Differential Equations Plus (Math 286)

H12 Determine all maximal solutions of $t^2\,y' = y^2$ and decide for which points $(t_0, y_0) \in \mathbb{R}^2$ the IVP $t^2\,y' = y^2 \wedge y(t_0) = y_0$ has no solution/exactly one solution/more than one solution.

H13 Determine the general solution of the following ODE's in terms of $y(0)$ (three answers suffice).

a) $\mathrm{d}y/\mathrm{d}t = e^{y+t}$;
b) $\mathrm{d}y/\mathrm{d}t = ty + y + t$;
c) $\mathrm{d}y/\mathrm{d}t = (\cos t)y + 4\cos t$;
d) $\mathrm{d}y/\mathrm{d}t = t^m y^n$ $(m, n \in \mathbb{Z})$.

H14 For the following ODE's, solve the corresponding IVP with $y(0) = 1$.

a) $\mathrm{d}y/\mathrm{d}t = -4ty$;
b) $\mathrm{d}y/\mathrm{d}t = t\,y^3$;
c) $(1+t)\mathrm{d}y/\mathrm{d}t = 4y$.

H15 Show that the graph of $y(t) = a/(d\,\mathrm{e}^{-at} + b)$ $(a, b, d > 0)$ is point-symmetric to its inflection point.

Hint: A superb way to solve this exercise is to observe that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.

H16

a) Explain how to adapt the analysis of the harvesting equation in the lecture to $y' = ay^2 + by + c$ with $a, b, c \in \mathbb{R}$ and $a > 0$.

b) Sketch the solution curves of (i) $y' = y^2 - y + 1$, (ii) $y' = y^2 + 2y + 1$, (iii) $y' = y^2 + y - 2$ without actually computing solutions. Steady-state solutions and inflection points (if any) should be drawn exactly.

H17 The ODE $y' = a(t)y - b(t)y^n$, $n \in \mathbb{R} \setminus \{0, 1\}$ is called *Bernoulli's differential equation.*

a) Show that for an appropriate choice of $\beta \in \mathbb{R}$ the substitution $z = y^\beta$ turns Bernoulli's differential equation into a linear 1st-order ODE (which can be solved by the usual methods).

b) Solve the IVP $y' = 4y - y^3 \wedge y(0) = 1$ by the method suggested in a).

c) Investigate the asymptotic stability of the steady-state solutions of the ODE in b).

H18 *Optional exercise*

a) Show that the general (real) solution of $y'' = y$ is $y(x) = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x}$, $c_1, c_2 \in \mathbb{R}$.
Hint: For a solution y the functions $y + y'$ and $y - y'$ satisfy linear 1st-order ODE's.

b) For $x \in \mathbb{R}$ let

$$F(x) = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t\,.$$

Show that

$$F'(x) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t \quad \text{for } x > 0.$$

Hint: Differentiate F under the integral sign and use $\int_0^\infty \sin(xt)/t\,\mathrm{d}t = \int_0^\infty \sin(t)/t\,\mathrm{d}t = \pi/2$ for $x > 0$.

c) Show that F solves $y'' = y$ on $(0, \infty)$.

d) Determine F from a), c) and $F(0)$, $F'(0+)$, and use the result to evaluate the integral

$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t\,.$$

H19 *Optional exercise*

The task of this exercise is to show the Cauchy-Hadamard formula

$$R = \frac{1}{L}, \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

(with the conventions $1/0 = \infty$, $1/\infty = 0$) for the radius of convergence R of a (complex) power series $\sum_{n=0}^\infty a_n(z-a)^n$. Here $L = \limsup_{n\to\infty} x_n \in [-\infty, +\infty]$ (*limit superior*) denotes the largest accumulation point of a real sequence (x_n), i.e., for every $\epsilon > 0$ there are only finitely many indexes n satisfying $x_n \geq L + \epsilon$ but no real number $L' < L$ has this property (with suitable modifications for $L = \pm\infty$).

a) If $L = \infty$ (i.e., $\sqrt[n]{|a_n|}$ is unbounded), show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges only for $z = a$.

b) If $L = 0$ (i.e., $\sqrt[n]{|a_n|}$ converges to zero), show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges for all $z \in \mathbb{C}$.

c) If $0 < L < \infty$, show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges for $|z-a| < 1/L$ and diverges for $|z-a| > 1/L$.

Due on Fri Oct 15, 6 pm

The optional exercises can be handed in until Fri Oct 22, 6 pm.

Solutions

12 We can rewrite this separable ODE as

$$\frac{dy}{y^2} = \frac{dt}{t^2} \quad (y, t \neq 0)$$

Integrating both sides of the above equation, we get

$$\int_{y_0}^{y} \frac{1}{\eta^2}\,\mathrm{d}\eta = \int_{t_0}^{t} \frac{1}{\tau^2}\,\mathrm{d}\tau$$
$$-\frac{1}{y} + \frac{1}{y_0} = -\frac{1}{t} + \frac{1}{t_0}$$

which gives

$$y(t) = \frac{t_0 y_0}{\frac{t_0 y_0}{t} - (y_0 - t_0)} = \frac{(t_0 y_0) t}{t_0 y_0 - (y_0 - t_0) t}.$$

For $t = 0$ we must have $y(t) = 0$ (from $t^2 y' = y^2$).
There is a constant solution $y(t) = 0$, $t \in \mathbb{R}$.

1) $t_0 = y_0 \neq 0$

$$y(t) = \begin{cases} t, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

And $y(0) = 0$ fits with the expression $y(t) = t$, so we can write the solution as $y(t) = t$, $t \in \mathbb{R}$.
There is only one solution.

2) $t_0 = y_0 = 0$
Every solution of the ODE defined at $t = 0$ must satisfy $y(0) = 0$ (see above). The non-constant (maximal) solutions are

$$y(t) = \frac{t}{1 - Ct}, \quad C \in \mathbb{R}.$$

with domain $\mathbb{R}$ if $C = 0$, $(-\infty, 1/C)$ if $C > 0$, and $(1/C, +\infty)$ if $C < 0$. In particular there are an infinite number of solutions.

3) $t_0 = 0, y_0 \neq 0$
This IVP contradicts $y(0) = 0$. Therefore, there is no solution.

4) $t_0 \neq 0, y_0 = 0$
The solution is

$$y(t) = 0$$

Therefore, there is only one solution.

5) $0 \neq t_0 \neq y_0 \neq 0$

$$y(t) = \frac{t}{1 - \frac{y_0 - t_0}{t_0 y_0} t}, \quad t \neq \frac{t_0 y_0}{y_0 - t_0}$$

Therefore, there is only one solution.

In conclusion, the IVP $t^2 y' = y^2 \wedge y(t_0) = y_0$ has

1) no solution when $t_0 = 0,\ y_0 \neq 0$;

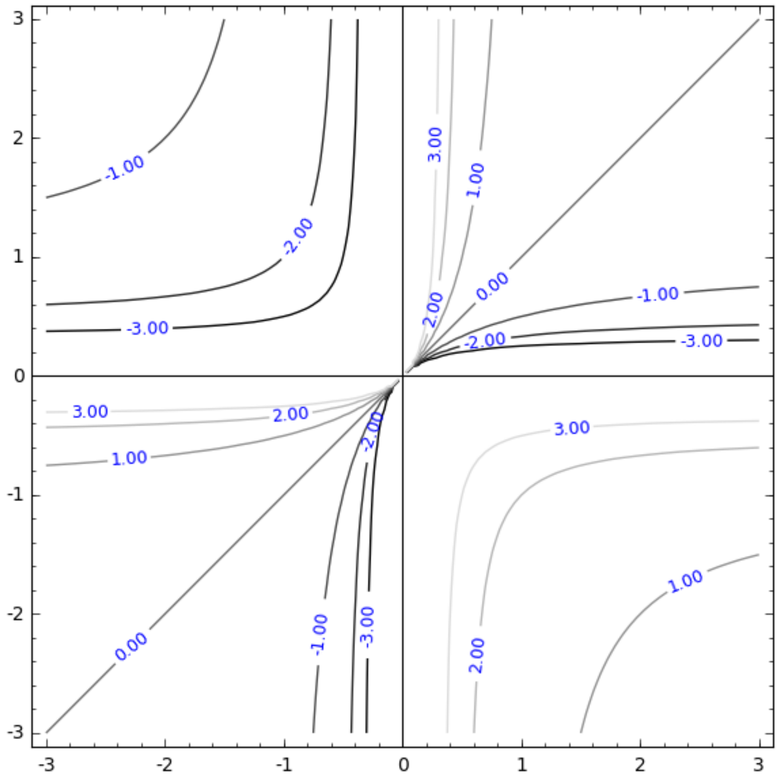
2) infinitely many (maximal) solutions when $t_0 = y_0 = 0$;

3) exactly one (maximal) solution otherwise.

Moreover, the maximal solutions are

$$\begin{aligned} y(t) &= 0, && t \in \mathbb{R}; \\ y(t) &= t, && t \in \mathbb{R}; \\ y_C^-(t) &= \frac{t}{1-Ct}, && t \in (-\infty, 1/C); \\ y_C^+(t) &= \frac{t}{1-Ct}, && t \in (1/C, +\infty). \end{aligned}$$

The 3rd and 4th type of solutions exist for any real number $C \neq 0$.
A better picture can be obtained by solving $y = t/(1-Ct)$ for C, which gives $C = (y-t)/(ty)$ and shows that $F(t,y) = (y-t)/(ty)$ provides a first integral for the given ODE.



From the contour plot of F you can see that all non-constant solutions that are defined at $t = 0$ share the same tangent at $(0,0)$. This also follows from $\frac{\mathrm{d}}{\mathrm{d}t}\frac{t}{1-Ct} = \frac{1}{(1-Ct)^2}$. Solutions $y_C^+(t)$ with $C > 0$ fill the 4th quadrant, solutions $y_C^-(t)$ with $C > 0$ fill the region above $y = t$ in the 1st and 3rd quadrant, etc. All these properties can also be derived with some effort from the formulas.

Remark: With the Existence and Uniqueness Theorems now at hand, we can easily get a complete qualitive picture. Rewriting $t^2y' = y^2$ as $y^2\,\mathrm{d}t - t^2\,\mathrm{d}y = 0$, we see that the origin $(t_0, y_0) = (0,0)$ is the only singular point, and hence that through any other point there passes precisely one integral curve (solution curve). For points $(0, y_0)$ with $y_0 \neq 0$ this is the curve $t = 0$, which cannot be seen from $t^2y' = y^2$, because it can be parametrized only as $t(y)$.

13 a)

$$e^{-y}\,\mathrm{d}y = e^t\,\mathrm{d}t$$

Integrating both sides of the equation, we get

$$\int_{y(0)}^{y} e^{-r}\,\mathrm{d}r = \int_0^t e^s\,\mathrm{d}s$$
$$-e^{-y} + e^{-y(0)} = e^t - 1$$

Finally, we obtain

$$y(t) = -\ln(e^{-y(0)} + 1 - e^t), \quad t < \ln(1 - e^{-y(0)}).$$

Remark: When determining the solution, one can also use indefinite integration $\int e^{-y}\,\mathrm{d}y = e^t\,\mathrm{d}t + C$ and determine C in terms of $y(0)$. This applies to the subsequent exercises as well.

b) Rewrite the ODE in the form of $y' = a(t)y + b(t)$:

$$y' = (t+1)y + t$$

According to the particular solution formula,

$$\begin{aligned}
y_p(t) &= e^{\frac{t^2}{2}+t} \int_0^t s\, e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s \\
&= e^{\frac{t^2}{2}+t} \left(\int_0^t (s+1)e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s - \int_0^t e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s \right) \\
&= -e^{\frac{t^2}{2}+t} \left(e^{-(\frac{t^2}{2}+t)} - 1 \right) - e^{\frac{1}{2}} \int_0^t e^{-\left(\frac{s^2}{2}+s+\frac{1}{2}\right)}\,\mathrm{d}s \\
&= e^{\frac{t^2}{2}+t} - 1 - e^{\frac{1}{2}} \int_0^t e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2}\,\mathrm{d}s,
\end{aligned}$$

and the "homogeneous solution" is

$$y_h(t) = y(0)\, e^{\frac{t^2}{2}+t}$$

Since $y_p(0) = 0$, the general solution in terms of $y(0)$ is

$$y(t) = y(0)e^{\frac{t^2}{2}+t} - 1 - e^{\frac{1}{2}} \int_0^t e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2}\,\mathrm{d}s\,.$$

Remark: It is not necessary to rewrite the integrand occuring in $y_p(t)$ in the particular form shown above, but at least this shows the relation with the incomplete Gauss integral (or the so-called error function). The simple answer is $y(t) = y_p(t) + y_h(t)$, $t \in \mathbb{R}$, with y_p, y_h as above.

c) According to the particular solution formula,

$$
\begin{aligned}
y_p(t) &= e^{\sin(t)} \int_0^t 4\cos(s) e^{-\sin(s)}\ \mathrm{d}s && (1)\\
&= -4e^{\sin(t)}(e^{-\sin(t)} - 1) && (2)\\
&= 4e^{\sin(t)} - 4, && (3)
\end{aligned}
$$

and the “homogeneous solution” is

$$y_h(t) = y(0)e^{\sin(t)}.$$

The general solution is then

$$y(t) = (y(0) + 4)e^{\sin(t)} - 4.$$

The general form $y(t) = C\mathrm{e}^{\sin t} - 4$, $C \in \mathbb{R}$, also follows from the observation that $y(t) \equiv -4$ is a particular solution.

d) There is the constant solution $y = 0$, and for $y \neq 0$ we can separate:

$$\frac{dy}{y^n} = t^m \,\mathrm{d}t\,.$$

Integrating both sides, we get

$$\int_{y(0)}^{y} \frac{1}{r^n}\,\mathrm{d}r = \int_0^t s^m \,\mathrm{d}s$$

$$-\frac{1}{(n-1)y^{n-1}} + \frac{1}{(n-1)y(0)^{n-1}} = \frac{t^{m+1}}{m+1}, \quad (n \neq 1, \quad m \neq -1).$$

Then, we obtain the general solution

$$y(t) = \left[(n-1)\left(\frac{1}{(n-1)y(0)^{n-1}} - \frac{t^{m+1}}{m+1}\right)\right]^{-\frac{1}{n-1}} \quad (n \neq 1, \quad m \neq -1).$$

Next, we deal with the special cases:

i) $n = 1, \quad m = -1$

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}t}{t}$$

$$\ln|y| = \ln|t| + C$$

Finally, we obtain, with a different parameter $C' \in \mathbb{R}$,

$$y(t) = C't, \quad t \in (-\infty, 0) \text{ or } t \in (0, +\infty).$$

$y(0)$ is not defined in this case.

ii) $n = 1, \quad m \neq -1$

$$\frac{\mathrm{d}y}{y} = t^m dt$$

Integrating both sides, we get

$$\int_{y(0)}^{y} \frac{1}{r}\,\mathrm{d}r = \int_0^t s^m\,\mathrm{d}s,$$

$$\ln|y| - \ln|y(0)| = \frac{t^{m+1}}{m+1}.$$

Finally, noting that $y(t)$ and $y(0)$ must have the same sign, we obtain

$$y(t) = y(0)e^{\frac{t^{m+1}}{m+1}}, \quad t \in \mathbb{R}.$$

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iii) $n \neq 1, \quad m = -1$

$$\frac{\mathrm{d}y}{y^n} = \frac{\mathrm{d}t}{t}$$

Integrate both sides, we get

$$-\frac{1}{(n-1)y^{n-1}} = \ln|t| + C$$

Finally, we obtain

$$y(t) = (-(n-1)(\ln|t| + C))^{-\frac{1}{n-1}}, \quad t < -e^{-C} \text{ or } t > e^{-C}.$$

$y(0)$ is not defined in this case.

14 a) $dy/dt = -4ty$

This is a homogeneous linear ODE, so we get

$$y(t) = Ce^{-2t^2}$$

Plugging into the IVP $y(0) = 1$, we can obtain the solution as

$$y(t) = e^{-2t^2}, \quad t \in \mathbb{R}.$$

b) $dy/dt = ty^3$

This is a separable ODE, so we can write

$$\frac{dy}{y^3} = tdt$$

$$\int_1^y \frac{1}{r^3}\,dr = \int_0^t s\,ds$$

The solution is

$$y(t) = (1 - t^2)^{-\frac{1}{2}}, \quad -1 < t < 1.$$

c) $(1+t)dy/dt = 4y$
Rewrite the ODE as
$$y' = \frac{4}{t+1}y.$$
We use the "homogeneous solution formula" to get
$$y(t) = Ce^{4\ln|t+1|} = C(t+1)^4.$$
Plugging into the IVP $y(0) = 1$, we obtain the solution as
$$y(t) = (t+1)^4, \quad t \in \mathbb{R}.$$

15 With the Hint, we want to prove that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.
The function of the mirror image is
$$g(t) = \frac{a}{b} - \frac{a}{de^{-a(2t_h-t)} + b}$$
where $t_h = (\ln d - \ln b)/a$.
First, we prove that $g(t)$ is a solution to the ODE $y' = ay - by^2$.
$$g'(t) = -\frac{a^2 de^{a(2t_h-t)}}{(de^{a(2t_h-t)} + b)^2}$$
and
$$\begin{aligned} ag(t) - bg^2(t) &= \frac{a^2}{b} - \frac{a^2}{de^{-a(2t_h-t)} + b} - b(\frac{a^2}{b^2} - \frac{2a^2}{b(de^{-a(2t_h-t)} + b)} + \frac{a^2}{(de^{-a(2t_h-t)} + b)^2}) \\ &= \frac{-a^2de^{-a(2t_h-t)} - a^2b + 2a^2b - a^2b}{(de^{-a(2t_h-t)} + b)^2} \\ &= -\frac{a^2de^{-a(2t_h-t)}}{(de^{-a(2t_h-t)} + b)^2} \end{aligned}$$
Thus, $g'(t) = ag(t) - bg^2(t)$, which means $g(t)$ is also a solution to the ODE $y' = ay - by^2$. Then, we will use the uniqueness of the solution of th IVP $y' = ay - by^2 \wedge y(t_h) = a/2b$. Since the original solution curve has the inflection point $(t_h, a/2b)$, it shares the same IVP with the mirror image $g(t)$. The logistic equation has a unique solution for any given IVP, so $y(t) = g(t)$.
Remark: The computation can be simplified a little by using the observation that $y(t)$ solves $y' = ay - by^2$ iff $t \mapsto y(t - t_0)$, $t_0 \in \mathbb{R}$, does.

16 a) It should be noted that the analysis in the lecture used the notation $y' = ay - by^2 - h$, where $a, b, h > 0$. However, the parabola $f(y) = y' = ay^2 + by + c$, $a > 0$, is a vertically flipped version of that considered in the lecture. This discrepancy will lead to different behaviors of the solution curves.
The discriminant is $\Delta = b^2 - 4ac$. For $\Delta \geq 0$, there are the steady-state solutions
$$y_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$y_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

where $0 < y_1 \leq y_2$.

i) $c < b^2/4a$
If the initial condition $y(t_0)$ satisfies $y_1 < y(t_0) < y_2$, then $y(t)$ decreases and $\lim_{t\to\infty} y(t) = y_1$.
If $y(t_0) > y_2$, then $y(t)$ increases to ∞. If $y(t_0) < y_1$, then $y(t)$ increases and $\lim_{t\to\infty} y(t) = y_1$.

ii) $c = b^2/4a$
If $y(t_0) > -b/2a$, then $y(t)$ increases to ∞. If $y(t_0) < -b/2a$, then $y(t)$ increases and $\lim_{t\to\infty} y(t) = y_1$.

iii) $c > b^2/4a$
Regardless of the initial condition, $y(t)$ will increase to ∞.

b) i) $y' = y^2 - y + 1$

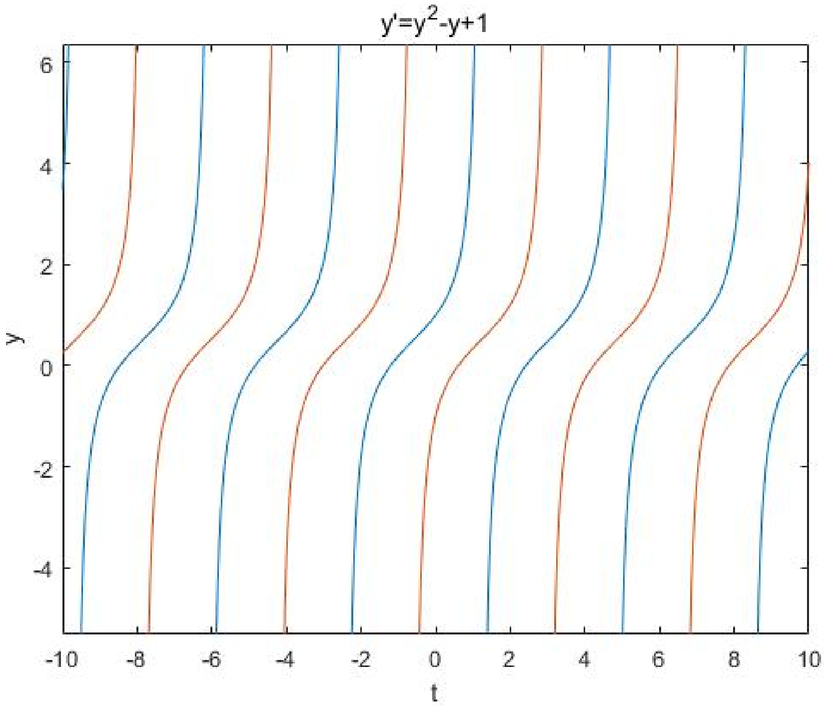


Figure 1: $y' = y^2 - y + 1$

ii) $y' = y^2 + 2y + 1$

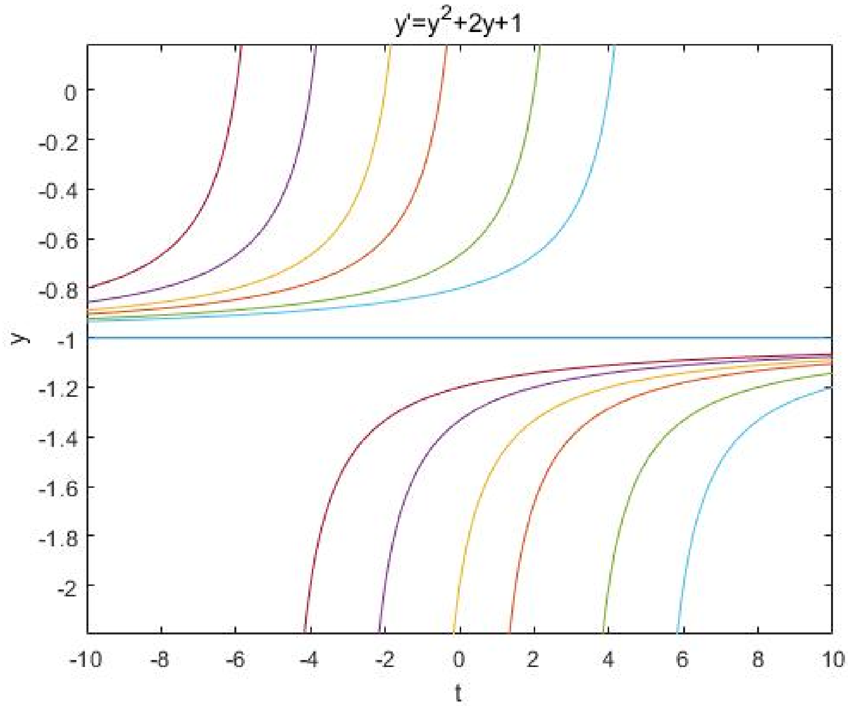


Figure 2: $y' = y^2 + 2y + 1$

iii) $y' = y^2 + y - 2$

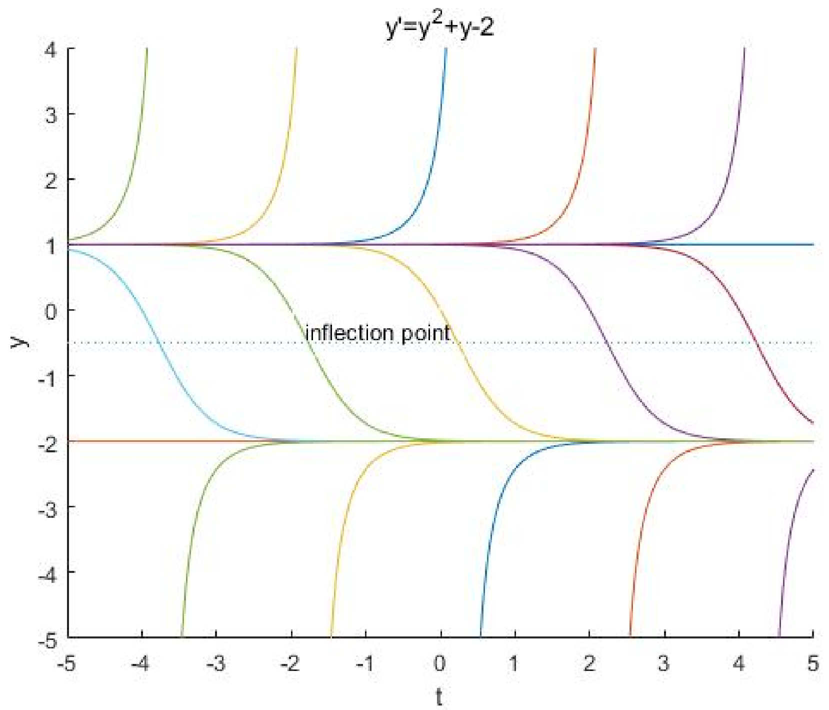


Figure 3: $y' = y^2 + y - 2$

17 a) We can write

$$\frac{dz}{dt} = \beta y^{\beta-1}\frac{dy}{dt}.$$

Then, we get

$$\frac{dy}{dt} = \frac{1}{\beta} y^{1-\beta} \frac{dz}{dt}.$$

Substituting the above expression into the ODE, we get

$$\begin{aligned} \frac{1}{\beta} y^{1-\beta} \frac{dz}{dt} &= a(t)y - b(t)y^n, \\ z' &= \beta a(t) y^\beta - \beta b(t) y^{n+\beta-1}, \\ z' &= \beta a(t) z - \beta b(t) y^{n+\beta-1}. \end{aligned}$$

Then, setting $\beta = 1 - n$, we can obtain the 1st-order linear ODE

$$z' = \beta a(t) z - \beta b(t)$$

for $z(t) = y(t)^{1-n}$. Depending on n, the 1-1 correspondence between solutions of both ODEs may only hold for a smaller domain, e.g., for general $n > 1$ we need to restrict to $y > 0$ (except for certain integers n).

b) Setting $\beta = 1 - 3 = -2$, we can rewrite the ODE as

$$z' = -8z + 2.$$

The corresponding IVP is $z(0) = y(0)^{-2} = 1$.
Then, we can get its solution as

$$z(t) = \frac{3}{4} e^{-8t} + \frac{1}{4}.$$

Since $z = y^\beta = y^{-2}$,

$$y(t) = \pm z(t)^{-\frac{1}{2}} = \pm \left(\frac{3}{4} e^{-8t} + \frac{1}{4} \right)^{-\frac{1}{2}}$$

Because $y(0) = 1$, we eliminate the negative solution, leaving

$$y(t) = \left(\frac{3}{4} e^{-8t} + \frac{1}{4} \right)^{-\frac{1}{2}} = \frac{2}{\sqrt{1 + 3\,\mathrm{e}^{-8t}}}, \quad t \in \mathbb{R}.$$

c) The steady-state solution is $z(t) = 1/4$, corresponding to $y(t) = \pm 2$.
The general solution to the ODE in b) is $y(t) \equiv 0$ and the non-constant solutions

$$y_1(t) = -\left[\left(y^{-2}(0) - \frac{1}{4} \right) e^{-8t} + \frac{1}{4} \right]^{-\frac{1}{2}},$$

and

$$y_2(t) = \left[\left(y^{-2}(0) - \frac{1}{4} \right) e^{-8t} + \frac{1}{4} \right]^{-\frac{1}{2}}.$$

$\lim_{t\to\infty} y_1(t) = -2$, and $\lim_{t\to\infty} y_1(t) = 2$.

If the initial condition is $y(0) = y_0 < 0$, then the solution will be $y_1(t)$, so $\lim_{t\to\infty} y(t) = -2$;
if the initial condition is $y(0) = y_0 > 0$, then the solution will be $y_2(t)$, so $\lim_{t\to\infty} y(t) = 2$.
This shows that both $y = -2$ and $y = 2$ are asymptotically stable.
The third steady state solution $y(t) \equiv 0$ is unstable. This follows from the cases $y(0) > 0$ and $y(0) < 0$ covered above.

18 a) We have

$$\begin{aligned}(y+y')' &= y' + y'' = y' + y = y + y',\\ (y-y')' &= y' - y'' = y' - y = -(y-y'),\end{aligned}$$

i.e., $z = y + y'$ satisfies $z' = z$ and $w = y - y'$ satisfies $w' = -w$. From the theory of 1st-order linear ODE's it follows that $z(x) = y(x)+y'(x) = c_1\mathrm{e}^x$, $w(x) = y(x)-y'(x) = c_2\mathrm{e}^{-x}$ for some $c_1, c_2 \in \mathbb{R}$. $\Longrightarrow y(x) = \frac{1}{2}(c_1\mathrm{e}^x + c_2\mathrm{e}^{-x}) = (c_1/2)\mathrm{e}^x + (c_2/2)\mathrm{e}^{-x}$, which is of the required form.

b) From the lecture recall that F is continuous on $\mathbb{R}$ and can be differentiated under the integral sign for $x > 0$. Thus for $x > 0$ we have

$$\begin{aligned}F'(x) &= -\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t = -\int_0^\infty \frac{t^2\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = -\int_0^\infty \frac{(t^2+1-1)\sin(xt)}{t(t^2+1)}\,\mathrm{d}t\\ &= -\int_0^\infty \frac{\sin(xt)}{t}\,\mathrm{d}t + \int_0^\infty \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t\,.\end{aligned}$$

The first integral is actually independent of x, since

$$\int_0^\infty \frac{\sin(xt)}{t} = \int_0^\infty \frac{\sin s}{(s/x)x}\,\mathrm{d}s = \int_0^\infty \frac{\sin s}{s}\,\mathrm{d}s, \qquad \text{(Subst. } s = xt,\ \mathrm{d}s = x\,\mathrm{d}t)$$

and has the value $\pi/2$, as we know from the Calculus III fnal exam.

c) Differentiating the expression in b) again under the integral sign, we obtain

$$F''(x) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = \int_0^\infty \frac{t\cos(xt)}{t(t^2+1)}\,\mathrm{d}t = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t = F(x).$$

This is justified, since

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(xt)}{t(t^2+1)}\right| = \frac{|\cos(xt)|}{t^2+1} \leq \frac{1}{t^2+1} = \Phi(t),$$

which is independent of x and integrable over $(0,\infty)$.

d) According to a) and c) we have

$$\begin{aligned}F(x) &= c_1\mathrm{e}^x + c_2\mathrm{e}^{-x},\\ F'(x) &= c_1\mathrm{e}^x - c_2\mathrm{e}^{-x}\end{aligned}$$

for some $c_1, c_2 \in \mathbb{R}$ and $x > 0$. Since F is continuous in 0, the first identity holds also for $x = 0$ and gives $c_1 + c_2 = F(0) = \int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \pi/2$.

Since

$$\left|\frac{\sin(xt)}{t(t^2+1)}\right| \le \frac{1}{t(t^2+1)} = \Phi(t),$$

which is independent of x and integrable over $(0,\infty)$, we get

$$F'(0+) = -\frac{\pi}{2} + \int_0^\infty \lim_{x\downarrow 0} \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = -\frac{\pi}{2} + \int_0^\infty 0\,\mathrm{d}t = -\frac{\pi}{2}$$

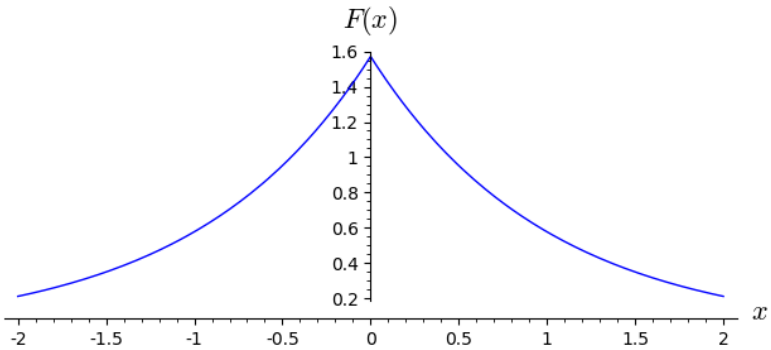
On the other hand, $F'(0+) = \lim_{x\downarrow 0}(c_1\mathrm{e}^{x} - c_2\mathrm{e}^{-x}) = c_1 - c_2$, so that $c_1 - c_2 = -\pi/2$. It follows that $c_1 = 0$, $c_2 = \pi/2$. Hence $F(x) = (\pi/2)\mathrm{e}^{-x}$ for $x \ge 0$ and

$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t = F(1) = \frac{\pi}{2\mathrm{e}}.$$

Remarks: This exercise is based on a video from the Youtube channel "Flammable Maths", who's author Jens Fehlau has shot several nice videos with quite nontrivial evaluations of interesting integrals.
Since F is even, we have $F(x) = (\pi/2)\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$. At $x = 0$ the function F is not differentiable, although the right-hand side of the integral representation

$$F'(x) = -\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t, \qquad \text{valid for } x \ne 0,$$

evaluates to zero at $x = 0$.



Numerically, $\pi/(2\mathrm{e}) \approx 0.5778636748954609$. This differs only slightly from the Euler-Mascheroni constant $\gamma = \lim_{n\to\infty}(1 + 1/2 + 1/3 + \cdots + 1/n - \ln(n)) \approx 0.5772156649015329$, so that perhaps someone who computes the integral $\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t$ numerically but doesn't know about the exact evaluation is mislead to conjecture that it has the value γ.

19 First a remark on the cases $L = \pm\infty$. If (x_n) is unbounded then (and only then) for every $R \in \mathbb{R}$ there exist infinitely many indexes n such that $x_n > R$, and hence it is natural to call $+\infty$ an accumulation point of (x_n) and set $L = +\infty$ in this case. On the other hand, if (x_n) diverges to $-\infty$ then (and only then) for every $R \in \mathbb{R}$ there exist only finitely many indexes n such that $x_n > R$, but of course infinitely many indexes n such that $x_n < R$, and hence it is natural to call $-\infty$ an accumulation point of (x_n) and set $L = -\infty$ in this case, since there is no other accumulation point. The case $L = -\infty$ doesn't occur for nonnegative sequences like $x_n = \sqrt[n]{|a_n|}$.

a) Suppose the power series converges for some $z_1 \neq a$ and set $r = |z_1 - a|$, which is then > 0. Since $\sum a_n(z_1 - a)^n$ converges, there exists a constant $M > 1$ such that $|a_n(z_1 - a)^n| = |a_n| r^n \leq M$ for all n. Hence

$$\sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{M}}{r} \leq \frac{M}{r} \quad \text{for all } n,$$

contradicting the unboundedness of $\sqrt[n]{|a_n|}$.

b) Assume $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon^n$ for $n > N$. Now let $z \in \mathbb{C} \setminus \{a\}$ be arbitary and $r = |z - a|$, i.e., $|a_n(z - a)^n| = |a_n| r^n$. Setting $\epsilon = 1/(2r)$ and denoting by N the corresponding response, we get

$$|a_n| r^n \leq \left(\frac{1}{2r}\right)^n r^n = \frac{1}{2^n} \quad \text{for } n > N.$$

Since $\sum 2^{-n}$ converges, the series $\sum a_n(z-a)^n$ converges absolutely by the comparison test. In particular $\sum a_n(z - a)^n$ converges for all $z \in \mathbb{C}$ (including $z = a$, of course).

c) Suppose first that $z \neq a$ satisfies $r = |z - a| < 1/L$. Then $L < 1/r$, and hence there exist $\theta \in (0, 1)$ and $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} \leq \theta/r$ for all $n > N$. (The number θ need only satisfy $L < \theta/r < 1/r$, i.e., $\theta \in (rL, 1)$. Then there can be only finitely many n such that $\sqrt[n]{|a_n|} > \theta/r$.) From this we obtain $|a_n| r^n \leq \theta^n$ for $n > N$ and can use the comparison test with the convergent series $\sum \theta^n$ to conclude that $\sum_{n=0}^{\infty} a_n(z - a)^n$ converges.

Next suppose $r = |z - a| > 1/L$. Then $1/r < L$, and hence $\sqrt[n]{|a_n|} > 1/r$ for infinitely many n. Thus $|a_n| r^n > 1$ for infinitely many n, implying the divergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$. (Since convergence requires $|a_n| r^n \to 0$.)