Differential Equations Plus (Math 286)

H63 Find the Laplace transforms of

a) $1+2t+3t^2$; b) e^{5t+3}; c) $\int_0^t \tau\sin\tau\,\mathrm{d}\tau$; d) $\sin^3 t$.

H64 Find inverse Laplace transforms of

a) $\dfrac{5}{s+6}$; b) $\dfrac{2s-1}{s^2+3}$; c) $\dfrac{1}{(s^2+1)(s^2+4)}$; d) $\dfrac{\mathrm{d}}{\mathrm{d}s}\dfrac{1-\mathrm{e}^{-5s}}{s}$;

e) $\ln\dfrac{s}{s-1}$; f) $\ln\dfrac{s^2+1}{(s-1)^2}$; g) $\dfrac{s+1}{s^2(s^2+1)}$; h) $\dfrac{\mathrm{e}^{-2s}-\mathrm{e}^{-4s}}{s}$;

i) $\operatorname{arccot}\dfrac{s}{\omega}$; j) $\dfrac{s^2-1}{(s^3+s^2-5s+3)(s^2-4)}$.

Six answers suffice.

H65 Solve the following initial value problems with the Laplace transform:

a) $y''-3\,y'+2\,y=6\,\mathrm{e}^{-t}$, $y(0)=9$, $y'(0)=6$;
b) $y''+2\,y'-3\,y=6\sinh(2t)$, $y(0)=0$, $y'(0)=4$;
c) $y'''+y''-5\,y'+3\,y=6\sinh(2t)$, $y(0)=y'(0)=0$, $y''(0)=4$.

H66 Find the Laplace transform of the Bessel function J_0 in two ways:

a) From the power series of J_0 and termwise integration of the Laplace integral.
Hint: The power series expansion

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n,\quad \text{valid for } |x|<1/4,$$

may help (but you should prove it first).

b) From the Bessel ODE of order $\nu=0$.

H67 Do Exercise 24 in [BDM17], Ch. 6.3, and use the result to verify that $\mathcal{L}\{|\sin t|\}=\frac{1}{s^2+1}\coth\frac{\pi s}{2}$ for $\mathrm{Re}(s)>0$; cp. also [BDM17], Ch. 6.3, Ex. 28.

Due on Wed Dec 8, 6 pm

Solutions (prepared by Zhang Zhuhaobo, Niu Yiqun, and TH)

63 a) $\mathcal{L}\{1+2t+3t^2\} = \mathcal{L}\{1\} + 2\,\mathcal{L}\{t\} + 3\,\mathcal{L}\{t^2\} = 1/s + 2/s^2 + 6/s^3$ for $\mathrm{Re}(s) > 0$;

b) $\mathcal{L}\{\mathrm{e}^{5t+3}\} = \mathrm{e}^3\mathcal{L}\{\mathrm{e}^{5t}\} = \mathrm{e}^3/(s-5)$ for $\mathrm{Re}(s) > 5$;

c) $\mathcal{L}\left\{\int_0^t \tau\sin\tau\,\mathrm{d}\tau\right\} = \frac{1}{s}\,\mathcal{L}\{t\sin t\} = -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathcal{L}\{\sin t\} = -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1} = -\frac{1}{s}\frac{-2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$. Alternatively (but more costly), evaluate the integral first using integration by parts, $\int_0^t \tau\sin\tau\,\mathrm{d}\tau = \sin t - t\cos t$, and then recall $\frac{1}{(s^2+1)^2} = \mathcal{L}\left\{\frac{1}{2}(\sin t - t\cos t)\right\}$ from the lecture.

d) From $\sin(3t) = \mathrm{Im}(\cos t + \mathrm{i}\sin t)^3 = 3\cos^2 t\sin t - \sin^3 t = 3\sin t - 4\sin^3 t$ we get $\mathcal{L}\left\{\sin^3 t\right\} = \mathcal{L}\left\{\frac{1}{4}(3\sin t - \sin(3t))\right\} = \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right) = \frac{6}{(s^2+1)(s^2+9)}$.

64 a) $\mathcal{L}^{-1}\left\{\frac{5}{s+6}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} = 5\,\mathrm{e}^{-6t}$;

b) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+3}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\} = 2\cos(\sqrt{3}\,t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t)$;

c) $\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \Longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) = \frac{1}{3}\sin t - \frac{1}{6}\sin(2t)$;

d) $\frac{1-\mathrm{e}^{-5s}}{s} = \mathcal{L}\{H(t) - H(t-5)\} \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}s}\frac{1-\mathrm{e}^{-5s}}{s} = \mathcal{L}\{-t\,H(t) + t\,H(t-5)\}$, i.e., $\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\frac{1-\mathrm{e}^{-5s}}{s}\right\} = -t\,H(t) + t\,H(t-5)$;

e) We have

$$\ln\frac{s}{s-1} = \ln\frac{1}{1-1/s} = -\ln(1-1/s) = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{3s^3} + \frac{1}{4s^4} + \cdots$$

for $|s| > 1$, and hence

$$\mathcal{L}^{-1}\left\{\ln\frac{s}{s-1}\right\} = 1 + \frac{t}{2} + \frac{t^2}{3\,2!} + \frac{t^3}{4\,3!} + \cdots = \sum_{n=1}^{\infty}\frac{t^{n-1}}{n!} = \frac{\mathrm{e}^t - 1}{t}.$$

f) Let $F(s) = \ln\frac{s^2+1}{(s-1)^2} = \ln(s^2+1) - 2\ln(s-1)$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$$\begin{aligned}
\Longrightarrow\quad & \mathcal{L}\{-t\,f(t)\} = F'(s) = \frac{2s}{s^2+1} - \frac{2}{s-1} = \mathcal{L}\left\{2\cos t - 2\,\mathrm{e}^t\right\} \\
\Longrightarrow\quad & -t\,f(t) = 2\cos t - 2\,\mathrm{e}^t \\
\Longrightarrow\quad & f(t) = \frac{2\,\mathrm{e}^t - 2\cos t}{t} \qquad (t \geq 0)
\end{aligned}$$

Since $\mathrm{e}^0 = \cos 0 = 1$ this is in fact an everywhere analytic function of t.

g) We have

$$\frac{s+1}{s^2(s^2+1)} = \frac{1}{s(s^2+1)} + \frac{1}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$
$$\Longrightarrow \mathcal{L}\left\{\frac{s+1}{s^2(s^2+1)}\right\} = 1 - \cos t + t - \sin t.$$

h) $\mathcal{L}^{-1}\left\{\left(\mathrm{e}^{-2s} - \mathrm{e}^{-4s}\right)/s\right\} = \mathcal{L}^{-1}\left\{\mathrm{e}^{-2s}/s\right\} - \mathcal{L}^{-1}\left\{\mathrm{e}^{-4s}/s\right\} = H(t-2) - H(t-4).$

i) From the lecture we know $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \operatorname{arccot} s$. Dilation in the domain gives

$$\mathcal{L}\left\{\frac{\sin(\omega t)}{\omega t}\right\} = \frac{1}{\omega}\operatorname{arccot}\frac{s}{\omega}. \qquad \Longrightarrow \qquad \mathcal{L}^{-1}\left\{\operatorname{arccot}\frac{s}{\omega}\right\} = \frac{\sin(\omega t)}{t}$$

j) We have

$$\frac{s^2-1}{s^3+s^2-5s+3} = \frac{s+1}{(s-1)(s+3)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+3}\right).$$
$$\Longrightarrow \mathcal{L}^{-1}\left\{\frac{s^2-1}{s^3+s^2-5s+3}\right\} = \frac{1}{2}\left(\mathrm{e}^{t} + \mathrm{e}^{-3t}\right).$$

65 As usual, we denote the Laplace transform of $y(t)$ by $Y(s)$

a) Applying $\mathcal{L}$ to both sides of the equation and inserting the initial conditions gives

$$\begin{aligned} s^2\,Y(s) - 9s - 6 - 3\big(s\,Y(s) - 9\big) + 2\,Y(s) &= \frac{6}{s+1} \\ (s^2-3s+2)Y(s) &= \frac{6}{s+1} + 9s - 21 = \frac{9s^2-12s-15}{s+1} \\ Y(s) &= \frac{9s^2-12s-15}{(s-1)(s-2)(s+1)} \end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$$

with

$$\begin{aligned} A &= (s-1)Y(s)\big|s = 1 = 9, \\ A &= (s-2)Y(s)\big|s = 2 = -1, \\ C &= (s+1)Y(s)\big|s = -1 = 1, \\ \Longrightarrow Y(s) &= \frac{9}{s-1} - \frac{1}{s-2} + \frac{1}{s+1} \\ \Longrightarrow y(t) &= \mathcal{L}^{-1}\big\{y(s)\big\} = 9\,\mathrm{e}^{t} - \mathrm{e}^{2t} + \mathrm{e}^{-t}. \end{aligned}$$

b) The Laplace transform of $\sinh t = \frac{1}{2}(\mathrm{e}^{t} - \mathrm{e}^{-t})$ is $F(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \frac{1}{s^2-1}$, from which $\mathcal{L}\{\sinh(2t)\} = \frac{1}{2}F\left(\frac{s}{2}\right) = \frac{1/2}{(s/2)^2-1} = \frac{2}{s^2-4}$.

$$\begin{aligned}
\implies s^2\,Y(s) - 4 + 2s\,Y(s) - 3\,Y(s) &= \frac{12}{s^2-4} \\
(s^2+2s-3)Y(s) &= \frac{12}{s^2-4} + 4 = \frac{4\,s^2-4}{s^2-4} \\
Y(s) = \frac{4(s^2-1)}{(s^2+2s-3)(s^2-4)} &= \frac{4(s+1)}{(s+3)(s-2)(s+2)}
\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{s+2}$$

with

$$\begin{aligned}
A &= (s+3)Y(s)\big|s = -3 = -8/5, \\
A &= (s-2)Y(s)\big|s = 2 = 3/5, \\
C &= (s+2)Y(s)\big|s = -2 = 1, \\
\implies Y(s) &= -\frac{8/5}{s+3} + \frac{3/5}{s-2} + \frac{1}{s+2} \\
\implies y(t) &= -\frac{8}{5}\,\mathrm{e}^{-3t} + \frac{3}{5}\,\mathrm{e}^{2t} + \mathrm{e}^{-2t}.
\end{aligned}$$

c)

$$\begin{aligned}
s^3\,Y(s) - 4 + s^2\,Y(s) - 5s\,Y(s) + 3\,Y(s) &= \frac{12}{s^2-4} \\
Y(s) = \frac{4(s^2-1)}{(s^3+s^2-5s+3)(s^2-4)} &= \frac{4(s+1)}{(s-1)(s+3)(s-2)(s+2)}
\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is (details omitted)

$$\begin{aligned}
Y(s) &= \frac{2}{5\,(s+3)} - \frac{1}{3\,(s+2)} - \frac{2}{3\,(s-1)} + \frac{3}{5\,(s-2)}. \\
\implies y(t) &= \frac{2}{5}\,\mathrm{e}^{-3t} - \frac{1}{3}\,\mathrm{e}^{-2t} - \frac{2}{3}\,\mathrm{e}^{t} + \frac{3}{5}\,\mathrm{e}^{2t}.
\end{aligned}$$

66 a) We have

$$\begin{aligned}
\mathrm{J}_0(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m (m!)^2}\,t^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m}\binom{2m}{m}\frac{t^{2m}}{(2m)!}. \\
\implies \mathcal{L}\{\mathrm{J}_0\} &= \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m}\binom{2m}{m}\frac{1}{s^{2m+1}} = \frac{1}{s}\sum_{m=0}^{\infty}\binom{2m}{m}\left(-\frac{1}{4s^2}\right)^m \\
&= \frac{1}{s}\frac{1}{\sqrt{1-4\left(-\frac{1}{4s^2}\right)}} \qquad \text{(using the hint)} \\
&= \frac{1}{\sqrt{s^2+1}}.
\end{aligned}$$

The computation is valid for $|s| > 1$, since the binomial series involved (see below) has radius of convergence 1; cf. the theorem about termwise integration of Laplace integrals in the lecture.

Finally we prove the asserted series expansion:

$$\begin{aligned}\binom{-1/2}{m} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2m-1}{2}\right)}{m!} = (-1)^m\frac{1\cdot 3\cdot 5\cdots(2m-1)}{m!\,2^m}\\ &= (-1)^m\frac{(2m)!}{m!\,2^m\cdot 2\cdot 4\cdot 6\cdots(2m)} = (-1)^m\frac{(2m)!}{(m!)^2 4^m} = \frac{(-1)^m}{4^m}\binom{2m}{m},\end{aligned}$$

and therefore

$$\sum_{m=0}^{\infty}\binom{2m}{m}x^m = \sum_{m=0}^{\infty}(-1)^m 4^m\binom{-1/2}{m}x^m = \sum_{m=0}^{\infty}\binom{-1/2}{m}(-4x)^m = (1-4x)^{-1/2},$$

using the binomial series.

J_0 is the solution of the IVP $t\,y'' + y' + t\,y = 0$, $y(0) = 1$, $y'(0) = 0$. Writing $Y(s) = \mathcal{L}\{\mathrm{J}_0(t)\}$ and taking the Laplace transform on both sides gives

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\,Y(s) - s\right) + s\,Y(s) - 1 - Y'(s) &= 0\\ -\left(s^2\,Y'(s) + 2s\,Y(s) - 1\right) + s\,Y(s) - 1 - Y'(s) &= 0\\ Y'(s) &= -\frac{s}{s^2+1}\,Y(s)\\ \Longrightarrow\ Y(s) = c\exp\int_0^s -\frac{1}{2}\ln(\sigma^2+1)\mathrm{d}\sigma &= \frac{c}{\sqrt{s^2+1}}\quad\text{for some constant } c.\end{aligned}$$

The constant c can be determined from

$$\mathcal{L}\{\mathrm{J}_0'(t)\} = s\,Y(s) - \mathrm{J}_0(0)$$

and the general fact that Laplace transforms tend to zero for $s \to \infty$. It follows that

$$c = \lim_{s\to\infty}\frac{cs}{\sqrt{s^2+1}} = \lim_{s\to\infty} s\,Y(s) = \mathrm{J}_0(0) = 1,$$

and hence $\mathcal{L}\{\mathrm{J}_0(t)\} = Y(s) = 1/\sqrt{s^2+1}$.

67 We have

$$\begin{aligned}\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t &= \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \int_T^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\\ &= \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \int_0^\infty f(T+\tau)\mathrm{e}^{-s(T+\tau)}\mathrm{d}\tau\\ &\qquad\qquad\text{(Subst. } \tau = t-T,\ \mathrm{d}\tau = \mathrm{d}t)\\ &= \int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \mathrm{e}^{-sT}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau.\quad \text{(Since } f(T+\tau) = f(\tau))\\ \Longrightarrow\ \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t &= \frac{1}{1-\mathrm{e}^{-sT}}\int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t\end{aligned}$$

In the special case $f(t) = |\sin t|$ the smallest period is π, so that

$$\begin{aligned}
\mathcal{L}\,|\sin t|\} &= \frac{1}{1-\mathrm{e}^{-\pi s}} \int_0^{\pi} \sin t\, \mathrm{e}^{-st}\, \mathrm{d}t = \frac{1}{1-\mathrm{e}^{-\pi s}} \left[\frac{\mathrm{e}^{-st}}{s^2+1}(-s\sin t - \cos t) \right]_0^{\pi} \\
&= \frac{\mathrm{e}^{-\pi s}+1}{1-\mathrm{e}^{-\pi s}}\, \frac{1}{s^2+1} = \frac{\mathrm{e}^{-\pi s/2}+\mathrm{e}^{\pi s/2}}{\mathrm{e}^{\pi s/2}-\mathrm{e}^{-\pi s/2}}\, \frac{1}{s^2+1} = \frac{1}{s^2+1}\, \coth \frac{\pi s}{2}.
\end{aligned}$$