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Zhejiang University – University of Illinois at Urbana-Champaign Institute

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ECE 210 Analog Signal Processing Spring 2022
Homework #9: Solution

1. Obtain the average power of the following signals:

   (a) $f(t) = 3 + 3e^{j4t} + 3e^{-j4t})$

   (b) $f(t) = 3 + 4\cos(2t) + 5\sin(5t)$

   Solution

   (a) $P = \sum_{n=-\infty}^{\infty} |F_n|^2 = 3^2 + 3^2 + 3^2 = 27.$

   (b) $P = \left(\frac{c_o}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{1}{2} c_n^2 = 3^2 + \frac{(4)^2}{2} + \frac{(5)^2}{2} = 29.5.$

2. Consider the periodic function $f(t) = \begin{cases} \sin(\pi t), & \text{for } t \in [0,2) \\ 0, & \text{for } t \in [2,4) \end{cases}$, where the signal period is $T = 4\,\text{s}$. Its corresponding Fourier series in exponential form is given by:

$$f(t) = \frac{-j}{4} e^{j\pi t} + \frac{j}{4} e^{-j\pi t} + \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{\pi(4-n^2)} e^{jn\frac{\pi}{2}t},$$

and in compact form:

$$f(t) = \frac{1}{2}\cos(\pi t - \frac{\pi}{2}) + \frac{4}{3\pi}\cos(\frac{\pi}{2}t) + \sum_{n=3, n \text{ odd}}^{\infty} \frac{4}{\pi(n^2-4)} \cos(\frac{n\pi}{2}t + \pi)$$

Let $f(t)$ be the input to an LTI system with frequency response $H(\omega) = \frac{\pi}{2}\left(1 + \frac{\pi^2}{4\omega^2}\right) e^{-j\omega}$ for $\omega \in [-7\pi/4, 7\pi/4]$ rad/s.

   (a) Obtain the corresponding steady state response $y_{ss}(t)$.

   Solution

   Given that $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$ (from the Fourier series), and the LTI system has frequency response $H(\omega)$ for $\omega \in [-7\pi/4, 7\pi/4]$ rad/s. Thus, we need to consider,

$$-7\pi/4 \leq n\omega_0 \leq 7\pi/4$$

   Thus we consider the coefficients for $n = \pm 1, \pm 2, \pm 3$ (note we do not need to consider n=0 since there's no input term with frequency 0).

$$F_1 = \frac{c_1}{2} = \frac{2}{3\pi} \quad F_{-1} = \frac{2}{3\pi} \quad F_2 = \frac{-j}{4} \; F_{-2} = \frac{j}{4} \; F_3 = -\frac{2}{5\pi} \; F_{-3} = -\frac{2}{5\pi}$$

$$\begin{aligned}
Y_1 &= F_1 \cdot H(\tfrac{\pi}{2}) = \tfrac{2}{3\pi} \cdot \tfrac{\pi}{2}(1 + \tfrac{\pi^2}{4 \cdot \frac{\pi^2}{4}}) e^{-j\frac{\pi}{2}} = \tfrac{2}{3} e^{-j\frac{\pi}{2}} \\
Y_{-1} &= F_{-1} \cdot H(-\tfrac{\pi}{2}) = \tfrac{2}{3\pi} \cdot \tfrac{\pi}{2}(1 + \tfrac{\pi^2}{4 \cdot \frac{\pi^2}{4}}) e^{j\frac{\pi}{2}} = \tfrac{2}{3} e^{j\frac{\pi}{2}} \\
Y_2 &= F_2 \cdot H(\pi) = -\tfrac{j}{4} \cdot \tfrac{\pi}{2}(1 + \tfrac{\pi^2}{4 \cdot \pi^2}) e^{-j\pi} = \tfrac{5\pi}{32} e^{j\frac{\pi}{2}} \\
Y_{-2} &= F_{-2} \cdot H(-\pi) = \tfrac{j}{4} \cdot \tfrac{\pi}{2}(1 + \tfrac{\pi^2}{4 \cdot \pi^2}) e^{j\pi} = \tfrac{5\pi}{32} e^{-j\frac{\pi}{2}} \\
Y_3 &= F_3 \cdot H(\tfrac{3\pi}{2}) = -\tfrac{2}{5\pi} \cdot \tfrac{\pi}{2}(1 + \tfrac{\pi^2}{9\pi^2}) e^{-j\frac{3\pi}{2}} = -\tfrac{2}{9} e^{-j\frac{3\pi}{2}} \\
Y_{-3} &= F_3 \cdot H(\tfrac{3\pi}{2}) = -\tfrac{2}{5\pi} \cdot \tfrac{\pi}{2}(1 + \tfrac{\pi^2}{9\pi^2}) e^{j\frac{3\pi}{2}} = -\tfrac{2}{9} e^{j\frac{3\pi}{2}}
\end{aligned}$$

   Therefore, we can represent $y_{ss}(t)$ by its compact form as,

$$y_{ss}(t) = \frac{4}{3}\cos(\frac{\pi}{2}t - \frac{\pi}{2}) + \frac{5\pi}{16}\cos(\pi t + \frac{\pi}{2}) - \frac{4}{9}\cos(\frac{3\pi}{2}t + \frac{\pi}{2}),$$

(b) Is $y_{ss}(t)$ periodic? If so, obtain its fundamental frequency.

Solution

Yes, it's periodic. Recall that a sinusoidal input to an LTI system will yield a sinusoidal output of the same frequency. This means that the output of an LTI system to a periodic input will also be periodic because no new frequencies can be introduced. It is possible though, like in this case, that some frequencies can be removed, potentially changing the fundamental frequency of the signal. However, in this case, the fundamental frequency won't change, $\omega_0 = \frac{\pi}{2}$.

(c) Calculate the ratio between the input signal power and the output signal power.

Solution

The given $f(t)$ should have half of the power of pure sinusoidal function $g(t) = \sin(\pi t) = \cos(\pi t - \frac{\pi}{2})$ since half period of the signal is cut to zero. Therefore, by Parseval's theorem,

$$\frac{P_f}{P_y} = \frac{\frac{1}{2}\left(\frac{1^2}{2}\right)}{\sum |Y_n|^2} = \frac{\frac{1}{4}}{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{5\pi}{32}\right)^2 + \left(\frac{5\pi}{32}\right)^2 + \left(\frac{\cancel{4}\,2}{9}\right)^2 + \left(\frac{\cancel{4}\,2}{9}\right)^2} \approx \cancel{0.142}\ 0.169$$

3. The input-output relation for a system with input $f(t)$ is given by

$$y(t) = 6f(t) + f^2(t) - f^3(t).$$

Obtain the total harmonic distortion (THD) of the system response to a pure cosine input of the form $f(t) = 2\cos(\omega_o t)$, where $\omega_o > 0$ is a real positive constant. The following trig. identities can be useful: $\cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)]$ and $\cos^3(\theta) = \frac{1}{4}[3\cos(\theta) + \cos(3\theta)]$.

Solution

Finding $f^2(t)$:

$$f^2(t) = 4\cos^2(\omega_o t) = 2[1 + \cos(2\omega_o t)].$$

Finding $f^3(t)$:

$$f^3(t) = 2[3\cos(\omega_o t) + \cos(3\omega_o t)]$$

Therefore, the output of this system can be written as

$$y(t) = 12\cos(\omega_o t) + 2[1 + \cos(2\omega_o t)] - 2[3\cos(\omega_o t) + \cos(3\omega_o t)].$$

Regrouping yields

$$y(t) = 2 + 6\cos(\omega_o t) + 2\cos(2\omega_o t) - 2\cos(3\omega_o t).$$

Therefore

$$\text{THD} = \frac{\sum_{n=2}^{\infty} \frac{1}{2} c_n^2}{\frac{1}{2} c_1^2} = \frac{c_2^2 + c_3^2}{c_1^2} = \frac{2^2 + 2^2}{6^2} = 22\%.$$

4. Obtain the Fourier transform of

(a) $f_1(t) = e^{-t}u(t-2)$

Solution

$$\begin{aligned} f_1(t) &= e^{-t}u(t-2) = e^{-2}e^{-(t-2)}u(t-2) \\ &\leftrightarrow e^{-(2+2j\omega)}\frac{1}{1+j\omega}, \end{aligned}$$

by amplitude scaling (7.1-1) and time shift (7.1-8) properties applying to fourier transform pair 7.2-1.

(b) $f_2(t) = \frac{1}{1+j(t-1)} + \frac{1}{1+j(t+1)}$

Solution

From the Fourier Transform pair:

$$\frac{1}{1+j\omega} \leftrightarrow e^{-t}u(t)]$$

we have

$$\frac{1}{1+jt} \quad \leftrightarrow \quad 2\pi[e^{\omega}u(-\omega)]$$

by transform pair 7.2-1, 7.2-2, additional property (7.1-2) and symmetry property (7.1-6), and hence

$$\begin{aligned} f_2(t) &= \frac{1}{1+j(t-1)} + \frac{1}{1+j(t+1)} \\ &\leftrightarrow 2\pi e^{\omega}[e^{-j\omega}u(-\omega) + e^{j\omega}u(-\omega)] = 4\pi e^{\omega}cos(\omega)u(-\omega), \end{aligned}$$

by time shift (7.1-8) property.

5. Let $f(t) = \text{rect}(\frac{t}{2})$ and let $g(t) = f\left(t - \frac{1}{2}\right)$

   (a) Obtain $F(\omega)$ and plot $|F(\omega)|$ and $\angle F(\omega)$ in the frequency range $-3\pi < \omega < 3\pi$ rad/s. Remember to keep the phase $\angle F(\omega)$ in the interval $(-\pi, \pi]$ rad on the vertical axis.

   Solution

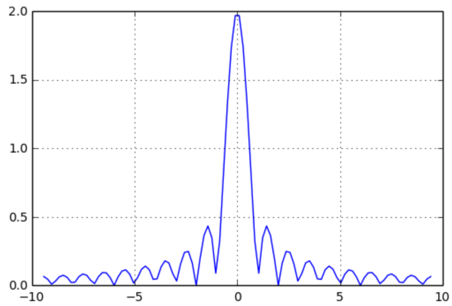
   From the Froutier Transform pair:

$$rect(\frac{t}{\tau}) \leftrightarrow sinc(\frac{\omega\tau}{2})$$

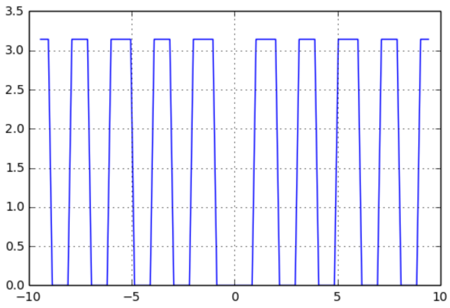
   where $\tau = 2$, we can easily obtain the Fourier transform of $f(t)$:

$$F(\omega) = 2\text{sinc}(\omega).$$

   Plotting $|F(\omega|)$:



   Plotting $\angle F(\omega)$:



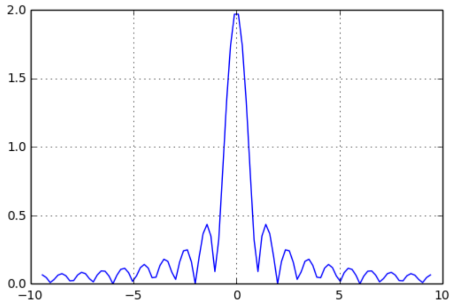
   (b) Obtain $G(\omega)$ and plot $|G(\omega)|$ and $\angle G(\omega)$ in the frequency range $-3\pi < \omega < 3\pi$ rad/s. Remember to keep the phase $\angle G(\omega)$ in the interval $(-\pi, \pi]$ rad on the vertical axis.

   Solution

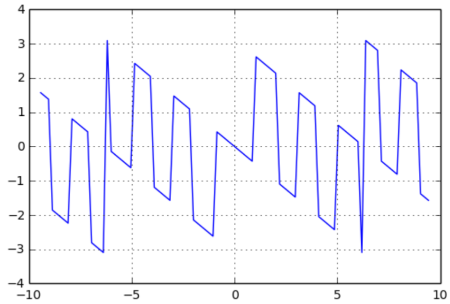
   Applying the time shift property of the F.T. we obtain $G(\omega)$:

$$G(\omega) = F(\omega)e^{-j\omega\frac{1}{2}} = 2e^{-j\frac{\omega}{2}}\text{sinc}(\omega).$$

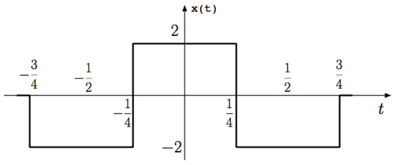
   Plotting $|G(\omega)|$:



Plotting $\angle G(\omega)$:



(c) Obtain the Fourier transform, $X(\omega)$, of the signal shown below



Solution

We notice that we can express $h(t)$ as the sum of two rect functions

$$h(t) = 4\text{rect}\left(\frac{t}{1/2}\right) - 2\text{rect}\left(\frac{t}{3/2}\right).$$

Then, using the transform pair $\text{rect}(t/\tau) \leftrightarrow \tau\text{sinc}\,(\omega\tau/2)$ (Table 7.2, item 7) or the answer from part (a), the Fourier transform takes the form

$$H(\omega) = 2\,\text{sinc}\left(\frac{\omega}{4}\right) - 3\text{sinc}\left(\frac{3\omega}{4}\right).$$

Another way to obtain the same result would be the sum of 3 rect functions

$$h(t) = 2\text{rect}\left(\frac{t}{1/2}\right) - 2\text{rect}\left(\frac{t+1/2}{1/2}\right) - 2\text{rect}\left(\frac{t-1/2}{1/2}\right),$$

which yields

$$\begin{aligned}
H(\omega) &= \text{sinc}\left(\frac{\omega}{4}\right) - \text{sinc}\left(\frac{\omega}{4}\right)e^{j\omega\frac{1}{2}} - \text{sinc}\left(\frac{\omega}{4}\right)e^{-j\omega\frac{1}{2}} = \text{sinc}\left(\frac{\omega}{4}\right)\left[1 - 2\cos\left(\frac{\omega}{2}\right)\right] \\
&= \text{sinc}\left(\frac{\omega}{4}\right) - \frac{\sin\left(\frac{\omega}{4}\right)}{\frac{\omega}{4}}\left[2\cos\left(\frac{\omega}{2}\right)\right]
\end{aligned}$$

Applying the trig. identity: $\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$, we obtain

$$\begin{aligned}
H(\omega) &= \text{sinc}\left(\frac{\omega}{4}\right) - \frac{\sin\left(\frac{\omega}{4}+\frac{\omega}{2}\right) + \sin\left(\frac{\omega}{4}-\frac{\omega}{2}\right)}{\frac{\omega}{4}} \\
&= \text{sinc}\left(\frac{\omega}{4}\right) - 3\text{sinc}\left(\frac{3\omega}{4}\right) + \frac{\sin\left(\frac{\omega}{4}\right)}{\omega/4} = 2\text{sinc}\left(\frac{\omega}{4}\right) - 3\text{sinc}\left(\frac{3\omega}{4}\right),
\end{aligned}$$

which is the same result we obtained before.

6. Let $f(t) = \frac{1}{2\pi}\text{sinc}\left(-\frac{t}{4}\right)\left[1 - 2\cos\left(-\frac{t}{2}\right)\right]$, with Fourier transform $F(\omega) = 2\text{rect}(2\omega) - 2\text{rect}(2\omega+1) - 2\text{rect}(2\omega-1)$. Let $G(\omega) = \text{sinc}\left(\frac{\omega}{4}\right)\left[1 - 2\cos\left(\frac{\omega}{2}\right)\right]$.

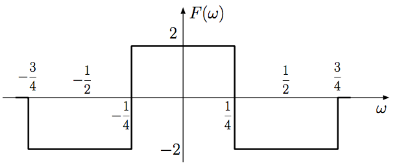
   (a) Obtain the inverse Fourier transform of $G(\omega)$, that is, obtain $g(t)$.
   Solution
   Applying the symmetry property $F(t) \leftrightarrow 2\pi f(-\omega)$, yields

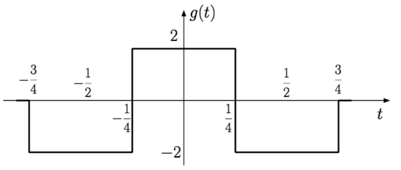
$$F(t) = 2\text{rect}(2t) - 2\text{rect}(2t+1) - 2\text{rect}(2t-1) \leftrightarrow 2\pi f(-\omega) = \text{sinc}\left(\frac{\omega}{4}\right)\left[1 - 2\cos\left(\frac{\omega}{2}\right)\right] = G(\omega).$$

   Therefore, since the Fourier transform pairing is unique, then we can write

$$g(t) = F(t) = 2\text{rect}(2t) - 2\text{rect}(2t+1) - 2\text{rect}(2t-1).$$

   (b) Plot $F(\omega)$ and $g(t)$.
   Solution





7. Given that $f(t) = 5\triangle^2(\frac{t}{4})$, evaluate the Fourier transform $F(\omega)$ at $\omega = 0$.
   Solution:
   We can express $f(t)$ as

$$f(t) = \begin{cases} 5(1 - \frac{|t|}{2})^2, & |t| \leq 2 \\ 0, & |t| > 2. \end{cases}$$

   Finding the Fourier transform at $\omega = 0$,

$$\begin{aligned} F(0) &= \int_{-\infty}^{\infty} f(t)e^{-j0t}dt = \int_{-\infty}^{\infty} f(t)dt \\ &= \int_{-2}^{2} 5(1 - \frac{|t|}{2})^2 dt = \frac{20}{3}. \end{aligned}$$