Differential Equations Plus (Math 286)

H80 Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Use this to solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = (0, 1, 0)^\mathsf{T}$, and determine $\lim_{t\to+\infty} \mathbf{y}(t)$ for the solution.

H81 Determine a fundamental system of solutions of $\mathbf{y}' = \mathbf{B}\mathbf{y}$ for the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & -2 & -1 & 2 \\ 5 & -2 & -3 & -2 & 3 \\ 14 & 3 & -12 & -5 & 9 \\ 13 & 3 & -8 & -8 & 8 \\ 16 & 3 & -10 & -6 & 7 \end{pmatrix}$$

H82 Consider again the matrix $\mathbf{A}$ from H80. Determine the matrix eponential function $\mathrm{e}^{\mathbf{A}t}$in two ways,

a) using the fundamental matrix $\mathbf{\Phi}(t)$ obtained in H80 and the formula $\mathrm{e}^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$;

b) using the "new method" for computing $\mathrm{e}^{\mathbf{A}t}$ discussed in Lecture 51.

H83 Consider the matrix

$$\mathbf{M} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

a) Show, without using the characteristic polynomial, that the eigenvalues of $\mathbf{M}$ are purely imaginary and have absolute value 1 (hence must be i or −i). *Hint:* Compute $\mathbf{M}^2$ first.

b) Determine the eigenvalues of $\mathbf{M}$ and their multiplicities from the trace of $\mathbf{M}$.

H84 a) If $\mathbf{A} \in F^{n\times n}$ is invertible, show that

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}.$$

b) Show that similar matrices have the same characteristic polynomial (and hence the same trace, the same determinant, and the same eigenvalues with the same multiplicities).

H85 *Optional Exercise*

a) Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \ell_i(\mathbf{A}),$$

where $\ell_i(X) = \prod_{j=1, j\neq i}^{n} \frac{X-\lambda_j}{\lambda_i - \lambda_j}$ are the corresponding Lagrange polynomials.

b) Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}},$$

where λ_i is the eigenvalue corresponding to $\mathbf{v}_i$. (Note that the vectors $\mathbf{v}_i$ are column vectors, and hence $\mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}$ are $n \times n$ matrices of rank 1.)

c) The matrix considered in H 80 and H 83 satisfies both conditions. Use a) and b) to give two further evaluations of its matrix exponential function.

H86 *Optional Exercise*

Suppose that $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{T} \in \mathbb{C}^{n\times n}$ satisfy $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.

a) Setting $\mathbf{T} = \mathbf{P} + \mathrm{i}\mathbf{Q}$ with $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$, show that $\mathbf{P}, \mathbf{Q}$ satisfy the matrix equation $\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{B}$.

b) Show that any matrix $\mathbf{S} = \mathbf{P} + \lambda\mathbf{Q}$, $\lambda \in \mathbb{R}$, satisfies this matrix equation as well.

c) Show that that there exists $\lambda \in \mathbb{R}$ such that $\mathbf{P} + \lambda\mathbf{Q}$ is invertible.
Hint: Show that $\lambda \mapsto \det(\mathbf{P} + \lambda\mathbf{Q})$ is a polynomial function.

d) Use a), b), c) to show that there exists $\mathbf{S} \in \mathbb{R}^{n\times n}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

H87 *Optional Exercise*

a) Suppose that $\mathbf{A} \in \mathbb{R}^{2\times 2}$ has two distinct real eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Show that for $t \to \pm\infty$ the tangent unit vector $\frac{\mathbf{y}'(t)}{|\mathbf{y}'(t)|}$ of any non-constant solution $\mathbf{y}(t)$ of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ approaches the direction of one of the four rays $\mathbb{R}^+(\pm\mathbf{v}_1)$, $\mathbb{R}^+(\pm\mathbf{v}_2)$ (i.e., $\pm\frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\pm\frac{\mathbf{v}_2}{|\mathbf{v}_2|}$).

b) Work out the four possible cases (including the explicit determination of the rays) for the system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{y}.$$

Due on Wed Dec 22, 6 pm

Optional exercises this time should be handed in simultaneously with the mandatory exercises, if you want them graded. Alternatively, you can do optional exercises later and use the published solution to validate your own solution.

Solutions

80 The eigenvalues are the roots of the characteristic equation.

$$\det|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix}$$
$$= -(\lambda+1)^2(\lambda+2)+2(\lambda+1) = -(\lambda+1)(\lambda^2+3\lambda) = -\lambda(\lambda+1)(\lambda+3) = 0.$$

Thus the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = -3$. To find the eigenvectors, we replace λ by each of the eigenvalues in turn.
For $\lambda_1 = 0$,

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{x}^{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{x}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_3 = -3$,

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{x}^{(3)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Therefore, the ODE system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ has the general solution

$$y(t) = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3e^{-3t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Then consider the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = (0,1,0)^{\mathsf{T}}$:

$$y(0) = \begin{pmatrix} c_1+c_2+c_3 \\ c_1-2c_3 \\ c_1-c_2+c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\implies c_1 = \frac{1}{3},\ c_2 = 0,\ c_3 = -\frac{1}{3}$$

Therefore, the solution of the initial value problem is

$$y(t) = \begin{pmatrix} \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{1}{3} + \frac{2}{3}e^{-3t} \\ \frac{1}{3} - \frac{1}{3}e^{-3t} \end{pmatrix}.$$

Moreover,

$$\lim_{t\to+\infty} \mathbf{y}(t) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

81 First we compute the characteristic polynomial of $\mathbf{B}$. According to my experience it is better not to follow the Gaussian elimination strategy strictly but look for elementary row/column operations which generate more than one zero entry and for polynomial factors that can be taken out of the determinant.

$$
\begin{aligned}
\chi_{\mathbf{B}}(X) &= (-1)^5 \begin{vmatrix} -X & 1 & -2 & -1 & 2 \\ 5 & -2-X & -3 & -2 & 3 \\ 14 & 3 & -12-X & -5 & 9 \\ 13 & 3 & -8 & -8-X & 8 \\ 16 & 3 & -10 & -6 & 7-X \end{vmatrix} \\
&= - \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ -X^2-2X+5 & -2-X & -2X-7 & -4-X & 7+2X \\ 14+3X & 3 & -6-X & -2 & 3 \\ 13+3X & 3 & -2 & -5-X & 2 \\ 16+3X & 3 & -4 & -3 & 1-X \end{vmatrix} \\
&= \begin{vmatrix} -X^2-2X+5 & -2X-7 & -4-X & 7+2X \\ 14+3X & -6-X & -2 & 3 \\ 13+3X & -2 & -5-X & 2 \\ 16+3X & -4 & -3 & 1-X \end{vmatrix} \\
&= \begin{vmatrix} -X^2-2X+5 & -2X-7 & -4-X & 0 \\ 14+3X & -6-X & -2 & -3-X \\ 13+3X & -2 & -5-X & 0 \\ 16+3X & -4 & -3 & -3-X \end{vmatrix} \\
&= -(X+3) \begin{vmatrix} -X^2-2X+5 & -2X-7 & -4-X & 0 \\ 14+3X & -6-X & -2 & 1 \\ 13+3X & -2 & -5-X & 0 \\ 16+3X & -4 & -3 & 1 \end{vmatrix} \\
&= -(X+3) \begin{vmatrix} -X^2-2X+5 & -2X-7 & -4-X & 0 \\ -2 & -2-X & 1 & 0 \\ 13+3X & -2 & -5-X & 0 \\ 16+3X & -4 & -3 & 1 \end{vmatrix} \\
&= -(X+3) \begin{vmatrix} -X^2-2X+5 & -2X-7 & -4-X \\ -2 & -2-X & 1 \\ 13+3X & -2 & -5-X \end{vmatrix} \\
&= -(X+3) \begin{vmatrix} -X^2-4X-3 & -2X-7 & -4-X \\ 0 & -2-X & 1 \\ 3+X & -2 & -5-X \end{vmatrix} \\
&= -(X+3)^2 \begin{vmatrix} -X-1 & -2X-7 & -4-X \\ 0 & -2-X & 1 \\ 1 & -2 & -5-X \end{vmatrix} \\
&= -(X+3)^2 \begin{vmatrix} 0 & -4X-9 & -X^2-7X-9 \\ 0 & -2-X & 1 \\ 1 & -2 & -5-X \end{vmatrix} \\
&= -(X+3)^2 \begin{vmatrix} -4X-9 & -X^2-7X-9 \\ -2-X & 1 \end{vmatrix} \\
&= -(X+3)^2(-X^3-9X^2-27X-27) \\
&= (X+3)^5.
\end{aligned}
$$

$$\mathbf{S} = \begin{pmatrix} 1 & -1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 \\ 2 & -1 & -1 & -1 & -1 \\ 1 & 1 & -6 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 \end{pmatrix}$$

Hence $\lambda = -3$ is the only eigenvalue of $\mathbf{B}$ and has algebraic multiplicity 5.
Next we compute the first few powers of $\mathbf{B} - \lambda\mathbf{I} = \mathbf{B} + 3\mathbf{I}$.

$$\mathbf{B} + 3\mathbf{I} = \begin{pmatrix} 3 & 1 & -2 & -1 & 2 \\ 5 & 1 & -3 & -2 & 3 \\ 14 & 3 & -9 & -5 & 9 \\ 13 & 3 & -8 & -5 & 8 \\ 16 & 3 & -10 & -6 & 10 \end{pmatrix},$$

$$(\mathbf{B} + 3\mathbf{I})^2 = \begin{pmatrix} 5 & 1 & -3 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 10 & 2 & -6 & -4 & 6 \\ 5 & 1 & -3 & -2 & 3 \\ 5 & 1 & -3 & -2 & 3 \end{pmatrix},$$

$$(\mathbf{B} + 3\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\Longrightarrow$ A fundamental matrix of $\mathbf{y}' = \mathbf{B}\mathbf{y}$ is

$$\begin{aligned}
\mathbf{\Phi}(t) &= \mathrm{e}^{-3t}\mathbf{I} + t\,\mathrm{e}^{-3t}(\mathbf{B} + 3\mathbf{I}) + \tfrac{1}{2}t^2\mathrm{e}^{-3t}(\mathbf{B} + 3\mathbf{I})^2 \\
&= \begin{pmatrix}
\mathrm{e}^{-3t} + 3t\,\mathrm{e}^{-3t} + \frac{5}{2}t^2\mathrm{e}^{-3t} & t\,\mathrm{e}^{-3t} + \frac{1}{2}t^2\mathrm{e}^{-3t} & -2t\,\mathrm{e}^{-3t} - \frac{3}{2}t^2\mathrm{e}^{-3t} & -t\,\mathrm{e}^{-3t} - t^2\mathrm{e}^{-3t} & 2t\,\mathrm{e}^{-3t} + \frac{3}{2}t^2\mathrm{e}^{-3t} \\
5t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t} + t\,\mathrm{e}^{-3t} & -3t\,\mathrm{e}^{-3t} & -2t\,\mathrm{e}^{-3t} & 3t\,\mathrm{e}^{-3t} \\
14t\,\mathrm{e}^{-3t} + 5t^2\mathrm{e}^{-3t} & 3t\,\mathrm{e}^{-3t} - t^2\mathrm{e}^{-3t} & \mathrm{e}^{-3t} - 9t\,\mathrm{e}^{-3t} - 3t^2\mathrm{e}^{-3t} & -5t\,\mathrm{e}^{-3t} - 2t^2\mathrm{e}^{-3t} & 9t\,\mathrm{e}^{-3t} + 3t^2\mathrm{e}^{-3t} \\
13t\,\mathrm{e}^{-3t} + \frac{5}{2}t^2\mathrm{e}^{-3t} & 3t\,\mathrm{e}^{-3t} + \frac{1}{2}t^2\mathrm{e}^{-3t} & -8t\,\mathrm{e}^{-3t} - \frac{3}{2}t^2\mathrm{e}^{-3t} & \mathrm{e}^{-3t} - 5t\,\mathrm{e}^{-3t} - t^2\mathrm{e}^{-3t} & 8t\,\mathrm{e}^{-3t} + \frac{3}{2}t^2\mathrm{e}^{-3t} \\
16t\,\mathrm{e}^{-3t} + \frac{5}{2}t^2\mathrm{e}^{-3t} & 3t\,\mathrm{e}^{-3t} + \frac{1}{2}t^2\mathrm{e}^{-3t} & -10t\,\mathrm{e}^{-3t} - \frac{3}{2}t^2\mathrm{e}^{-3t} & -6t\,\mathrm{e}^{-3t} - t^2\mathrm{e}^{-3t} & \mathrm{e}^{-3t} + 10t\,\mathrm{e}^{-3t} + \frac{3}{2}t^2\mathrm{e}^{-3t}
\end{pmatrix}.
\end{aligned}$$

For this note that the generalized eigenspace for $\lambda = -3$ is the whole of $\mathbb{C}^5$, and the columns of $\mathbf{\Phi}(t)$ contain the solutions $\mathbf{y}_j(t) = \sum_{k=0}^{4}(\mathbf{B} + 3\mathbf{I})^k\mathbf{e}_j$. (The sums terminate early, since $(\mathbf{B} + 3\mathbf{I})^k = 0$ for $k \geq 3$.)
Remark: Since $\mathbf{\Phi}(0) = \mathbf{I}_5$, we have in fact $\mathbf{\Phi}(t) = \mathrm{e}^{\mathbf{B}t}$. This can also be seen as follows

$$\begin{aligned}
\mathrm{e}^{\mathbf{B}t} &= \mathrm{e}^{-3t\mathbf{I}}\mathrm{e}^{(\mathbf{B}+3\mathbf{I})t} && \text{(Since } -3t\mathbf{I} \text{ and } \mathbf{B} + 3\mathbf{I})t \text{ commute)} \\
&= \mathrm{e}^{-3t}\left(\mathbf{I} + t(\mathbf{B} + 3\mathbf{I})t + \tfrac{1}{2}t^2(\mathbf{B} + 3\mathbf{I})^2\right). && \text{(Since } (\mathbf{B} + 3\mathbf{I})^3 = \mathbf{0}\text{)}
\end{aligned}$$

Alternatively, a fundamental system can be obtained by using "chains" of generalized eigenvectors as discussed in the lecture. Since $(\mathbf{B} + 3\mathbf{I})^2\mathbf{e}_2 \neq \mathbf{0}$, say, we can use the chain $\mathbf{e}_2$, $(\mathbf{B} + 3\mathbf{I})\mathbf{e}_2 = (1, 1, 3, 3, 3)^\mathsf{T}$, $(\mathbf{B} + 3\mathbf{I})^2\mathbf{e}_2 = (1, 0, 2, 1, 1,)^\mathsf{T}$ to obtain the three fundamental solutions

$$\begin{aligned}
\mathbf{y}_1(t) &= \mathrm{e}^{-3t}\mathbf{e}_2 + t\,\mathrm{e}^{-3t}(\mathbf{B}+3\mathbf{I})\mathbf{e}_2 + t^2\mathrm{e}^{-3t}(\mathbf{B}+3\mathbf{I})^2\mathbf{e}_2 \\
&= \mathrm{e}^{-3t}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + t\,\mathrm{e}^{-3t}\begin{pmatrix}1\\1\\3\\3\\3\end{pmatrix} + \tfrac{1}{2}t^2\mathrm{e}^{-3t}\begin{pmatrix}1\\0\\2\\1\\1\end{pmatrix}, \\
\mathbf{y}_2(t) &= \mathrm{e}^{-3t}\begin{pmatrix}1\\1\\3\\3\\3\end{pmatrix} + t\,\mathrm{e}^{-3t}\begin{pmatrix}1\\0\\2\\1\\1\end{pmatrix}, \\
\mathbf{y}_3(t) &= \mathrm{e}^{-3t}\begin{pmatrix}1\\0\\2\\1\\1\end{pmatrix}.
\end{aligned}$$

Now there are two possibilities: Either the eigenspace V_{-3} is 3-dimensional, in which case two further fundamental solutions of the form $\mathbf{y}(t) = \mathrm{e}^{-3t}\mathbf{v}$ can be found, or V_{-3} is 2-dimensional and there exists a vector $\mathbf{w}$ such that $(\mathbf{B}+3\mathbf{I})\mathbf{w} \in V_{-3}$ is linearly independent of the first eigenvector $(1,0,2,1,1)^\mathsf{T}$, in which case $\mathbf{y}_4(t) = \mathrm{e}^{-3t}\mathbf{w} + t\,\mathrm{e}^{-3t}(\mathbf{B}+3\mathbf{I})\mathbf{w}$, $\mathbf{y}_5(t) = \mathrm{e}^{-3t}(\mathbf{B}+3\mathbf{I})\mathbf{w}$ are two further fundamental solutions. By computing the rank of $\mathbf{B}+3\mathbf{I}$ we can decide which of the two possibilities holds:

$$\begin{pmatrix}3&1&-2&-1&2\\5&1&-3&-2&3\\14&3&-9&-5&9\\13&3&-8&-5&8\\16&3&-10&-6&10\end{pmatrix} \to \begin{pmatrix}3&1&-2&-1&2\\2&0&-1&-1&1\\5&0&-3&-2&3\\4&0&-2&-2&2\\7&0&-4&-3&4\end{pmatrix} \to \begin{pmatrix}3&1&-2&-1&2\\2&0&-1&-1&1\\-1&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$\Longrightarrow$ $\mathrm{rk}(\mathbf{B}+3\mathbf{I}) = 3$ and the 2nd possibility holds. A suitable vector $\mathbf{w}$ is then easy to find: We must pick $\mathbf{w}$ from the right kernel of $(\mathbf{B}+3\mathbf{I})^2$, which has rank 1, and check that $(\mathbf{B}+3\mathbf{I})\mathbf{w} \notin \mathbb{R}(1,0,2,1,1)^\mathsf{T}$. A valid choice is, e.g., $\mathbf{w} = (0,2,0,1,0)^\mathsf{T}$, for which $(\mathbf{B}+3\mathbf{I})\mathbf{w} = (1,0,1,1,0)^\mathsf{T}$. (As a sanity check, obeserve that $(1,0,1,1,0)^\mathsf{T}$ indeed solves $(\mathbf{B}+3\mathbf{I})\mathbf{x} = \mathbf{0}$.)

$\Longrightarrow$ Two further fundamental solutions are

$$\begin{aligned}
\mathbf{y}_4(t) &= \mathrm{e}^{-3t}\begin{pmatrix}0\\2\\0\\1\\0\end{pmatrix} + t\,\mathrm{e}^{-3t}\begin{pmatrix}1\\0\\1\\1\\0\end{pmatrix}, \\
\mathbf{y}_5(t) &= \mathrm{e}^{-3t}\begin{pmatrix}1\\0\\1\\1\\0\end{pmatrix}.
\end{aligned}$$

Remark: The matrix $\mathbf{S}$ with columns $\mathbf{e}_2$, $(\mathbf{B}+3\mathbf{I})\mathbf{e}_2$, $(\mathbf{B}+3\mathbf{I})^2\mathbf{e}_2$, $\mathbf{w}$, $(\mathbf{B}+3\mathbf{I})\mathbf{w}$ in this order, i.e.,

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 3 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 3 & 1 & 0 & 0 \end{pmatrix},$$

transforms $\mathbf{A}$ into Jordan canonical form, i.e., $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$ with

$$\mathbf{J} = \left(\begin{array}{ccc|cc} -3 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right).$$

82 a) From the solution of H 80 we have

$$\mathbf{\Phi}(t) = \begin{pmatrix} 1 & \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ 1 & 0 & -2\mathrm{e}^{-3t} \\ 1 & -\mathrm{e}^{-t} & \mathrm{e}^{-3t} \end{pmatrix}, \quad \mathbf{\Phi}(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \quad \mathbf{\Phi}(0)^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

$$\Longrightarrow \mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 1 & \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ 1 & 0 & -2\mathrm{e}^{-3t} \\ 1 & -\mathrm{e}^{-t} & \mathrm{e}^{-3t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{3} + \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} \\ \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3} + \frac{2}{3}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} \\ \frac{1}{3} - \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3} + \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} \end{pmatrix}$$

b) We apply the new method with $a(X) = \chi_{\mathbf{A}}(X) = X(X+1)(X+3)$; cf. the solution of H 80. A fundamental system of solutions of $\chi_{\mathbf{A}}(\mathrm{D})y = 0$ is 1, e^{-t}, e^{-3t}, which has Wronski matrix

$$\mathbf{W}(t) = \begin{pmatrix} 1 & \mathrm{e}^{-t} & \mathrm{e}^{-3t} \\ 0 & -\mathrm{e}^{-t} & -3\,\mathrm{e}^{-3t} \\ 0 & \mathrm{e}^{-t} & 9\,\mathrm{e}^{-3t} \end{pmatrix}.$$

The special fundamental system $c_0(t)$, $c_1(t)$, $c_2(t)$ required for the computation of $\mathrm{e}^{\mathbf{A}t}$ is obtained by multiplying the first row of $\mathbf{W}(t)$ with $\mathbf{W}(0)^{-1}$. The standard method for matrix inversion gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 6 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{6} & \frac{1}{6} \end{array}\right].$$

$$\Longrightarrow \big(c_0(t), c_1(t), c_2(t)\big) = \big(1, \mathrm{e}^{-t}, \mathrm{e}^{-3t}\big) \begin{pmatrix} 1 & \frac{4}{3} & \frac{1}{3} \\ 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$
$$= \big(1, \tfrac{4}{3} - \tfrac{3}{2}\mathrm{e}^{-t} + \tfrac{1}{6}\mathrm{e}^{-3t}, \tfrac{1}{3} - \tfrac{1}{2}\mathrm{e}^{-t} + \tfrac{1}{6}\mathrm{e}^{-3t}\big)$$

$$\implies \mathrm{e}^{\mathbf{A}t} = c_0(t)\mathbf{I}_3 + c_1(t)\mathbf{A} + c_2(t)\mathbf{A}^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(\tfrac{4}{3} - \tfrac{3}{2}\mathrm{e}^{-t} + \tfrac{1}{6}\mathrm{e}^{-3t}\right) \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} + \left(\tfrac{1}{3} - \tfrac{1}{2}\mathrm{e}^{-t} + \tfrac{1}{6}\mathrm{e}^{-3t}\right) \begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} + \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} \\ \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3} + \frac{2}{3}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} \\ \frac{1}{3} - \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} & \frac{1}{3} - \frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3} + \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{-3t} \end{pmatrix}.$$

The computation is facilitated by the symmetry properties of $\mathbf{I}_3$, $\mathbf{A}$, $\mathbf{A}^2$, which are all of the form $\left(\begin{smallmatrix} a & b & c \\ b & d & b \\ c & b & a \end{smallmatrix}\right)$, leaving only 4 entries of $\mathrm{e}^{\mathbf{A}t}$ to be determined.

83 a) The eigenvalues λ and corresponding eigenvectors $\mathbf{x}$ satisfy the equation

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x}.$$

Applying $\mathbf{M}$ to both sides of the equation gives

$$\mathbf{M}^2\mathbf{x} = \mathbf{M}(\mathbf{M}\mathbf{x}) = \mathbf{M}\lambda\mathbf{x} = \lambda\mathbf{M}\mathbf{x} = \lambda^2\mathbf{x}.$$

On the other hand, we have

$$\mathbf{M}^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\mathbf{I}_4.$$

Hence $\mathbf{M}^2\mathbf{x} = -\mathbf{x}$. (This holds for all $\mathbf{x} \in \mathbb{C}^4$.) Putting both equations together gives $(\lambda^2 + 1)\mathbf{x} = \mathbf{0}$. Since eigenvectors are non-zero, this implies $\lambda^2 + 1 = 0$, so that the possible eigenvalues of $\mathbf{M}$ are $\pm\mathrm{i}$.

b) Set $\lambda_1 = \mathrm{i}$, $\lambda_2 = -\mathrm{i}$ and let m_1, m_2 be the corresponding algebraic multiplicities as eigenvalues of $\mathbf{M}$. (If λ_i is not an eigenvalue of $\mathbf{M}$ then $m_i = 0$.) Then $m_1 + m_2 = 4$ and

$$0 = \operatorname{trace}(\mathbf{M}) = m_1\mathrm{i} + m_2(-\mathrm{i}) = (m_1 - m_2)\mathrm{i}.$$

$\implies m_1 = m_2 = 2.$

Remarks: The matrix $\mathbf{M}$ is both skew-symmetric and orthogonal, giving $\mathbf{I} = \mathbf{M}\mathbf{M}^\mathsf{T} = \mathbf{M}(-\mathbf{M}) = -\mathbf{M}^2$. From this one sees that skew-symmetric, orthogonal matrices are characterized by the matrix equation $\mathbf{M}^2 = -\mathbf{I}$. Such matrices are diagonalizable, since their minimum polynomial divides $X^2 + 1$ and hence is squarefree. So in the special case under consideration the geometric multiplicities of the eigenvalues are also equal to 2.

Alternatively, one can argue as follows: Since $\mathbf{M}$ is real, its non-real eigenvalues must occur in complex conjugate pairs, which have the same (algebraic or geometric) multiplicity. Thus the algebraic multiplicity of i and $-$i must be 2. For diagonalizability one needs stronger assumptions (e.g., "skew-symmetric" or "orthogonal") as the example

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

shows, which is not diagonalizable.

84 a) From $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$ and the multiplicativity of the determinant we have

$$\det(\mathbf{A})\det(\mathbf{A}^{-1}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{I}_n) = 1.$$

This implies the stated formula.

b) If $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ then

$$X\,\mathbf{I}_n - \mathbf{B} = X\,\mathbf{I}_n - \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{S}^{-1}(X\,\mathbf{I}_n)\mathbf{S} - \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{S}^{-1}(X\,\mathbf{I}_n - \mathbf{A})\mathbf{S}.$$

$$\Longrightarrow \det(X\,\mathbf{I}_n - \mathbf{B}) = \det(\mathbf{S}^{-1})\det(X\,\mathbf{I}_n - \mathbf{A})\det(\mathbf{S}) = \frac{1}{\det(\mathbf{S})}\det(X\,\mathbf{I}_n - \mathbf{A})\det(\mathbf{S}) = \det(X\,\mathbf{I}_n - \mathbf{A}).$$

This says $\chi_{\mathbf{B}}(X) = \chi_{\mathbf{A}}(X)$.

Remark: The computation uses matrices with entries in the polynomial ring $F[X]$. If you wonder whether all the matrix arithmetic you have learned remains valid in this more general setting, here is a hopefully convincing argument that this is true: The usual laws for matrix arithmetic hold for matrices over any field in place of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. (This fact you have to accept.) The ring $F[X]$ can be embedded in the field $F(X)$ of rational functions in one indeterminate over F (analogous to the embedding of $\mathbb{Z}$ into $\mathbb{Q}$), and hence the above computations involving matrices over $F[X]$ are just instances of matrix arithmetic over the field $F(X)$. For the same reason, when computing a characteristic polynomial $\chi_{\mathbf{A}}(X) = \det(X\mathbf{I} - \mathbf{A})$, we can do Gaussian elimination over $F(X)$, i.e., as the first step add $\frac{a_{21}}{X-a_{11}}\times$ the first row to the 2nd row, which eliminates the $(2,1)$ entry. However, it is usually better to avoid proper rational functions as entries (just like determinnats of integral matrices are best computed with pure integer arithmetic).