HW4

a) $\mathcal{L}\{1+2t+3t^2\} = \mathcal{L}\{1\} + \mathcal{L}\{2t\} + \mathcal{L}\{3t^2\} = \frac{1}{s} + \frac{2}{s^2} + \frac{6}{s^3}$

b) $\mathcal{L}\{e^{5t+3}\} = e^3 \mathcal{L}\{e^{5t}\} = \frac{e^3}{s-5}$

c) $\int_0^t \tau \sin\tau \, d\tau = \sin t - t\cos t$

$\frac{1}{(s^2+1)^2} = \mathcal{L}\{\frac{1}{2}(\sin t - t\cos t)\}$

$\Rightarrow \mathcal{L}\{\sin t - t\cos t\} = \frac{2}{(s^2+1)^2}$

d) $\operatorname{Im}(e^{it})^3 = \operatorname{Im}(\cos t + i\sin t)^3 = 3\cos^2 t \sin t - \sin t^3 = 3\sin t - 4\sin^3 t$

$\Rightarrow \mathcal{L}\{\sin^3 t\} = \mathcal{L}\{\frac{1}{4}(3\sin t - \sin 3t)\} = \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)$

HW5

a) $\mathcal{L}^{-1}\{\frac{5}{s+6}\} = 5e^{-6t}$

b) $\mathcal{L}^{-1}\{\frac{2s-1}{s^2+3}\} = 2\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t$

c) $\mathcal{L}^{-1}\{\frac{1}{(s^2+1)(s^2+4)}\} = \mathcal{L}^{-1}\{\frac{1}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4})\} = \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} - \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s^2+4}\}$

$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$

d) $\frac{1-e^{-5s}}{s} = \mathcal{L}\{H(t) - H(t-5)\}$

$\Rightarrow \frac{d}{ds}\frac{1-e^{-5s}}{s} = \mathcal{L}\{-tH(t) + tH(t-5)\} \Rightarrow \mathcal{L}^{-1}\{\frac{d}{ds}\frac{1-e^{-5s}}{s}\} = -tH(t) + tH(t-5)$

h) $\mathcal{L}^{-1}\{\frac{e^{-2s} - e^{-4s}}{s}\} = H(t-2) - H(t-4)$

i) $\mathcal{L}^{-1}\{\operatorname{arccot}\frac{s}{\omega}\} = \frac{\sin(\omega t)}{t}$

HW6

a) ∵ $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$\Rightarrow s^2 Y(s) - s\cdot 9 - 6 - 3\cdot s\cdot Y(s) + 27 + 2\cdot Y(s) = \frac{6}{1+s}$

$\Rightarrow (s^2 - 3s + 2)Y(s) + 21 - 9s = \frac{6}{1+s}$

$\Rightarrow Y(s) = \frac{9s^2 - 12s - 15}{(s-1)(s-2)(s+1)}$

$\Rightarrow Y(s) = \frac{9}{s-1} + \frac{1}{s+1} - \frac{1}{s-2}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = 9e^t - e^{2t} + e^{-t}$

b) $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$

$\Rightarrow \mathcal{L}\{\sinh(2t)\} = \frac{2}{s^2-4}$ (查表 .... orz)

$\Rightarrow s^2 Y(s) - \cancel{s\cdot f(0)} - f'(0) + 2(sY(s) - \cancel{f(0)}) - 3Y(s) = \frac{12}{s^2-4}$

∵ $y(0) = 0 \quad y'(0) = 4$

$s^2 Y(s) - 4 + 2sY(s) - 3Y(s) = \frac{12}{s^2-4}$

$(s^2 + 2s - 3)Y(s) = \frac{4s^2 - 4}{s^2 - 4}$

$Y(s) = \frac{4s^2 - 4}{(s+2)(s-2)(s+3)(s-1)} = \frac{4s+4}{(s+2)(s-2)(s+3)}$

$\frac{A}{s+2} + \frac{B}{s-2} + \frac{C}{s+3} \Rightarrow A(s^2+s-6) + B(s^2+5s+6) + C(s^2-4) = 4s+4$

$\Rightarrow \begin{cases} A+B+C=0 \\ A+5B=4 \\ -6A+6B-4C=4 \end{cases} \Rightarrow \begin{cases} -2A+10B=4 \\ 2A+10B=8 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=3/5 \\ C=-\frac{8}{5} \end{cases}$

$\Rightarrow Y(s) = \frac{1}{s+2} + \frac{3/5}{s-2} - \frac{8/5}{s+3}$

$\Rightarrow y(t) = e^{-2t} + \frac{3}{5}e^{2t} - \frac{8}{5}e^{-3t}$

(c) $\mathcal{L}\{f'''(t)\} = s^3 Y(t) - \cancel{s^2 f(0)} - \cancel{sf'(0)} - f''(0)$

$(s^3 Y(s) - 4) + (s^2 Y(s)) - 5sY(s) + 3Y(s) = \frac{12}{s^2-4}$

$(s^3 + s^2 - 5s + 3)Y(s) = \frac{4s^2 - 4}{s^2 - 4}$

$Y(s) = \frac{4(s+1)(s-1)}{(s^3+s^2-5s+3)(s+2)(s-2)}$

$$\begin{array}{r} s^2+2s-3 \\ (s-1)\overline{)\,s^3+s^2-5s+3} \\ \underline{s^3-s^2} \\ 2s^2-5s+3 \\ \underline{2s^2-2s} \\ -3s+3 \\ \underline{-3s+3} \\ 0 \end{array}$$

$\Rightarrow s^2 + 2s - 3 = (s+3)(s-1)$

$\Rightarrow Y(s) = \frac{4(s+1)}{(s+3)(s-1)(s+2)(s-2)}$

$= \frac{A}{s+3} + \frac{B}{s-1} + \frac{C}{s+2} + \frac{D}{s-2}$

$\Rightarrow (s-1)(s^2-4)A + B(s^2-4)(s+3) + C(s^2+2s-3)(s-2) + D(s^2+2s-3)(s+2)$

$\Rightarrow A(s^3 - 4s - s^2 + 4) + B(s^3 - 4s + 3s^2 - 12) + C(s^3 + \cancel{2s^2} - 3s - \cancel{2s^2} - 4s + 6)$

$+ D(s^3 + 2s^2 - 3s + 2s^2 + 4s - 6) = 4s + 4$

$\Rightarrow A(s^3 - s^2 - 4s + 4) + B(s^3 + 3s^2 - 4s - 12) + C(s^3 - 7s + 6) + D(s^3 + 4s^2 + s - 6)$

$\Rightarrow \begin{cases} (A+B+C+D) = 0 \\ -A+3B+4D=0 \\ -4A-4B-7C+D=4 \\ 4A-12B+6C-6D=4 \end{cases} \Rightarrow \begin{cases} A=\frac{2}{5} \\ B=-\frac{2}{3} \\ C=-\frac{1}{3} \\ D=\frac{3}{5} \end{cases}$

$\Rightarrow y(t) = \frac{2}{5}e^{-3t} - \frac{2}{3}e^{t} - \frac{1}{3}e^{-2t} + \frac{3}{5}e^{2t}$

HW7

a) ∵ $J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \cdot \frac{(2n!)}{(n!)\cdot n!} \frac{x^{2n}}{(2n!)}$

$\Rightarrow J_0(x) = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \cdot \binom{2n}{n} \frac{t^{2n}}{(2n)!}$

$\Rightarrow \mathcal{L}\{J_0(x)\} = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \binom{2n}{n} \frac{1}{\cancel{(2n)!}} \cdot \frac{\cancel{(2n!)}}{s^{2n+1}} = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \binom{2n}{n} \frac{1}{s^{2n+1}}$

$= \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{-1}{4s^2}\right)^n \binom{2n}{n}$

$= \frac{1}{s\sqrt{1 - 4\frac{-1}{4s^2}}} = \frac{1}{\sqrt{1+s^2}}$

b) $ty'' + y' + ty = 0$

$\Rightarrow -\frac{d}{ds}(s^2 Y(s) - s) + sY(s) - 1 - Y'(s) = 0$

$\Rightarrow -2sY(s) - s^2 Y'(s) \cancel{+1} + sY(s) \cancel{-1} - Y'(s) = 0$

$\Rightarrow (s^2+1)Y'(s) = -sY(s)$

$\Rightarrow Y(s) = ce^{-\int \frac{s}{s^2+1} ds} = \frac{c}{\sqrt{s^2+1}}$

∵ $\mathcal{L}\{J_0'(x)\} = sY(s) - J_0(0)$

$c = \lim_{s\to\infty} \frac{cs}{\sqrt{s^2+1}} = J_0(0) =$

$\Rightarrow \mathcal{L}\{J_0(x)\} = \frac{1}{\sqrt{s^2+1}}$

HW8

24. Let f satisfy $f(t+T) = f(t)$ for all $t \geq 0$ and for some fixed positive number T; f is said to be **periodic with period** T on $0 \leq t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}.$$

In each of Problems 25 through 28, use the result of Problem 24 to find the Laplace transform of the given function.

$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

$= \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t)e^{-st}dt$

$= \int_0^T f(t)e^{-st}dt + \int_0^\infty f(t+T)e^{-s(t+T)}dt$

$= \int_0^T f(t)e^{-st}dt + \int_0^T f(t)e^{-s(t+T)}dt + \int_0^\infty f(t+T)e^{-s(t+2T)}dt$

$= \int_0^T f(t)e^{-st}dt + e^{-sT}\int_0^T f(t)e^{-s(t+T)}dt + \cdots$

$= \sum_{m=0}^{\infty} \int_0^T f(t)e^{-s(t+m\cdot T)}dt$

$\Rightarrow \mathcal{L}\{f(t)\} = \frac{\int_0^T f(t)e^{-st}dt}{1 - e^{-sT}}$. Q.E.D.

$\mathcal{L}\{|\sin t|\}$:

$\int_0^\pi \sin t\, e^{-st}dt = \frac{1}{1-s^2}\cos t e^{-st} + \frac{s\cdot \sin t e^{-st}}{1-s^2}\Big|_0^\pi = \frac{1}{1-s^2}(-e^{-s\pi} - 1)$

$\mathcal{L}\{|\sin t|\} = \Rightarrow \frac{e^{-s\pi}+1}{(s^2+1)(e^{-s\pi}-1)} = \frac{1}{s^2+1}\coth\frac{\pi s}{2}$