ANALOG SIGNAL PROCESSING

ECE 210 & 211

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Objectives

- Fourier transform of impulse response and power signals
- Sampling of Analog Signals
- Analog Signal Reconstruction
- Impulse Response of LTI Systems
- BIBO Stability

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Power signals and Impulse response

- Recall that in the previous lectures, we discussed the Fourier transform of signals having a finite energy

$$W = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

- After introduced with the impulse $\delta(t)$, we can extend our approach of FT to infinite energy signals

- Finite instantaneous power $|f(t)|^2$ of signals like $cos(\omega_o t)$ can be represented in terms of impulse $\delta(\omega)$ in the Fourier domain at any instant t

Power signals FT – Example 1

- **Such signals are called power signals opposed to energy signals having a finite *W***

- ***Let's look into some examples***

Question: **Show that $1 \leftrightarrow 2\pi\delta(\omega)$ and $e^{j\omega_o t} \leftrightarrow 2\pi\delta(\omega - \omega_o)$ are valid Fourier pairs.**

Solution: **We know that $e^{j\omega_o t}$ is the IFT of**

$$e^{j\omega_o t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_o)e^{j\omega t}\,d\omega$$

Power signals FT – Example 1

- This statement is valid because by using the shifting property of FT, the right hand side is reduced to

$$\frac{1}{2\pi} 2\pi e^{j\omega_o t} = e^{j\omega_o t}$$

- The Fourier pair $1 \leftrightarrow 2\pi\delta(\omega)$ is just a special case of $e^{j\omega_o t} \leftrightarrow 2\pi\delta(\omega - \omega_o)$ when $\omega_o = 0$.

Power signals FT – Example 2

Question: **Show that**

$$\cos(\omega_o t) \leftrightarrow \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

and

$$\sin(\omega_o t) \leftrightarrow j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$$

are valid Fourier pairs.

Power signals FT – Example 2

Solution: **We will make use of *Euler's identity* to rewrite $\cos(\omega_o t)$ and the result of Example 1**

$$(e^{j\omega_o t} \leftrightarrow 2\pi\delta(\omega-\omega_o)\ ).$$

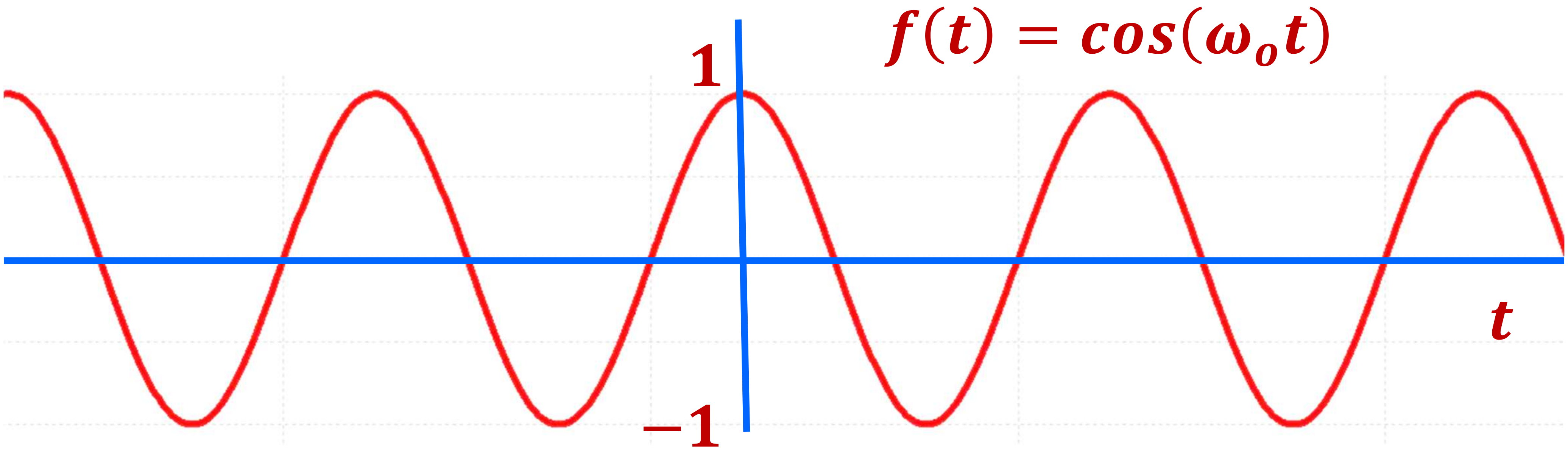
$$\cos(\omega_o t)=\frac{1}{2}\left(e^{j\omega_o t}+e^{-j\omega_o t}\right)\leftrightarrow \pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$$

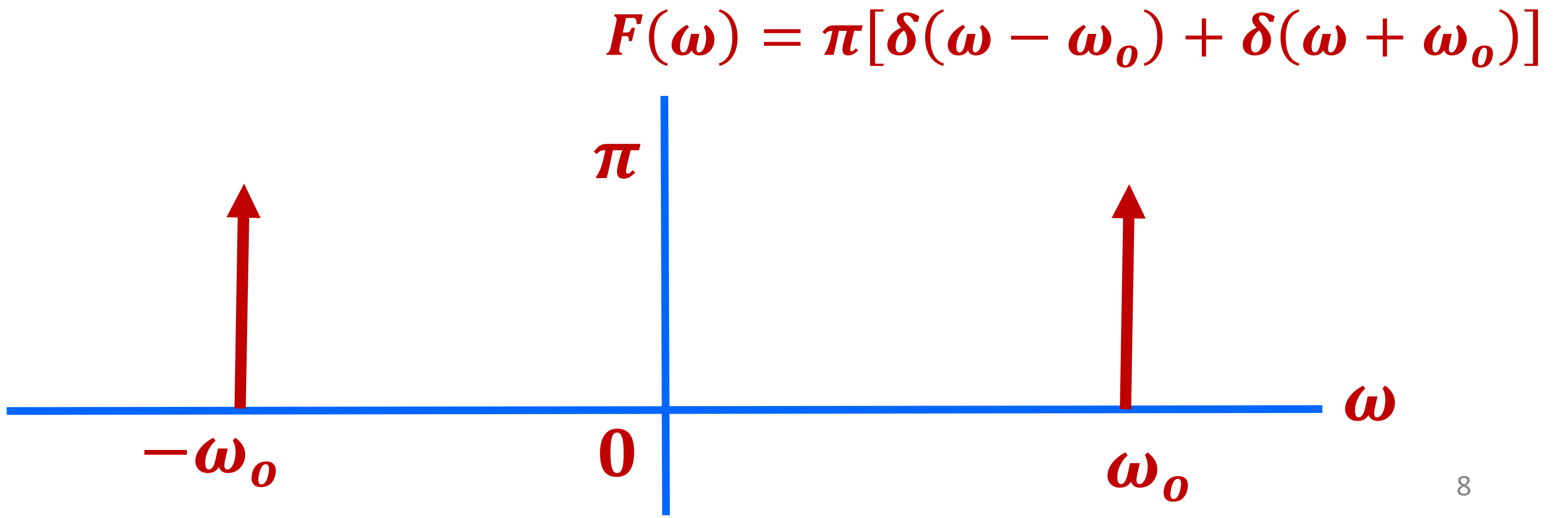
and

$$\sin(\omega_o t)=\frac{j}{2}\left(e^{-j\omega_o t}-e^{j\omega_o t}\right)\leftrightarrow j\pi[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)]$$

- **Graphical explanations of power signals of *cosine, sine* and *DC* are shown**

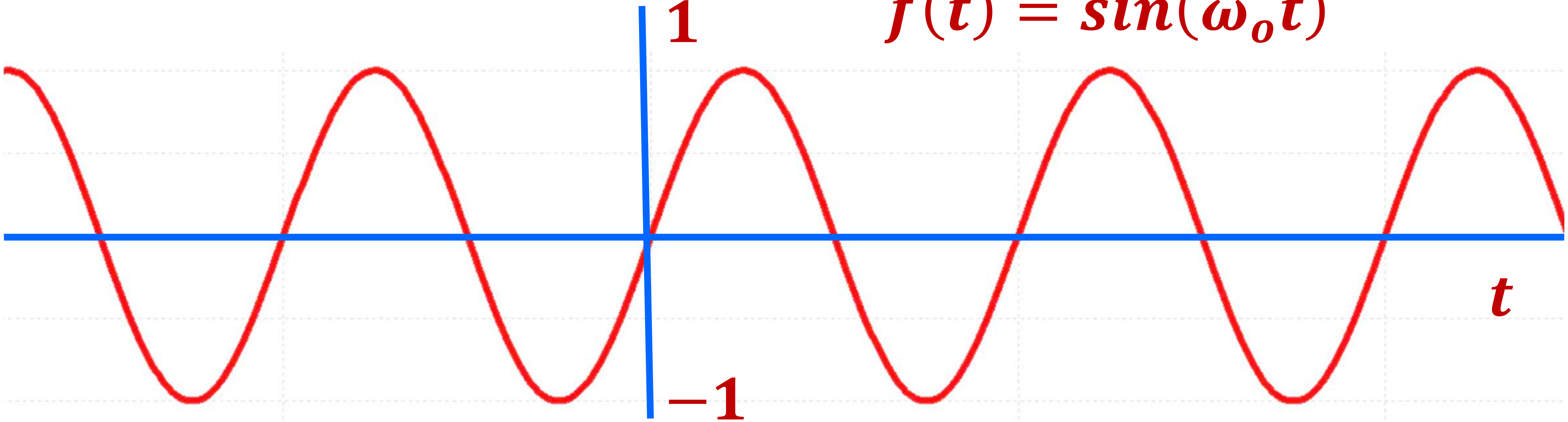
Power signals FT – Example 2



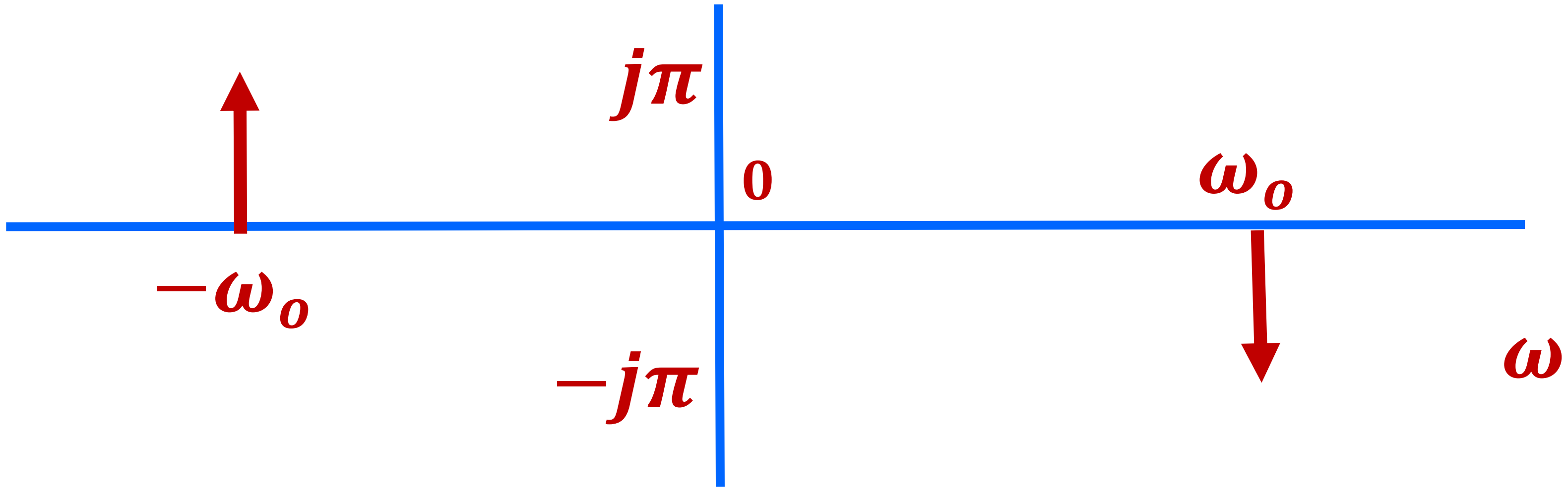


Power signals FT – Example 2

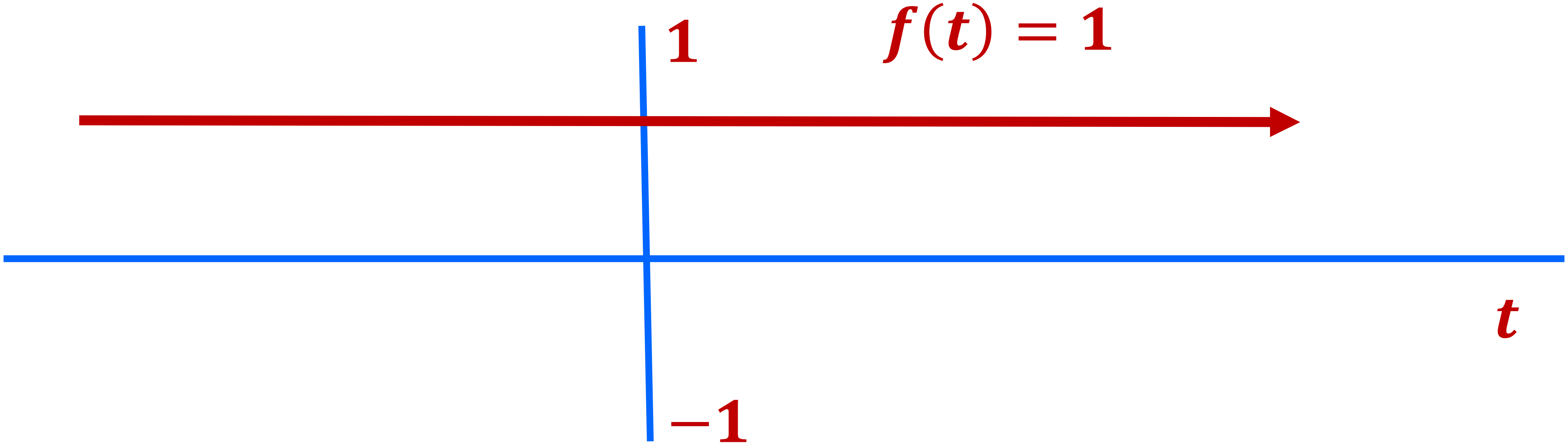
$$f(t) = sin(\omega_o t)$$

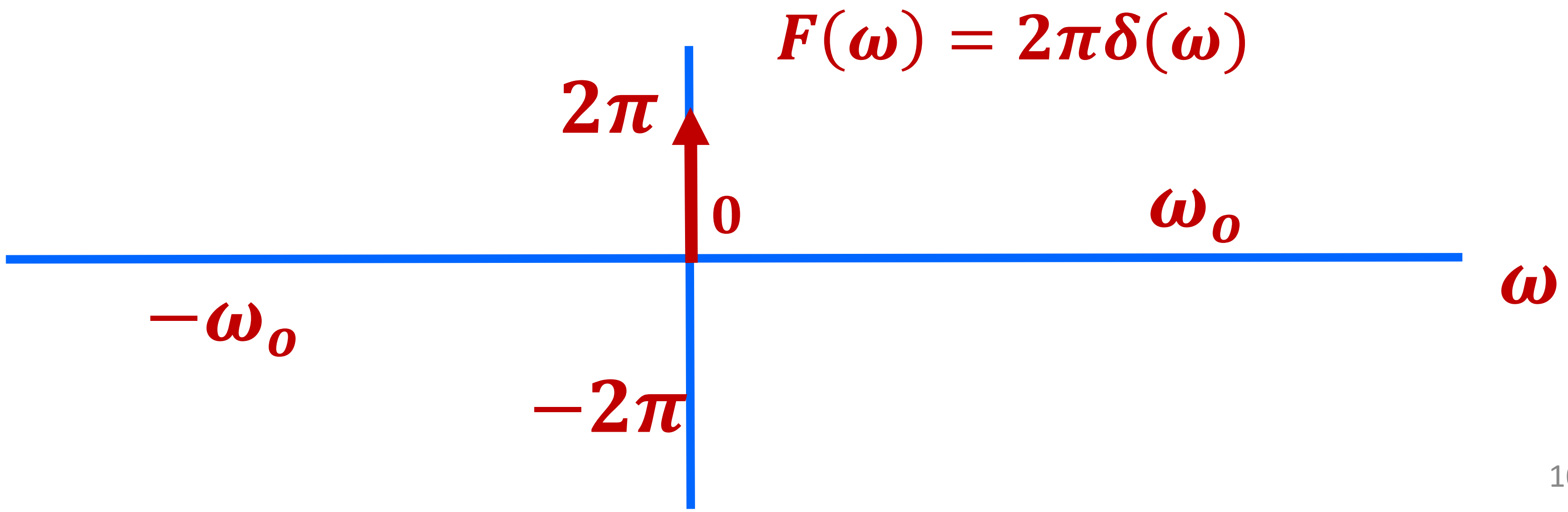


$$F(\omega) = j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$$



Power signals FT – Example 2





Power signals FT – Example 3

Question: **Given that $f(t) \leftrightarrow F(\omega)$, determine the Fourier transform of $f(t)sin(\omega_o t)$ by using the Fourier frequency convolution.**

Solution: **The Fourier frequency convolution property states that**

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega)$$

Using this property with

$$g(t) = sin(\omega_o t)$$

Power signals FT – Example 3

Taking the Fourier of $g(t)$,

$$G(\omega) = j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$$

we obtain

$$f(t)sin(\omega_o t) \leftrightarrow \frac{1}{2\pi}F(\omega) * j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$$

$$= \frac{j}{2}[F(\omega + \omega_o) - F(\omega - \omega_o)]$$

Power signals FT – Example 4

Question: **Find the Fourier transform of an arbitrary periodic signal**

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}$$

with Fourier coefficients F_n.

Power signals FT – Example 4

Solution: **Since**

$$e^{jn\omega_o t} \leftrightarrow 2\pi\delta(\omega - n\omega_o)$$

Application of Fourier addition property yields

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \Longleftrightarrow F(\omega) = \sum_{n=-\infty}^{\infty} 2\pi F_n \delta(\omega - n\omega_o)$$

- **It is an infinite sum of *weighted* and *shifted* frequency domain impulses $2\pi F_n\delta(\omega - n\omega_o)$, placed at *integer multiples* of fundamentals frequency $\omega_o = \frac{2\pi}{T}$.**

Power signals FT – Example 5

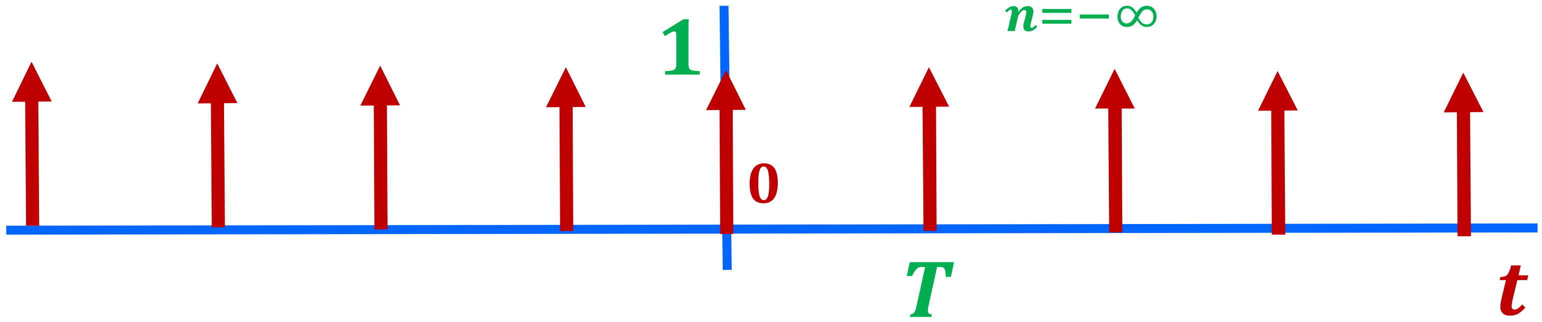
Question: **Validate the corresponding Fourier pair of an *impulse train* given by**

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\frac{2\pi}{T})$$

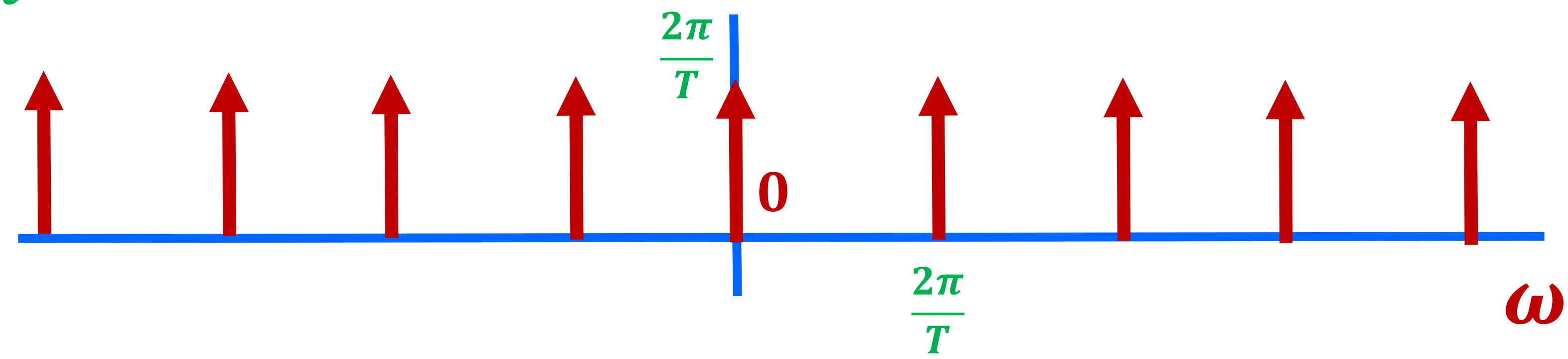
by first determining the Fourier series of the *impulse train*.

Power signals FT – Example 5

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{T}\right)$$



Power signals FT – Example 5

Solution: **The impulse train**

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

is a periodic signal with period T and having fundamental frequency $\frac{2\pi}{T}$. Its exponential Fourier transform is given by

$$F_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \left\{ \sum_{m=-\infty}^{\infty} \delta(t - mT) \right\} e^{-jn\frac{2\pi}{T}t} dt$$

Power signals FT – Example 5

$$F_n = \frac{1}{T}\sum_{m=-\infty}^{\infty}\int_{\frac{-T}{2}}^{\frac{T}{2}}\delta(t-mT)e^{-jn\frac{2\pi}{T}t}dt$$

$$F_n = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}}\delta(t)e^{-jn\frac{2\pi}{T}t}dt = \frac{1}{T} \qquad (\forall n)$$

By equating the impulse train to its Fourier series, we obtain

$$\sum_{n=-\infty}^{\infty}\delta(t-nT) \leftrightarrow \sum_{n=-\infty}^{\infty}\frac{1}{T}e^{jn\frac{2\pi}{T}t}$$

Power signals FT – Example 5

By using the Fourier transform pair,

$$e^{jn\frac{2\pi}{T}t} \leftrightarrow 2\pi\delta(\omega - n\frac{2\pi}{T})$$

we obtain

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\frac{2\pi}{T})$$

as required and it's also a direct evidence of frequency convolution property.

Power signals FT – Example 6

Question: **Suppose that a function generator produces a periodic signal**

$$f(t) = 4\cos(4t) + 2\cos(8t)$$

Assume a spectrum analyzer multiplies the finite length segment of $f(t)$ based upon a selection window,

$$w(t) = rect\left(\frac{t}{T_0}\right)$$

and then displays the squared magnitude of FT of $f(t)w(t)$. How does the output look like at $T_o = 10\ s$ & $20\ s$?

Power signals FT – Example 6

Solution: **Let $g(t) = f(t)w(t)$. Accordingly, the frequency convolution property results in**

$$G(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega)$$

where

$$F(\omega) = 4\pi[\delta(\omega - 4) + \delta(\omega + 4)] + 2\pi[\delta(\omega - 8) + \delta(\omega + 8)]$$

and $W(\omega)$ is the FT of $w(t)$. Therefore,

$$G(\omega) = 2[\delta(\omega - 4) + \delta(\omega + 4)] * W(\omega)$$

$$+ [\delta(\omega - 8) + \delta(\omega + 8)] * W(\omega)$$

Power signals FT – Example 6

$$G(\omega) = 2W(\omega - 4) + 2W(\omega + 4) + W(\omega - 8) + W(\omega + 8)$$

From the FT pair

$$rect\left(\frac{t}{T_o}\right) \leftrightarrow T_o sinc\left(\frac{\omega T_o}{2}\right)$$

We have

$$W(\omega) = T_o sinc\left(\frac{\omega T_o}{2}\right)$$

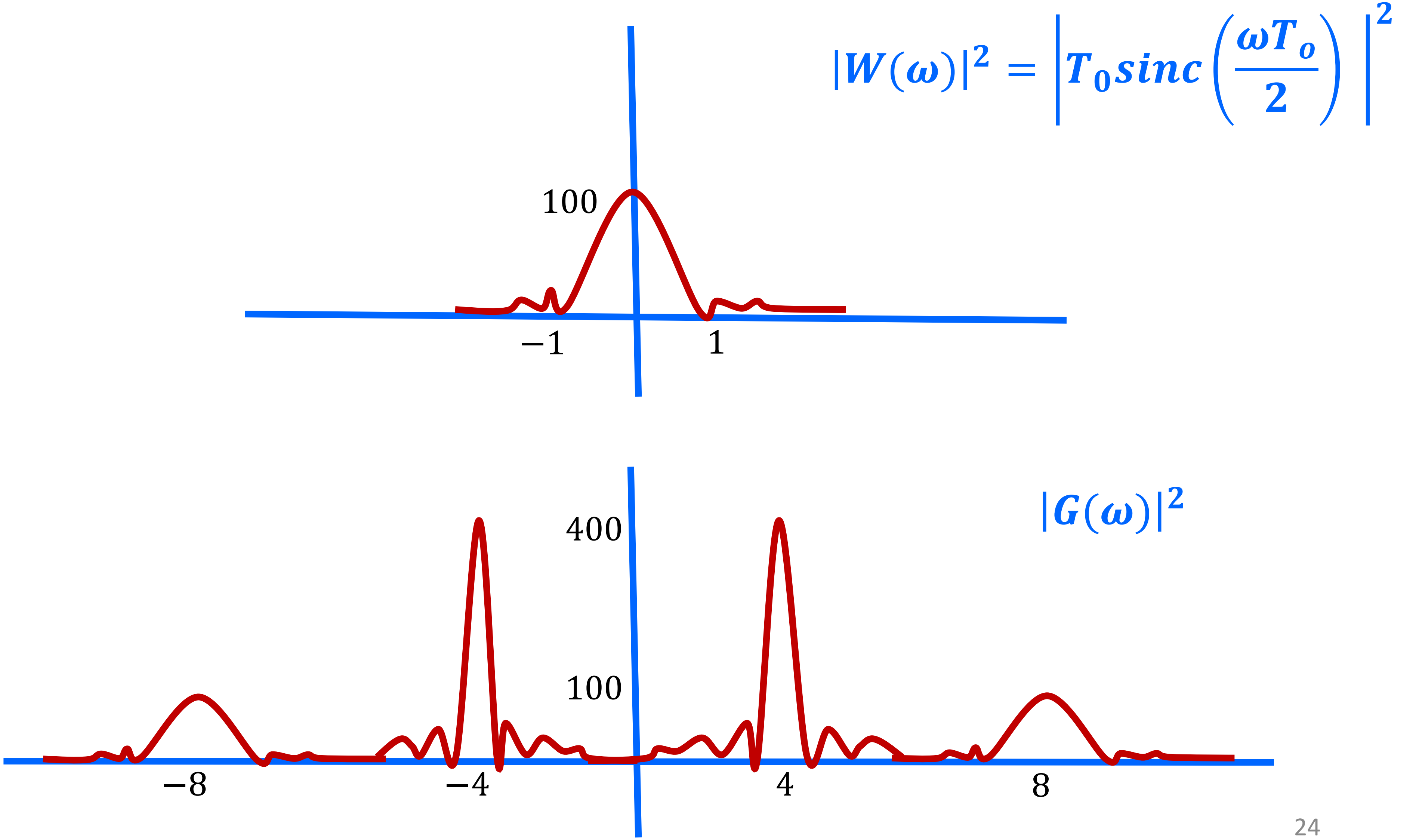
The resultant squared magnitude of FT follows as

$$|G(\omega)|^2 \approx 4|W(\omega - 4)|^2 + 4|W(\omega + 4)|^2$$

$$+ |W(\omega - 8)|^2 + |W(\omega + 8)|^2$$

Power signals FT – Example 6

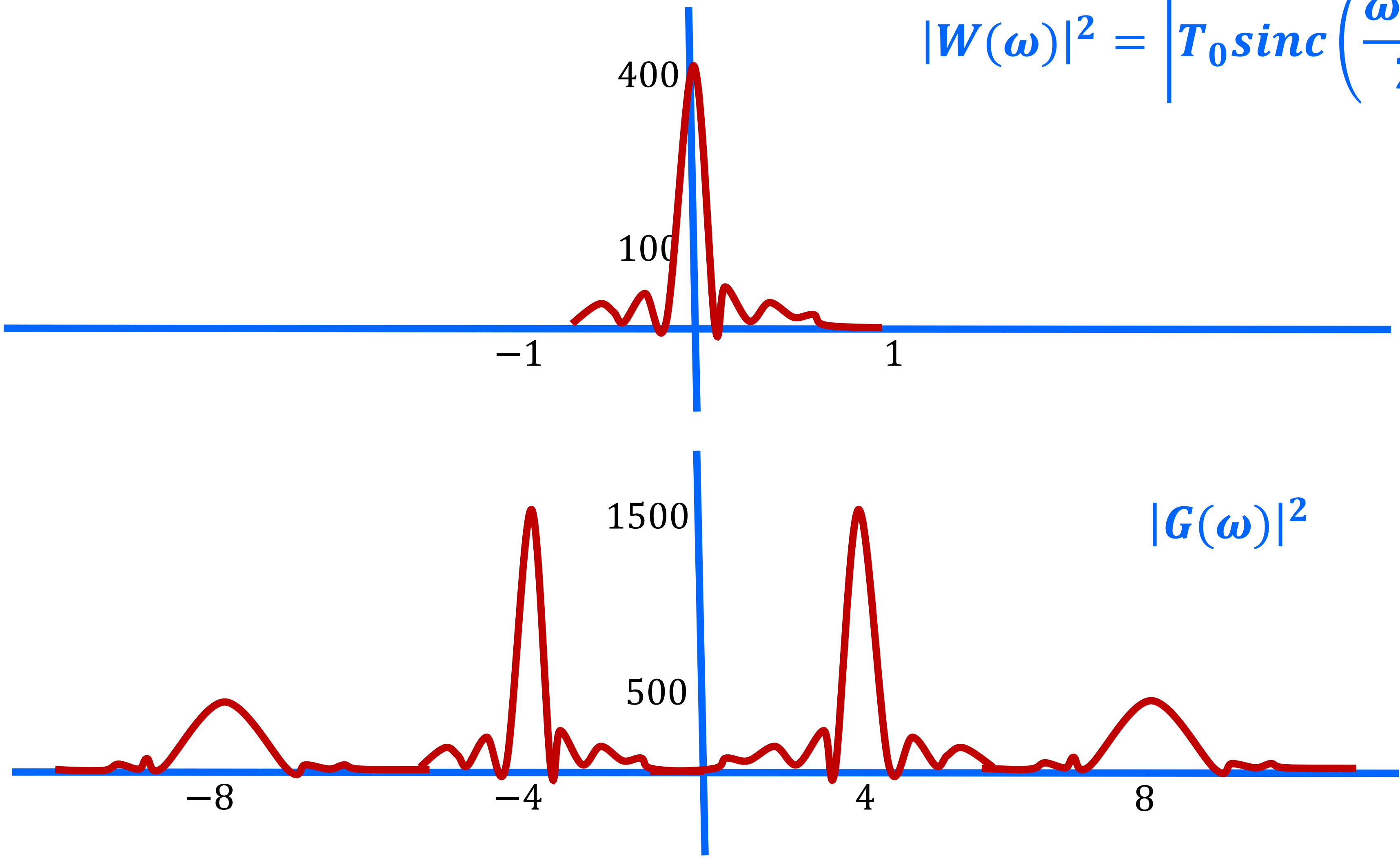
The resultant energy spectrum for $T_o = 10\ s$



Power signals FT – Example 6

The resultant energy spectrum for $T_o = 20\ s$

$$|W(\omega)|^2 = \left| T_0 sinc\left(\frac{\omega T_o}{2}\right) \right|^2$$



Power signals FT – Example 6

- **In both cases, the 90% of the bandwidth $\frac{2\pi}{T}$ of $w(t)$ is less than the shifted frequencies of 4 and 8 rad/s relevant for $G(\omega)$**

- **The various components of $G(\omega)$ have *little overlap***

- **The longer analysis windows (*Larger* T_o) produces *higher resolution estimate* of the spectrum of $f(t)$**

Power signals FT – Example 7

Question: **An incoming radio signal**

$$y(t) = \big(f(t+\alpha)\big)cos(\omega_c t)$$

is mixed with a signal $cos(\omega_c t)$, and the result $p(t)$ is filtered with an ideal low-pass filter $H(\omega)$

- **The filter bandwidth is less than ω_c, but larger than the bandwidth *Ω* of the LP message signal $f(t)$**
- **In addition, $\omega_c \ll \Omega$,**
- **What is the output $q(t)$ of the low-pass filter?**

Power signals FT – Example 7

Question: **Let**

$$p(t) = y(t)cos\omega_c t = (f(t) + \alpha)(cos(\omega_c t))^2$$

$$= (f(t) + \alpha)\frac{1}{2}(1 + cos(2\omega_c t))$$

Using Fourier frequency-convolution property, we find that FT of $p(t)$ as

$$P(\omega) = \frac{1}{4\pi}\big(F(\omega) + \alpha 2\pi\delta(\omega)\big) * [2\delta(\omega) + \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)]$$

$$= \frac{1}{2}\big(F(\omega) + \alpha 2\pi\delta(\omega)\big) + \frac{1}{4}[F(\omega - 2\omega_c) + \alpha 2\pi\delta(\omega - 2\omega_c)]$$

$$+ \frac{1}{4}[F(\omega + 2\omega_c) + \alpha 2\pi\delta(\omega + 2\omega_c)]$$

Power signals FT – Example 7

- **Only the first term of the expression lies within the passband of the low-pass filter $H(\omega)$; therefore, it follows that**

$$Q(\omega) = H(\omega)P(\omega)$$

implying an output,

$$q(t) = \frac{1}{2}(f(t) + \alpha)$$

as expected in coherent demodulation of a given AM signal.

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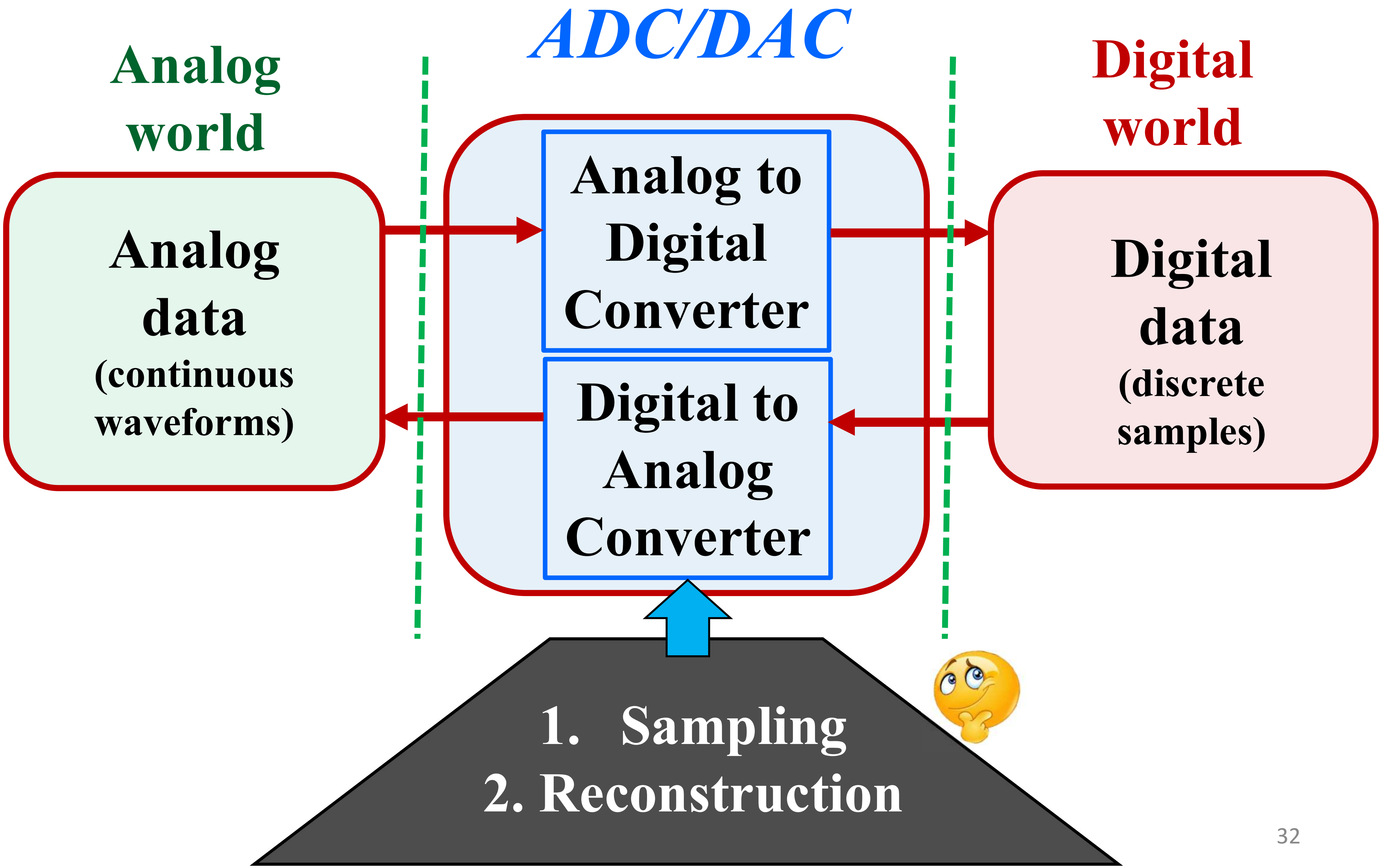
Sampling of Analog Signals

- **After learned how to generate impulse train for desired time period *T*, we are ready to implement impulse train for *sampling* and *reconstruction of signals***

- **Both techniques work as bridge between analog and digital data analysis and processing**

- **We will study *Nyquist criterion,* which constrains the sampling rate for data conversion and reconstruction**

Sampling of Analog Signals



Sampling of Analog Signals

- Consider a ***bandlimited signal***

$$f(t) \leftrightarrow F(\omega)$$

having ***bandwidth*** B, so that $F(\omega) = 0$ outside the frequency interval

$$|\omega| \leq 2\pi B \ \ rad/s$$

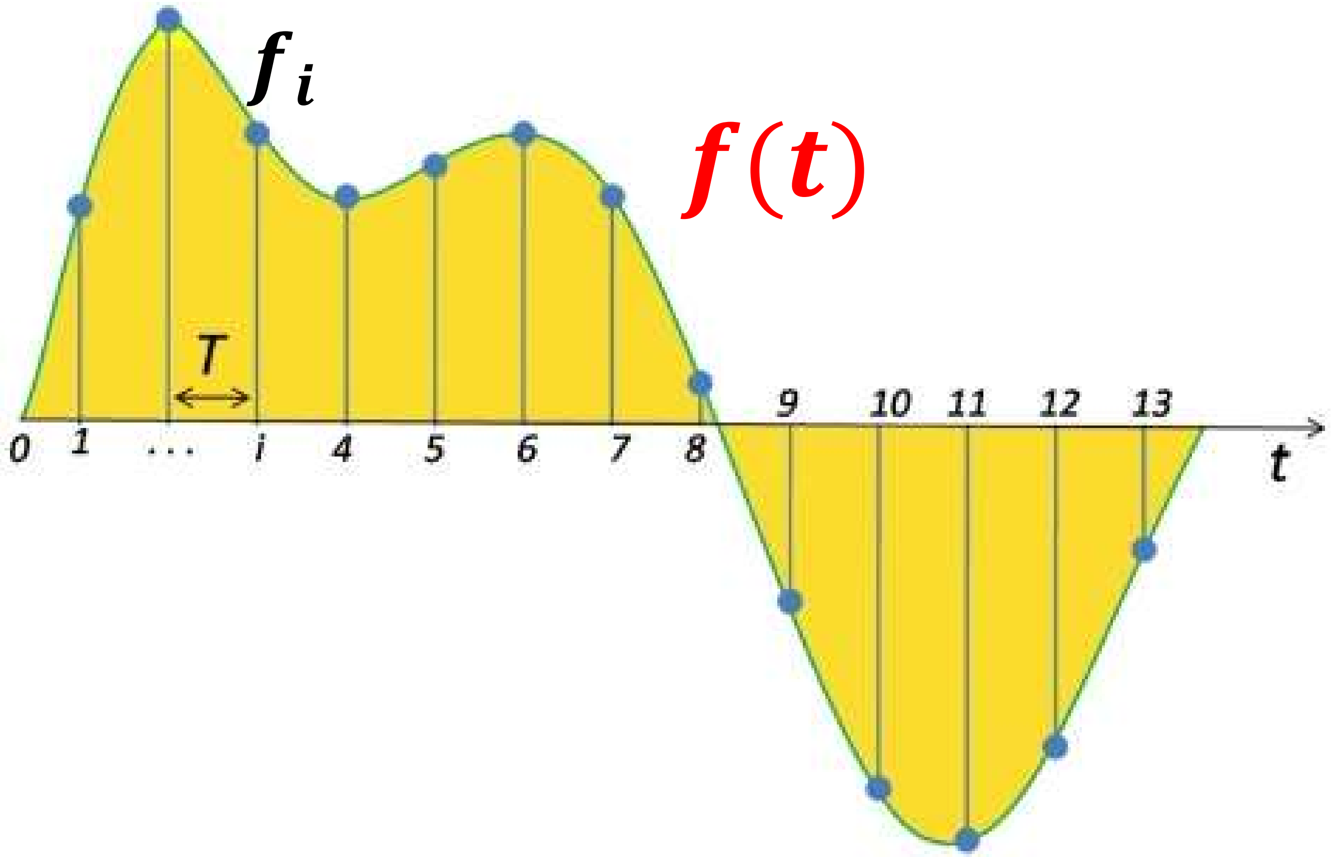
Suppose that only discrete samples of $f(t)$ are available, defined by

$$f_n = f(nT) \qquad -\infty < n < \infty$$

have ***equally-spaced*** samples at integer multiple of T ***(sampling interval)***

Sampling of Analog Signals

An Example: *A typical analog signal and sampling*



Sampling of Analog Signals

- ***Question arises here*: whether the signal $f(t)$ can be reconstructed with full fidelity after sampled with f_n**

- **Nyquist says yes! if sampling interval T is small enough, compared with the reciprocal of the bandwidth *(Nyquist criterion)***

$$T < \frac{1}{2B} \quad \text{or} \quad \frac{1}{T} > 2B$$

- **The sampling frequency $\frac{1}{T}$ must be larger than the twice of highest frequency *B (Hz)* in the signal being sampled**

Sampling of Analog Signals

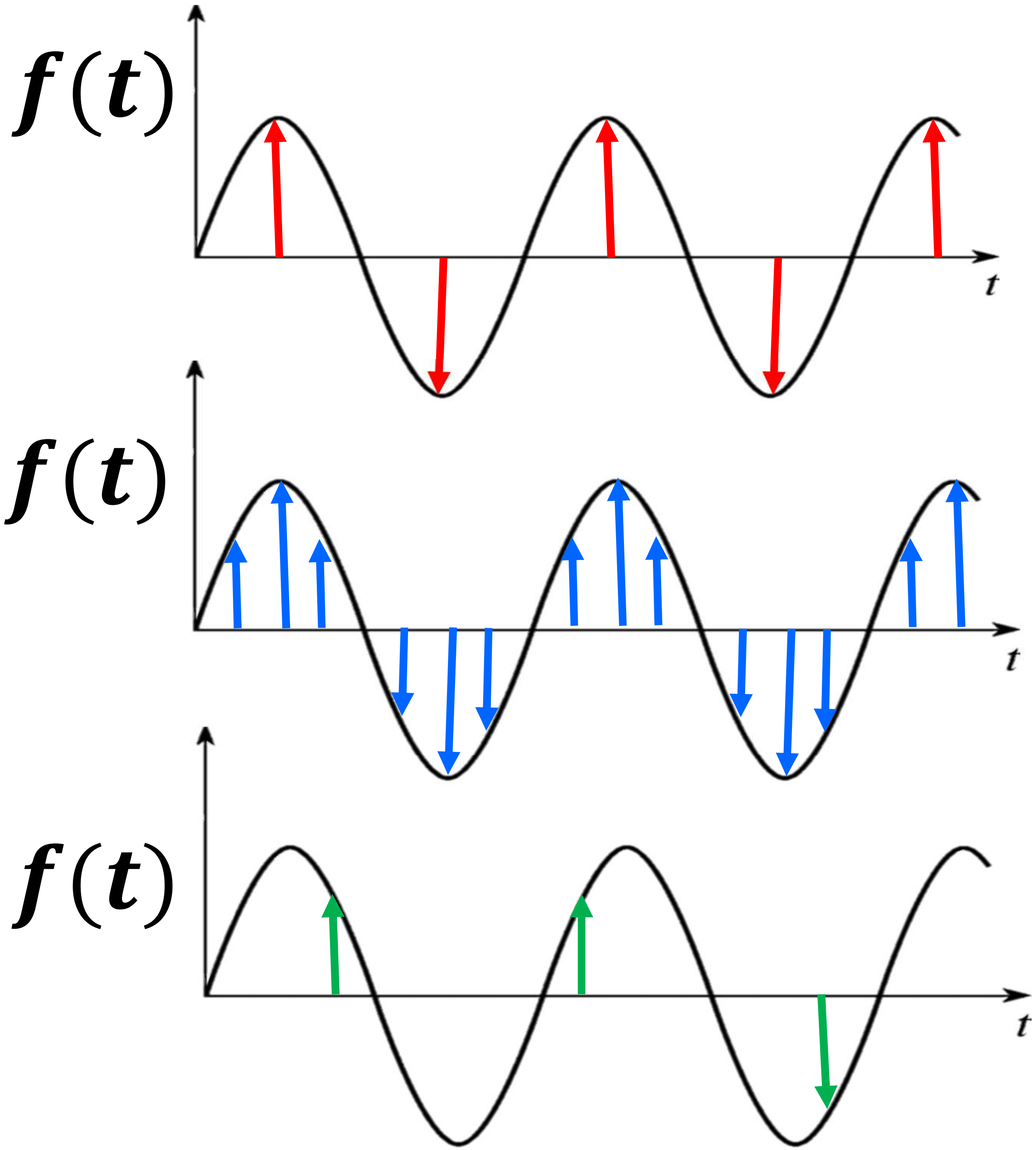
- **Alternatively, each frequency component in $f(t)$ must be sampled at a rate of at least *two samples per period***

- ***Reconstruction formula* makes theoretically possible to reconstruct analog signal identically, written as**

$$\boxed{f(t) = \sum_{n} f_n sinc\left(\frac{\pi}{T}(t - nT)\right)}$$

- **If the Nyquist criterion is violated, then this formula is invalid; the sum of RHS of this formula converges to an *aliased signal* rather than the original signal**

Sampling of Analog Signals



Nyquist Sampling

Over Sampling

Under Sampling

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Verification of reconstruction formula

- **The impulse train identity**

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\frac{2\pi}{T})$$

and frequency convolution property of Fourier transform imply that the product

$$f(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

has the Fourier transform

Verification of reconstruction formula

$$\frac{1}{2\pi}F(\omega) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T}\right) = \sum_{n=-\infty}^{\infty}\frac{1}{T}F\left(\omega - n\frac{2\pi}{T}\right)$$

Hence,

$$\sum_{n=-\infty}^{\infty} f(t)\delta(t - nT) \leftrightarrow \sum_{n=-\infty}^{\infty}\frac{1}{T}F\left(\omega - n\frac{2\pi}{T}\right)$$

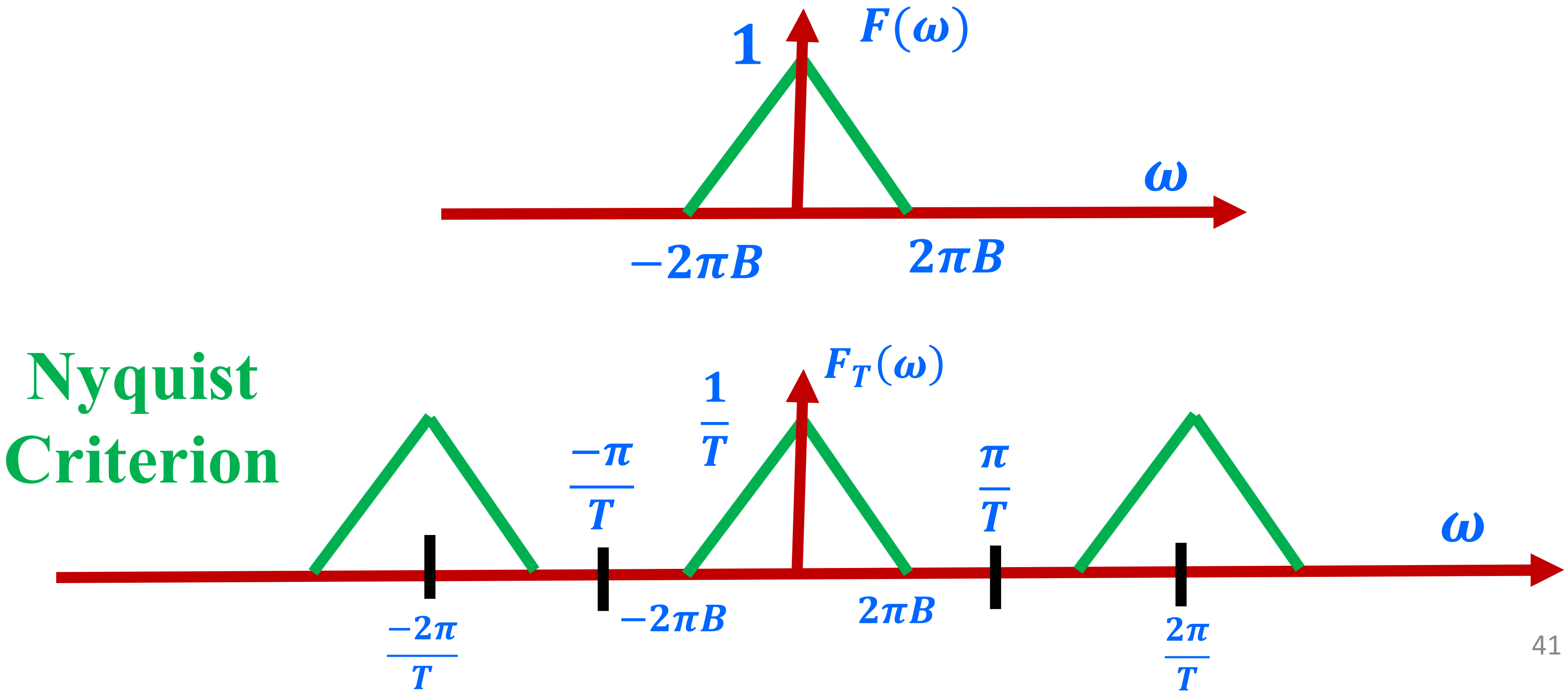
and also (in view of sampling of shifted impulse)

$$\sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT) \leftrightarrow \sum_{n=-\infty}^{\infty}\frac{1}{T}F\left(\omega - n\frac{2\pi}{T}\right)$$

Graphical depiction of Nyquist criterion

Let's interpret the graphical impact of Nyquist criterion. For this, let's pick the Fourier transform on the right side namely,

$$F_T(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F\left(\omega - n\frac{2\pi}{T}\right)$$

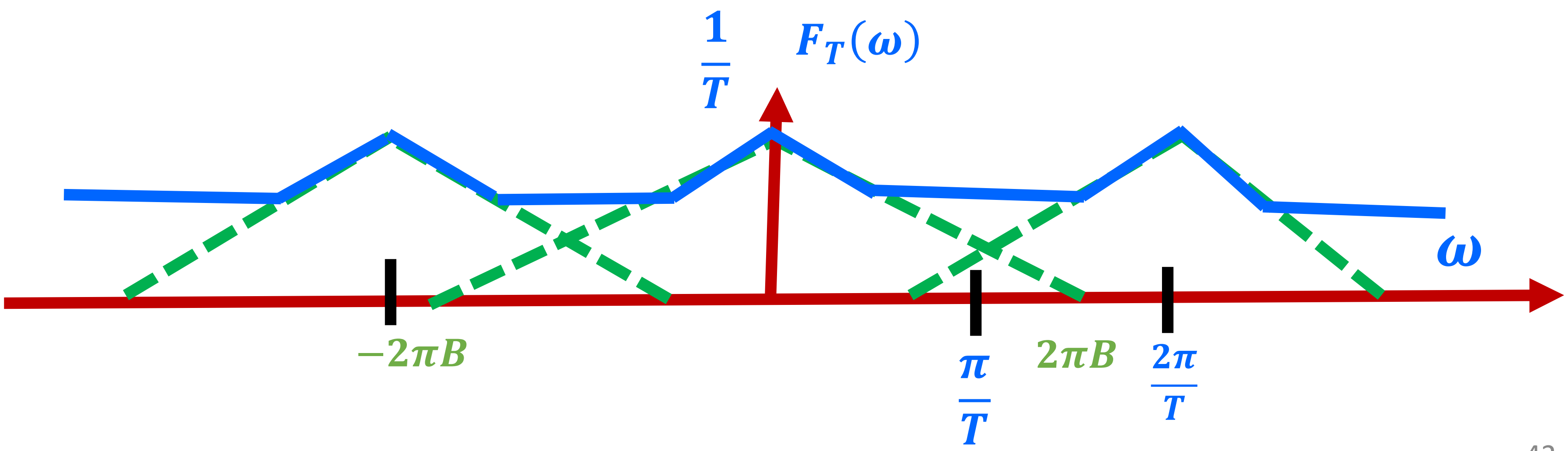


Graphical depiction of Nyquist criterion

Nyquist Criterion violated!

Reconstructed aliasing signal is not the actual signal!

- $H(\omega) = T\,rect\left(\frac{\omega}{\frac{2\pi}{T}}\right)$ **cannot recover actual signal as in the case of Nyquist *safe zone* does, rather ends up with aliased signal *of no worth***



Digital to analog converter

- The DAC is hardware that mimics the verification of reconstruction formula just discussed

- This type of circuit creates a weighted pulse train

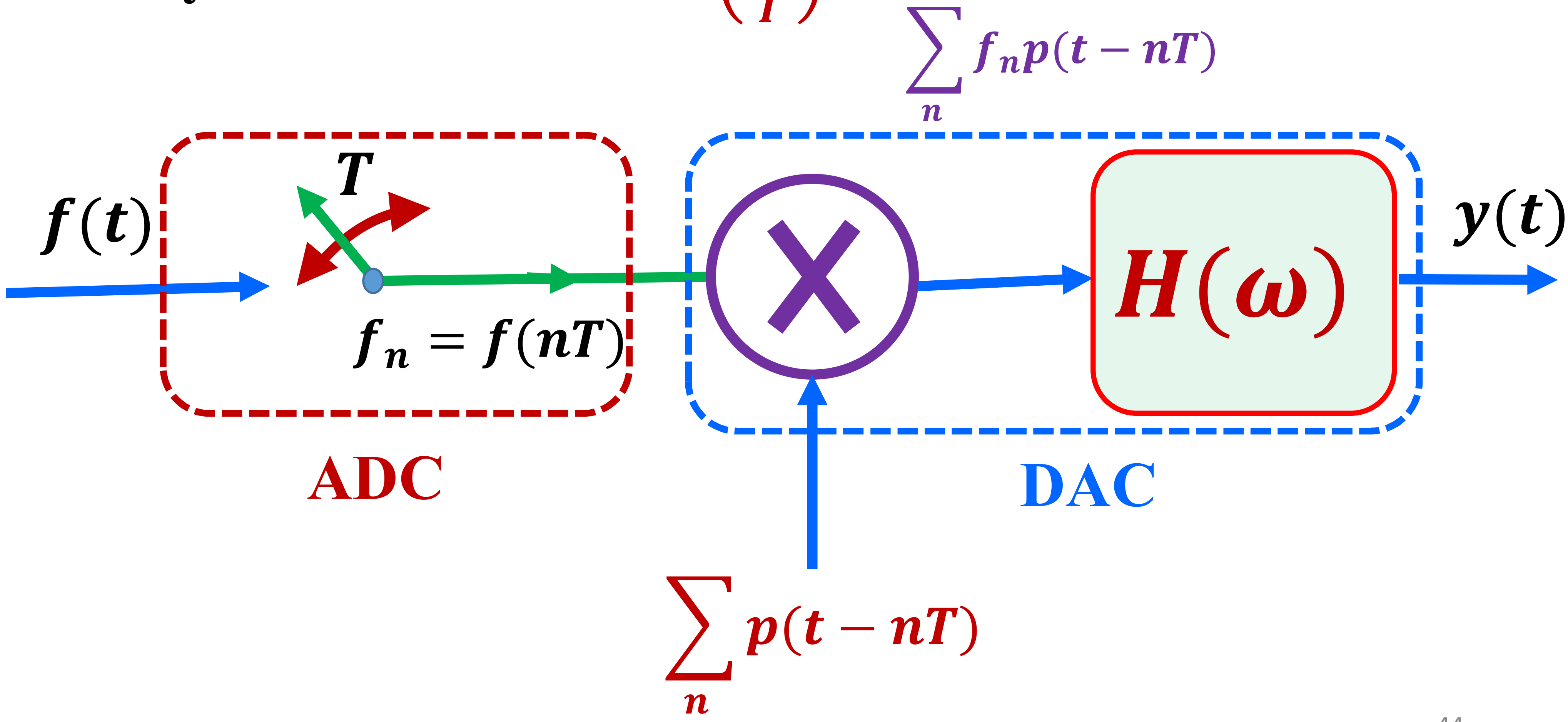
$$\sum_n f_n p(t - nT)$$

where $p(t)$ is a rectangular pulse having width T and then low-pass filtering this pulse train with a suitable LTI system

$$h(t) \leftrightarrow H(\omega)$$

Digital to Analog Converter

➢ The reconstruction is nearly ideal if $h(t)$ is designed so that $h(t) * p(t)$ is a good approximation to some delayed version of $sinc\left(\frac{\pi t}{T}\right)$



Digital to Analog Converter

- **The system, shown previously, should generate output $y(t)$ exactly the same as input $f(t)$ if there is no signal manipulation in whole ADC/DAC conversion**

- **But to introduce *digital signal processing (DSP),* one can convert $f(t)$ to new, desirable analog output $y(t)$ *by replacing* the samples f_n *with newly computed sequence y_n,* prior to reconstruction**

Digital to Analog Converter

- **Example of simplest signal processing (sample manipulation) is**

$$y_n = \frac{1}{2}(f_n + f_{n-1})$$

which is *a simple smoothing (averaging)* digital low pass filter

and, $$y_n = \frac{1}{2}(f_n - f_{n-1})$$

which is *a simple digital high pass filter* which emphasizes variation in the sample

- **In modern computing, the output is generated with the present and past inputs and sometimes, past outputs (memory circuits)**

Digital to Analog Converter

Question: **If we want to digitize the human voice, what would be bit rate assuming *8 bits per sample*?**

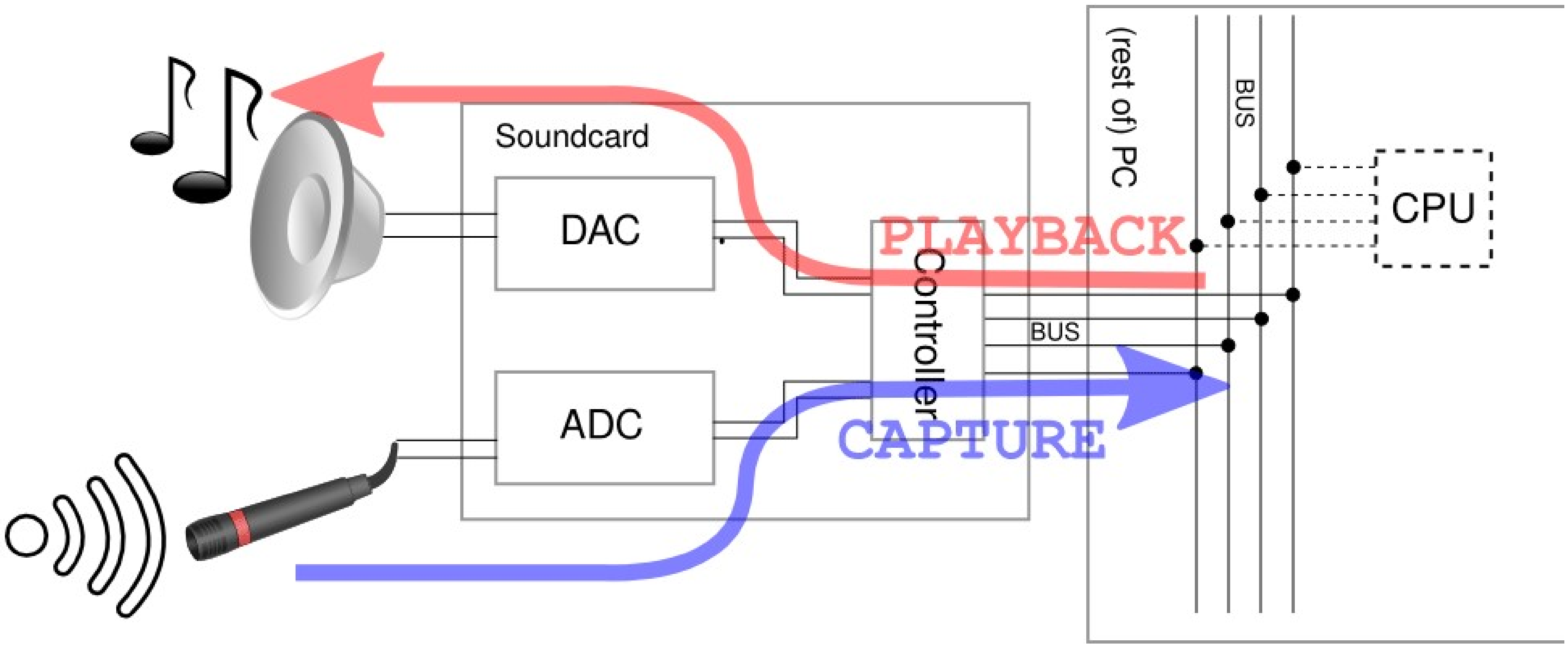
Solution: **The human voice normally contains frequencies ranging from 0 to 4000 Hz**

$$Sampling\ rate = 4000 \times 2 = 8000\ samples\ per\ second$$

$$Bit\ rate = Sampling\ rate \times No.of\ bits\ in\ one\ sample$$
$$= 8000 \times 8 = 64000\ bps = 64\ Kbps$$

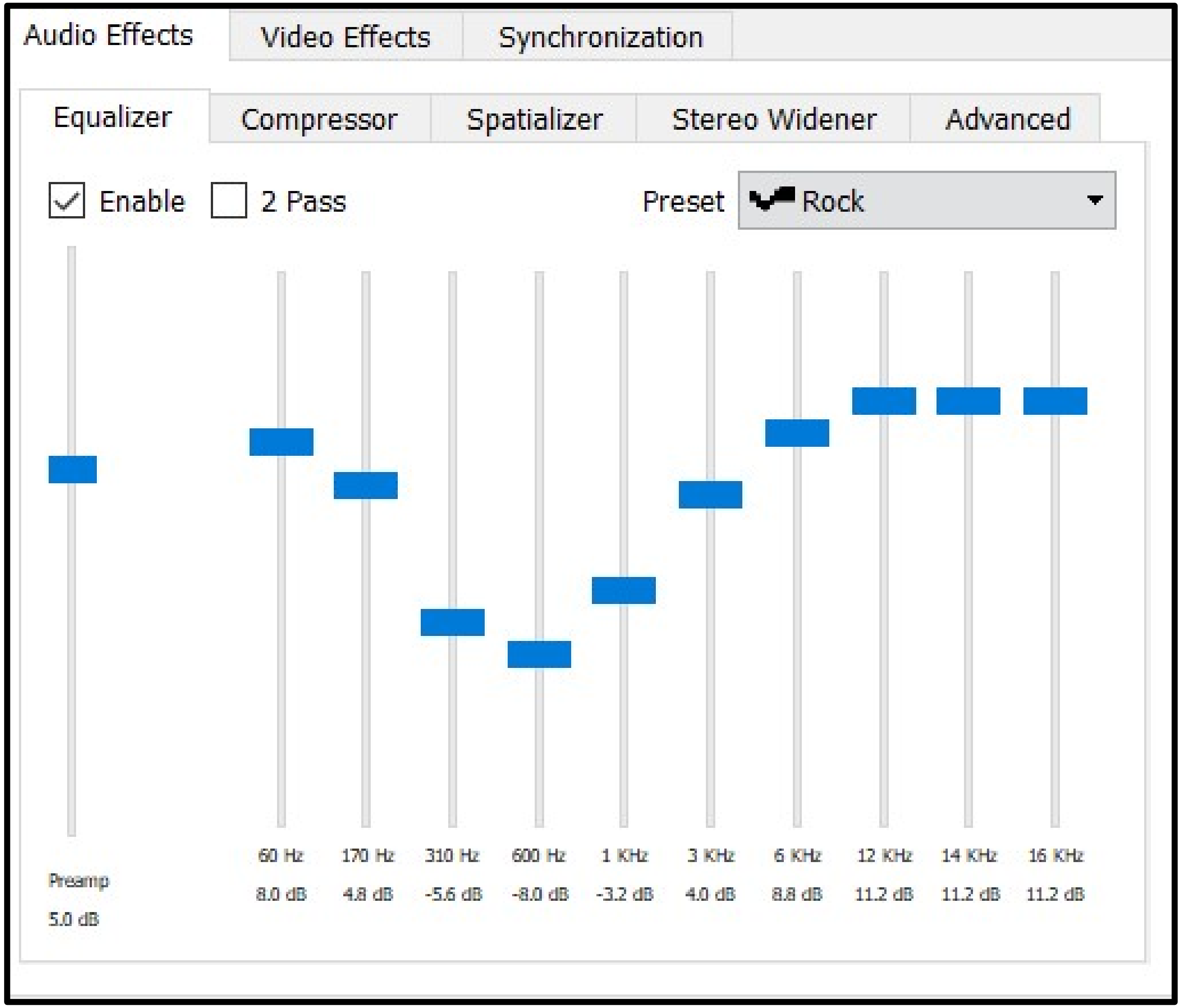
DAC/ADC combined…

A typical sound card interface



DAC/ADC combined…

A digital sound mixing and adjustment interface in Windows®



Calculation of $F(\omega)$

- From the previous section,

$$F_T(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F\left(\omega - n\frac{2\pi}{T}\right)$$

is the Fourier transform of time signal,

$$f_T(t) = \sum_n f(nT)\delta(t - nT)$$

Also recall that if $f(t)$ is bandlimited and T satisfies the Nyquist criterion, then

$$f_T(\omega) = \frac{1}{T} F(\omega) \quad for\ |\omega| < \frac{\pi}{T}$$

Calculation of $F(\omega)$

- So, if we can find a way of calculating $F_T(\omega)$, we then have a way for calculating $F(\omega)$

- *This is achievable*

- Knowing that,

$$\delta(t - nT) \leftrightarrow e^{-j\omega nT}$$

then transforming the preceding term for $f_T(t)$, we get

$$F_T(\omega) = \sum_n f(nT) e^{-j\omega nT}$$

which is an alternative formula for $F_T(\omega)$

Calculation of $F(\omega)$

- This formula enables us to compute Fourier transform $F(\omega)$ of a bandlimited signal $f(t)$ by only using its sample data $f(nT)$ as

$$F(\omega) = T\,F_T(\omega) = T\sum_{n} f(nT)e^{-j\omega nT}, \quad |\omega| < \frac{\pi}{T}$$

where $\frac{\pi}{T}$ is normally known as Nyquist frequency.

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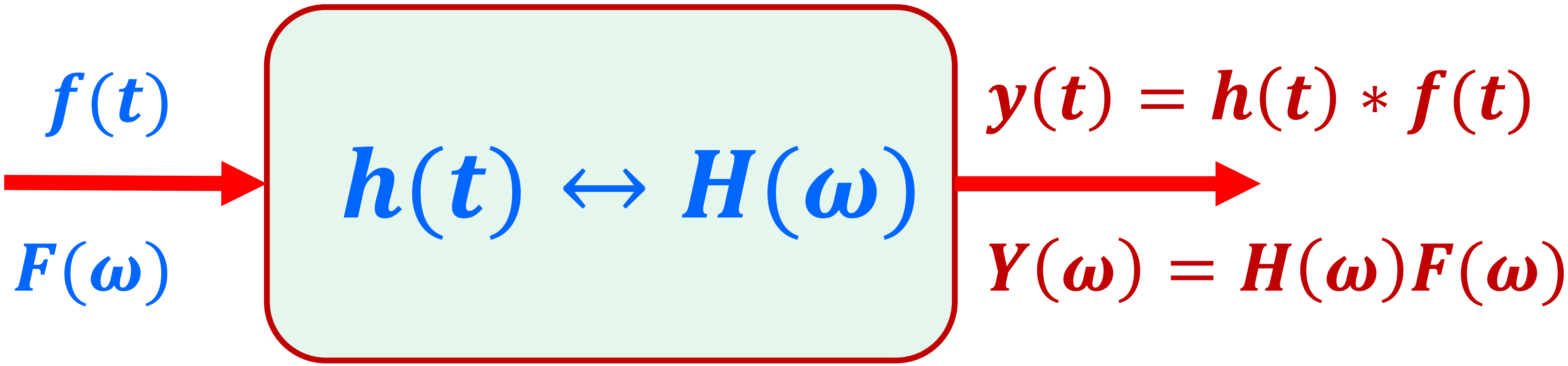
Impulse and *zero – state* response

- In the previous section, we discussed about the **zero-state response** $y(t)$ of an LTI system $H(\omega)$ to an arbitrary input calculated in the time domain using the **convolution formula**

$$y(t) = h(t) * f(t)$$



- $h(t)$ is the zero state response of system to $\delta(t)$

Impulse and *zero – state* response

- **This formula is always valid for LTI systems, as long as the integral converges and the input is continuous over time**

- **But what happens if FT of an impulse response does not exist like in the case of $h(t) = e^{t}u(t)$**

- **We will extend this discussion to verify the universality of implementation of the convolution formula for some *lab measureable signals* – Starting with the measurement methods of $h(t)$**

Measuring $h(t)$ of an LTI System

- **Previously, we evaluated $h(t)$ by taking the IFT of frequency response $H(\omega)$**

- **There are two alternative methods to obtain $h(t)$**

Method 1

Recall the identity, $\delta(t) * h(t) = h(t)$

is a symbolic shorthand for

$$\lim_{\epsilon \to 0}\{p_\epsilon(t) * h(t)\} = h(t)$$

where $p_\epsilon(t)$ is a pulse at t = 0, having a unity area and a width ϵ

Measuring $h(t)$ of an LTI System

- **Now, if we were to apply an input $p_\epsilon(t)$ to a system in the lab with an unknown impulse response $h(t)$, we can measure the output as $p_\epsilon(t) * h(t)$**

- **Taking a sequence of measurements while decreasing width ϵ, we should see the output $p_\epsilon(t) * h(t)$ converges to $h(t)$**

- **Keep reducing ϵ until further changes in the output were too small to be observed. Then we obtain $h(t)$**

- **If the output doesn't converge, it's better to choose the second method**

Measuring $h(t)$ of an LTI System

Method 2

➢ **Excite the system with a unit step input to obtain the *unit step response***

$$y(t) = h(t) * u(t) = g(t)$$

in symbolic terms

$$u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) * u(t) = g(t)$$

Then, differentiating and using the time derivative property of convolution, we obtain

Measuring $h(t)$ of an LTI System

$$\frac{dg}{dt} = h(t) * \frac{du}{dt} = h(t) * \delta(t) = h(t)$$

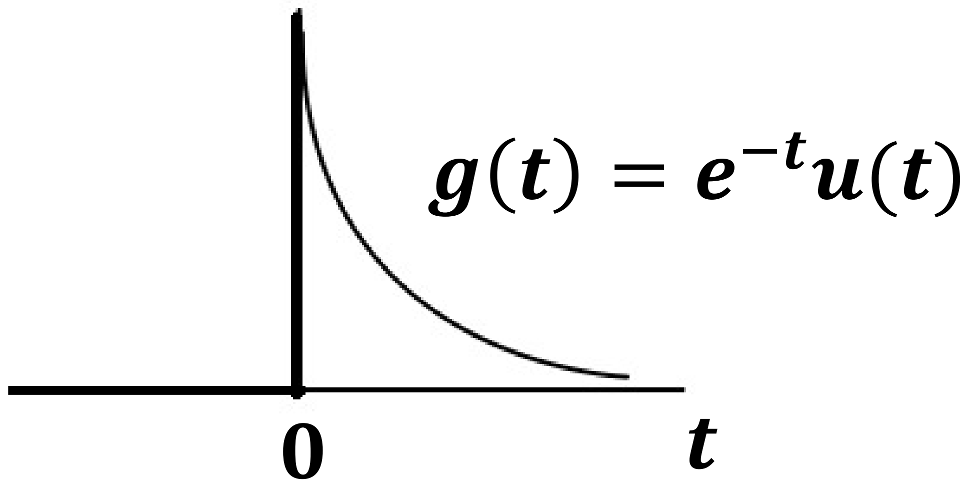
Therefore, the second method for finding the impulse response $h(t)$ is to differentiate the system's unit step response $g(t)$ which can be measured with a single input

Measuring $h(t)$ – Example 1

Question: Suppose that measurement in the lab indicates that the unit step response of a certain circuit is

$$g(t) = e^{-t}u(t)$$

what is the system's impulse response $h(t)$? Can we measure the impulse response by method 1?



Measuring $h(t)$ – Example 1

Solution: We find $h(t)$ by differentiating $g(t)$:

$$h(t) = \frac{dg}{dt} = \frac{d}{dt}\left(e^{-t}u(t)\right) = -e^{-t}u(t) + e^{-t}\frac{du}{dt}$$

$$= -e^{-t}u(t) + e^{-t}\delta(t) = \delta(t) - e^{-t}u(t)$$

where we used sampling property of the impulse response to simplify $e^{-t}\delta(t)$

When we apply the first method, the system response to input $p_\epsilon(t)$ is

$$h(t) * p_\epsilon(t) = \left(e^{-t}\delta(t) - e^{-t}u(t)\right) * p_\epsilon(t)$$

$$= p_\epsilon(t) - e^{-t}u(t) * p_\epsilon(t)$$

Measuring $h(t)$ – Example 1

As ϵ is kept reducing, the second term of the output will converge to $e^{-t}u(t)$, because

$$\lim_{\epsilon \to 0}\left(e^{-t}u(t) * p_{\epsilon}(t)\right) = e^{-t}u(t) * \delta(t) = e^{-t}u(t)$$

However, the first term $p_{\epsilon}(t)$ will not converges (i.e. stop changing) as ϵ is kept reducing

- **Even if we guess that an impulse is appearing in the output as *ϵ is made small,* it would be difficult to estimate the area of the impulse**
- ***The first method is not workable* in practice**
- **The problem here is that $h(t)$ *contains an impulse***

Measuring $h(t)$ – Example 2

Question: **Measurements in the lab indicate that the unit step response of a certain circuit is**

$$g(t) = te^{-t}u(t)$$

what is the system's impulse response $h(t)$? Can we measure the impulse response by method 1?

Measuring $h(t)$ – Example 2

Solution: **We find $h(t)$ by differentiating $g(t)$:**

$$h(t) = \frac{dg}{dt} = \frac{d}{dt}\left(te^{-t}u(t)\right) = (1-t)e^{-t}u(t) + te^{-t}\frac{du}{dt}$$

$$= (1-t)e^{-t}u(t) + te^{-t}\delta(t) = (1-t)e^{-t}u(t)$$

As you can see, there is no any impulse in the output; hence, the first method will also work

Measuring $h(t)$ – Example 3

Question: What would be the frequency response of the system described in Example 2?

Solution: Given that $h(t) = (1 - t)e^{-t}u(t)$

the Fourier transform is

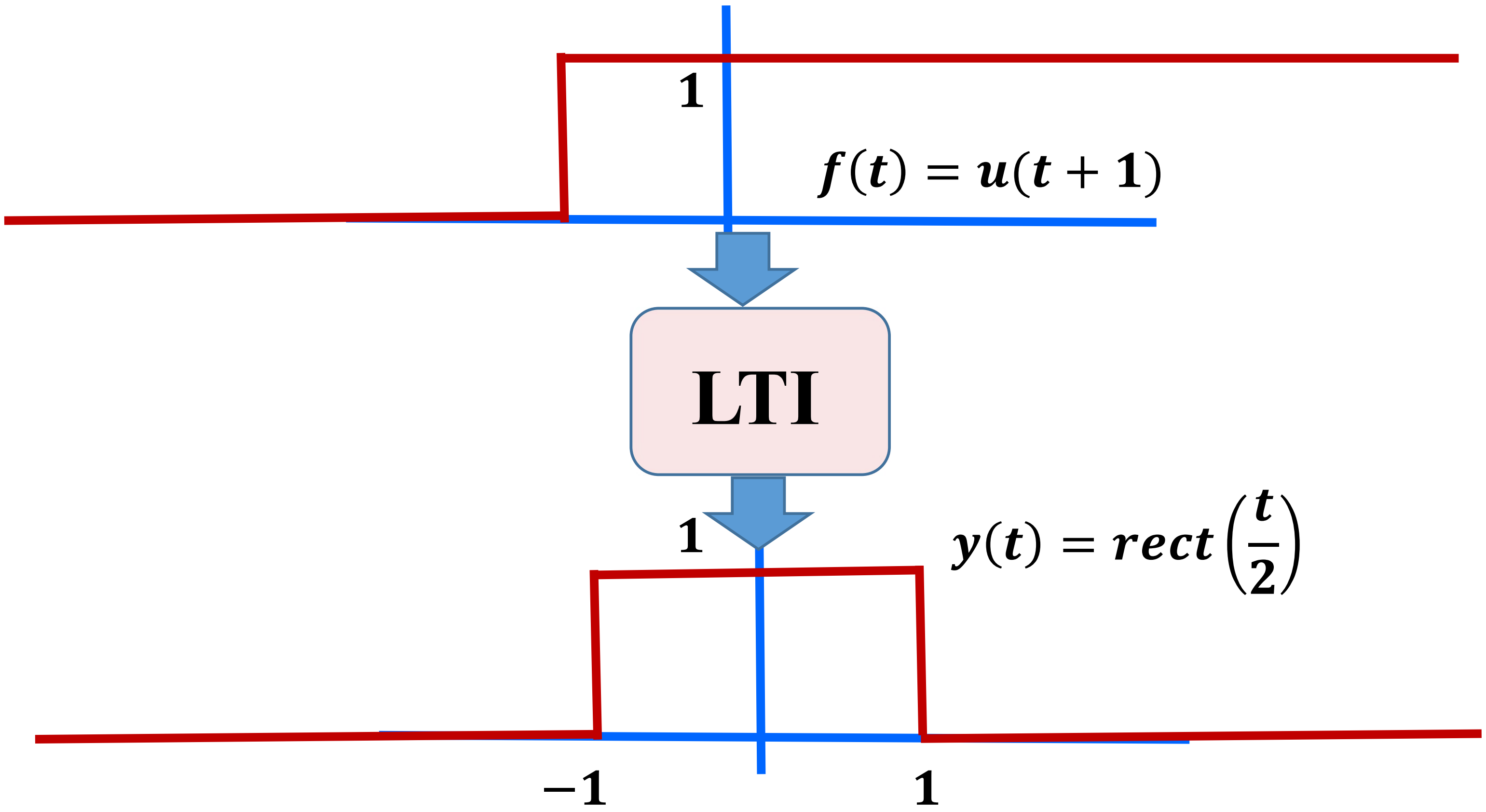
$$H(\omega) = \frac{1}{1 + j\omega} - \left(\frac{1}{1 + j\omega}\right)^2$$

$$H(\omega) = \frac{j\omega}{(1 + j\omega)^2}$$

must be the corresponding frequency response.

Measuring $h(t)$ – Example 4

Question: **An LTI system responds to an input $u(t+1)$ with the output $rect\left(\frac{t}{2}\right)$. What will be the system response $y(t)$ to the input $f(t) = rect(t)$? (Solve the problem by first finding the system impulse response)**



Measuring $h(t)$ – Example 4

Solution: Since the system is time-invariant, the information

$$u(t+1) \longrightarrow \boxed{\text{LTI}} \longrightarrow rect\left(\frac{t}{2}\right)$$

implies that

$$u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow rect\left(\frac{t-1}{2}\right) = u(t) - u(t-2)$$

$$g(t) = u(t) - u(t-2)$$

so that

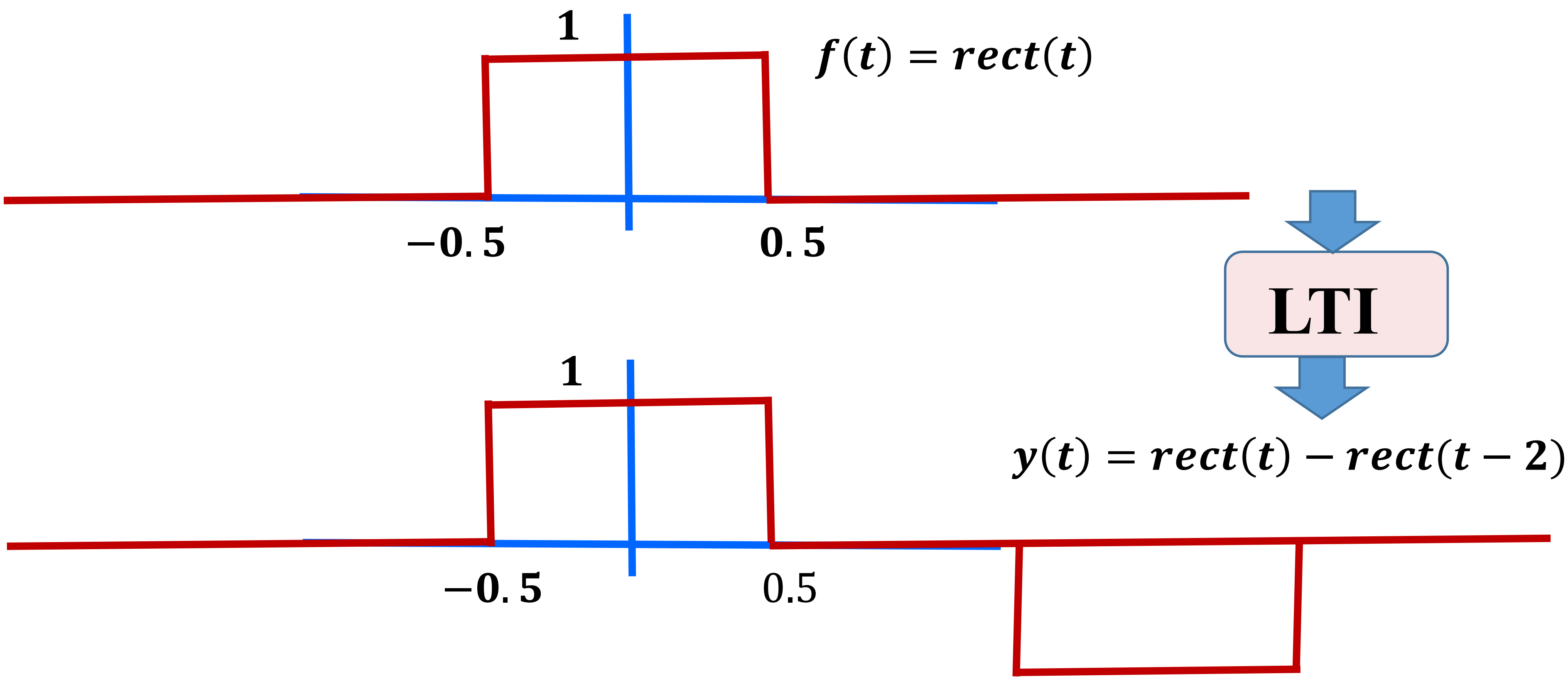
$$h(t) = g'(t) = \delta(t) - \delta(t-2)$$

Measuring $h(t)$ – Example 4

Consequently, the response to input $f(t) = \text{rect}(t)$ is

$$y(t) = h(t) * f(t) = [\delta(t) - \delta(t-2)\,] * rect(t)$$

$$= rect(t) - rect(t-2)$$



Testing whether System is LTI

- **If a system is LTI, then the relationship between $f(t)$ and $y(t)$ can be expressed in terms of convolution form as**

$$y(t) = f(t) * h(t)$$

- **In this case, $h(t)$ will not depend on choice of $f(t)$**

- **If the system is not LTI, then either linearity or time-invariance must be violated**

Testing system's LTI – Example 5

Question: **For a system with input $f(t)$, the output is given by**

$$y(t) = f(t+T)$$

is this system LTI?

Solution: **Because we can write**

$$y(t) = f(t+T) = \delta(t+T) * f(t)$$

This system is LTI with the impulse response

$$h(t) = \delta(t+T)$$

and the system satisfies zero-state linearity and time-invariance

Testing system's LTI – Example 6

Question: Suppose a system has the input-output relation,

$$\boldsymbol{y(t) = f^2(t+T)}$$

is this system LTI?

Solution: We can write

$$\boldsymbol{y(t) = f^2(t+T) = (\delta(t+T) * f(t)\,)^2}$$

This system is not LTI, as it is not in the form

$$\boldsymbol{y(t) = h(t) * f(t)}$$

Testing system's LTI – Example 7

Question: **Is the system,**

$$\boldsymbol{y(t) = f^2(t+T)}$$

time invariant?

Solution: **We already know that system is not LTI, but still it could be time invariant; to test this, we feed the system with new input**

$$\boldsymbol{f_1(t) = f(t-t_o)}$$

and observe that new output is

$$\boldsymbol{y_1(t) = f_1^2(t+T) = f^2(t+T-t_o) = f^2\big((t-t_o)+T\big) = y(t-t_o)}$$

Testing system's LTI – Example 7

- **New output is a delayed version of the original output**

- **The system is time invariant, *but not LTI***

- **It is non-linear**

- **It is evident that a doubling of the input does not doubles the output –*nonlinearity***

Objectives

- Fourier transform of impulse response and power signals
- Sampling of Analog Signals
- Analog Signal Reconstruction
- Impulse Response of LTI Systems
- **BIBO Stability**

BIBO Stability

Stability

- The stability of a system can be thought as a continuity in its dynamic behavior

- If a small perturbation arises in the system inputs or initial conditions, a stable system will produce small modifications in its response

Instability

- The instability arise when even a small change at the input fluctuates a lot in its output that ends up on system saturation or disintegration

BIBO Stability

- **Recall that response of an LTI system is composed of**

Response to initial conditions + **Response to inputs**

- **The concept of *input-output stability* refers to stability of the response to the inputs only, assuming zero initial conditions**

- **BIBO Stability: A system is BIBO (Bounded Input Bounded Output) stable if every bounded input produces a bounded output**

BIBO Stability

- **The key to BIBO stability turns out to be *absolute integrability* of the impulse response $h(t)$**

- **An LTI system is stable if and only if its impulse response $h(t)$ is absolute integrable, satisfying**

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

BIBO Stability – Example 8

Question: **Determine whether the systems given are BIBO stable or not?**

$$a)\quad h_1(t) = 2u(t-1)$$

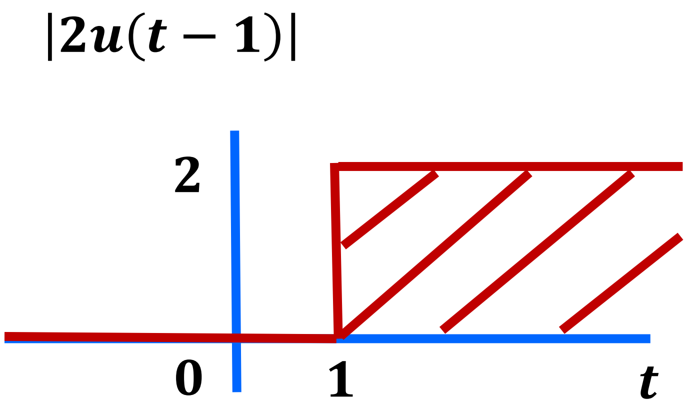
$$b)\quad h_2(t) = e^{2t}u(t)$$

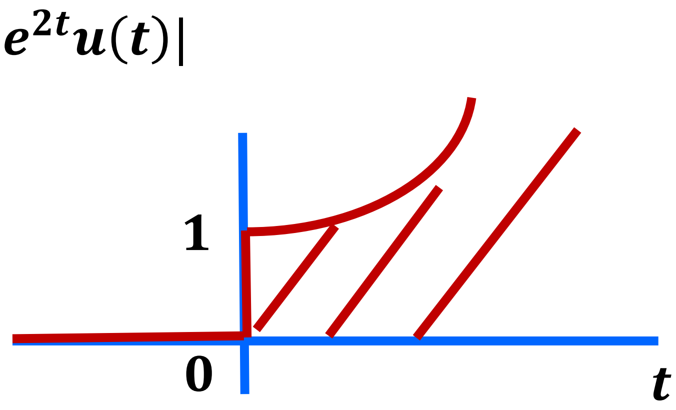
BIBO Stability – Example 8

Solution:

- **a)** **The system $h_1(t) = 2u(t-1)$ is not BIBO stable, because the area under $|2u(t-1)|$ is infinite**

- **b)** **The system $h_2(t) = e^{2t}u(t)$is not BIBO stable, because the area under $\left|e^{2t}u(t)\right|$ is infinite**





Summary

- **Nyquist Criterion: The sampling frequency must be larger than the twice of highest frequency *B (Hz)* in the signal being sampled**

- **Each frequency component in $f(t)$ must be sampled at a rate of at least *two samples per period***

- **By using reconstruction formula, we can reconstruct actual signal with a proper sample rate used**

- **In modern ADC/DAC, signal manipulation can be possible by changing the impulse train sequence**

Summary

- **With the application of impulse response, FT of power signals can be determined**

- **Infinite energy signals-refereed to as power signals-can be Fourier transformed by taking instantaneous values from impulse response convolved with power signal**

- **The selection of filter window can be made better by using FT analysis of power signals**

Summary

- **For any LTI systems, if the impulse response exists, the system always has a pre-determined output-input relation**

- **For all those systems, we can evaluate $h(t)$ by exciting it with a unit step input and then differentiating the unit step response**

- **BIBO Stability: A system is BIBO (Bounded Input Bounded Output) stable if every bounded input produces a bounded output**

Further reading

1. Ch. 9 (page 325-332), Ch. 10 (page 337-350), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.
2. Ch. 7 (page 522-547), A. V. Oppenheim, ***Signals and Systems***, 2nd ed., Prentice Hall, 1996.

Preview:

1. Ch. 11 (page 361-375), E. Kudeki and D. C. Munson, ***Analog Signals and Systems,*** Prentice Hall, 2008.

Homework 12

Deadline: 10:00 PM, 18th May, 2022

Thank you!