Differential Equations Plus (Math 286)

H68 Find the following convolutions and their Laplace transforms (three answers suffice):

a) $t^2 * t^3$; b) $\mathrm{J}_0 * \mathrm{J}_0$; c) $\sin t * \cos(2t)$; d) $\mathrm{u}(t-1) * t$.

H69 Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$, $a \in [-\infty, \infty)$. Show that $\lim_{s\to+\infty} F(s) = 0$; cp. Exercise 24 in [BDM17], Ch. 6.1.
Hint: Use the uniform convergence of $\int_0^\infty f(t)\mathrm{e}^{-st}$ on $\mathrm{Re}(s) \geq a+1$ (resp., for $a = -\infty$ on $\mathrm{Re}(s) \geq 0$).

H70 Solve the following IVP's with the Laplace transform:

a) $y'' + y' + y = \mathrm{u}_\pi(t) - \mathrm{u}_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0;$

b) $y'' + 2y' + y = \begin{cases} \sin(2t) & \text{if } 0 \leq t \leq \pi/2, \\ 0 & \text{if } t > \pi/2, \end{cases} \quad y(0) = 1,\ y'(0) = 0.$

H71 Do Exercise 18 in [BDM17], Ch. 6.5.

H72 *Optional Exercise*

Repeat Exercises 20, 21 in [BDM17], Ch. 6.6, for the integro-differential equation

$$\phi'(t) = \sin t + \int_0^t \phi(t-\xi)\cos\xi \,\mathrm{d}\xi, \quad \phi(0) = 2.$$

Hint: It may be helpful to use the commutativity of the convolution product.

H73 Do Exercises 11 and 20 in [BDM17], Ch. 7.1.

H74 Find $\mathbf{S}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix for

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Show that $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$ for $k \in \mathbb{N}$, and use this to obtain explicit formulas for the entries of $\mathbf{A}^k$.

H75 *Optional Exercise*

a) Show that $\displaystyle\int_0^\infty \ln t\,\mathrm{e}^{-t}\,\mathrm{d}t = -\gamma = -0.577\ldots$. For this recall that the Euler-Mascheroni constant γ was defined as $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$
Hint: Relate the integral to the Gamma function. Gauss's formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)} \qquad (x \neq 0, -1, -2, \ldots),$$

which you don't need to prove, may help.

PLEASE TURN OVER

b) Use a) to find the Laplace transform of $t \mapsto \ln t$ and the inverse Laplace transform of $s \mapsto \frac{\ln s}{s}$ ($\operatorname{Re} s > 0$).

H76 *Optional Exercise*

Suppose V is a vector space over a field F.

a) Using the vector space axioms, prove the *scalar zero law*

$$0_F\, v = 0_V \quad \text{for all } v \in V.$$

b) Similarly, prove the *vector zero law*

$$a\, 0_V = 0_V \quad \text{for all } a \in F.$$

c) Prove that $(-1)x = -x$ for all $x \in V$.

H77 *Optional Exercise*

In each of the following cases, let S be the set of vectors $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ satisfying the given condition. Decide whether S is a subspace of $\mathbb{C}^3/\mathbb{C}$ and, if so, determine the dimension of S.

a) $\alpha = 0$;
b) $\alpha\beta = 0$;
c) $\alpha + \beta = 1$;
d) $\alpha + \beta = 0$;
e) $\alpha = 3\beta \wedge \beta = (2 - \mathrm{i})\gamma$;
f) $\alpha \in \mathbb{R}$.

H78 *Optional Exercise*

Let P_3 be the vector space (over $\mathbb{R}$) of polynomials $p(X) \in \mathbb{R}[X]$ of degree at most 3. Repeat the previous exercise for the sets $S \subseteq P_3$ defined by each of the following conditions:

a) $p(X)$ has degree 3;
b) $2p(0) = p(1)$;
c) $p(t) \geq 0$ for $0 \leq t \leq 1$;
d) $p(t) = p(1 - t)$ for all $t \in \mathbb{R}$.

H79 *Optional Exercise*

a) Write down the linear system of equations satisfied by a classical 3×3 magic square and transform this system into row-echelon form. (What is the magic number in this case?)

b) Use the equations in a) to show that up to obvious symmetries there exists exactly one classical 3×3 magic square.

Due on Thu Dec 16, 7:30 pm

The optional exercises can be handed in until Wed Dec 22, 6 pm.

Solutions (prepared by Zhang Zhuhaobo, Niu Yiqun, and TH)

68 a) The Laplace transform of $t^2 * t^3$ is

$$\mathcal{L}\{t^2 * t^3\} = \mathcal{L}\{t^2\} \cdot \mathcal{L}\{t^3\} = \frac{2}{s^3}\,\frac{6}{s^4} = \frac{12}{s^7} = \mathcal{L}\left\{\frac{12}{6!}\,t^6\right\} = \mathcal{L}\left\{\frac{1}{60}\,t^6\right\}.$$

It follows that $t^2 * t^3 = \frac{1}{60}\,t^6$.

Remark: With the same reasoning one can show

$$t^m * t^n = \frac{m!\,n!}{(m+n+1)!}\,t^{m+n+1} = \frac{1}{(m+n+1)\binom{m+n}{n}}\,t^{m+n+1} \quad \text{for integers } m,n \geq 0.$$

This also yields an evaluation of so-called Beta integrals; cf. [BDM17], Ch. 6.6., Ex. 10.

b) $\mathcal{L}\{\mathrm{J}_0 * \mathrm{J}_0\} = \mathcal{L}\{\mathrm{J}_0\}^2 = \left(1/\sqrt{s^2+1}\right)^2 = 1/(s^2+1) = \mathcal{L}\{\sin t\} \Longrightarrow \mathrm{J}_0 * \mathrm{J}_0 = \sin t$, a rather curious formula!

c) Here it helps less to use the convolution theorem, but we use it nevertheless because this way is more fun than the direct computation of the convolution:

$$\begin{aligned}\mathcal{L}\{\sin t * \cos(2t)\} &= \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{\cos(2t)\} = \frac{1}{s^2+1}\,\frac{s}{s^2+4} = \frac{1}{3}\left(\frac{s}{s^2+1} - \frac{s}{s^2+4}\right)\\ &= \mathcal{L}\left\{\frac{1}{3}\cos t - \frac{1}{3}\cos(2t)\right\}\end{aligned}$$

$\Longrightarrow \sin t * \cos(2t) = \frac{1}{3}\cos t - \frac{1}{3}\cos(2t)$.

d) Here we do the computation in both ways:

1. Direct way:

$$\begin{aligned}\mathrm{u}(t-1) * t &= \int_0^t \tau\,\mathrm{u}(t-\tau-1)\mathrm{d}\tau = \int_0^{t-1} \tau\mathrm{d}\tau\\ &= \begin{cases} 0 & \text{for } 0 \leq t \leq 1,\\ (t-1)^2/2 & \text{for } t \geq 1.\end{cases}\end{aligned}$$

2. Using the convolution theorem:

$$\mathcal{L}\{\mathrm{u}(t-1) * t\} = \mathcal{L}\{\mathrm{u}(t-1)\} \cdot \mathcal{L}\{t\} = \frac{\mathrm{e}^{-s}}{s}\,\frac{1}{s^2} = \frac{\mathrm{e}^{-s}}{s^3} = \mathcal{L}\left\{\mathrm{u}_1(t)\frac{(t-1)^2}{2}\right\}$$

$$\Longrightarrow\ \mathrm{u}(t-1) * t = \mathrm{u}_1(t)\frac{(t-1)^2}{2} \quad (t \geq 0).$$

This is the same as above.

69 The assertion "$\lim_{s\to+\infty} F(s) = 0$", which tacitly assumes $s \in \mathbb{R}$, can in fact be strenghtened to $\lim_{\mathrm{Re}(s)\to+\infty} F(s) = 0$, as the subsequent proof shows. But the complex limit $\lim_{|s|\to\infty} F(s)$ need not exist, because near the line of convergence $F(s)$ may be unbounded.
Let $\epsilon > 0$ be given. Since the Laplace integral converges uniformly for $\mathrm{Re}\, s \geq a+1$ ($\mathrm{Re}\, s \geq 0$), as shown in the lecture, we can find $R > 0$ such that $\left|\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| < \epsilon/2$ for all such s. Assuming that f is piecewise continuous, hence bounded on $[0, R]$, there exists $M > 0$ such $|f(t)| \leq M$ for $t \in [0, R]$. Writing $s = x + \mathrm{i}y$, we then have

$$\left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| \leq \int_0^R |f(t)|\,\mathrm{e}^{-xt}\,\mathrm{d}t \leq M\int_0^R \mathrm{e}^{-xt}\,\mathrm{d}t = \frac{M(1-\mathrm{e}^{-xR})}{x} \leq \frac{M}{x},$$

provided that $x > 0$. For $x > 2M/\epsilon$ the right-hand side is $\epsilon/2$.

$$\begin{aligned}\Longrightarrow |F(s)| &= \left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| \\ &\leq \left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| + \left|\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

provided that $\mathrm{Re}(s) > \max\{a+1, 0, 2M/\epsilon\}$. This shows $\lim_{\mathrm{Re}(s)\to+\infty} F(s) = 0$.

70 a) The transformed ODE is

$$s^2\,Y(s) - s + s\,Y(s) - 1 + Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s}$$

$$Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s(s^2+s+1)} + \frac{s+1}{s^2+s+1} = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s} + \left(1 - \mathrm{e}^{-\pi s} + \mathrm{e}^{2\pi s}\right)\frac{s+1}{s^2+s+1}$$

using $\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{s+1}{s^2+s+1}$. The first summand has inverse Laplace transform $\mathrm{u}_\pi(t) - \mathrm{u}_{2\pi}(t)$. The second summand can be rewritten as

$$\left(1 - \mathrm{e}^{-\pi s} + \mathrm{e}^{2\pi s}\right)\frac{s + \frac{1}{2} + \frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

and has inverse Laplace transform $y_1(t) - \mathrm{u}_\pi(t)y_1(t-\pi) + \mathrm{u}_{2\pi}(t)y_1(t-2\pi)$ with

$$y_1(t) = \mathrm{e}^{-t/2}\cos\frac{\sqrt{3}\,t}{2} + \frac{1}{\sqrt{3}}\,\mathrm{e}^{-t/2}\sin\frac{\sqrt{3}\,t}{2}.$$

$\Longrightarrow$ The solution is

$$\begin{aligned} y(t) &= y_1(t) + \mathrm{u}_\pi(t)\big(1 - y_1(t-\pi)\big) + \mathrm{u}_{2\pi(t)}\big(y_1(t-2\pi) - 1\big) \\ &= \begin{cases} y_1(t) & \text{if } 0 \leq t \leq \pi, \\ 1 + y_1(t) - y_1(t-\pi) & \text{if } \pi \leq t \leq 2\pi, \\ y_1(t) - y_1(t-\pi) + y_1(t-2\pi) & \text{if } t \geq 2\pi. \end{cases}\end{aligned}$$

Remark: The solution on $[0, \pi]$, viz. $y_1(t)$, is the solution of the IVP $y'' + y' + y = 0$, $y(0) = 1$, $y'(0) = 0$, as can also be checked using our earlier discussion of higher order linear ODE's with constant coefficients. The solution on $[\pi, 2\pi]$ is obtained by adding to $y_1(t)$ the solution of the IVP $y'' + y' + y = 1$, $y(\pi) = y'(\pi) = 0$ (to fit the initial conditions at $t = \pi$), which is $y_2(t) = 1 - y_1(t-\pi)$. The solution on $[2\pi, \infty)$ is obtained by adding to this in turn the solution of the IVP $y'' + y' + y = -1$, $y(2\pi) = y'(2\pi) = 0$ (to fit the initial conditions at $t = 2\pi$), which is $y_3(t) = y_1(t-2\pi) - 1$.

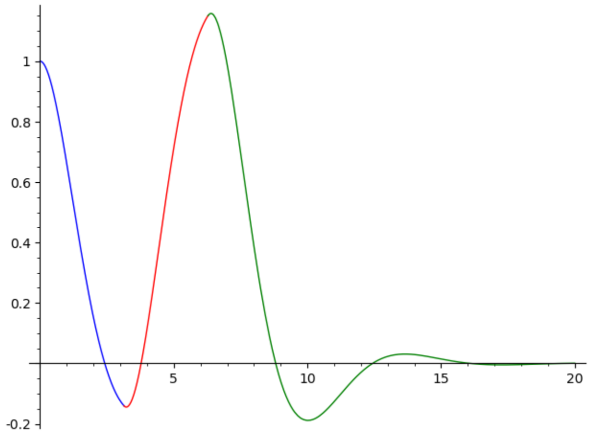

Figure 1: The solution $y(t)$ of H70 a)

b) Here the forcing function is $\sin(2t) - \mathrm{u}_{\pi/2}(t)\sin(2t) = \sin(2t) + \mathrm{u}_{\pi/2}(t)\sin\bigl(2(t-\pi/2)\bigr)$ and the transformed ODE is

$$s^2\,Y(s) - s + 2\bigl(s\,Y(s) - 1\bigr) + Y(s) = \frac{2}{s^2+4}\,\bigl(1 + \mathrm{e}^{-\pi s/2}\bigr)$$
$$Y(s) = \frac{2 + 2\,\mathrm{e}^{-\pi s/2}}{(s^2+4)(s+1)^2} + \frac{s+2}{(s+1)^2}$$

The real partial fractions decomposition of $\frac{1}{(s^2+4)(s+1)^2}$ is

$$\frac{1}{(s^2+4)(s+1)^2} = -\frac{2\,s+3}{25\,(s^2+4)} + \frac{2}{25\,(s+1)} + \frac{1}{5\,(s+1)^2}.$$
$$\implies Y(s) = \bigl(2 + 2\,\mathrm{e}^{-\pi s/2}\bigr)\left(-\frac{2\,s+3}{25\,(s^2+4)} + \frac{2}{25\,(s+1)} + \frac{1}{5\,(s+1)^2}\right) + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$
$$\begin{aligned}
\implies y(t) &= -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{4}{25}\mathrm{e}^{-t} + \frac{2}{5}t\mathrm{e}^{-t}\\
&\quad - \frac{4}{25}\mathrm{u}_{\pi/2}(t)\cos(2t-\pi) - \frac{3}{25}\mathrm{u}_{\pi/2}(t)\sin(2t-\pi) + \frac{4}{25}\mathrm{u}_{\pi/2}(t)\mathrm{e}^{-(t-\pi/2)}\\
&\quad + \frac{2}{5}\mathrm{u}_{\pi/2}(t)\,(t-\pi/2)\,\mathrm{e}^{-(t-\pi/2)}\\
&\quad + \mathrm{e}^{-t} + t\mathrm{e}^{-t}\\
&= -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{29}{25}\mathrm{e}^{-t} + \frac{7}{5}t\mathrm{e}^{-t}\\
&\quad + \frac{4}{25}\mathrm{u}_{\pi/2}(t)\cos(2t) + \frac{3}{25}\mathrm{u}_{\pi/2}(t)\sin(2t) + \frac{(4-5\pi)\mathrm{e}^{\pi/2}}{25}\mathrm{u}_{\pi/2}(t)\mathrm{e}^{-t} + \frac{2\mathrm{e}^{\pi/2}}{5}\mathrm{u}_{\pi/2}(t)t\,\mathrm{e}^{-t}\\
&= \begin{cases} -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{29}{25}\,\mathrm{e}^{-t} + \frac{7}{5}\,t\,\mathrm{e}^{-t} & \text{if } t \le \pi/2,\\ \frac{29+(4-5\pi)\mathrm{e}^{\pi/2}}{25}\,\mathrm{e}^{-t} + \frac{7+2\mathrm{e}^{\pi/2}}{5}\,t\,\mathrm{e}^{-t} & \text{if } t \ge \pi/2. \end{cases}
\end{aligned}$$

71

73 Ex. 11 From Figure (b) we can easily get the forces acting on each block. Assume the positive direction of the motion is to the right.

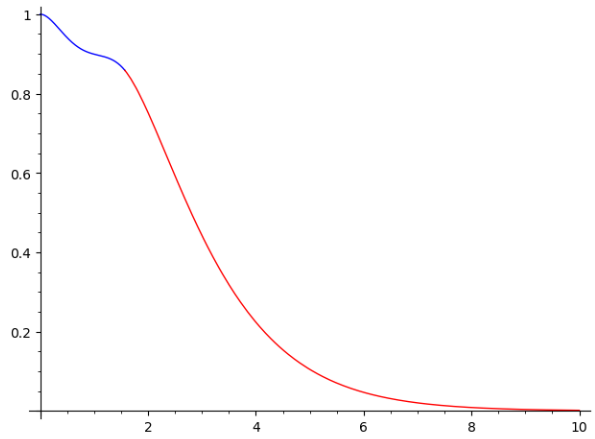

Figure 2: The solution $y(t)$ of H70 b)

For Block 1: $a = \frac{d^2x_1}{dt^2}$,

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= m_1a = -k_1x_1 + F_1(t) + k_2(x_2 - x_1) \\ &= -(k_1 + k_2)x_1 + k_2x_2 + F_1(t) \end{aligned}$$

Similarly, for Block 2: $a = \frac{d^2x_2}{dt^2}$,

$$\begin{aligned} m_2\frac{d^2x_2}{dt^2} &= m_2a = -k_2(x_2 - x_1) + F_2(t) - k_3x_2 \\ &= k_2x_1 - (k_2 + k_3)x_2 + F_2(t) \end{aligned}$$

Ex. 20 According to the current-voltage relation for each element in the circuit, we have

$$\begin{aligned} V_L &= L\,\frac{\mathrm{d}I}{\mathrm{d}t}, \\ I_C &= C\frac{\mathrm{d}V}{\mathrm{d}t}, \\ V_1 &= I_1R_1, \\ V_2 &= I_2R_2. \end{aligned}$$

Applying Kirchhoff's voltage law to the L-R_1-R_2 loop gives

$$L\,\frac{\mathrm{d}I}{\mathrm{d}t} + I_1R_1 + I_2R_2 = 0.$$

But $I_1 = I$, $I_2R_2 = V_2 = V$, and hence $L\,\frac{\mathrm{d}I}{\mathrm{d}t} + R_1I + V = 0$, proving the first equation. Applying Kirchhoff's current law to the lower node gives

$$I_1 = I_2 + I_C = \frac{V_2}{R_2} + C\,\frac{\mathrm{d}V}{\mathrm{d}t}.$$

Since $I_1 = I$, $V_2 = V$, the second equation follows.

74 The eigenvalues λ and corresponding eigenvectors $\mathbf{x}$ satisfy the equation

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}, \quad \text{i.e.}$$

$$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The eigenvalues are the solutions of the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0.$$

Thus the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. To find the eigenvectors, we replace λ by each of the eigenvalues in turn.
For $\lambda_1 = 1$,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \mathbf{x}^{(1)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(up to scalar multiples)}$$

For $\lambda_2 = 3$,

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \mathbf{x}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{(up to scalar multiples)}$$

The corresponding transformation matrix $\mathbf{S}$ and its inverse $\mathbf{S^{-1}}$ are

$$\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{S^{-1}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \qquad \text{(Note that } \mathbf{S}^2 = 2\mathbf{S}.\text{)}$$

By construction,

$$\mathbf{D} := \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

is a diagonal matrix. The powers of $\mathbf{A}$ satisfy

$$\mathbf{A^k} = \underbrace{(\mathbf{SDS^{-1}})(\mathbf{SDS^{-1}})\cdots(\mathbf{SDS^{-1}})}_{k \text{ times}} = \mathbf{SD^kS^{-1}}.$$

Calculate the corresponding matrix $\mathbf{D^k}$:

$$\mathbf{D^k} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \cdots = \begin{pmatrix} 1^k & 0 \\ 0 & 3^k \end{pmatrix}$$

Therefore, $\mathbf{A^k}$ can be obtained explicitly as

$$\mathbf{A^k} = \mathbf{SD^kS^{-1}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{3^k}{2} & \frac{1}{2} - \frac{3^k}{2} \\ \frac{1}{2} - \frac{3^k}{2} & \frac{1}{2} + \frac{3^k}{2} \end{pmatrix}.$$