Differential Equations Plus (Math 286)

H27 Use the phase line to investigate the stability of the equilibrium solutions of the following autonomous ODE's.

a) $y' = 2(1-y)(1-\mathrm{e}^y)$; b) $y' = (1-y^2)(4-y^2)$; c) $y' = \sin^2 y$.

H28 For the following ODE's $y' = f(y)$, use the Existence and Uniqueness Theorem to determine the points $(t_0, y_0) \in \mathbb{R}^2$ such that the initial value problem $y' = f(y) \wedge y(t_0) = y_0$ has a unique solution near (t_0, y_0). Then solve the ODE, sketch the integral curves, and compare with your prediction.

a) $y' = |y|$; b) $y' = \sqrt{|y - y^2|}$.

H29 Use Picard-Lindelöf iteration to compute the solution $\phi = (\phi_1, \phi_2)^\mathsf{T}$ of the system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$

with initial condition $\phi(0) = (1, 0)^\mathsf{T}$.

H30 Suppose that $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies locally a Lipschitz condition, and that

$$f(-t, y) = -f(t, y) \quad \text{for all } (t, y) \in \mathbb{R}^2.$$

Show that any solution $\phi\colon [-r, r] \to \mathbb{R}$, $r > 0$, of $y' = f(t, y)$ is its own mirror image with respect to the y-axis.

H31 Compute the norms $\|\mathbf{A}\|$ of the following matrices $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and compare them with their Frobenius norms $\|\mathbf{A}\|_\mathrm{F}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \pm 1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

H32 Solve the initial value problem

$$y'' + |y| = 0, \quad y(0) = 0,\ y'(0) = 1.$$

Your solution should have the (maximal) domain $\mathbb{R}$.

Does the Existence and Uniqueness Theorem apply to this ODE?

H33 *Optional Exercise*

Let M be a set and $d\colon M \times M \to \mathbb{R}$ a function satisfying $d(a,a) = 0$ for $a \in M$, $d(a,b) \neq 0$ for $a, b \in M$ with $a \neq b$, and $d(a,b) \leq d(b,c)) + d(c,a)$ for $a, b, c \in M$.

a) Show that d is a metric.

b) Does this conclusion also hold if $d(a,b) \leq d(b,c)) + d(c,a)$ is replaced by the ordinary triangle inequality $d(a,b) \leq d(a,c)) + d(c,b)$?

H34 *Optional Exercise*

Let (M,d) be a metric space and $(a,b) \in M \times M$.

a) Show that the metric d is *continuous* in the following sense:
For every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x,a) < \delta \wedge d(y,b) < \delta$ implies $|d(x,y) - d(a,b)| < \epsilon$.
Hint: First derive the so-called *quadrangle inequality* $|d(x,y) - d(a,b)| \leq d(x,a) + d(y,b)$.

b) Using a), show in detail that $x_n \to a$ and $y_n \to b$ implies $d(x_n, y_n) \to d(a,b)$. (A special case of this, viz. $d(x_n, b) \to d(a,b)$, was used in the proof of Part (2) of Banach's Fixed-Point Theorem.)

H35 *Optional Exercise*

a) Show that a closed subset N of a complete metric space (M,d) is complete in the induced metric $d|_N\colon N \times N \to \mathbb{R}$, $(x,y) \mapsto d(x,y)$.

b) Conversely, show that a subset of a metric space that is complete in the induced metric must be closed.

H36 *Optional Exercise*

a) Prove that $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathbf{A} \mapsto \|\mathbf{A}\|$ satisfies (N1)–(N4).

b) Repeat a) for the Frobenius norm $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathbf{A} \mapsto \|\mathbf{A}\|_{\mathrm{F}}$.

c) Show that $\|\mathbf{A}\| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for all matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ or, equivalently, $|\mathbf{Ax}| \leq \|\mathbf{A}\|_{\mathrm{F}}\, |\mathbf{x}|$ for all $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.
Hint: Use $\|\mathbf{A}\| = \max\bigl\{|\mathbf{Ax}|\,; \mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| = 1\bigr\}$ and the Cauchy-Schwarz Inequality for vectors in $\mathbb{R}^n$.

Due on Fri Oct 29, 6 pm

The phase line of an autonomous ODE (required for H27) will be discussed in the lecture on Wed Oct 27 (cf. also [BDM17], Ch. 2.5); Picard-Lindelöf iteration (required for H29) in the lecture on Mon Oct 25 (cf. also [BDM17], Ch. 2.8). The optional exercises can be handed in until Fri Nov 5, 6 pm.

Solutions

27 a) Setting $y' = 2(1-y)(1-e^y) = 0$ gives the two equilibrium solutions $y_1 = 0$, $y_2 = 1$. The graph of y' versus y is shown below. So $y_1 = 0$ is an asymptotically stable stable

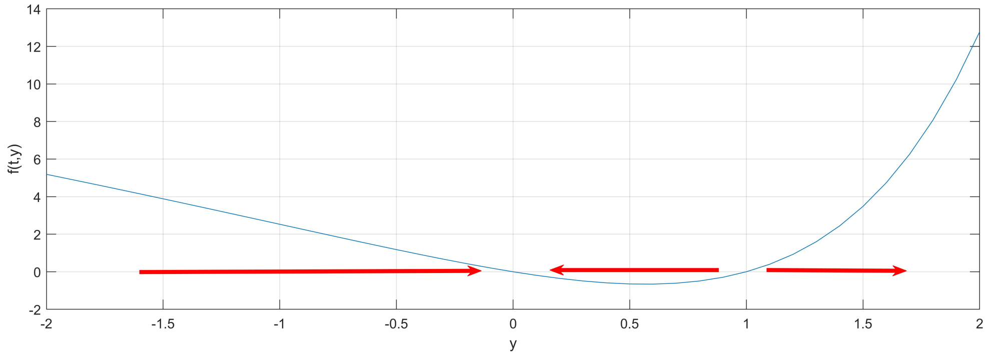

Figure 1: H27 a)

equilibrium, while $y_2 = 1$ is an unstable equilibrium.

b) Setting $y' = (1-y^2)(4-y^2) = 0$ gives the four equilibria $y_1 = -2, y_2 = -1, y_3 = 1, y_4 = 2$. The graph of y' versus y is shown below. So $y_1 = -2$, $y_3 = 1$ are asymptotically

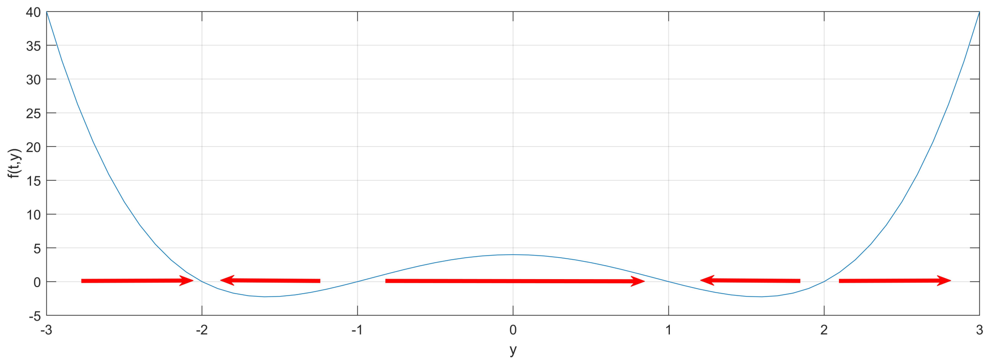

Figure 2: H27 b)

stable solutions, while $y_2 = -1, y_4 = 2$ are unstable solutions.

c) Setting $y' = \sin^2 y = 0$ gives infinitely many equilibrium solutions, viz. $y_k = k\pi$ ($k \in \mathbb{Z}$). The graph of y' versus y is shown below. So all equilibria are semistable (asymptotically stable from below, unstable from above).

28 Note that solutions of all three ODE's must have non-negative derivative and hence cannot decrease anywhere strictly.

a) The function $f(t,y) = |y|$ is continuous and trivially satisfies a Lipschitz condition with respect to y (with Lipschitz constant $L = 1$, since $|f(t,y_1) - f(t,y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|$. Hence solutions exist and are unique everywhere. The general solution is

$$y_C(t) = \begin{cases} C\,\mathrm{e}^t & \text{if } C \geq 0, \\ C\,\mathrm{e}^{-t} & \text{if } C < 0, \end{cases}$$

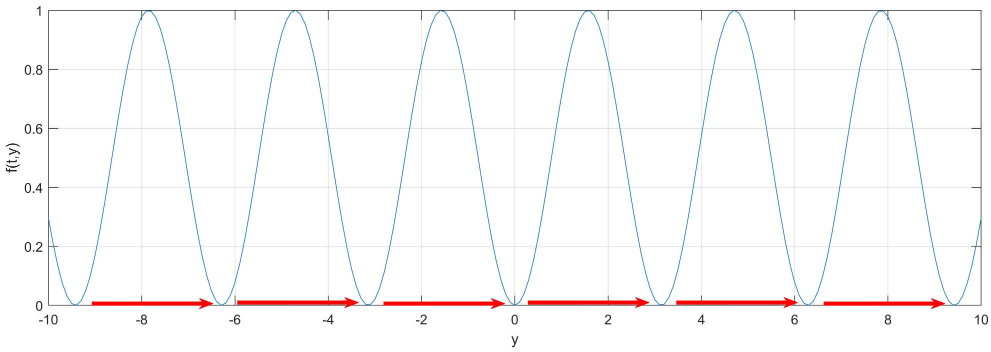

Figure 3: H27 c)

where C can be any real number. This follows by considering the three cases $y > 0$, $y = 0$, $y < 0$ separately.

b) $f(t, y) = \sqrt{|y - y^2|}$ is C^1 on the three plane regions $y < 0$, $0 < y < 1$, $y > 1$, and does not satisfy a Lipschitz condition with respect to y locally at points of the separating lines $y = 0$ and $y = 1$. The latter follows from the fact that the derivative $\frac{\partial f}{\partial y}$ is unbounded near $y = 0$ and $y = 1$. For example, for $0 < y < 1$ we have

$$|f(t, y) - f(t, 1)| = \left|\frac{\partial f}{\partial y}(t, \eta)\right| |y - 1| = \left|\frac{1 - 2\eta}{2\sqrt{\eta - \eta^2}}\right| |y - 1|$$

for some $\eta \in (y, 1)$, and for y (and hence η) close to 1 the factor $\left|\frac{1-2\eta}{2\sqrt{\eta-\eta^2}}\right|$ becomes arbitarily large.

The Existence and Uniqueness Theorem gives that solutions exist and are unique locally at points within the three regions. At points (t, y) with $y \in \{0, 1\}$ solutions are not unique as the following explicit solution shows.

$0 < y < 1$: $\mathrm{d}y/\sqrt{y - y^2} = 2\,\mathrm{d}y\,/\sqrt{1 - (2y - 1)^2} = 1 \Longrightarrow \arcsin(2y - 1) = t + C \Longrightarrow$
$y = \frac{1}{2}\bigl(1 + \sin(t + C)\bigr) = \frac{1}{2}\bigl(1 + \cos(t + C')\bigr)$

$y > 1$: $\mathrm{d}y/\sqrt{y^2 - y} = 2\,\mathrm{d}y\,/\sqrt{(2y - 1)^2 - 1} = 1 \Longrightarrow \operatorname{arcosh}(2y - 1)$
$= t + C \Longrightarrow y = \frac{1}{2}\bigl(1 + \cosh(t + C)\bigr)$

$y < 0$: $\mathrm{d}y/\sqrt{y^2 - y} = 2\,\mathrm{d}y\,/\sqrt{(1 - 2y)^2 - 1} = 1 \Longrightarrow -\operatorname{arcosh}(1 - 2y)$
$= t + C \Longrightarrow y = \frac{1}{2}\bigl(1 - \cosh(-t + C')\bigr)$

Solutions from the 3 cases can be glued together at $y = 0$ and $y = 1$ to satisfy the same initial conditions as the constant solutions. One particular example is

$$y(t) = \begin{cases} \frac{1}{2}(1 - \cosh(-t - \pi) & \text{for } t \leq -\pi, \\ \frac{1}{2}(1 + \cos t) & \text{for } -\pi \leq t \leq 0, \\ \frac{1}{2}(1 + \cosh t) & \text{for } t \geq 0; \end{cases}$$

see Figure 4. When constructing solutions, there is more degree of freedom, e.g., we can make solutions follow the line $y = 0$ for a while, then branch off and flow into the line $y = 1$, follow this line for another while, etc.

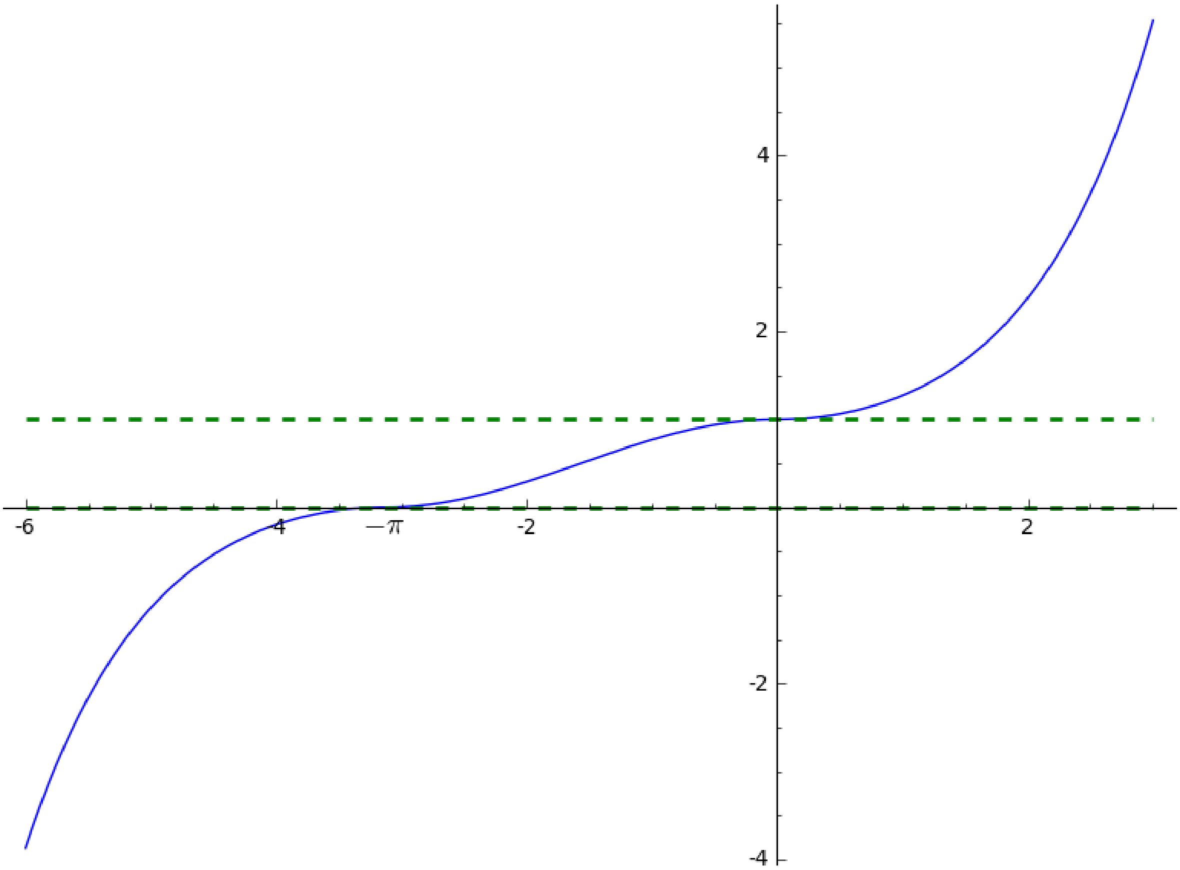

Figure 4: The solution $y(t)$ from H28c)

29 According to Picard-Lindelöf iteration, we have

$$\phi_{k+1}(t) = y_0 + \int_0^t f(s, \phi_k(s))\, ds, \quad k = 0, 1, 2, \ldots$$

Note that in this case $\phi_k(t)$ and y_0 are vectors in $\mathbb{R}^2$, and the notation used is somewhat inconsistent with "$\phi = (\phi_1, \phi_2)^\mathsf{T}$" in the statement of the exercise (but preferred for its simplicity).

Since $\phi_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = y_0$, we have

$$\begin{aligned}
\phi_1(t) &= y_0 + \int_0^t f(s, \phi_0(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}, \\
\phi_2(t) &= y_0 + \int_0^t f(s, \phi_1(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{t^2}{2} \\ t \end{pmatrix} = \begin{pmatrix} 1 - \frac{t^2}{2} \\ t \end{pmatrix}, \\
\phi_3(t) &= y_0 + \int_0^t f(s, \phi_2(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ 1 - \frac{s^2}{2} \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{t^2}{2} \\ t - \frac{t^3}{6} \end{pmatrix} = \begin{pmatrix} 1 - \frac{t^2}{2} \\ t - \frac{t^3}{6} \end{pmatrix}, \\
\phi_4(t) &= y_0 + \int_0^t f(s, \phi_3(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{s^3}{6} - s \\ 1 - \frac{s^2}{2} \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{t^4}{24} - \frac{t^2}{2} \\ t - \frac{t^3}{6} \end{pmatrix} = \begin{pmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{24} \\ t - \frac{t^3}{6} \end{pmatrix}, \\
&\;\;\vdots \\
\phi_{2k-1}(t) &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^{k-1} \frac{t^{2k-2}}{(2k-2)!} \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!} \end{pmatrix}, \\
\phi_{2k}(t) &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^{k} \frac{t^{2k}}{2k!} \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!} \end{pmatrix}.
\end{aligned}$$

$$\Longrightarrow \phi(t) = \begin{pmatrix} \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2k!} \\ \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

30 Consider the function $\psi(t) = \phi(-t)$, also defined for $t \in [-r, r]$. We have $\psi(0) = \phi(0) = y_0$, say, and

$$\psi'(t) = -\phi'(-t) = -f\big(-t, \phi(-t)\big) = f\big(t, \phi(-t)\big) = f\big(t, \psi(t)\big).$$

Hence both ϕ and ψ solve the IVP $y' = f(t, y) \wedge y(0) = y_0$. Since f satisfies the assumptions in the Existence and Uniqueness Theorem(s), it follows that $\phi = \psi$, i.e., $\phi(t) = \phi(-t)$ for $t \in [-r, r]$. This is the indicated symmetry property.

Remark: It is sufficient to assume that f satisfies locally a Lipschitz condition with respect to y, which is weaker than "Lipschitz condition per se".

31 Set $\mathbf{x} = \begin{pmatrix} \sin x & \cos x \end{pmatrix}^\mathrm{T}$ for $x \in [0, 2\pi)$.

In what follows, all maxima are taken over $x \in [0, 2\pi)$ (or over $\mathbb{R}$, which amounts to the same).

a) The norms of $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{A}\mathbf{x}|\} = \max\left\{\left|\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}\right|\right\} = \max\left\{\sqrt{\sin x^2 + \cos x^2}\right\} = 1$$

$$\|\mathbf{A}\|_F = \sqrt{1^2 + 0^2 + 0^2 + 1^2} = \sqrt{2}$$

Therefore for $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\|\mathbf{A}\| < \|\mathbf{A}\|_F$.

b) The norms of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{Ax}|\} = \max\left\{\left|\begin{pmatrix} 2\sin x + 2\cos x \\ 2\sin x + 2\cos x \end{pmatrix}\right|\right\} = \max\left\{\sqrt{2\left(2\sin x + 2\cos x\right)^2}\right\} = 4$$
$$\|\mathbf{A}\|_F = \sqrt{2^2 + 2^2 + 2^2 + 2^2} = 4$$

Therefore for $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $\|\mathbf{A}\| = \|\mathbf{A}\|_F$.

c) The norms of $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{Ax}|\} = \max\left\{\left|\begin{pmatrix} 2\sin x \\ -3\cos x \end{pmatrix}\right|\right\} = \max\left\{\sqrt{2\left(\sin x\right)^2 + \left(-3\cos x\right)^2}\right\} = 3$$
$$\|\mathbf{A}\|_F = \sqrt{2^2 + 0^2 + 0^2 + 3^2} = \sqrt{13}$$

Therefore for $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, $\|\mathbf{A}\| < \|\mathbf{A}\|_F$.

d) The norms of $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \pm 1 \\ 0 & \frac{1}{2} \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{Ax}|\} = \max\left\{\left|\begin{pmatrix} \frac{1}{2}\sin x \pm \cos x \\ \frac{1}{2}\cos x \end{pmatrix}\right|\right\}$$
$$= \max\left\{\sqrt{\left(\frac{1}{2}\sin x \pm \cos x\right)^2 + \left(\frac{1}{2}\cos x\right)^2}\right\} = \sqrt{\frac{3}{4} + \frac{\sqrt{2}}{2}} = \frac{1+\sqrt{2}}{2} \approx 1.207,$$
$$\|\mathbf{A}\|_F = \sqrt{\frac{1}{2}^2 + 1^2 + 0^2 + \frac{1}{2}^2} = \frac{\sqrt{6}}{2} \approx 1.225$$

(For the former, using the Calculus I machinery one finds that $x \mapsto \left(\frac{1}{2}\sin x \pm \cos x\right)^2 + \left(\frac{1}{2}\cos x\right)^2 = \frac{1}{4} + \cos^2 x \pm \sin x \cos x$ is maximized at $x_1 = \pm\pi/8$ and $x_2 = \pm 5\pi/8$ with value $\frac{3}{4} + \frac{1}{2}\sqrt{2} = \frac{3+2\sqrt{2}}{4}$.)

Therefore for $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \pm 1 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\|\mathbf{A}\| < \|\mathbf{A}\|_F$.

32 When $y \geq 0$, $y'' + y = 0$. The characteristic polynomial is $X^2 + 1 = 0$ with roots $\lambda_1 = i, \lambda_2 = -i$.
The general real solution is $y(t) = c_1 \cos t + c_2 \sin t$, $c_1, c_2 \in \mathbb{R}$.

$$\because \begin{cases} y(0) = c_1 = 0 \\ y'(0) = c_2 = 1 \end{cases} \quad \therefore y = \sin t \quad (t \in [0, \pi])$$

When $y \leq 0$, $y'' - y = 0$. The characteristic polynomial is $X^2 - 1 = 0$ with roots $\lambda_1 = 1, \lambda_2 = -1$.
$\Longrightarrow$ The general real solution is $y(t) = c_1 e^t + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$.

$$\because \begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = c_1 - c_2 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{cases} \quad \therefore y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} \quad (t \leq 0)$$

In order to get maximal domain $\mathbb{R}$, we impose for $t \geq \pi$ the new initial conditions $y(\pi) = \sin(\pi) = 0$, $y'(\pi) = \cos(\pi) = -1$, which are satisfied by the already defined solution on $(-\infty, \pi]$. Since $y(t) < 0$ for $t \downarrow \pi$, we must fit the general solution for $y \leq 0$, viz. $y(t) = c_1 e^t + c_2 e^{-t}$, to the new initial conditions.

$$\because \begin{cases} y(\pi) = c_1 e^{\pi} + c_2 e^{-\pi} = 0 \\ y'(\pi) = c_1 e^{\pi} - c_2 e^{-\pi} = -1 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{1}{2e^{\pi}} \\ c_2 = \frac{1}{2e^{-\pi}} \end{cases} \quad \therefore y(t) = -\frac{1}{2e^{\pi}}\,\mathrm{e}^t + \frac{1}{2e^{-\pi}}\,e^{-t} \quad (t \geq \pi)$$

Since this function is negative for all $t > \pi$, it also provides a solution of $y'' + |y| = 0$ on $[\pi, +\infty)$.
The final solution is

$$y(t) = \begin{cases} \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t & \text{for } t \leq 0, \\ \sin t & \text{for } 0 \leq t \leq \pi, \\ -\frac{1}{2}\,\mathrm{e}^{t-\pi} + \frac{1}{2}\,e^{-(t-\pi)} = -\sinh(t - \pi) & \text{for } t \geq \pi. \end{cases}$$

The function $y(t)$ is differentiable also at $t = 0, \pi$, because the one-sided derivatives exist there and coincide. (In fact $y(t)$ is even C^2, but not C^3.)

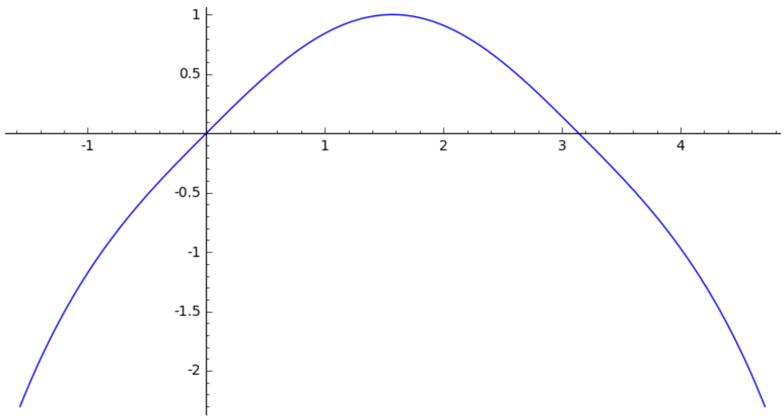

Figure 5: The solution $y(t)$ to H29

The Existence and Uniqueness Theorem applies to $y'' + |y| = 0$, because it is equivalent to the explicit ODE $y'' = f(t, y, y')$ with $f(t, y_0, y_1) = -|y_0|$. The function $f(t, y_0, y_1)$is continuous and satisfies

$$|f(t, y_0, y_1) - f(t, z_0, z_1)| = |-|y_0| + |z_0|| = \pm\big(|y_0| - |z_0|\big) \leq |y_0 - z_0| \leq \sqrt{(y_0 - z_0)^2 + (y_1 - z_1)^2}$$

for all $\mathbf{y} = (y_0, y_1)$, $\mathbf{z} = (z_0, z_1) \in \mathbb{R}^2$, i.e., a global Lipschitz condition with $L = 1$. As shown in the lecture, the (trivially continuous) 1st-order system obtained from $y'' = f(t, y, y')$ by order-reduction then satisfies such a Lipschitz condition as well (perhaps with slightly larger Lipschitz constant), so that the Existence and Uniqueness Theorem can be applied.

33 a) We need to show the missing properties of d required for a metric, i.e., $d(a,b) \geq 0$ and $d(a,b) = d(b,a)$ for all $a,b \in M$. Then we can conclude $d(b,a) = d(a,b) \leq d(b,c)) + d(c,a)$, i.e., the ordinary triangle inequality also holds.

Setting $c = a$ in the postulated "triangle inequality" gives $d(a,b) \leq d(b,a) + d(a,a) = d(b,a)$ for all $a,b \in M$. But then interchanging a and b also yields $d(b,a) \leq d(a,b)$, and so we must have $d(a,b) = d(b,a)$ for all $a,b \in M$. Further, setting $b = a$ in the "triangle inequality" gives $d(a,a) \leq d(a,c) + d(c,a)$, which on account of the already proved symmetry of d reduces to $0 \leq 2\,d(a,c)$. Thus we also have $d(a,c) \geq 0$ for $a,c \in M$, completing the proof.

b) No. A counterexample is $M = \{a,b\}$ with d defined by $d(a,a) = d(b,b) = 0$, $d(a,b) = 1$, $d(b,a) = -1$. In this case the ordinary triangle inequality has 8 instances:

$$\begin{aligned}
d(a,a) &\leq d(a,a) + d(a,a),\\
d(a,a) &\leq d(a,b) + d(b,a),\\
d(b,b) &\leq d(b,b) + d(b,b),\\
d(b,b) &\leq d(b,a) + d(a,b),\\
d(a,b) &\leq d(a,a) + d(a,b),\\
d(a,b) &\leq d(a,b) + d(b,b),\\
d(b,a) &\leq d(b,a) + d(a,a),\\
d(b,a) &\leq d(b,b) + d(b,a).
\end{aligned}$$

The only nontrivial relation obtained from these is $d(a,b) + d(b,a) \geq 0$, which is also true in our example. Hence the example satisfies all assumptions made in b).

Remark: This funny exercise is taken from the Book *Set Theory and Metric Spaces* by *Irving Kaplansky*, which is higly recommended for studying if you are interested in the underlying mathematical theory.

34 a) Applying the triangle inequality twice, we have

$$\begin{aligned}
d(x,y) &\leq d(x,a) + d(a,y)\\
&\leq d(x,a) + d(a,b) + d(b,y).\\
\implies d(x,y) - d(a,b) &\leq d(x,a) + d(y,b)
\end{aligned}$$

Interchanging x,a as well as y,b in this inequality turns the left-hand side into $d(a,b) - d(x,y)$ and preserves the right-hand side, so that we also have $d(a,b) - d(x,y) \leq d(x,a) + d(y,b)$. Thus $\pm\big(d(x,y) - d(a,b)\big) \leq d(x,a) + d(y,b)$, which is equivalent to the quadrangle inequality.

With the quadrangle inequality at hand the continuity of d is easy to prove: Just choose $\delta = \epsilon/2$ as response to ϵ.

b) Let $\epsilon > 0$ be given. There exists $N_1 \in \mathbb{N}$ such that $d(x_n,a) < \epsilon/2$ for all $n > N_1$, and $N_2 \in \mathbb{N}$ such that $d(y_n,b) < \epsilon/2$ for all $n > N_2$. Using the qudrangle inequality, we then have

$$|d(x_n,y_n) - d(a,b)| \leq d(x_n,a) + d(y_n,b) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n \geq \max\{N_1,N_2\}$. Thus $N = \max\{N_1,N_2\}$ can serve as response to ϵ in a proof of $d(x_n,y_n) \to d(a,b)$.

35 a) If (x_n) is a Cauchy sequence in N, it is a fortiori a Cauchy sequence in M and hence converges to some $a \in M$, since (M, d) is complete. But "N closed" means that N contains all limit points of sequences in N, so $a \in N$ and (x_n) converges in $(N, d|_N)$, which is therefore complete as well.

b) Using the notation in a), let (x_n) be a sequence in N, which has a limit in M, say a. Then (x_n) must be a Cauchy sequence, and hence convergent in N, since $(N, d|_N)$ is complete. Since limits of sequences are unique (the easily proved analogue for metric spaces of Exercise W30 of Worksheet 7 in Calculus III), this implies $a \in N$. Thus N contains all limit points of sequences in N and hence is closed.

Remarks: By the term "limit point" I mean just "limit", but some people would interpret "limit points" as "accumulation points" of not necessarily convergent sequences. In fact, since closed subsets are also characterized as subsets containing all their accumulation points, both views are admitted for this exercise.

Note that in b) the completeness of M is not required, and hence b) holds also for complete subspaces of incomplete metric spaces.

36 a) (N1), (N2) follow from the corresponding properties of the Euclidean length. For (N3) this is also true, but here we give a detailed proof: The triangle inequality for $|\cdot|$ yields for $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x}| = 1$ the estimate

$$|(\mathbf{A}+\mathbf{B})\mathbf{x}| = |\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{x}| \leq |\mathbf{A}\mathbf{x}| + |\mathbf{B}\mathbf{x}| \leq \|\mathbf{A}\| + \|\mathbf{B}\|\,.$$

Taking the maximum over all such vectors $\mathbf{x}$ then gives

$$\|\mathbf{A}+\mathbf{B}\| = \max\bigl\{|(\mathbf{A}+\mathbf{B})\mathbf{x}|\,;\mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| = 1\bigr\} \leq \|\mathbf{A}\| + \|\mathbf{B}\|\,.$$

For (N4) we can argue as follows:

$$|(\mathbf{AB})\mathbf{x}| = |\mathbf{A}(\mathbf{B}\mathbf{x})| \leq \|\mathbf{A}\|\,|\mathbf{B}\mathbf{x}| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|\,|\mathbf{x}| \implies \frac{|(\mathbf{AB})\mathbf{x}|}{|\mathbf{x}|} \leq \|\mathbf{A}\|\,\|\mathbf{B}\| \quad \text{for } \mathbf{x} \neq \mathbf{0}$$

Taking the maximum over all vectors $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ then gives $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$. (Alternatively we can restrict the above computation to vectors $\mathbf{x}$ of length 1, resulting in $|(\mathbf{AB})\mathbf{x}| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$, and then take the maximum over those vectors as in the proof of (N3).)

b) Since the Frobenius norm is a matrix analogue of the Euclidean length on $\mathbb{R}^{n^2}$, it clearly satisfies (N1)–(N3). For the proof of (N4) we write $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$, so that $\mathbf{AB} = (c_{ij}) = \bigl(\sum_{k=1}^n a_{ik}b_{kj}\bigr)_{i,j=1}^n$. Denoting the i-th row of $\mathbf{A}$ by $\mathbf{a}_i$ and the j-th column of $\mathbf{B}$ by $\mathbf{b}_j$, we have

$$\begin{aligned} c_{ij} &= \mathbf{a}_i \cdot \mathbf{b}_j\,, \\ c_{ij}^2 &\leq |\mathbf{a}_i|^2\,|\mathbf{b}_j|^2\,. && \text{(Cauchy-Schwarz Inequality)} \end{aligned}$$

Summing these inequalities over i, j gives

$$\|\mathbf{AB}\|_{\mathrm{F}}^2 \leq \sum_{i,j=1}^n |\mathbf{a}_i|^2\,|\mathbf{b}_j|^2 = \left(\sum_{i=1}^n |\mathbf{a}_i|^2\right)\left(\sum_{j=1}^n |\mathbf{b}_j|^2\right) = \|\mathbf{A}\|_{\mathrm{F}}^2\,\|\mathbf{B}\|_{\mathrm{F}}^2\,,$$

and (N4) follows.

c) It suffices to show $|\mathbf{A}\mathbf{x}| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x}| = 1$. Using the notation introduced in b) we have

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_n \cdot \mathbf{x} \end{pmatrix},$$

$$(\mathbf{a}_i \cdot \mathbf{x})^2 \leq |\mathbf{a}_i|^2 \, |\mathbf{x}|^2 = |\mathbf{a}_i|^2 .$$

$$\Longrightarrow \; |\mathbf{A}\mathbf{x}|^2 = \sum_{i=1}^{n} (\mathbf{a}_i \cdot \mathbf{x})^2 \leq \sum_{i=1}^{n} |\mathbf{a}_i|^2 = \|\mathbf{A}\|_{\mathrm{F}}^2$$

This proves $|\mathbf{A}\mathbf{x}| \leq \|\mathbf{A}\|_{\mathrm{F}}$ and implies the desired inequality $\|\mathbf{A}\| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Remark: The matrix norms considered so far and their properties remain true if $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$ and the Euclidean length on $\mathbb{R}^n$ by $|\mathbf{x}| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$. The above proofs remain valid for $\mathbb{C}^n$, provided we change squares of real numbers to squared absolute values of complex numbers, e.g., $(\mathbf{a}_i \cdot \mathbf{x})^2$ becomes $|\mathbf{a}_i \cdot \mathbf{x}|^2$.