Math 286 Differential Equations Plus (Prof. Honold) **Midterm Exam** 2021/04/01

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2 \colon I \to \mathbb{R}$ satisfying $y_1(t_0) = y_2(t_0)$ for some $t_0 \in I$?

   ☐ $y' = \sin(t\,y^2)$ ☐ $yy' = 0$ ☐ $y' = |ty|$ ☐ $y' = y\sqrt{t}$ ☐ $y' = \sqrt{|ty|}$

2. $(x^4y^2 - y)\,\mathrm{d}x + (x^2y^4 - x)\,\mathrm{d}y = 0$ has the integrating factor

   ☐ $1/(xy)$ ☐ 1 ☐ $1/(xy)^2$ ☐ $1/(xy^2)$ ☐ $1/(x^2y)$

3. The family of curves $y = c/x^2$, $c \in \mathbb{R}$ satisfies the ODE

   ☐ $\mathrm{d}y = x^{-2}\,\mathrm{d}x$ ☐ $\mathrm{d}y = 2x^{-3}\,\mathrm{d}x$ ☐ $2xy\,\mathrm{d}x + x^2\,\mathrm{d}y = 0$ ☐ $\mathrm{d}x = \mathrm{d}y$
   ☐ $2yx^{-3}\,\mathrm{d}x - x^{-2}\,\mathrm{d}y = 0$

4. For the solution $y(t)$ of the IVP $y' = y^3 - 7y + 6$, $y(0) = 0$ the limit $\lim_{t\to+\infty} y(t)$ equals

   ☐ -3 ☐ -2 ☐ -1 ☐ 1 ☐ 2

5. For the solution $y(t)$ of the IVP $y' = (y/t) - 1$, $y(1) = \ln 2$ the value $y(2)$ is equal to

   ☐ 0 ☐ 1 ☐ 2 ☐ $\ln 2$ ☐ $2\ln 2$

6. For the solution $y(t)$ of the IVP $y' = y^2\,\mathrm{e}^{-t}$, $y(0) = -1$ the value $y(-1)$ is equal to

   ☐ e ☐ $1/(\mathrm{e}-2)$ ☐ $\mathrm{e}+2$ ☐ $1/(\mathrm{e}+2)$ ☐ $\mathrm{e}-2$

7. For the solution $y\colon (0,+\infty) \to \mathbb{R}$ of the IVP $t^2y'' + 2t\,y' - 2y = 1$, $y(1) = 0$, $y'(1) = 1$ the value $y(2)$ is equal to

   ☐ $\frac{13}{8}$ ☐ 1 ☐ $\frac{13}{24}$ ☐ $\frac{19}{8}$ ☐ $\frac{19}{24}$

8. The power series $\sum_{k=1}^{\infty} 2^k z^{k^2}$ has radius of convergence

   ☐ 0 ☐ $\frac{1}{2}$ ☐ 1 ☐ 2 ☐ ∞

9. The smallest integer s such that $f_s(x) = \sum_{k=1}^{\infty} \frac{\cos(k^2x)}{k^s}$ is differentiable on $\mathbb{R}$ is equal to

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

10. If $y(t)$ solves $y' = t^2y + t\,y^2$ then $z = 1/y(t)$ solves

    ☐ $z' = -t^2z$ ☐ $z' = -z^2 + t$ ☐ $z' = t^2/z + t/z^2$ ☐ $z' = -z^2$
    ☐ $z' = -t^2z - t$

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11. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ of Picard-Lindelöf iterates for the IVP $\quad y' = y + 2t,\ \ y(0) = -2$ has $\phi_2(t)$ equal to

☐ $-2 + 2t + \frac{1}{3}t^3$ ☐ $-t^2 + \frac{1}{3}t^3$ ■ $-2 - 2t + \frac{1}{3}t^3$ ☐ $t^2 + \frac{1}{3}t^3$
☐ $1 + t + \frac{3}{2}t^2 + \frac{1}{3}t^3$

12. $y'' - 4y' + 4y = 2t + \mathrm{e}^{2t}$ has a particular solution $y_p(t)$ of the form

☐ $c_0 + c_1 t + c_2 \mathrm{e}^{2t}$ ■ $c_0 + c_1 t + c_2 t^2 \mathrm{e}^{2t}$ ■ $c_0 t + c_1 t^2 \mathrm{e}^{2t}$ ☐ $c_0 + c_1 t$
☐ $c_0 t + c_1 \mathrm{e}^{2t}$

13. Maximal solutions of $\quad y' = y^2 + y \quad$ satisfying $y(0) > 0 \quad$ are defined on an interval of the form

☐ (a, b) ☐ $[a, b]$ ☐ $(a, +\infty)$ ■ $(-\infty, b)$ ■ $(-\infty, +\infty)$
with $a, b \in \mathbb{R}$.

14. For $\quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad$ the matrix $\mathrm{e}^{\mathbf{A}t}$ is equal to

☐ $\begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}$ ■ $\begin{pmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \mathrm{e}^t & \mathrm{e}^{-t} \end{pmatrix}$ ☐ $\begin{pmatrix} \mathrm{e}^t & \mathrm{e}^t \\ \mathrm{e}^{-t} & \mathrm{e}^{-t} \end{pmatrix}$
☐ $\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$ ■ $\begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}$

15. The matrix norm of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is equal to

☐ 0 ☐ 1 ■ $\sqrt{2}$ ■ 2 ☐ 4

Time allowed: 60 min CLOSED BOOK ***Good luck!***

Notes

1. (A) and (D) are of the form $y' = f(t,y)$ with $\frac{\partial f}{\partial y}$ continuous; (C) is of the same form with $|f(t,y_1) - f(t,y_2)| = |t|\,(|y_1| - |y_2|) \leq |t|\,|y_1 - y_2| \leq L|y_1 - y_2|$, provided that $t \in [-L,L]$ (local Lipschitz condition w.r.t. y, neighborhoods can be taken as $[-L,L] \times \mathbb{R}$); (B) is in implicit form, so that the Existence and Uniqueness Theorem (EUT) doesn't apply, but obviously it's solutions are precisely the constant functions and cannot satisfy the said condition. To (E) the EUT doesn't apply as well (cp. the similar $y' = \sqrt{|y|}$ discussed in the lecture), and indeed $y_1 \equiv 0$ and $y_2(t) = \frac{1}{9}t^3$ are distinct solutions on $I = \mathbb{R}$ satisfying $y_1(0) = y_2(0) = 0$.

2. Multiplying the ODE with $1/(xy)^2$ turns it into $\left(x^2 - \frac{1}{x^2 y}\right)\mathrm{d}x + \left(y^2 - \frac{1}{xy^2}\right) = 0$, which is of the form $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$ with $P_y = 1/(xy)^2 = Q_x$, and hence exact. The other factors do not yield an ODE with $P_y = Q_x$.

3. $y(t) = c\big/x(t)^2$ is contained in the c-contour of $f(x,y) = x^2 y$ and hence (implicit differentiation!) satisfies $2x(t)y(t)x'(t) + x(t)^2 y(t)y'(t) = 0$, i.e., $2xy\,\mathrm{d}x + x^2 y\,\mathrm{d}y = 0$. The other ODE's aren't satisfied (at least not for all $c \in \mathbb{R}$).

4. The phaseline should be used to answer this question. We have $y^3 - 7y + 6 = (y-1)(y-2)(y+3)$ with roots $z_1 = -3$, $z_2 = 1$, $z_3 = 2$. Since $t_0 = 0 \in (z_1, z_2)$ and f is positive in (z_1, z_2), we must have $\lim_{\to +\infty} y(t) = z_2 = 1$; cf. lecture.

5. This is a 1st-order inhomogeneous linear ODE and can be solved with the standard method: The solution of the associated homogeneous ODE $y' = y/t$ is $y_h(t) = ct$, $c \in \mathbb{R}$, and variation of parameters gives $y(t) = t\int -t^{-1}\,\mathrm{d}t = -t\ln t + ct$ as general solution. The initial condition $y(1) = c = \ln 2$ determines $y(t) = (\ln 2)t - t\ln t$, which has $y(2) = 0$.

6. This is a separable ODE and can be solved with the standard method, giving $y(t) = 1/(\mathrm{e}^{-t} + C)$, $C \in \mathbb{R}$, as general solution. The initial condition $y(0) = -1$ gives $C = -2$, so that the solution of the IVP is $y(t) = 1/(\mathrm{e}^{-t} - 2)$, which has $y(-1) = 1/(\mathrm{e} - 2)$.

7. This is an inhomogeneous Euler equation with $\alpha = 2$, $\beta = -2$, and particular solution $y_p(t) = -1/2$. Since $r^2 + (\alpha - 1)r + \beta = r^2 + r - 2 = (r-1)(r+2)$ has roots $r_1 = 1$, $r_2 = -2$, the general (real) solution is $y(t) = c_1 t + c_2 t^{-2} - 1/2$ with $c_1, c_2 \in \mathbb{R}$. The given initial conditions give the system $c_1 + c_2 - 1/2 = 0$, $c_1 - 2c_2 = 1$, which is solved by $c_2 = -1/6$, $c_1 = 2/3$, so that $y(t) = (2/3)t - (1/6)t^{-2} - 1/2$ and $y(2) = 19/24$.

8. This is $\sum_{n=0}^{\infty} a_n z^n$ with $a_n = 2^k = 2^{\sqrt{n}}$ if $n = k^2$ is a perfect square and $a_n = 0$ otherwise. Since
$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[k^2]{2^k} = 2^{1/k} & \text{for } n = k^2, \\ 0 & \text{for } n \neq k^2, \end{cases}$$
and hence $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{k\to\infty} 2^{1/k} = 1$, the radius of convergence is 1.

9. For checking the differentiability of $f_s(x)$ one has to look at the series of derivatives, which is
$$\sum_{k=1}^{\infty} \frac{-\sin(k^2 x)}{k^{s-2}},$$
and prove that this series converges uniformly on $\mathbb{R}$ (or on all intervals of the form $[-R,R]$, $R > 0$). For $s = 4$ we can apply the Weierstrass test with $M_k = 1/k^2$ and conclude that the series of derivatives converges uniformly on $\mathbb{R}$. The Differentiation Theorem then gives that f_4 is differentiable. For $s = 3$ we obtain the ugly series $\sum_{k=1}^{\infty} \frac{\sin(k^2 x)}{k}$, and for $s = 2$ the even uglier $\sum_{k=1}^{\infty} \sin(k^2 x)$, which don't converge for some x (i.e., take $x = \pi/2$), let alone converge uniformly. Hence the Differentiation Theorem cannot be applied in these cases, and the only reasonable conclusion (if your professor isn't a bad guy) is that f_2 and f_3

are not differentiable at some points x. For f_0, f_1, which are not defined at $x = 0$, this is trivially true.

10. This is a Bernoulli equation with $n = 2$, $a(t) = t^2$, $b(t) = -t$, and can be transformed into a 1st-order linear ODE with the indicated substitution (cf. H17). If $y\colon I \to \mathbb{R}$ is a nonzero solution (a slight inaccuracy in the statement of the question), then $z(t) = 1/y(t)$ is defined for $t \in I$ and satisfies $z' = -y'/y^2 = -(t^2y + t\,y^2)/y^2 = -t^2/y - t = -t^2z - t$.

11. $\phi_0(t) = -2$; $\phi_1(t) = -2 + \int_0^t \phi_0(s) + 2s\,\mathrm{d}s = -2 + \int_0^t -2 + 2s\,\mathrm{d}s = -2 + \left[-2s + s^2\right]_0^t = -2 - 2t + t^2$; $\phi_2(t) = -2 + \int_0^t \phi_1(s) + 2s\,\mathrm{d}s = -2 + \int_0^t -2 + s^2\,\mathrm{d}s = -2 + \left[-2s + s^3/3\right]_0^t = -2 - 2t + t^3/3$.

12. The correct „Ansatz“ for obtaining a particular solution of $y'' - 4y' + 4y = 2t$ is $y(t) = c_0 + c_1t$, and that for obtaining a particular solution of $y'' - 4y' + 4y = \mathrm{e}^{2t}$ is $y(t) = c_2t^2\mathrm{e}^{2t}$ (because $\mu = 2$ is a root of multiplicity 2 of the characteristic polynomial $X^2 - 4X + 4 = (X-2)^2$). Superposition then gives a solution of $y'' - 4y' + 4y = 2t + \mathrm{e}^{2t}$ of the form $c_0 + c_1t + c_2t^2\,\mathrm{e}^t$. The actual particular solution found turns out to be $y_p(t) = \frac{1}{2}\left(1 + t + t^2\mathrm{e}^{2t}\right)$. Since the general solution then is $y(t) = y_p(t) + y_h(t) = \frac{1}{2}\left(1 + t + t^2\mathrm{e}^{2t}\right) + d_0\,\mathrm{e}^{2t} + d_1\,t\,\mathrm{e}^{2t}$, none of the other forms offered can yield a solution.

13. The equilibrium solutions of this autonomous ODE are $y \equiv 0$ and $y \equiv -1$. Integrating gives
$$\ln\left|\frac{y}{y+1}\right| = \int \left(\frac{1}{y} - \frac{1}{y+1}\right)\mathrm{d}y = \int \frac{\mathrm{d}y}{y(y+1)} = \int \mathrm{d}t = t + C.$$
A maximal solution y with $y(0) > 0$ must be > 0 everywhere and satisfy $\ln\frac{y}{y+1} = t + C$ for all points (t, y) on its graph. Since ln maps $(0,1) \to (-\infty, 0)$, it follows that y has domain $(-\infty, -C)$.

14. The matrix $\mathbf{A}$ satisfies $\mathbf{A}^2 = \mathbf{0}$.
$$\Longrightarrow\ \mathrm{e}^{\mathbf{A}t} = \mathbf{I}_2 + t\,\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}.$$

15. Since $|\mathbf{Ax}|^2 = (x_1 - x_2)^2 + (x_1 + x_2)^2 = 2(x_1^2 + x_2^2) = 2\,|\mathbf{x}|^2$, the map $\mathbf{x} \mapsto \mathbf{Ax}$ scales lengths by $\sqrt{2}$. This implies $\|\mathbf{A}\| = \sqrt{2}$.