Differential Equations Plus (Math 286)

H45 For $\alpha, \beta \in \mathbb{C}$ consider the explicit so-called Euler equation

$$y'' + \frac{\alpha}{t}\,y' + \frac{\beta}{t^2}\,y = 0 \qquad (t > 0). \tag{1}$$

a) Show that $\phi\colon \mathbb{R}^+ \to \mathbb{C}$ is a solution of (1) iff $\psi\colon \mathbb{R} \to \mathbb{C}$ defined by $\psi(s) = \phi(\mathrm{e}^s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0. \tag{2}$$

b) Using a), determine the general solution of (1) for $(\alpha, \beta) = (6, 4)$ and $(3, 1)$.

H46 The solution to this exercise provides an easy method for computing $\mathrm{e}^{\mathbf{A}t}$ for a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbb{R}^{2\times 2}$ (or $\mathbb{C}^{2\times 2}$). We assume throughout the exercise that $\mathbf{A}$ is not a scalar multiple of the identity matrix $\mathbf{I}_2$.

a) Show $\mathbf{A}^2 - (a + d)\mathbf{A} + (ad - bc)\mathbf{I}_2 = \mathbf{0}$ (the all-zero 2×2 matrix).

b) Use a) to show that there exist uniquely determined functions $c_0, c_1\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{e}^{\mathbf{A}t} = c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A} \quad \text{for } t \in \mathbb{R}. \tag{$\star$}$$

Further, show that c_0, c_1 are at least twice differentiable.

c) Show that c_0, c_1 solve the homogeneous linear ODE of order 2 with characteristic polynomial $X^2 - (a + d)X + ad - bc$ and satisfy the initial conditions $c_0(0) = 1$, $c_0'(0) = 0$ and $c_1(0) = 0$, $c_1'(0) = 1$.
Hint: Differentiate $(\star)$ twice.

d) By solving the IVP's in c) determine $\mathrm{e}^{\mathbf{A}t}$ for $\mathbf{A} = \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}$.

H47 Solve the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\mathbf{y} + \begin{pmatrix} t \\ \sin t \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Due on Fri Nov 12, 6 pm

The matrix exponential function (relevant for H46) will be discussed in the lecture on Mon Nov 8. You are advised to do H45 and H46 before the midterm, because Euler equations and simple (2×2) instances of the matrix exponential function can be the subject of a midterm question; cf. the online midterm samples. Exercise H47, which is computationally intensive, can be safely considered only after the midterm.

Solutions

45 a) If $\psi(s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0,$$

we can use the variable substitution $\psi(s) = \phi(e^s)$ and get the first and second derivative of ψ as

$$\psi'(s) = \phi'(e^s)e^s,$$
$$\psi''(s) = [\phi'(e^s)e^s]' = \phi''(e^s)e^{2s} + \phi'(e^s)e^s.$$

This gives

$$\psi(s)'' + (\alpha - 1)\psi(s)' + \beta\psi(s) = 0,$$
$$[\phi''(e^s)e^{2s} + \phi'(e^s)e^s] + (\alpha - 1)\phi'(e^s)e^s + \beta\phi(s) = 0.$$

This simplifies to

$$e^{2s}\phi''(e^s) + \alpha e^s\phi'(e^s) + \beta\phi(e^s) = 0.$$

Since $t > 0$ is assumed, we can make the variable transformation $t = e^s$, i.e. $s = \ln(t)$, and obtain

$$t^2\phi''(t) + \alpha t\phi'(t) + \beta\phi(t) = 0,$$
$$\phi''(t) + \frac{\alpha}{t}\phi'(t) + \frac{\beta}{t^2}\phi(t) = 0.$$

Therefore, the function $\phi(t)$ is a solution of Equation (1).

Conversely, if $y(t) = \phi(t)$ is a solution of Equation (1), we can make the substitution $t = e^s$ to obtain a function $y(s) = \psi(s)$. The derivatives can be represented as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s},$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s}) = \frac{1}{t^2}(\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}).$$

Hence the Euler equation is converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}s} + \beta y = 0$$

.

b) 1) When $(\alpha, \beta) = (6, 4)$, the Euler equation can be converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + 5\frac{\mathrm{d}y}{\mathrm{d}s} + 4y = 0.$$

The corresponding characteristic equation is $X^2 + 5X + 4 = 0$, and hence the general solution of Equation (2) in this case is

$$y(s) = C_1e^{-s} + C_2e^{-4s}, \quad s \in \mathbb{R}.$$

Therefore, the solution of Equation (1) is

$$y(t) = \frac{C_1}{t} + \frac{C_2}{t^4}, \quad t > 0.$$

2) When $(\alpha, \beta) = (3, 1)$, the Euler equation can be converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + 2\frac{\mathrm{d}y}{\mathrm{d}s} + y = 0.$$

The corresponding characteristic equation is $X^2 + 2X + 1 = (X + 1)^2 = 0$, and hence the general solution of Equation (2) in this case is

$$y(s) = C_1 e^{-s} + C_2 s e^{-s}, \quad s \in \mathbb{R}.$$

Therefore, the solution of Equation (1) is

$$y(t) = \frac{C_1}{t} + \frac{C_2 \ln t}{t}, \quad t > 0.$$

46 a) We have

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix} \\
&= (a+d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a^2 + bc - (a+d)a & 0 \\ 0 & d^2 + bc - (a+d)d \end{pmatrix} \\
&= (a+d)\mathbf{A} - (ad - bc)\mathbf{I}_2.
\end{aligned}$$

b) The relation $\mathbf{A}^2 = \alpha \mathbf{I}_2 + \beta \mathbf{A}$ ($\alpha = -(ad - bc)$, $\beta = a + d$) can be used to express any power $\mathbf{A}^k$ as a linear combination of $\mathbf{I}_2$ and $\mathbf{A}$. It follows that

$$\sum_{k=0}^{n} \frac{t^k}{k!} \mathbf{A}^k = f_n(t)\mathbf{I}_2 + g_n(t)\mathbf{A}$$

for certain functions $f_n, g_n \colon \mathbb{R} \to \mathbb{R}$. Since the left-hand side converges to $\mathrm{e}^{\mathbf{A}t}$, so does the right-hand side, and hence the function sequences (f_n), (g_n) converge (point-wise) to functions f resp. g such that $\mathrm{e}^{\mathbf{A}t} = f(t)\mathbf{I}_2 + g(t)\mathbf{A}$ for $t \in \mathbb{R}$. This representation is unique, since $\mathbf{I}_2$ and $\mathbf{A}$ are assumed to be linearly independent.

From the lecture we know that $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}\mathrm{e}^{\mathbf{A}t}$. This can be iterated to yield $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}^2\mathrm{e}^{\mathbf{A}t}$ (and likewise $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}^k\mathrm{e}^{\mathbf{A}t}$ for all $k \in \mathbb{N}$). Hence the functions f, g, which are the coordinate functions of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ with respect to the "matrix basis" $\mathbf{I}, \mathbf{A}$, are twice differentiable as well (even of class C^∞).

Thus the assertion holds with $c_0 = f$, $c_1 = g$.

c) Using the observation made about the derivatives of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ in b), we obtain

$$\begin{aligned}
\mathrm{e}^{\mathbf{A}t} &= c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A}, \\
\mathbf{A}\mathrm{e}^{\mathbf{A}t} &= c_0'(t)\mathbf{I}_2 + c_1'(t)\mathbf{A}, \\
\mathbf{A}^2\mathrm{e}^{\mathbf{A}t} &= c_0''(t)\mathbf{I}_2 + c_1''(t)\mathbf{A}
\end{aligned} \tag{$\star\star$}$$

for $t \in \mathbb{R}$. Together with a) this yields

$$\begin{aligned}
\mathbf{0} &= \mathbf{A}^2 2\mathrm{e}^{\mathbf{A}t} - (a+d)\mathbf{A}\mathrm{e}^{\mathbf{A}t} + (ad - bc)\mathrm{e}^{\mathbf{A}t} \\
&= \big(c_0''(t) - (a+d)c_0'(t) + (ad - bc)c_0(t)\big)\mathbf{I}_2 + \big(c_1''(t) - (a+d)c_1'(t) + (ad - bc)c_1(t)\big)\mathbf{A}
\end{aligned}$$

for all $t \in \mathbb{R}$, which can only hold if the coefficient functions of $\mathbf{I}_2$ and $\mathbf{A}$ vanish, i.e., c_0, c_1 solve the homogeneous linear ODE with characteristic polynomial $X^2 - (a + d)X + ad - bc$. The asserted initial conditions follow by substituting $t = 0$ into the 1st and 2nd equation of $(\star\star)$ and comparing coefficients of $\mathbf{I}_2, \mathbf{A}$.

d) According to a) the given matrix satisfies $\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}_2 = \mathbf{0}$.
$\implies c_0, c_1$ solve $y'' - y' - 6y = 0$, which has characteristic polynomial $X^2 - X - 6 = (X + 2)(X - 3)$. The general solution of this ODE is $y(t) = a_1\,\mathrm{e}^{-2t} + a_2\,\mathrm{e}^{3t}$, with initial conditions $y(0) = a_1 + a_2$, $y'(0) = -2\,a_1 + 3\,a_2$. A short computation yields $c_0(t) = \frac{3}{5}\,\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{3t}$, $c_1(t) = -\frac{1}{5}\,\mathrm{e}^{-2t} + \frac{1}{5}\,\mathrm{e}^{3t}$, and hence

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \left(\frac{3}{5}\,\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{3t}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \left(-\frac{1}{5}\,\mathrm{e}^{-2t} + \frac{1}{5}\,\mathrm{e}^{3t}\right)\begin{pmatrix}0 & 6\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{5}\begin{pmatrix}3\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{3t} & -6\,e^{-2t} + 6\,\mathrm{e}^{3t}\\ -\mathrm{e}^{-2t} + \mathrm{e}^{3t} & 2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{3t}\end{pmatrix}. \end{aligned}$$

47 The identity $\mathbf{A}^2 = 7\mathbf{A}$ is easily verified. It is a special case of H46 a). From it we obtain $\mathbf{A}^3 = 7\mathbf{A}^2 = 7^2\mathbf{A}$, $\mathbf{A}^4 = 7^2\mathbf{A}^2 = 7^3\mathbf{A}$, etc., and in general $\mathbf{A}^k = 7^{k-1}\mathbf{A}$ for $k \in \mathbb{N}$ by induction.

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathbf{I}_2 + t\,\mathbf{A} + \frac{t^2}{2!}\,\mathbf{A}^2 + \frac{t^3}{3!}\,\mathbf{A}^3 + \cdots\\ &= \mathbf{I}_2 + t\,\mathbf{A} + \frac{7t^2}{2!}\,\mathbf{A} + \frac{7^2t^3}{3!}\,\mathbf{A} + \cdots\\ &= \mathbf{I}_2 + \frac{\mathrm{e}^{7t} - 1}{7}\,\mathbf{A} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\frac{1}{7}\,(\mathrm{e}^{7t} - 1) & \frac{2}{7}\,(\mathrm{e}^{7t} - 1)\\ \frac{3}{7}\,(\mathrm{e}^{7t} - 1) & \frac{6}{7}\,(\mathrm{e}^{7t} - 1)\end{pmatrix} = \begin{pmatrix}\frac{1}{7}\,(\mathrm{e}^{7t} + 6) & \frac{2}{7}\,(\mathrm{e}^{7t} - 1)\\ \frac{3}{7}\,(\mathrm{e}^{7t} - 1) & \frac{1}{7}\,(6\,\mathrm{e}^{7t} + 1)\end{pmatrix}. \end{aligned}$$

The particular solution $\mathbf{y}(t)$ of the inhomogeneous system satisfying $\mathbf{y}0 = \mathbf{0}$ is of the form $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t)$ with

$$\begin{aligned} \mathbf{c}(t) &= \int_0^t \mathrm{e}^{-\mathbf{A}s}\,\mathbf{b}(s)\,\mathrm{d}s \qquad\qquad (\text{since } \mathbf{c}(0) = \mathbf{0})\\ &= \int_0^t \begin{pmatrix}\frac{1}{7}\,(\mathrm{e}^{-7s} + 6) & \frac{2}{7}\,(\mathrm{e}^{-7s} - 1)\\ \frac{3}{7}\,(\mathrm{e}^{-7s} - 1) & \frac{1}{7}\,(6\,\mathrm{e}^{-7s} + 1)\end{pmatrix}\begin{pmatrix}s\\ \sin s\end{pmatrix}\mathrm{d}s\\ &= \frac{1}{7}\int_0^t \left[\begin{pmatrix}6 & -2\\ -3 & 1\end{pmatrix} + \mathrm{e}^{-7s}\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}\right]\begin{pmatrix}s\\ \sin s\end{pmatrix}\mathrm{d}s\\ &= \frac{1}{7}\int_0^t \begin{pmatrix}6s - 2\sin s + s\,\mathrm{e}^{-7s} + 2\sin s\,\mathrm{e}^{-7s}\\ -3s + \sin s + 3s\,\mathrm{e}^{-7s} + 6\sin s\,\mathrm{e}^{-7s}\end{pmatrix}\mathrm{d}s\\ &= \frac{1}{7}\begin{pmatrix}3\,t^2 - \frac{1}{49}\,(7\,t + 1)e^{(-7\,t)} - \frac{1}{25}\,(\cos(t) + 7\,\sin(t))e^{(-7\,t)} + 2\,\cos(t) - \frac{2376}{1225}\\ -\frac{3}{2}\,t^2 - \frac{3}{49}\,(7\,t + 1)e^{(-7\,t)} - \frac{3}{25}\,(\cos(t) + 7\,\sin(t))e^{(-7\,t)} - \cos(t) - \frac{2376}{1225}\end{pmatrix}. \end{aligned}$$

$$\implies \mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t) = \begin{pmatrix}\frac{3}{7}\,t^2 - \frac{1}{49}\,t + \frac{7}{25}\,\cos(t) + \frac{74}{8575}\,e^{(7\,t)} - \frac{1}{25}\,\sin(t) - \frac{99}{343}\\ -\frac{3}{14}\,t^2 - \frac{3}{49}\,t - \frac{4}{25}\,\cos(t) + \frac{222}{8575}\,e^{(7\,t)} - \frac{3}{25}\,\sin(t) + \frac{46}{343}\end{pmatrix}$$

For the last two steps the computer algebra system SageMath was used.