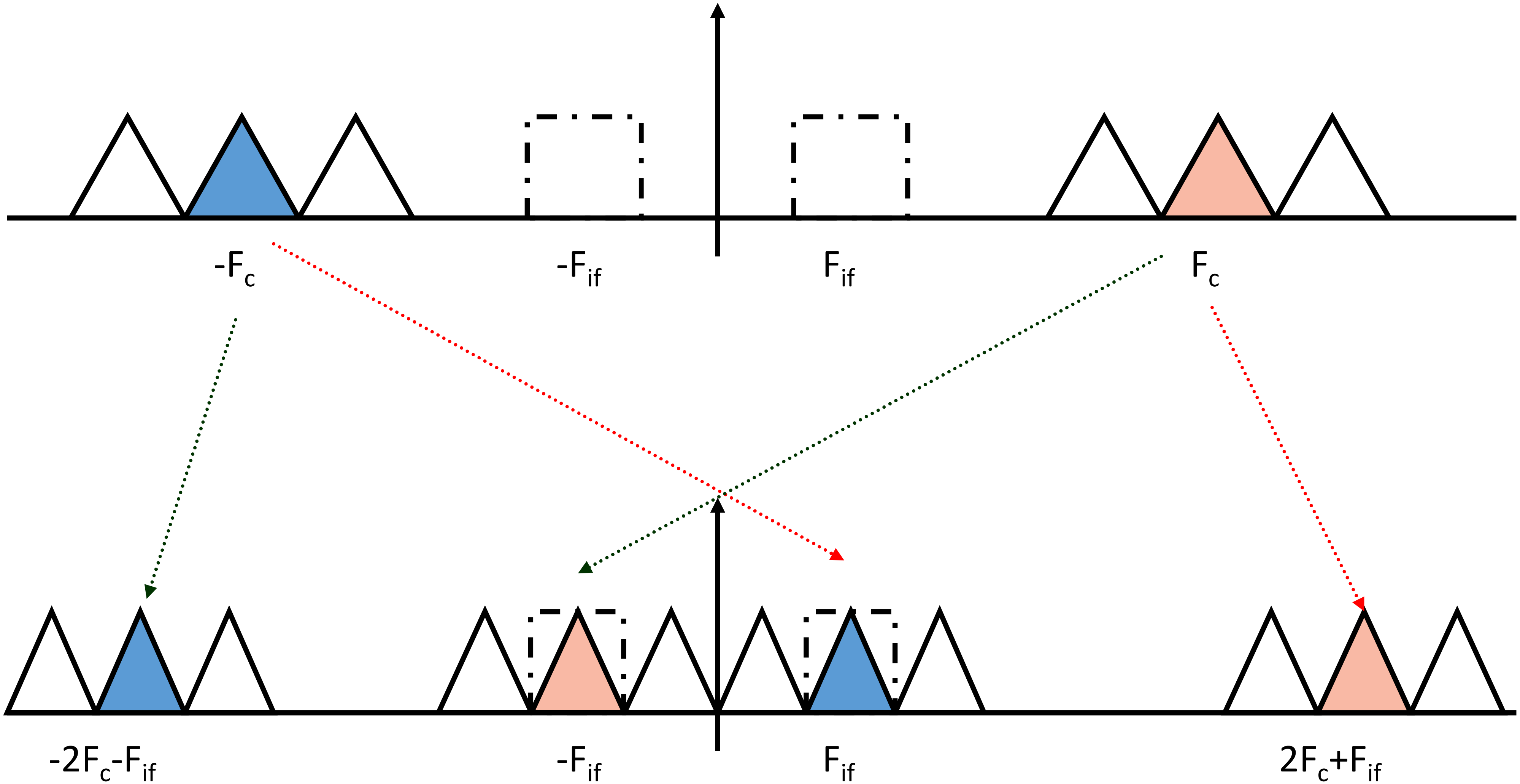
- **Let's have a look at this process**

Intermediate Frequency

- It is fixed frequency located at 455 kHz
- The IF filter is band-pass with center frequency of 455 kHz and bandwidth equal to the bandwidth of one AM channel approximately =10 kHz



Intermediate Frequency –Up conversion



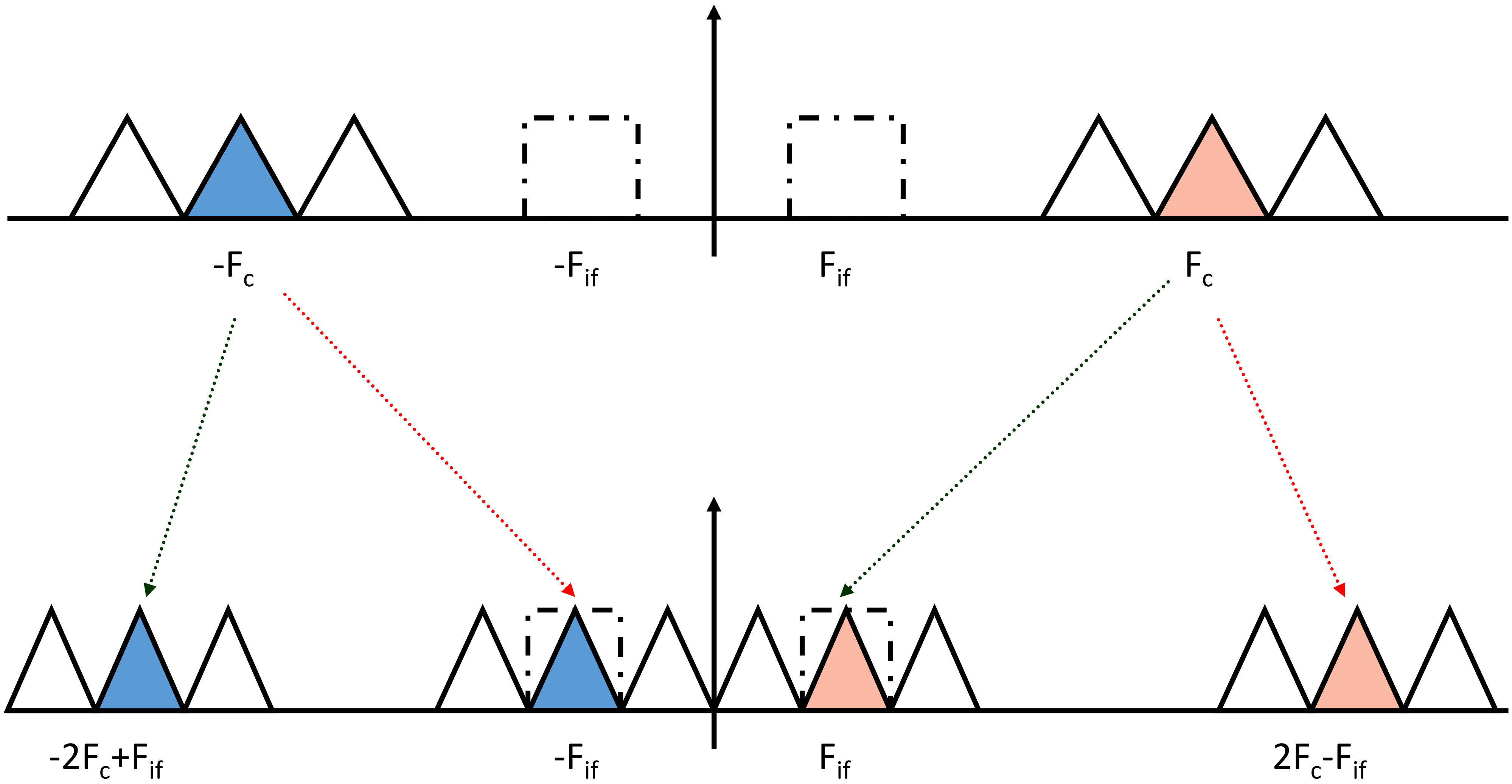
Second Image

Subtract Fc+Fif

First Image

Add Fc+Fif

Intermediate Frequency –Down conversion



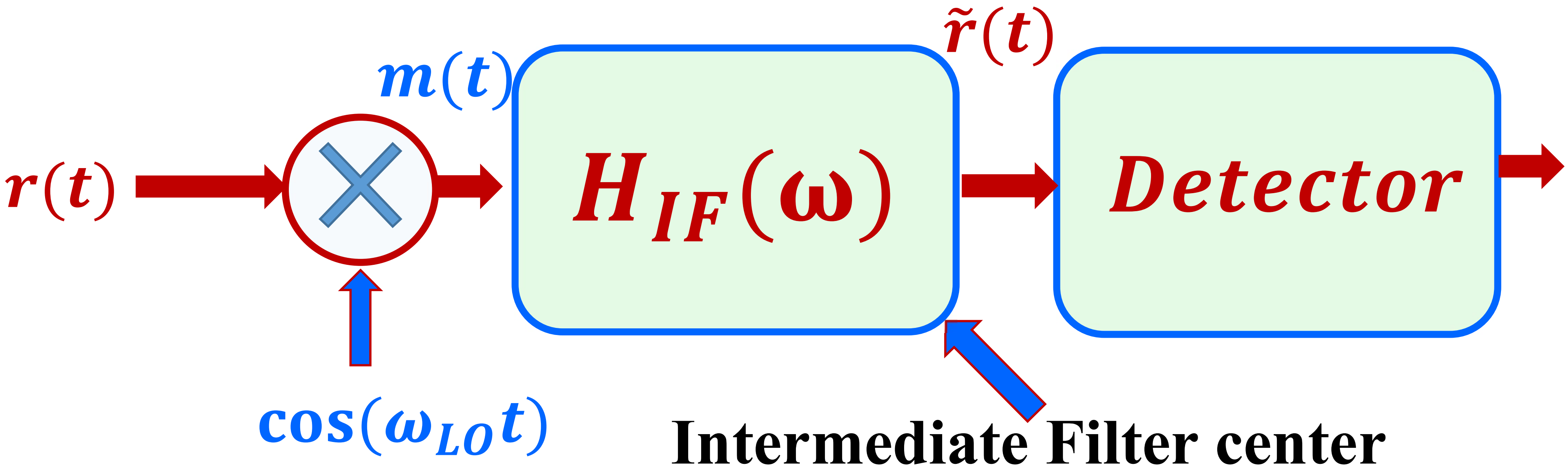
Second Image

Subtract Fc-Fif

First Image

Add Fc-Fif

Superheterodyne AM receiver with Env. Det.



Intermediate Filter center frequency = ω_{IF}

$$m(t) = \frac{1}{2}f(t)\cos\big((\omega_{LO} - \omega_c)t\big) + \frac{1}{2}f(t)\cos\big((\omega_{LO} + \omega_c)t\big)$$

choosing $\omega_{IF} = \omega_{LO} - \omega_c$,

$$m(t) = \frac{1}{2}f(t)\cos(\omega_{IF}t) + \frac{1}{2}f(t)\cos\big((\omega_{IF} + 2\omega_c)t\big)$$

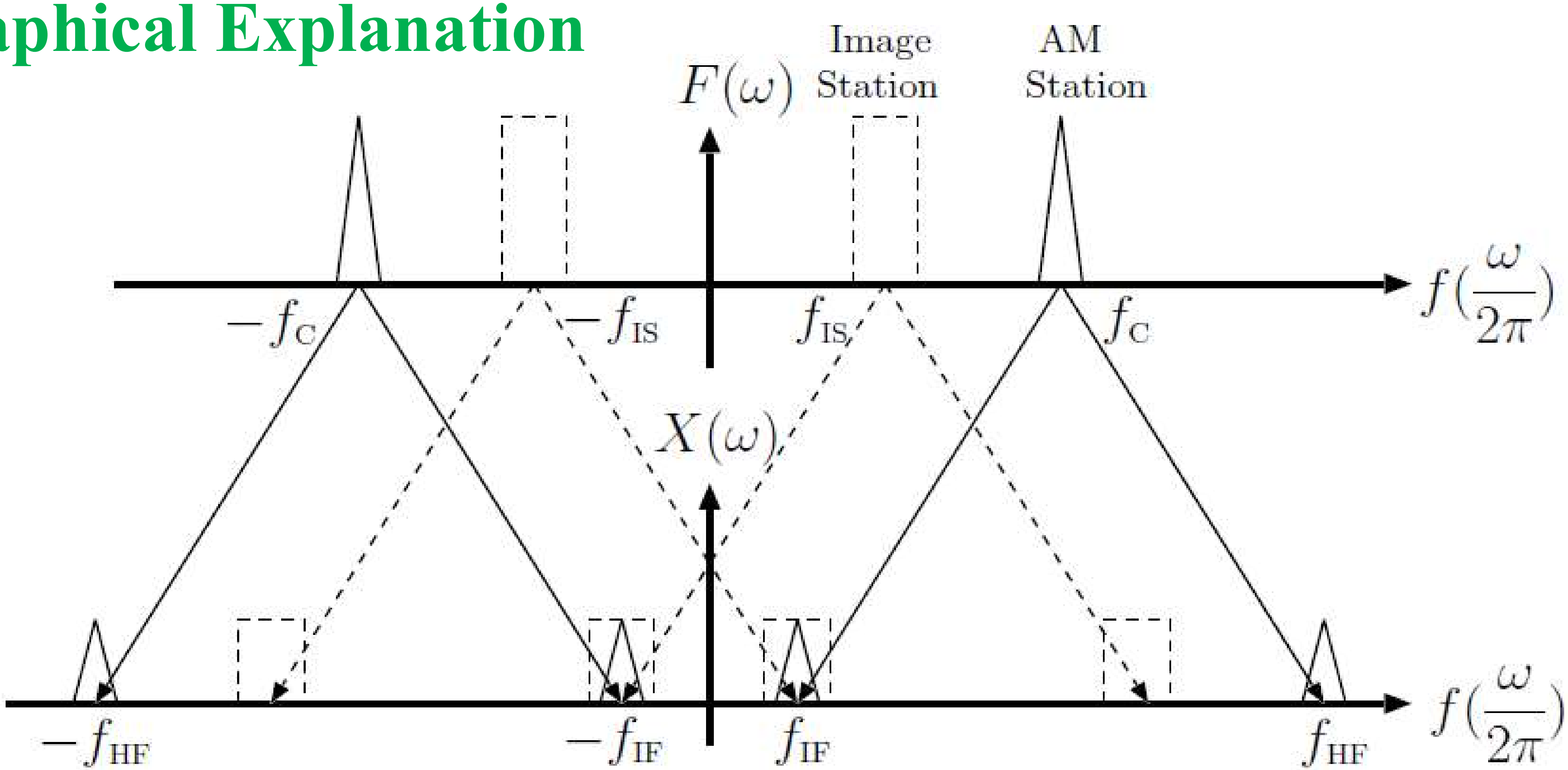
Superheterodyne AM receiver with Env. Det.

$$m(t) = \underbrace{\boxed{\frac{1}{2} f(t)\cos(\omega_{IF} t)}}_{\text{lies in Passband of IF filter}} + \underbrace{\boxed{\frac{1}{2} f(t)\cos\big((\omega_{IF} + 2\omega_c)t\big)}}_{\text{lies in Stopband of IF filter}}$$

- So, $\tilde{r}(t)$ contains only first term of $m(t)$ and detector output will be DC offset that includes desired signal of AM broadcaster

Superheterodyne AM receiver with Env. Det.

Graphical Explanation

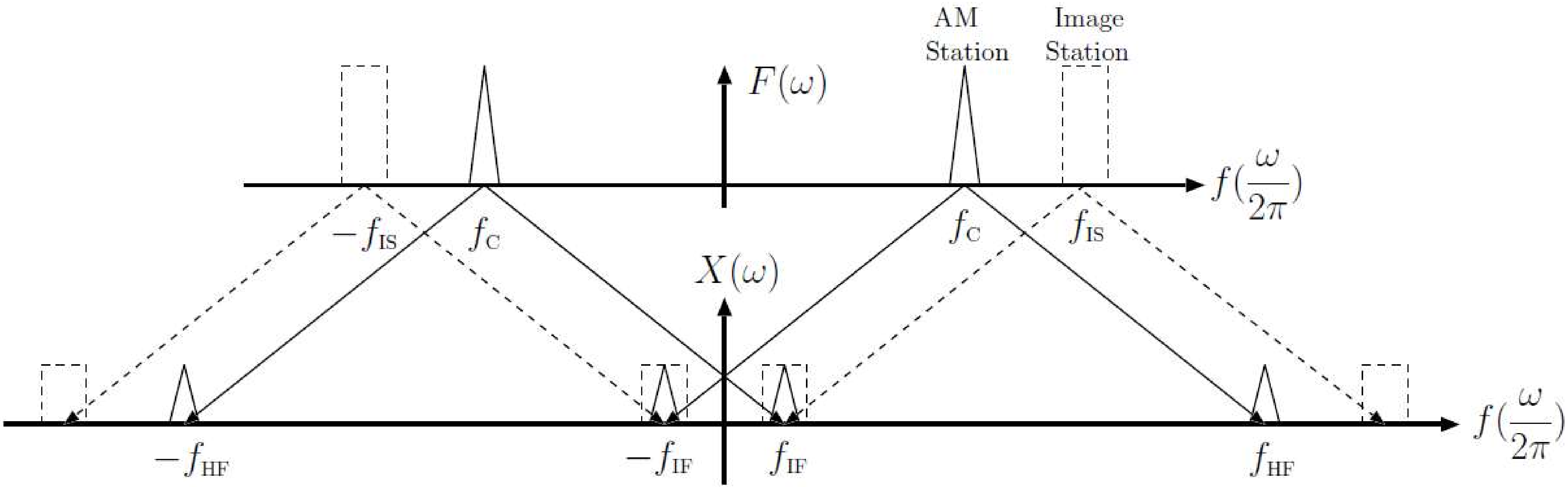


Modulation Property:

$$X(\omega) = \frac{1}{2}(F(\omega - \omega_{LO}) + F(\omega + \omega_{LO}))$$

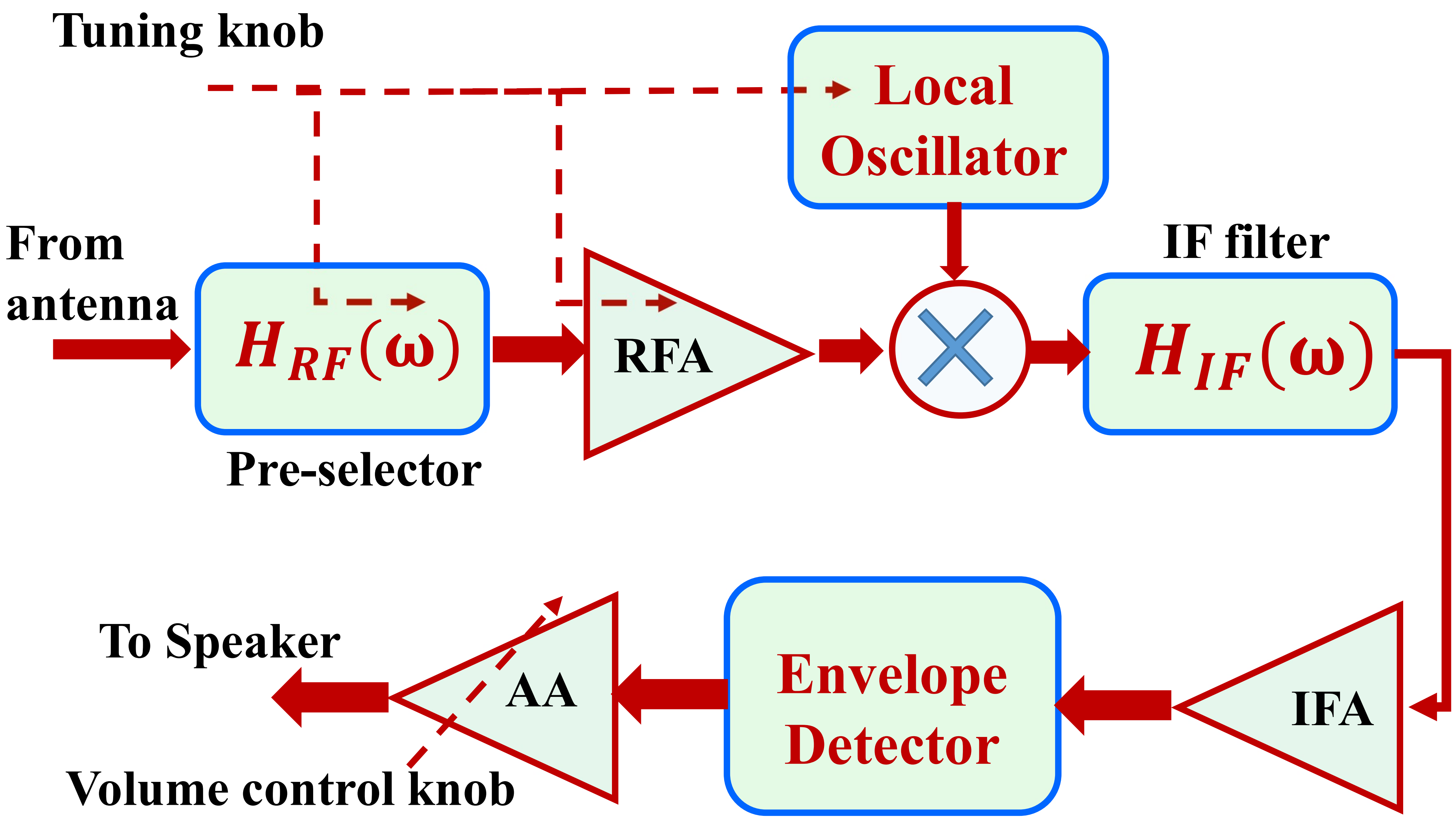
Superheterodyne AM receiver with Env. Det.

Graphical Explanation – Frequency Spectrum



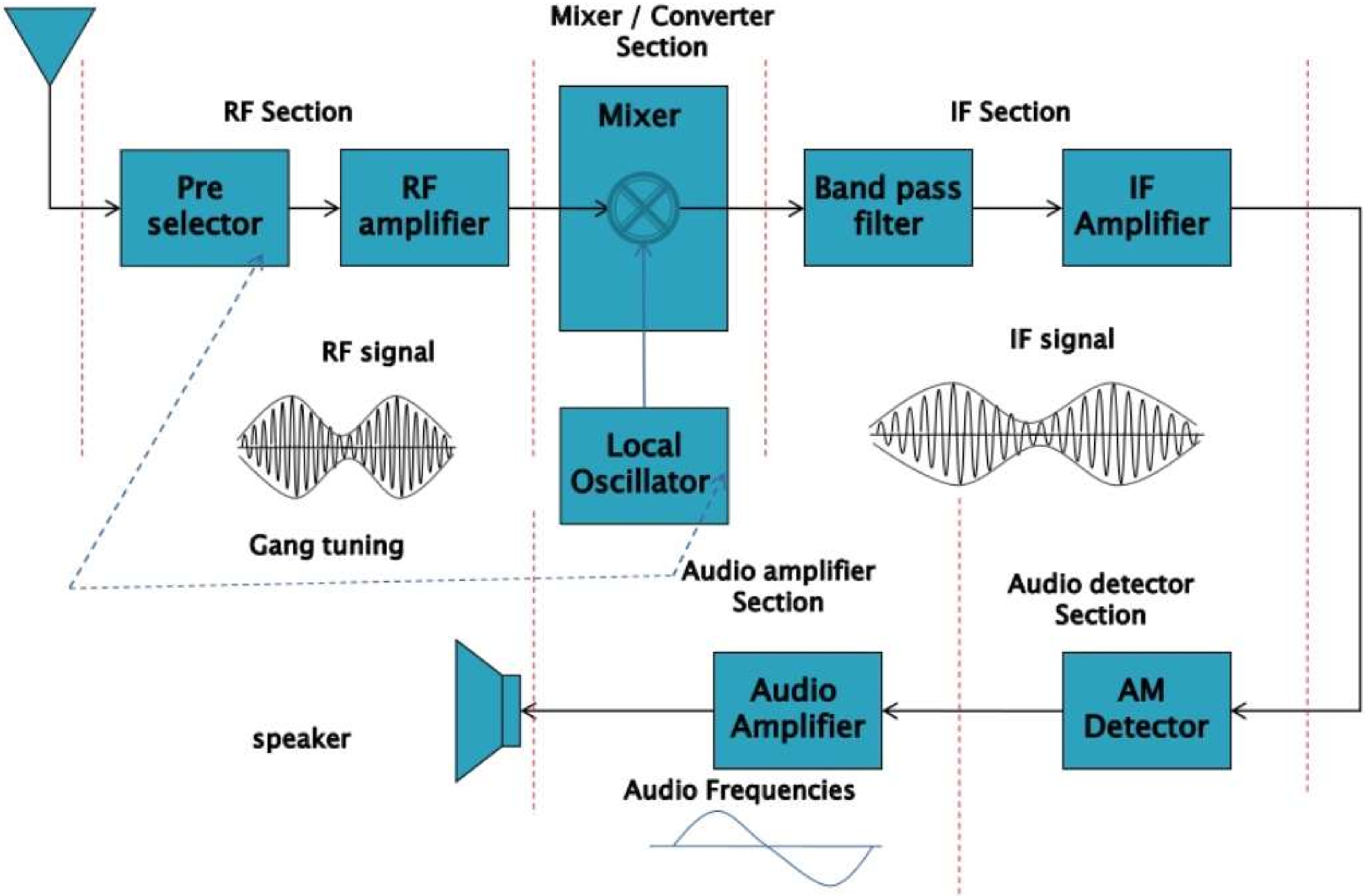
- **The circuit should be able to reject image station spectrum by carefully designing IF filter**

Superheterodyne AM receiver with Env. Det.



RFA: Radio Frequency amplifier, ***IFA***: intermediate Frequency amplifier
AA: Audio amplifier

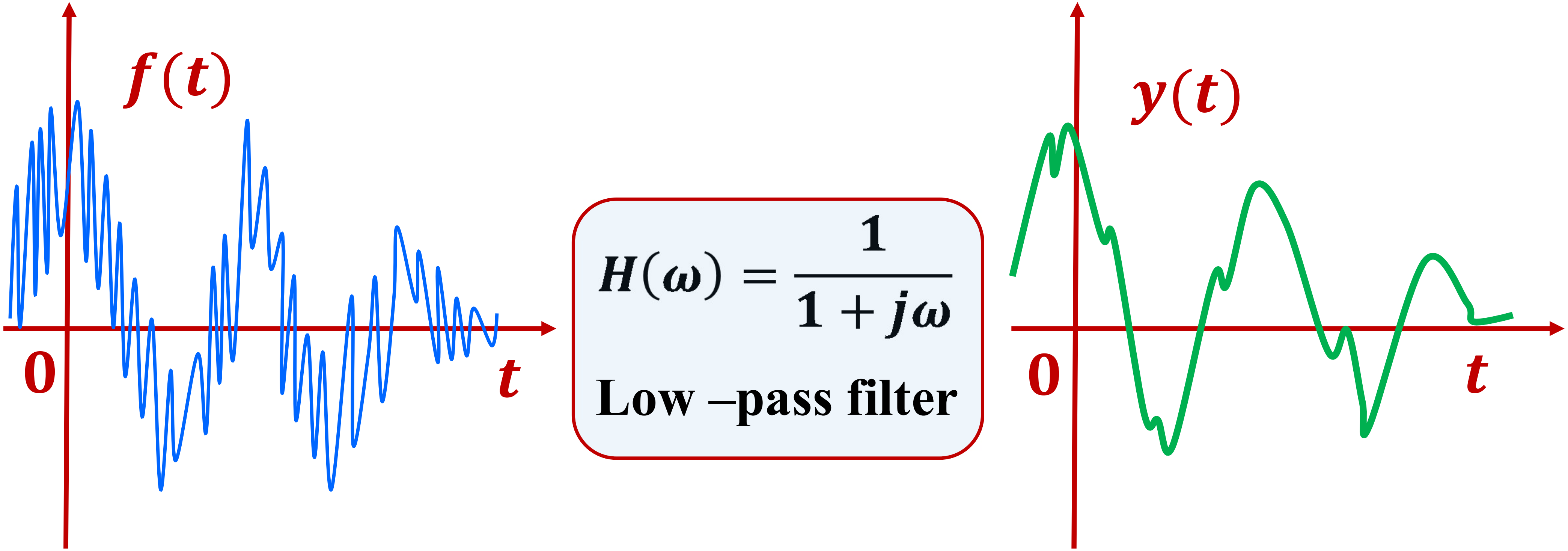
Superheterodyne AM receiver – Signals



Objectives

- Envelope detection
- Super heterodyne receiver with envelope detection
- Convolution
- The Fourier properties of convolution

Convolution – An LTI system



- **Previously, we have studied LTI system response of any signal, $f(t)$**
- **The LTI system *scales* the amplitudes of co-sinusoids with amplitude response of $|H(\omega)|$ and show phase *shift* in phase response $\angle H(\omega)$**

Convolution

- **Let's see the same process in time domain**

- **Carefully observing the $f(t)$ and $y(t)$, it seems that output signal after passing from *LPF* is more "noise free" signal with *rejection* of high frequency component**

- **The $y(t)$ is a weighted linear superposition of past and present values of $f(t)$, where the average of all controlled by $h(t) \longleftrightarrow H(\omega)$**

- **$h(t)$ is the system *impulse response* and describes the *convolution* in time domain**

Convolution – Example 1

Question: Let $y(t) \equiv h(t) * f(t)$, the convolution of $h(t)$ and some unit step function $u(t)$. Express $y(t)$ in terms of $h(t)$ only?

Solution: Since,

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

and,

$$u(t-\tau) = \begin{cases} 1, & t < \tau \\ 0, & t > \tau \end{cases}$$

So,

$$y(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Convolution – Example 2

Question: **Determine the function,**

$$y(t) = u(t) * u(t)$$

Solution: **Using the result of Example 1, with $h(t) = u(t)$, we get,**

$$y(t) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} 0, & t < 0, \\ \int_{0}^{t} d\tau = t & t > 0, \end{cases} = t\,u(t)$$

- **So the convolution of unit step with another unit step function results in a ramp function (accumulation)**

Objectives

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Fourier convolution properties

Commutative property of convolution

- **We can show that the convolution in time domain is equal to multiplication in frequency domain**

- **if $h(t) \leftrightarrow H(\omega)$, then time convolution will be,**

$$h(t) * f(t) \leftrightarrow H(\omega)F(\omega)$$

- **We, first take FT of $h(t) * f(t)$**

$$\int_{-\infty}^{\infty} \{h(t) * f(t)\} e^{-j\omega t} dt = \int_{t=-\infty}^{\infty} \left\{ \int_{\tau=-\infty}^{\infty} h(\tau) f(t-\tau) d\tau \right\} e^{-j\omega t} dt$$

Fourier convolution properties

➢ **Changing the order of t and τ,**

$$\int_{-\infty}^{\infty}\{h(t)*f(t)\}e^{-j\omega t}dt=\int_{\tau=-\infty}^{\infty}\left\{\int_{t=-\infty}^{\infty}h(\tau)f(t-\tau)dt\right\}e^{-j\omega t}d\tau$$

Hence,

$$\int_{\tau=-\infty}^{\infty}h(\tau)F(\omega)\,e^{-j\omega\tau}d\tau=F(\omega)\int_{\tau=-\infty}^{\infty}h(\tau)\,e^{-j\omega\tau}d\tau$$

$$=F(\omega)H(\omega)$$

So as claimed,

$$h(t)*f(t)\leftrightarrow H(\omega)F(\omega)$$

Fourier convolution properties

Similarly, it is true for Fourier transforms,

$$h(t) * f(t) = \int_{\tau=-\infty}^{\infty} f(\tau)h(t-\tau)\, e^{-j\omega\tau} d\tau$$

$$F(\omega)H(\omega) = H(\omega)F(\omega)$$

indicates commutative property of convolution

- **Similarly, for the Fourier *frequency convolution*,**

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$$

and,

$$F(\omega) * G(\omega) = G(\omega) * F(\omega)$$

Fourier convolution properties

These can be replaced with each other as commutation is valid for IFT as well:

$$\boldsymbol{F(\omega) * G(\omega) = \int_{-\infty}^{\infty} F(\Omega) G(\omega - \Omega) d\Omega}$$

$$\boldsymbol{G(\omega) * F(\omega) = \int_{-\infty}^{\infty} G(\Omega) F(\omega - \Omega) d\Omega}$$

Fourier convolution properties

Distributive property of convolution

$$f(t) * \big(g(t) + h(t)\big) = \int_{-\infty}^{\infty} f(\tau)\big(g(t-\tau) + h(t-\tau)\big)d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau + \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

$$= f(t) * g(t) + f(t) * h(t)$$

Fourier convolution properties

Associative property of convolution

$$f(t) * \big(g(t) * h(t)\big) \leftrightarrow F(\omega)\big(G(\omega)H(\omega)\big) = \big(F(\omega)G(\omega)\big)H(\omega)$$

$$\big(f(t) * g(t)\big) * h(t) \leftrightarrow \big(F(\omega)G(\omega)\big)H(\omega)$$

using the uniqueness of FT,

$$f(t) * \big(g(t) * h(t)\big) = \big(f(t) * g(t)\big) * h(t)$$

Fourier convolution properties

Shifting property of convolution

The convolution property also holds for shifting,

$$f(t-t_o) \leftrightarrow F(\omega)e^{-j\omega t_0}$$

And it follows that,

$$h(t)*f(t-t_o) \leftrightarrow H(\omega)F(\omega)e^{-j\omega t_0} = Y(\omega)e^{-j\omega t_0}$$

where $Y(\omega) = H(\omega)F(\omega)$ has IFT $y(t) = h(t)*f(t)$,
but IFT of $Y(\omega)e^{-j\omega t_0}$ is $y(t-t_o)$, so that

$$h(t)*f(t-t_o) = y(t) \quad \text{and} \quad h(t-t_o)*f(t) = y(t)$$

Fourier convolution properties

Derivative property of convolution

To verify derivative property,

$$\frac{d}{dt}y(t) = \frac{d}{dt}[h(t) * f(t)] \leftrightarrow j\omega[H(\omega)F(\omega)]$$
$$= H(\omega)[j\omega F(\omega)]$$

which implies that,

$$h(t) * \frac{df}{dt} = \frac{dy}{dt}$$

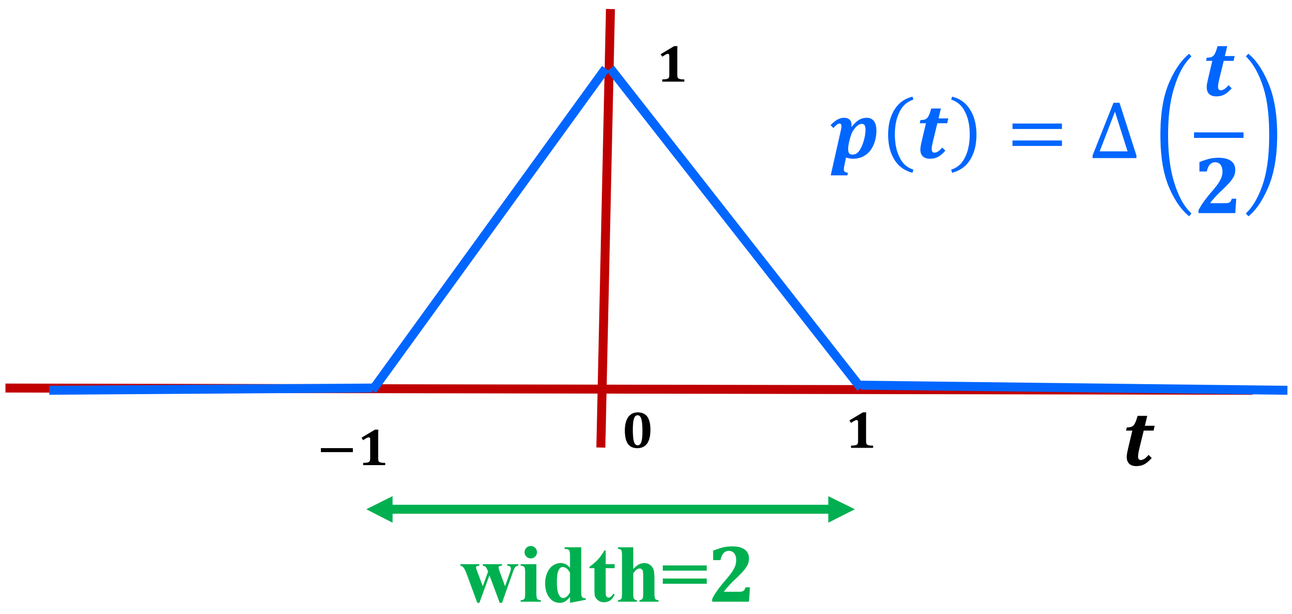
likewise, due to commutative property,

$$\frac{dh}{dt} * f(t) = \frac{dy}{dt}$$ **is also true**

Fourier convolution – Example 3

Question: if $f(t) * g(t) = p(t)$ where $p(t) = \Delta\left(\frac{t}{2}\right)$

Determine and plot, $c(t) = f(t) * (g(t) - g(t-2))$



Where $\Delta\left(\frac{t}{2}\right)$, t denotes the *center point* and denominator value 2 denotes the width of the base of triangle

Fourier convolution – Example 3

Solution: Using the distributive property, we can find that,

$$c(t) = f(t) * (g(t) - g(t-2))$$

$$c(t) = f(t) * g(t) - f(t) * g(t-2)$$

Since, $f(t) * g(t) = p(t)$, the shifting property states that,

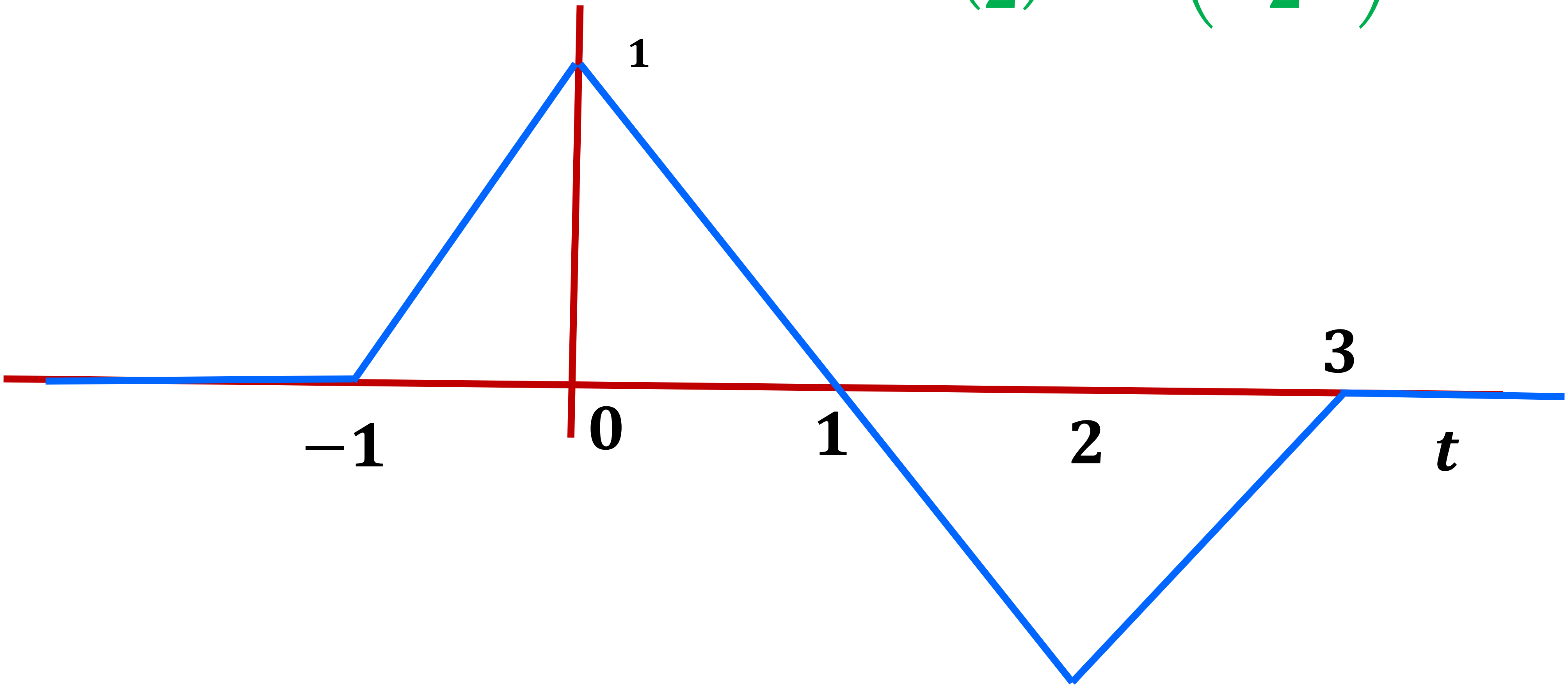
$$f(t) * g(t-2) = p(t-2)$$

Therefore,

$$c(t) = p(t) - p(t-2) = \Delta\left(\frac{t}{2}\right) - \Delta\left(\frac{t-2}{2}\right)$$

Fourier convolution – Example 3

$$p(t) = \Delta\left(\frac{t}{2}\right) - \Delta\left(\frac{t-2}{2}\right)$$

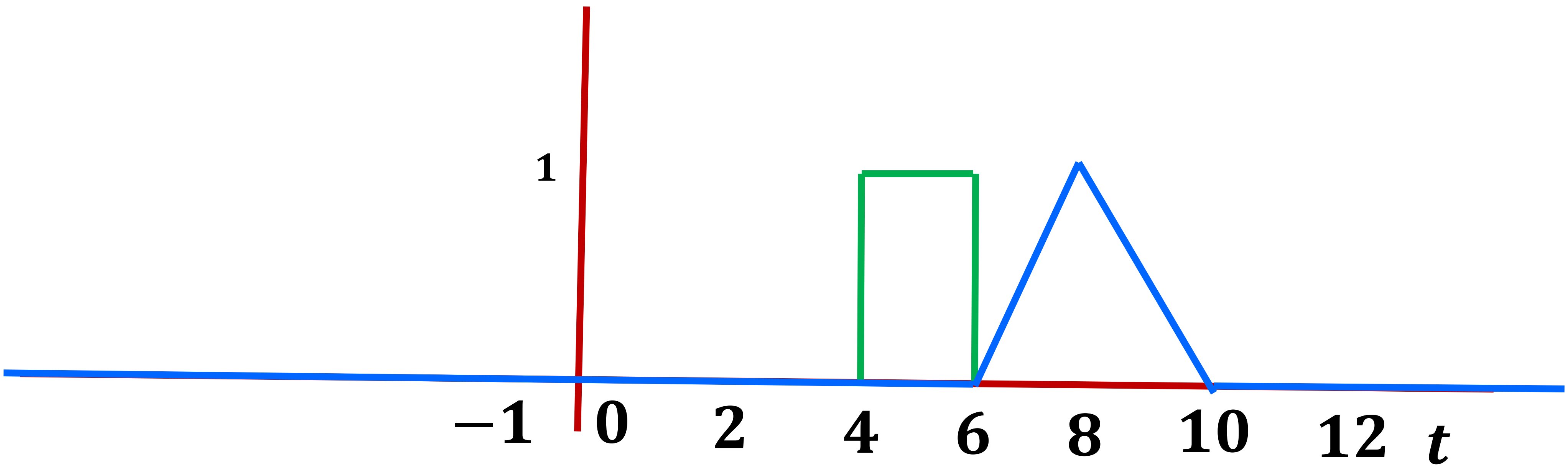


Fourier convolution – Example 4

Question: Consider the following signal $c(t)$ given by,

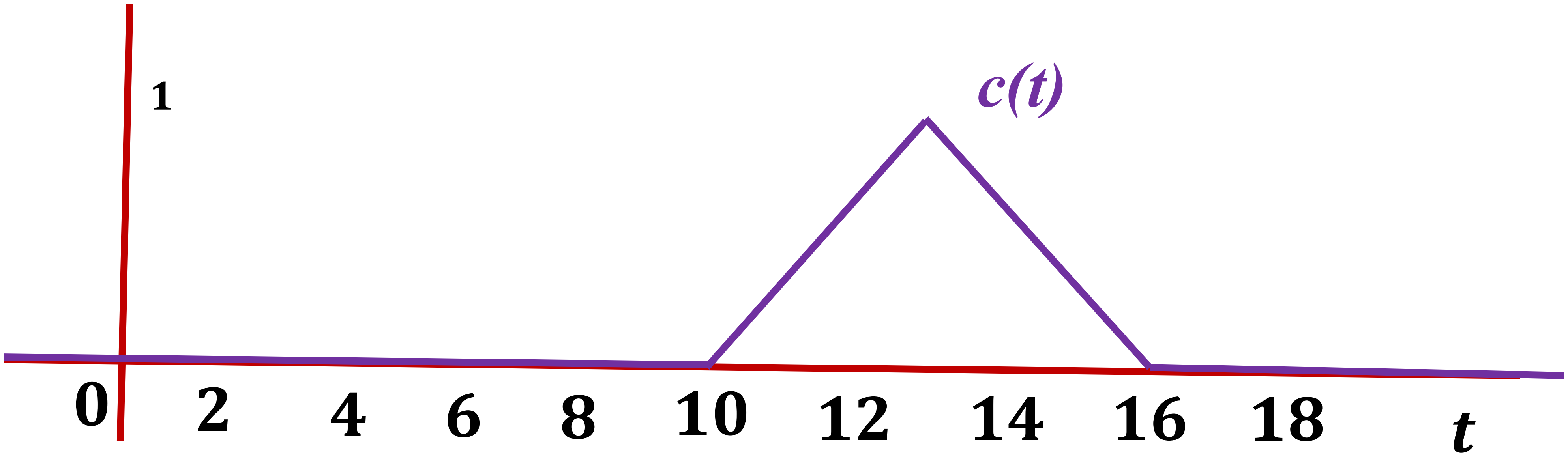
$$c(t) = rect\left(\frac{t-5}{2}\right) * \Delta\left(\frac{t-8}{4}\right)$$

Determine the width and start time of $c(t)$?



Fourier convolution – Example 4

Solution: **As the respective widths of** $\boldsymbol{rect}\left(\frac{t-5}{2}\right)$ **&** $\Delta\left(\frac{t-8}{4}\right)$ **are 2 and 4, the width of** ***c(t)*** **is expected to be 6. However, since the start times of** $\boldsymbol{rect}\left(\frac{t-5}{2}\right)$ **&** $\Delta\left(\frac{t-8}{4}\right)$ **are 4 and 6 respectively, the start time of** ***c(t)*** **becomes 10**

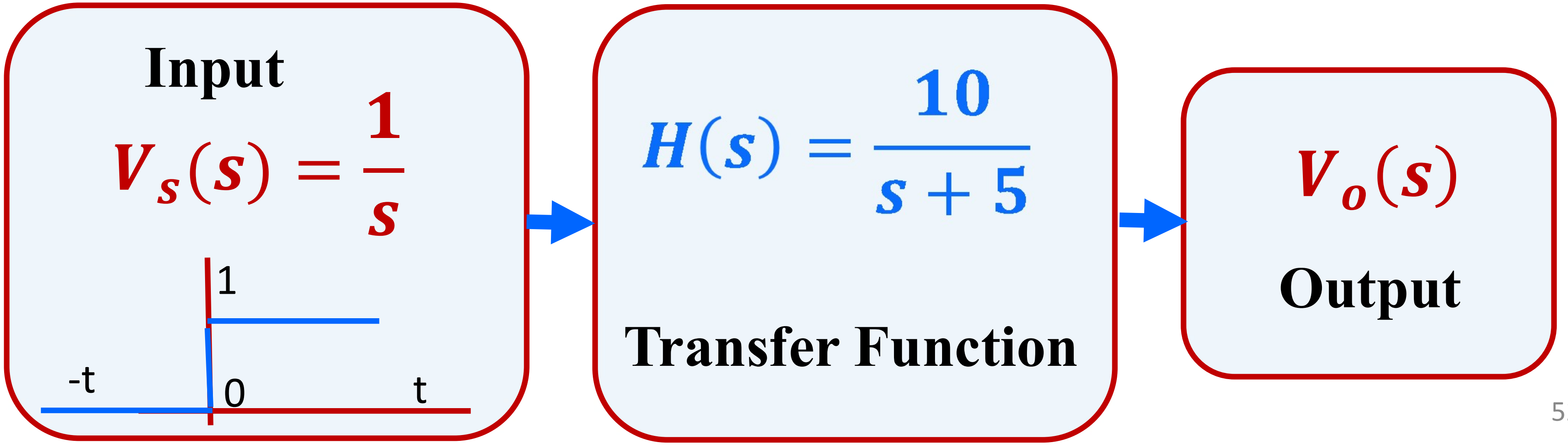


Fourier convolution – Example 5

Question: **The transfer function of a network is given by,**

$$H(s) = \frac{V_o(s)}{V_s(s)} = \frac{10}{s+5}$$

The input is a unit step function $V_s(s) = \frac{1}{s}$, use convolution to determine output $v_o(t)$ in the network?



Fourier convolution – Example 5

Solution: Since $H(s) = \frac{10}{s+5}$, $h(t) = 10e^{-5t}$

therefore,

$$v_o(t) = \int_0^t 10\, u(\lambda) e^{-5(t-\lambda)} d\lambda$$

$$= 10e^{-5t} \int_0^t e^{5(\lambda)} d\lambda$$

$$= \frac{10e^{-5t}}{5} \left[e^{5t} - 1 \right]$$

$$v_o(t) = 2\left[1 - e^{-5t}\right] u(t)\ V$$

Fourier convolution – Example 5

This can be verified by the partial fraction expansion,

$$V_o(s) = H(s)V_s(s)$$

$$V_o(s) = \frac{10}{s(s+5)} = \frac{K_o}{s} + \frac{K_1}{s+5}$$

evaluating the constants and replacing with values,

$$V_o(s) = \frac{2}{s} - \frac{2}{s+5}$$ **and hence,**

$$v_o(t) = 2\left[1 - e^{-5t}\right]u(t)\ V$$

Fourier convolution – Example 5

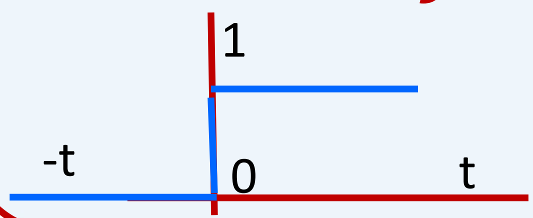
Consider an integrator of an LTI system,

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

The input of the system is unit step and transfer function is given by

Input

$$X(j\omega) = \frac{1}{j\omega}$$



$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Transfer Function

$$Y(j\omega)$$

Output

Fourier convolution – Example 5

Using the integration property of Fourier transform for

$$\boldsymbol{y(t) = \int_{-\infty}^{t} x(\tau)d\tau}$$

taking the Fourier of both sides gives,

$$\boldsymbol{Y(j\omega) = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(j\omega)}$$

due to unit step function $\boldsymbol{X(j\omega) = 1}$, we conclude that

$$\boldsymbol{Y(j\omega) = \frac{1}{j\omega} + \pi\delta(j\omega)}$$

Fourier convolution – Example 5

we know that,

Output = (Frequency Response) (Input)

$$Y(j\omega) = H(j\omega)F(j\omega)$$

$$Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

which is consistent with the integration property

Summary

- A DC offset is introduced in the transmitted signal to avoid noise inclusion and help detector circuit to extract message signal efficiently

- AM envelope detection is insensitive to phase shift of the carrier signal (non-linearity)

- Selection of desired radio station is done by shifting the incoming signal frequency to IF and passing it through IF band pass filter to reject image stations

- Superheterodyne receivers have almost 100% selectivity for picking up desired AM signal from whole spectrum of AM

Summary

- **Convolution in time domain is the multiplication in frequency domain**

- **Convolution holds the properties of commutative, associative, distributive, shifting, differentiation and integration**

- **Solution of circuits using convolution saves time due to reduction of taking FT and IFT multiple times**

- $\boldsymbol{h(t)}$ **is the system *impulse response* that describes the *convolution* in the time domain**

Further reading

1. Ch. 8 (page 267-278), Ch. 9 (page 281-288), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.
2. Ch. 8 (page 582-623), C. K. Allan V. Oppehnheim, ***Signals and Systems*** , 5th ed., Prentice hall,1996.
3. Ch. 15 (page 782-785), J. D. Irwin, and R. M. Nelms. ***Basic Engineering Circuit Analysis,*** 10th ed. Wiley, 2011.

Preview:

1. Ch. 9 (page 289-302), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.

Homework 10

Deadline: 10:00 PM, 4th May, 2022

Thank you!