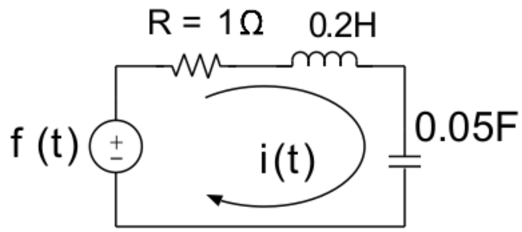
ZJU-UIUC INSTITUTE

Zhejiang University – University of Illinois at Urbana-Champaign Institute

---

ECE-210 Analog Signal Processing Spring 2022
Homework #7: Solution

1. Consider the circuit drawn below, where the frequency response of the circuit is: $H(\omega) = \frac{I}{F}$.



(a) What is the resonant frequency of this circuit?

(b) Plot $|H(\omega)|$, and label the resonant frequency on this plot;

(c) Plot $Re\{H(\omega)\}$and $Im\{H(\omega)\}$. You can use Matlab, Mathematica, etc.

(d) Explain why this circuit might be called a "bandpass" filter.

(e) Repeat (a) and (b) for resistor values of $10\,\Omega$ and $0.1\,\Omega$.

(f) Based on (d), how does the resistor value relate to the passband of the filter (e.g., does a larger value for the resistor give a narrower or wider passband)?

Solution

The phasor equivalent circuit has an inductance impedance $j\omega L$ and a capacitor impedance $\frac{1}{j\omega C}$. Applying KCL yields

$$I(R + j\omega L + \frac{1}{j\omega C}) = F.$$

Therefore, the frequency response of the system is

$$H(\omega) = \frac{I}{F} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}},$$

and the magnitude response

$$|H(\omega)| = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

In terms of physical meaning, $H(\omega)$ is the reciprocal of impedence. For this standard RLC circuit,

- When $\omega = 0$, $H(\omega) \to 0$. Intuitively, it is because capacitor performs as an open circuit for DC.
- When $\omega = \omega_{resonant}$, $H(\omega) = \frac{1}{R} = \frac{1}{Z_{min}}$. $H(\omega)$ achieves its maximum value.
- When $\omega \to \infty$, $H(\omega) \to 0$. Intuitively, it is because inductor acts as an open circuit.

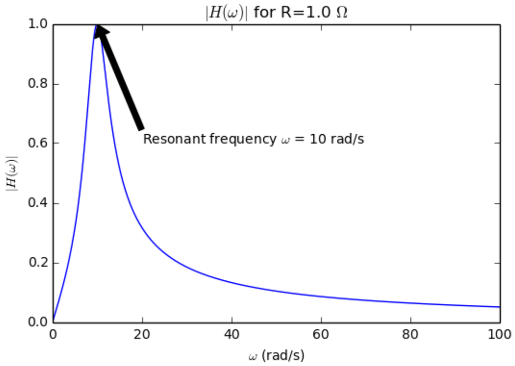
(a) To find the resonant frequency of this circuit we need to find a frequency ω that maximizes $|H(\omega)|$, which is the same as finding the ω that minimizes the denominator. The square root is a monotonically increasing function, so minimizing $\sqrt{(R + j\omega L + \frac{1}{j\omega C})}$ is the same as minimizing its argument

$$G(\omega) = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2 + \frac{L^2}{\omega^2}\left(\omega^2 - \frac{1}{LC}\right)^2.$$

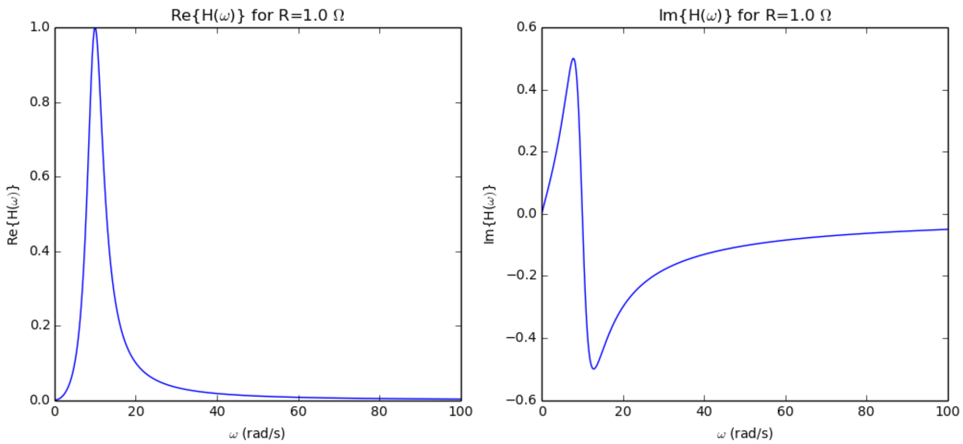
The function $G(\omega)$ has a minimum ($|H(\omega)|$ has a maximum) at $\omega_o = \sqrt{\frac{1}{LC}}$ rad/s. Plugging in the values we find that the resonant frequency is

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.2 \times 0.05}} = \sqrt{\frac{1}{0.01}} = 10\,\text{rad/s}.$$

(b) Plotting $|H(\omega)|$ on a linear scale for a resistor $R = 1\,\Omega$:



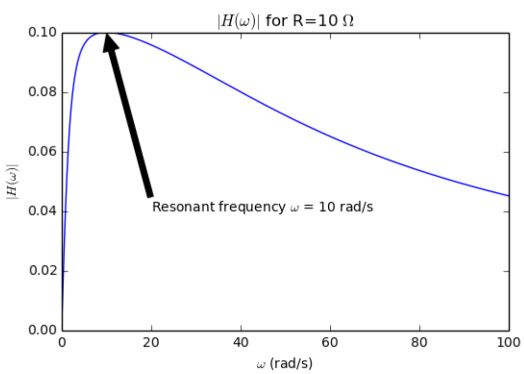
(c) Plotting $Re\{H(\omega)\}$ and $Im\{H(\omega)\}$ on a linear scale:



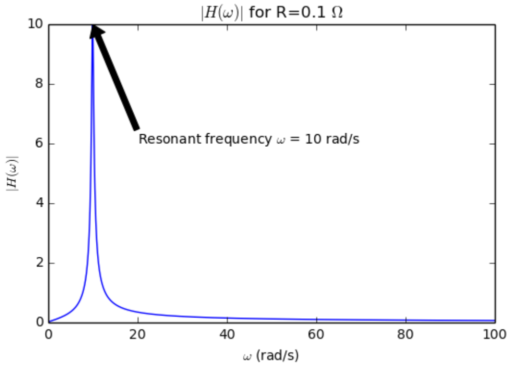
(d) This circuit might be called a "bandpass" circuit because it attenuates signals with frequencies outside a frequency band centered at $\omega = 10\,\text{rad/s}$. This band is called the passband and has a bandwidth that can be measured using for instance the $-3\,\text{dB}$ (half-power) criterion.

(e)

i. For $R = 10\,\Omega$, and based on the analysis done in part (a), the circuit will still have a resonant frequency of $\omega_o = 10\,\text{rad/s}$. Plotting $|H(\omega)|$ on a linear scale for a resistor $R = 10\,\Omega$:

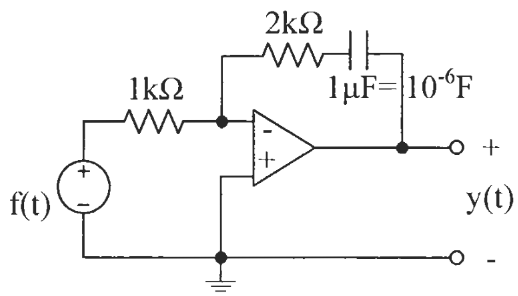


ii. For a $R = 0.1\,\Omega$ resistor, the resonant frequency will still be $\omega_o = 10\,\text{rad/s}$. Plotting $|H(\omega)|$ on a linear and a log-log scale for a resistor $R = 0.1\,\Omega$:



(f) From the results above, we can infer that a lower resistance corresponds to a narrower passband, which is to say that a higher resistance corresponds to a wider passband.

2. Determine the frequency response $H(\omega)$ of the following circuit.



Solution

The phasor equivalent circuit has an inductance impedance $j\omega L$ and a capacitor impedance $\frac{1}{j\omega C}$. Applying KCL,

$$\frac{F}{1k\Omega} = \frac{-Y}{2k\Omega + \frac{1}{j\omega C}}.$$

Therefore the frequency response of the system is,

$$H(\omega) = \frac{Y}{F} = -2 - \frac{1000}{j\omega}.$$

3. A linear system with input $f(t)$ and output $y(t)$ is described by the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y(t) = \frac{df}{dt} + 2\frac{d^2f}{dt^2}.$$

(a) Determine the frequency response $H(\omega)$ of the system.

(b) Determine and plot the magnitude response $|H(\omega)|$ for $0 < \omega < 20$ rad/s. You can use Matlab, Mathematica, etc.

(c) Determine if this filter is lowpass, bandpass, highpass, or none of these; and indicate why.

(d) Determine and plot the phase response $\angle H(\omega)$ for $0 < \omega < 20$ rad/s. You can use Matlab, Mathematica, etc.

Solution:

(a) We know that the time-varying signal corresponding to a phasor Y with frequency ω is

$$y(t) = \text{Re}\left\{Ye^{j\omega t}\right\},$$

and its corresponding first and second derivatives can be expressed as

$$\begin{aligned}
\frac{dy(t)}{dt} &= \mathrm{Re}\left\{Y\frac{d}{dt}\left(e^{j\omega t}\right)\right\} = \mathrm{Re}\left\{\underbrace{j\omega Y}_{\text{phasor of } \frac{dy}{dt}} e^{j\omega t}\right\} \\
\frac{d^2y(t)}{dt^2} &= \mathrm{Re}\left\{Y\frac{d^2}{dt^2}\left(e^{j\omega t}\right)\right\} = \mathrm{Re}\left\{\underbrace{(j\omega)^2 Y}_{\text{phasor of } \frac{d^2y}{dt^2}} e^{j\omega t}\right\}
\end{aligned}$$

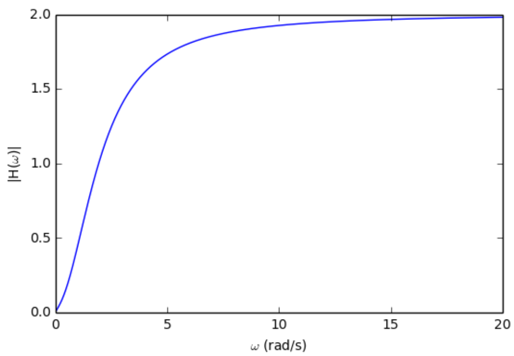
Writing the ODE in its phasor equivalent

$$(j\omega)^2Y + 4(j\omega)Y + 4Y = (j\omega)F + 2(j\omega)^2F,$$

yields

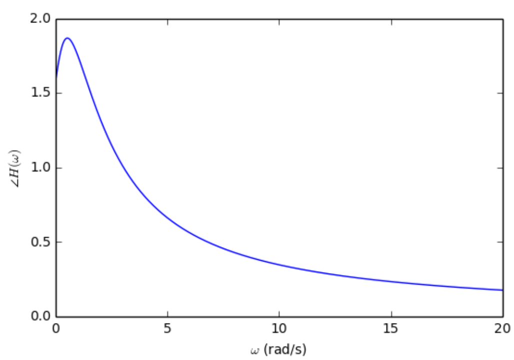
$$H(\omega) = \frac{Y}{F} = \frac{j\omega - 2\omega^2}{4 - \omega^2 + j\omega 4}.$$

(b) Plotting $|H(\omega)|$:



(c) The filter is a high-pass filter since at low frequency region, the signal is strongly attenuated.

(d) Plotting the phase angle of $\angle H(\omega)$:



4. A linear system with input $f(t)$ and output $y(t)$ is described by the ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = \frac{df}{dt}.$$

(a) Determine the frequency response $H(\omega)$ of the system.

(b) Determine and plot the magnitude response $|H(\omega)|$ for $0 < \omega < 20$ rad/s. You can use Matlab, Mathematica, etc.

(c) Determine if this filter is lowpass, bandpass, highpass, or none of these; and indicate why.

(d) Determine and plot the phase response $\angle H(\omega)$ for $0 < \omega < 20$ rad/s. You can use Matlab, Mathematica, etc.

Solution:

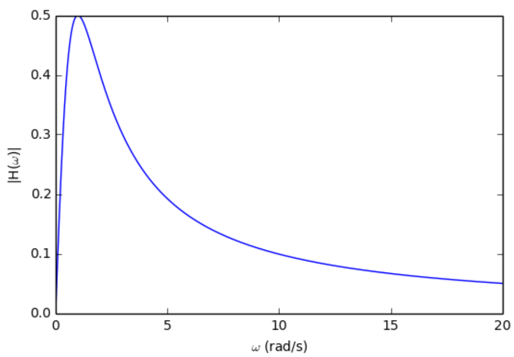
(a) Using the same method as the previous question, we have

$$H(\omega) = \frac{Y}{F} = \frac{j\omega}{1 - \omega^2 + j\omega 2}.$$

(b) Using the same method as the previous question, we have

$$\begin{aligned} |H(\omega)| &= \frac{|\omega|}{\sqrt{(1-\omega^2)^2 + (2\omega)^2)}} \\ &= \frac{|\omega|}{\sqrt{1 - 2\omega^2 + \omega^4 + 4\omega^2}} = \frac{|\omega|}{\sqrt{\omega^4 + 2\omega^2 + 1}} \\ &= \frac{|\omega|}{\omega^2 + 1}. \end{aligned}$$

Plotting the magnitude reponse $|H(\omega)|$, we have:



(c) The filter is a band-pass filter since at both low frequency region and high frequency region, the input is strongly attenuated.

(d) We approach the answer by two ways:

- Method 1: Phase of the numerator minus phase of the denominator:

$$\begin{aligned} \angle j\omega &= \begin{cases} \pi/2, & \text{for } \omega > 0, \\ -\pi/2, & \text{for } \omega < 0, \end{cases} = \frac{\omega}{|\omega|}\frac{\pi}{2} = \mathrm{sgn}(\omega)\frac{\pi}{2} \\ \angle\left(1 - \omega^2 + j\omega 2\right) &= \begin{cases} \arctan\left(\frac{2\omega}{1-\omega^2}\right), & \text{for } -1 < \omega < 1, \text{ (quadrants I \& II)} \\ \arctan\left(\frac{2\omega}{1-\omega^2}\right) + \mathrm{sgn}(\omega)\pi, & \text{for } |\omega| > 1, \text{ (quadrant II \& III).} \end{cases} \end{aligned}$$

Notice that an angle of $\pm\pi$ has been added for $|\omega| > 1$ to keep the angle in the quadrants II & III. The sign of ω, $\mathrm{sgn}(\omega)$ has been included to confine the angle in the range $[-\pi, \pi]$. Finally, the phase of the frequency response is given by

$$\begin{aligned} \angle H(\omega) &= \angle j\omega - \angle\left(1 - \omega^2 + j\omega 2\right) = \begin{cases} \mathrm{sgn}(\omega)\frac{\pi}{2} - \arctan\left(\frac{2\omega}{1-\omega^2}\right), & \text{for } -1 < \omega < 1, \\ -\mathrm{sgn}(\omega)\frac{\pi}{2} - \arctan\left(\frac{2\omega}{1-\omega^2}\right), & \text{for } |\omega| > 1, \end{cases} \\ &= \mathrm{sgn}\left(1 - |\omega|\right)\mathrm{sgn}(\omega)\frac{\pi}{2} - \arctan\left(\frac{2\omega}{1-\omega^2}\right) \end{aligned}$$

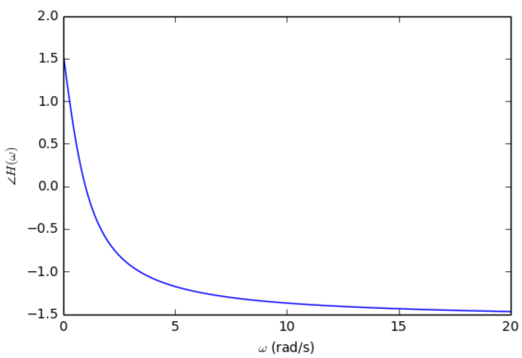
- Method 2: Multiplying the numerator and denominator of $H(\omega)$ by the conjugate of the denominator:

$$\begin{aligned} H(\omega) &= \frac{j\omega}{(1-\omega^2+j\omega 2)}\frac{(1-\omega^2-j\omega 2)}{(1-\omega^2-j\omega 2)} = \frac{2\omega^2 + j\omega\left(1-\omega^2\right)}{1 + 2\omega^2 + \left(\omega^2\right)^2} \\ &= \frac{2\omega^2 + j\omega(1-\omega^2)}{(\omega^2+1)^2}. \end{aligned}$$

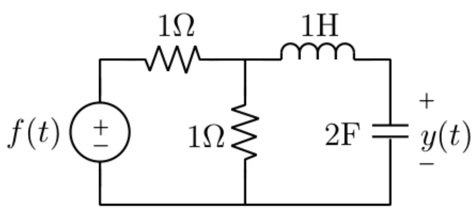
In this case, since the real part of $H(\omega)$ is always positive, Hence,

$$\angle H(\omega) = \arctan(\frac{1-\omega^2}{2\omega}).$$

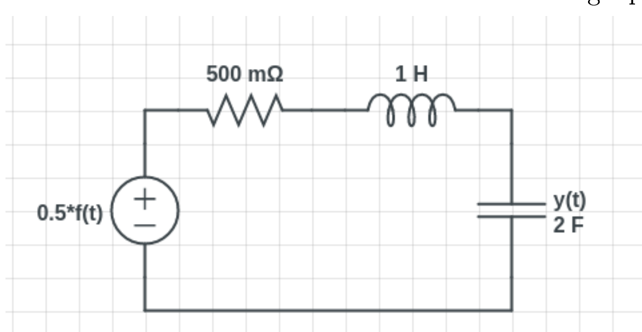
Plotting $\angle H(\omega)$, we have:



5. In the following circuit, the input is $f(t) = 4 + \cos(2t)$. Determine the steady-state output $y(t)$ of the circuit.



Solution:
The phasor equivalent circuit has an inductance impedance $j\omega L$ and a capacitor impedance $\frac{1}{j\omega C}$. Applying source transformations and we obtain the following equivalent circuit:



Applying voltage division we obtain

$$Y = \frac{F}{2}\frac{\frac{1}{j2\omega}}{\frac{1}{2} + j\omega + \frac{1}{j2\omega}},$$

which yields the frequency response of the system

$$H(\omega) = \frac{Y}{F} = \frac{\frac{1}{2}}{1 - 2\omega^2 + j\omega}.$$

We can break the input $f(t)$ into three components with frequencies 0 and 2 rad/s, and apply superposition to obtain the output $y(t)$, i.e.

$$y(t) = 4H(0) + |H(2)|\cos\left(t + \angle H(2)\right).$$

Hence, we need to find $H(0)$ and $H(2)$

$$\begin{aligned} H(0) &= \frac{1}{2} \\ H(2) &= \frac{\frac{1}{2}}{1 - 8 + 2j} = \frac{\frac{1}{2}}{-7 + 2j} = \frac{\frac{1}{2}}{\sqrt{53}e^{j(\pi - \arctan(\frac{2}{7}))}} = \frac{1}{2\sqrt{53}}e^{j(arctan(\frac{2}{7}) - \pi)}. \end{aligned}$$

Finally

$$y(t) = 2 + \frac{1}{2\sqrt{53}} \cos\left(2t + \arctan(\frac{2}{7}) - \pi\right) = 2 - \frac{1}{2\sqrt{53}} \cos(2t + \arctan(\frac{2}{7})).$$

6. Given an input $f(t) = 2e^{-j2t} + (2+j2)e^{-jt} + (2-j2)e^{jt} + 2e^{j2t}$ and $H(\omega) = \frac{1+j\omega}{2+j\omega}$ determine the steady-state response $y(t)$ of the system $H(\omega)$ and express it as a real valued signal.

Solution:

Applying the eigenfunction property ($e^{j\omega t}$ is the eigenfunction, and $H(\omega)$ is the eigenvalue) of LTI systems we can write the output as

$$y(t) = H(-2)2e^{-j2t} + H(-1)(2+j2)e^{-jt} + H(1)(2-j2)e^{jt} + H(2)2e^{j2t}.$$

We know that $H(\omega)$ is the frequency response of an LTI system. Hence it has the conjugate symmetry property:

$$\begin{aligned} H(-\omega) &= H^*(\omega) \\ &= |H(\omega)| e^{-j\angle H(\omega)} \end{aligned}$$

Therefore we can simplify our expression as

$$\begin{aligned} y(t) &= |H(2)|\, 2e^{-j(2t+\angle H(2))} + |H(1)|\, 2\sqrt{2}e^{-j\left(t-\frac{\pi}{4}+\angle H(1)\right)} \\ &\quad + |H(1)|\, 2\sqrt{2}e^{j\left(t-\frac{\pi}{4}+\angle H(1)\right)} + |H(2)|\, 2e^{j(2t+\angle H(2))} \\ &= 4\,|H(2)| \cos\left(2t + \angle H(2)\right) + 4\sqrt{2}\,|H(1)| \cos\left(t - \frac{\pi}{4} + \angle H(1)\right). \end{aligned}$$

Obtaining $H(2)$, and $H(1)$:

$$\begin{aligned} H(2) &= \frac{1+j2}{2+j2} = \frac{\sqrt{5}}{2\sqrt{2}}e^{j\left(\arctan(2)-\frac{\pi}{4}\right)} \\ H(1) &= \frac{1+j}{2+j} = \frac{\sqrt{2}}{\sqrt{5}}e^{j\left(\frac{\pi}{4}-\arctan(1/2)\right)}. \end{aligned}$$

Finally the output is

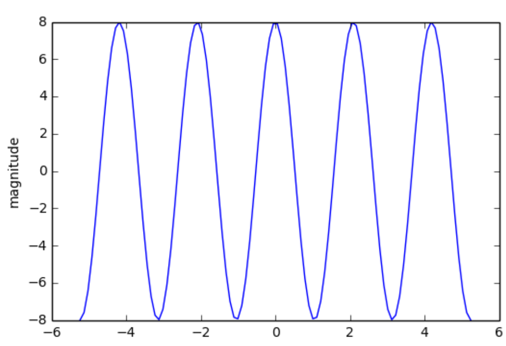
$$\begin{aligned} y(t) &= \sqrt{10}\cos\left(2t + \arctan(2) - \frac{\pi}{4}\right) + \frac{8}{\sqrt{5}}\cos\left(t - \arctan(1/2)\right) \\ &\approx 3.162\cos\left(2t + 0.322\right) + 3.578\cos\left(t - 0.464\right) \end{aligned}$$

(a) Consider the function $f(t) = Re\{4e^{j3t} + 4e^{-j3t}\}$. Find its period, T_0, its fundamental frequency, ω_o, and plot it over at least two periods.

Solution:

$$f(t) = Re\{4e^{j3t} + 4e^{-j3t}\} = 8\cos(3t)$$

Thus $\omega_o = 3\text{rad/s}$ and $T_0 = \frac{2\pi}{3}\text{s}$. By plotting $f(t)$, we have



7. For each one of the following functions of t, indicate whether they are periodic or not. If periodic, indicate its period, and if not periodic, indicate why. Assume n is a positive integer.

   (a) $\sin(t) + \sin\left(\frac{t}{2}\right) + \sin\left(\frac{t}{3}\right)$
   (b) $\sin(\pi t) + \cos\left(\sqrt{2}\,t\right)$
   (c) $\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{3\pi t}{2}\right) + \sin\left(\frac{2\pi t}{5}\right)$
   (d) $|\sin(nt)|$
   (e) $\cos(\pi t) + \cos\left(\frac{\pi}{n}t\right)$
   (f) $\cos(nt)\sin(nt)$

   Solution

   (a) Periodic, $T = LCM\{2\pi, 4\pi, 6\pi\} = 12\pi$
   (b) Not periodic.
   (c) Periodic, $T = LCM\{8, \frac{4}{3}, 5\} = 40$
   (d) Periodic, $T = \frac{1}{2} \times \frac{2\pi}{n} = \frac{\pi}{n}$
   (e) Periodic, $T = LCM\{2, 2n\} = 2n$
   (f) Periodic, $T = \frac{\pi}{n}$