Differential Equations Plus (Math 286)

H20 Find integrating factors for the following ODE's and determine their integral curves.

a) $e^x(x+1)\,dx + (y\,e^y - x\,e^x)\,dy = 0$;

b) $y(y+2x+1)\,dx - x(2y+x-1)\,dy = 0$.

H21 An ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be *homogeneous* if M and N are homogeneous functions of the same degree, i.e., there exists $d \in \mathbb{R}$ such that $M(\lambda x, \lambda y) = \lambda^d M(x,y)$ and $N(\lambda x, \lambda y) = \lambda^d N(x,y)$ for all x, y, and λ.

a) Show that the substitution $z = y/x$ (or $z = x/y$) transforms any homogeneous ODE into a separable ODE.

b) Solve the following ODE's in implicit form (answering two of (i)–(iii) suffices):

(i) $(x+y)\,dx - (x+2y)\,dy = 0$; (ii) $(x-2y)\,dx + y\,dy = 0$;
(iii) $(x^2+y^2)\,dx + 3xy\,dy = 0$; (iv) $(x-y-1)\,dx + (x+4y-6)\,dy = 0$.

H22 Analyze the alternative model $dy/dt = ay - by^2 - Ey$ $(a,b,E > 0)$ for harvesting a population (individuals are removed at a rate proportional to the current size of the population). Which rates E are sustainable? How to choose E in order to maximize the *yield* Ey in the long run?

H23 a) Assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ without resorting to the evaluation of $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$.
Hint: Add the two series.

b) Show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

H24 Evaluate the two series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} \quad \text{for } x \in \mathbb{R},$$

in a way similar to the evaluation of $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ in the lecture, and use this in turn to evaluate the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} \pm \cdots,$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \cdots$$

H25 *Optional exercise*

For $s \in \mathbb{C}$ consider the *binomial series*

$$\mathrm{B}_s(z) = \sum_{n=0}^{\infty} \binom{s}{n} z^n = \sum_{n=0}^{\infty} \frac{s(s-1)\cdots(s-n+1)}{1 \cdot 2 \cdots n} z^n.$$

a) Show that for $s \notin \{0, 1, 2, \dots\}$ the binomial series has radius of convergence $R = 1$.

b) Show that $\mathrm{B}_s(x) = (1+x)^s$ for $s \in \mathbb{C}$ and $-1 < x < 1$.
Hint: $x \mapsto (1+x)^s = \mathrm{e}^{s \ln(1+x)}$ is a solution of the IVP $y' = \frac{s}{1+x} y$, $y(0) = 1$. Show that the same is true of $x \mapsto \mathrm{B}_s(x)$; cf. also [Ste16], Ch. 11.10, Ex. 85.

c) Show $\mathrm{B}_s(z) = (1+z)^s$ for $s, z \in \mathbb{C}$ with $|z| < 1$.
Hint: Probably the easiest way to solve this part is to use the same idea as in b): Show that $z \mapsto \mathrm{B}_s(z)$ and $z \mapsto (1+z)^s = \mathrm{e}^{s \log(1+z)}$ both satisfy $y' = \frac{s}{1+z} y$ for $|z| < 1$ and $y(0) = 1$, and that the solution of this complex IVP is unique. Since we haven't discussed complex differentiation and ODE's in any depth, it is important that you justify carefully every step of your solution.

H26 *Optional exercise*

The system of equations

$$\begin{aligned} x &= 0.01\,x^2 + \sin(y) \\ y &= \cos(x) + 0.01\,y^2 \end{aligned}$$

has a unique solution (x^*, y^*) with $0.5 \leq x^* \leq 1$, $\pi/6 \leq y^* \leq 1$. Prove this statement and compute (x^*, y^*)

a) with simple fixed-point iteration, i.e., by reading the system as $(x, y) = T(x, y)$ and using the iteration $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$;

b) with Newton Iteration.

Indicate the speed of convergence of the two iterations.

Due on Fri Mar 18, 6 pm

Exercises H25 and H26 can be handed in until Fri Mar 25, 6 pm. Contractions of metric spaces and Banach's Fixed Point Theorem (required for H26) will be discussed in the lectures of next week.