# Differential Equations Plus (Math 286)

**H1** We have considered the ODE $y' = -x/y$ as an example in the lecture. Actually there are four ODE's, viz. $y' = \pm x/y$ and $y' = \pm y/x$, which look very similar. Draw direction fields for the other three ODE's and determine their solutions in both implicit and explicit form (if possible).

**H2** Determine all points $(t_0, y_0) \in \mathbb{R}^2$ such that there is a unique solution on $[t_0, \infty)$ of the IVP $y' = \sqrt{|y|}$, $y(t_0) = y_0$ ("the value at time $t_0$ determines the values at all future times $t > t_0$").

**H3** Let $t_0, y_0, y_1 \in \mathbb{R}$. Show that the IVP

$$y'' = -y, \quad y(t_0) = y_0,\ y'(t_0) = y_1$$

has a unique solution.

**H4** For each of the following ODE's, determine at least one nonzero solution by using the "Ansatz" $y(x) = a\,\mathrm{e}^{\alpha x}$ or $y(x) = b\,x^{\beta}$.

a) $y'' = y^2$;
b) $y'' - 5y' + 6y = 0$;
c) $y'' - 5y' + 6y = \mathrm{e}^x$;
d) $y'' - \frac{1}{2x}\,y' + \frac{1}{2x^2}\,y = 0$;
e) $(2x+1)y'' + (4x-2)y' - 8y = 0$;
f) $x^2(1-x)y'' + 2x(2-x)y' + 2(1+x)y = 0$.

**H5** Do two of the three exercises on the pendulum equation in [BDM17], Ch. 1.3 (Exercises 23–25 in the 11th global edition).

## Due on Fri Sep 24, 6 pm

**Instructions** For your homework it is best to maintain 2 notebooks, which are handed in on alternate Fridays. You may also use A4 sheets, provided they are firmly stapled together.

Don't forget to write your name (English and Chinese) and your student ID on the first page.

Homework is handed in on Fridays before the discussion session starts (late homework won't be accepted!) and will be returned on the next Friday.

Answers to exercises must be justified; it is not sufficient to state only the final result of a computation.

Answers must be written in English.

For a full homework score it is sufficient to solve ca. 80 % of the mandatory homework exercises. Optional exercises contribute to the homework score, but they are usually more difficult and you should work on them only if you have sufficient spare time.

# Solutions (prepared by TA's and TH)

**1** (A) $y' = x/y$: Rewriting the ODE as $yy' - x = 0$ and integrating gives $\frac{1}{2}\,y^2 - \frac{1}{2}\,x^2 = C \in \mathbb{R}$. Replacing

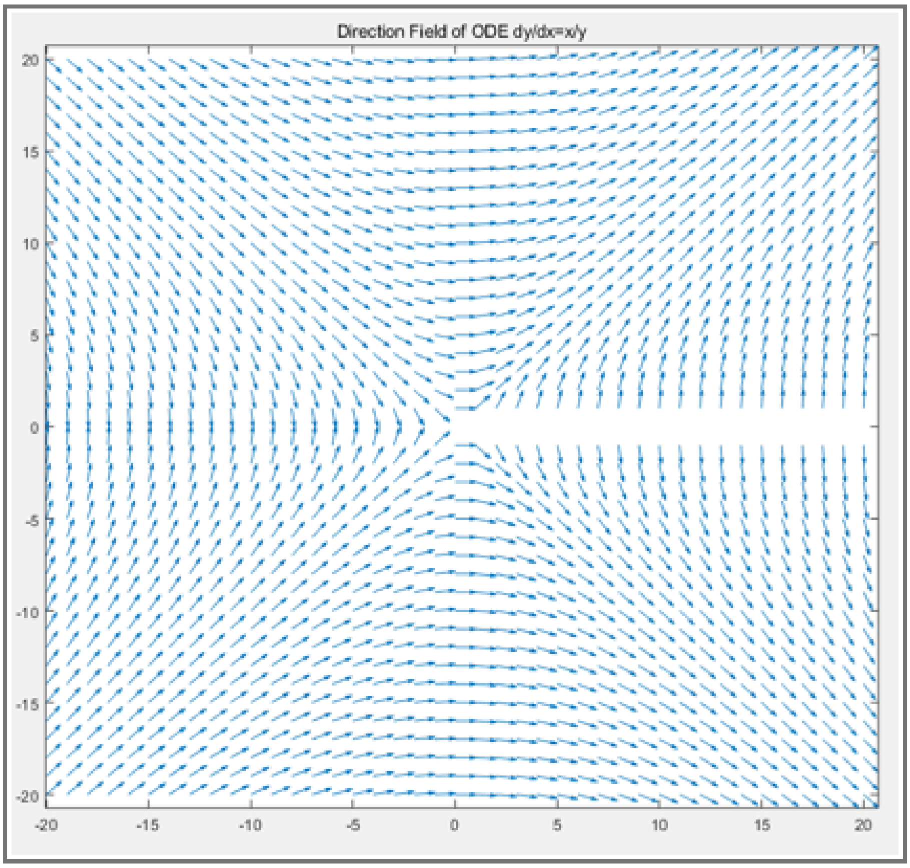


Figure 1: Direction field of $y' = x/y$

$C$ by $C/2$ turns this into $y^2 - x^2 = C$ (implicit form), $y(x) = \pm\sqrt{x^2 + C}$ (explicit form). Explicit solutions have domain $\mathbb{R}$ if $C > 0$ and domains $\left(-\infty, -\sqrt{-C}\right)$, $\left(\sqrt{-C}, \infty\right)$ if $C \leq 0$. (For $C = 0$ the solutions, viz. $y(x) = \pm x$, formally are not defined at 0, since the domain of $y' = x/y$ excludes the $x$-axis.)

(B) $y' = y/x$: The solutions are $y(x) = Cx$, $C \in \mathbb{R}$, with domains $(-\infty, 0)$, $(0, \infty)$. Again the exclusion of $x = 0$ is artificial and due to the special form of the ODE. (It would be included if we rewrite the ODE as $xy' - y = 0$.) That all solutions have been found, follows from $(y/x)' = (xy' - y)/x^2 = 0$, which implies $y/x = C$ is a constant.

(C) $y' = -y/x$: The solutions are $y(x) = C/x$, $C \in \mathbb{R}$, with domains $(-\infty, 0)$, $(0, \infty)$. That these are all solutions follows from $(xy)' = y + xy' = 0$, which implies $xy = C$ is a constant.

**2** As discussed in the lecture, the solutions of $y' = \sqrt{|y|}$ form the 2-parameter family

$$y(t) = \begin{cases} -\frac{1}{4}(t - c_1)^2 & \text{if } t < c_1, \\ 0 & \text{if } c_1 \leq t \leq c_2, \\ \frac{1}{4}(t - c_2)^2 & \text{if } t > c_2. \end{cases}$$

One or both of $c_1, c_2$ may be infinite ($c_1 = -\infty$, $c_2 = \infty$).
*Claim:* Solutions are uniquely determined for all $t \geq t_0$ iff $y_0 = y(t_0) > 0$.
If $y_0 > 0$ then near $t_0$ the solution must have the form $y(t) = \frac{1}{4}(t - c_2)^2$ with $c_2$ determined from $\frac{1}{4}(t_0 - c_2)^2 = y_0$, i.e, $t_0 - c_2 = 2\sqrt{y_0}$ and $c_2 = t_0 - 2\sqrt{y_0}$. (Note that $c_2$ must be smaller than $t_0$ in this case.) Thus $y(t)$ is determined for all $t \geq t_0$.

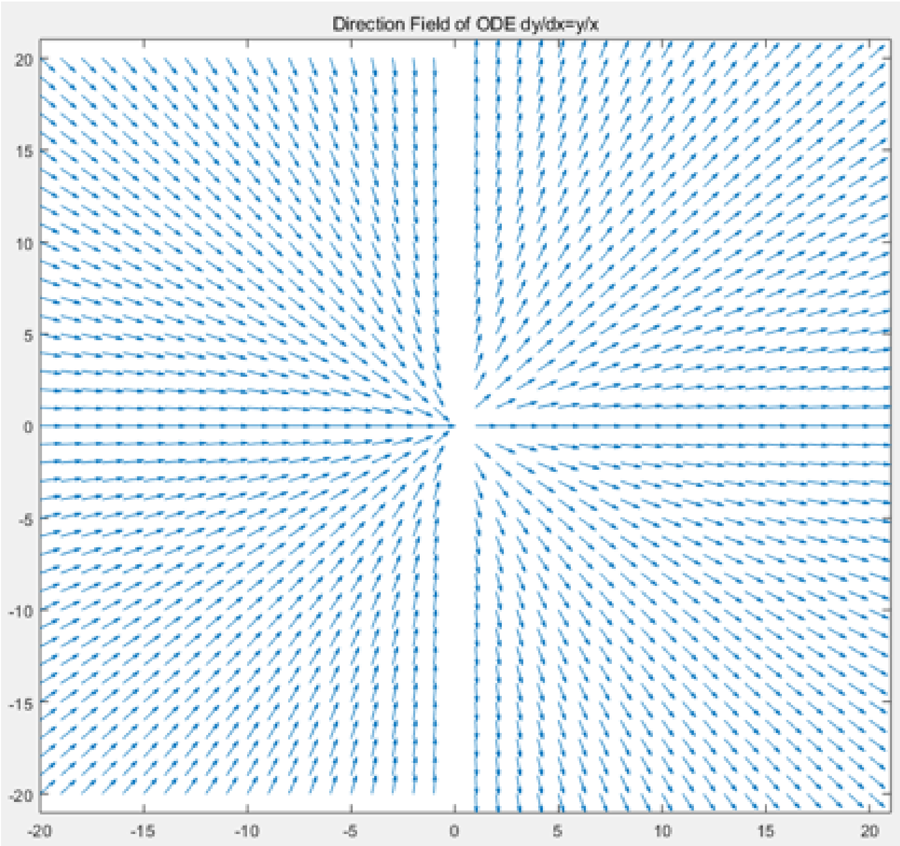


Figure 2: Direction field of $y' = y/x$

If $y_0 = 0$ then $y(t)$ is not uniquely determined, since $y_1(t) = 0$ for $t \geq t_0$ and $y_2(t) = \frac{1}{4}(t - t_0)^2$ for $t \geq t_0$ are two different solutions.
If $y_0 < 0$, then near $t_0$ the solution has the form $y(t) = -\frac{1}{4}(t - c_1)^2$ form some $c_1 > t_0$ (determined as in the case $y_0 > 0$), and hence $y(c_1) = 0$. Thus we are back in the previous case and the solution is not unique.

**3** If $y(t)$ (with domain $\mathbb{R}$) solves the given IVP then $z(t) = y(t + t_0)$ solves the IVP $z'' = -z$, $z(0) = y_0$, $z'(0) = y_1$. From Example 10 in the lecture we know that the unique solution of this IVP is $z(t) = y_0 \cos t + y_1 \sin t$. Hence

$$y(t) = z(t - t_0) = y_0 \cos(t - t_0) + y_1 \sin(t - t_0)$$

is unique as well.

**4** a) $y(x) = bx^\beta \implies y''(x) = b\beta(\beta - 1)x^{\beta-2} \doteq b^2 x^{2\beta}$
The only nonzero solution is $\beta = -2$, $b = 6$, i.e., $y(x) = 6x^{-2}$. (For this we use that $b_1 x^{\beta_1} = b_2 x^{\beta_2}$ holds for all $x$ in an intervall of positive length iff $b_1 = b_2 \wedge \beta_1 = \beta_2$, provided that both $b_1, b_2$ are nonzero.
The other "Ansatz" doesn't work, because $\mathrm{e}^{\alpha x}$ and $(\mathrm{e}^{\alpha x})^2 = \mathrm{e}^{2\alpha x}$ aren't scalar multiples of each other if $\alpha \neq 0$.

b) The Ansatz $y(x) = b\,x^\beta$ doesn't work, because in this case $y'' - 5y' + 6y$ involves $x^\beta$, $x^{\beta-1}$, $x^{\beta-2}$, which don't cancel.
For $y(x) = a\mathrm{e}^{\alpha x}$ we obtain

$$y'' - 5y' + 6y = \alpha^2 a\mathrm{e}^{\alpha x} - 5\alpha a\mathrm{e}^{\alpha x} + 6a\mathrm{e}^{\alpha x} = (\alpha^2 - 5\alpha + 6)a\mathrm{e}^{\alpha x},$$

which can be made zero by taking $\alpha$ as a root of $X^2 - 5X + 6$, i.e., $\alpha \in \{2, 3\}$. For $a = 1$ this gives the two solutions $y_1(x) = \mathrm{e}^{2x}$, $y_2(x) = \mathrm{e}^{3x}$. Every linear combination $y(x) = a_1\,\mathrm{e}^{2x} + a_2\,\mathrm{e}^{3x}$, $a_1, a_2 \in \mathbb{R}$, is then a solution as well.

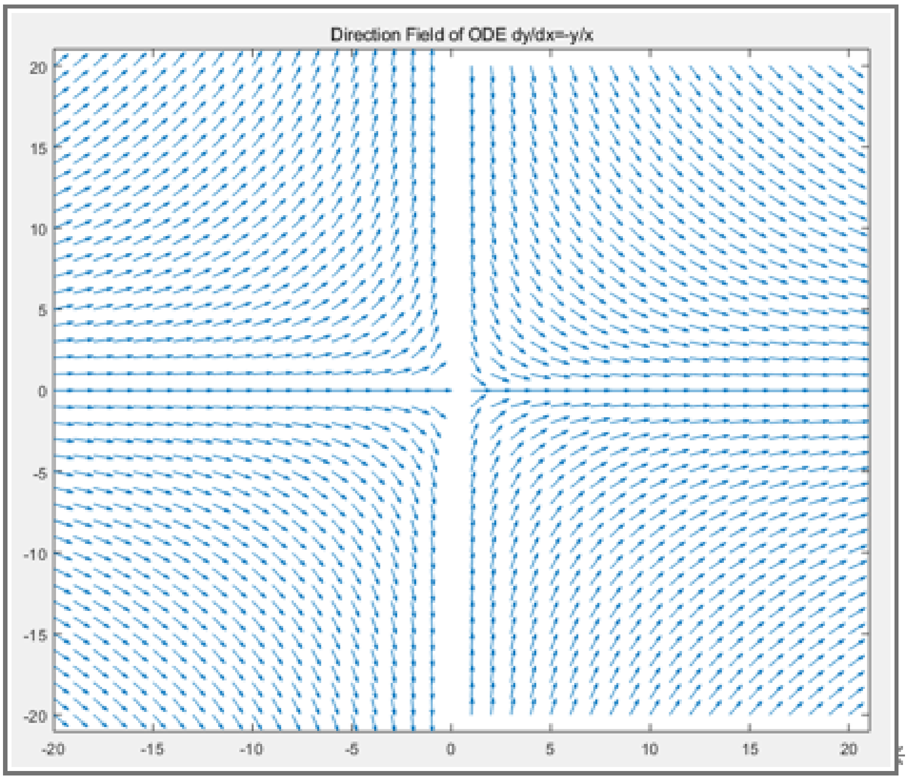


Figure 3: Direction field of $y' = -y/x$

c) Again it is clear that $y(x) = b\,x^\beta$ doesn't work. For $y(x) = a\mathrm{e}^{\alpha x}$ we see from b) that we need to solve

$$(\alpha^2 - 5\alpha + 6)a\mathrm{e}^{\alpha x} = \mathrm{e}^x.$$

This can be done: Set $\alpha = 1$ and solve $(\alpha^2 - 5\alpha + 6)a = 2a = 1$ for $a$, i.e., $a = 1/2$. A solution is therefore $y(x) = \frac{1}{2}\,\mathrm{e}^x$.

d) Clearly only the 2nd Ansatz can work, and in fact it does (w.l.o.g. set $b = 1$):

$$y'' - \frac{1}{2x}\,y' + \frac{1}{2x^2}\,y = \beta(\beta-1)x^{\beta-2} - \frac{\beta x^{\beta-1}}{2x} + \frac{x^\beta}{2x^2} = \left(\beta^2 - \frac{3}{2}\beta + \frac{1}{2}\right)x^{\beta-2} = 0$$

if $\beta$ is a root of $X^2 - \frac{3}{2}X + \frac{1}{2}$, i.e., $\beta \in \{1, \frac{1}{2}\}$. Thus $y_1(x) = x$, $y_2(x) = \sqrt{x}$ are two solutions, and, more generally, $y(x) = a_1 x + a_2\sqrt{x}$, $a_1, a_2, \in \mathbb{R}$ are solutions.

e) Here $y(x) = \mathrm{e}^{\alpha x}$ works (the constant $a$ is arbitrary and can be set to 1):

$$(2x+1)\alpha^2\mathrm{e}^{\alpha x} + (4x-2)\alpha\mathrm{e}^{\alpha x} - 8\mathrm{e}^{\alpha x} = \big((2x+1)\alpha^2 + (4x-2)\alpha - 8\big)\mathrm{e}^{\alpha x} = 0.$$

Since $\mathrm{e}^{\alpha x} \neq 0$, the polynomial

$$(2x+1)\alpha^2 + (4x-2)\alpha - 8 = (2\alpha^2 + 4\alpha)x + \alpha^2 - 2\alpha - 8$$

must be zero, since it vanishes for infinitely many $x$, and hence

$$2\alpha^2 + 4\alpha = \alpha^2 - 2\alpha - 8 = 0.$$

The unique solution is $\alpha = -2$, showing that $y(x) = \mathrm{e}^{-2x}$ solves the given ODE.

f) Here $y(x) = x^\beta$ works (the constant $b$ is arbitrary and can be set to 1):

$$\begin{aligned} &x^2(1-x)\beta(\beta-1)x^{\beta-2} + 2x(2-x)\beta x^{\beta-1} + 2(1+x)x^\beta \\ &\qquad = \big(x^2(1-x)\beta(\beta-1) + 2x(2-x)\beta x + 2(1+x)x^2\big)x^{\beta-2} = 0. \end{aligned}$$

$\Longrightarrow$ The first factor, which is

$$(-\beta^2 - \beta + 2)x^3 + (\beta^2 + 3\beta + 2)x^2 = -(\beta - 1)(\beta + 2)x^3 + (\beta + 1)(\beta + 2)x^2,$$

must be the zero polynomial. The unique solution is $\beta = -2$, giving the solution $y(x) = x^{-2}$ of the ODE.

**5 Ex. 23** (a) The relation between angular, angular velocity and linear velocity is:

$$v = \omega R = R\frac{d\theta}{dt}$$

Therefore, the kinetic energy T can be represented as:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(R\frac{d\theta}{dt})^2 = \frac{1}{2}mL^2\frac{d\theta}{dt}$$

(b) The potential energy V of the pendulum can be represented as:

$$V = mgh = mg(L - Lcos\theta) = mgL(1 - cos\theta)$$

(c) The total energy E can be represented as:

$$E = T + V = \frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mgL(1 - cos\theta)$$

Hence,

$$\frac{dE}{dt} = \frac{1}{2}mL^2(2\frac{d\theta}{dt})\frac{d^2\theta}{dt^2} - mgL(-sin\theta)\frac{d\theta}{dt}$$

The total energy E is invariant with time. Therefore,

$$\frac{dE}{dt} = \frac{1}{2}mL^2(2\frac{d\theta}{dt})\frac{d^2\theta}{dt^2} - mgL(-sin\theta)\frac{d\theta}{dt} = 0$$

Simplify the above equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}sin\theta = 0$$

**Ex. 24**

(a) According to the definition of angular momentum, it can be represented as:

$$M = L(mv) = mL(R\frac{d\theta}{dt}) = mL^2\frac{d\theta}{dt}$$

(b) According to the principle of angular momentum conservation,

$$\frac{dM}{dt} = -(mgsin\theta)L$$

Therefore,

$$\frac{dM}{dt} = mL^2\frac{d^2\theta}{dt^2} = -(mgsin\theta)L$$

Simplify the above equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}sin\theta = 0$$

**Ex. 25** (a) The free-body diagram is shown below:

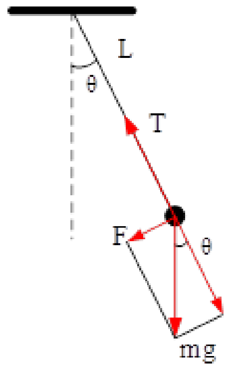


Figure 4: pendulum

(b)(c) Apply Newton's Law of motion in the direction tangential to the circular arc.

$$F = mgsin\theta = ma$$

$$a = -\frac{dv}{dt} = -\frac{d}{dt}(\omega L) = -L\frac{d\omega}{dt} = -L\frac{d^2\theta}{dt^2}$$

Therefore,

$$F = mgsin\theta = -m(L\frac{d^2\theta}{dt^2})$$

Simplify the above equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}sin\theta = 0$$