Differential Equations Plus (Math 286)

H71 Find the following convolutions and their Laplace transforms (three answers suffice):

a) $t^2 * t^3$; b) $\mathrm{J}_0 * \mathrm{J}_0$; c) $\sin t * \cos(2t)$; d) $\mathrm{u}(t-1) * t$.

a) $\mathcal{L}\{t^2\} \cdot \mathcal{L}\{t^3\} = \mathcal{L}\{t^2 * t^3\}$

$\Rightarrow \frac{2!}{s^3} \cdot \frac{3!}{s^4} = \frac{12}{s^7} = \frac{12}{6!} \frac{6!}{s^7}$

$\Rightarrow t^2 * t^3 = \frac{1}{60} t^6$

b) $\mathcal{L}\{J_0 * J_0\} = \mathcal{L}\{J_0\} \cdot \mathcal{L}\{J_0\}$

$= \frac{1}{s^2+1} \Rightarrow \mathcal{L}\{J_0 * J_0\} = \mathcal{L}\{\sin t\} \Rightarrow J_0 * J_0 = \sin t$

c) $\mathcal{L}\{\sin t * \cos(2t)\}$

$= \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{\cos(2t)\}$

$= \frac{1}{s^2+1} \cdot \frac{s}{s^2+4} = \frac{s}{(s^2+1)(s^2+4)}$

$= \frac{As}{(s^2+1)} + \frac{Bs}{(s^2+4)}$

$\Rightarrow As(s^2+4) + Bs(s^2+1) = s$

$\Rightarrow 4A + B = 1 \Rightarrow A = \frac{1}{3} \quad B = -\frac{1}{3}$

$\Rightarrow \mathcal{L}\{\sin t * \cos(2t)\}$

$= \frac{1}{3}\left(\frac{s}{s^2+1} - \frac{s}{s^2+4}\right)$

$\Rightarrow \sin t * \cos 2t = \frac{1}{3}(\cos t - \cos 2t)$

d) $u(t-1) * t = \int_0^t \lambda u(t-\lambda-1)\, d\lambda$

$$\begin{cases} = \int_0^{t-1} \lambda \, d\lambda = \frac{1}{2}(t-1)^2 & t > 1 \\ = 0 & t \in (0,1) \end{cases}$$

H72 Solve the following IVP's with the Laplace transform:

a) $y'' + y' + y = \mathrm{u}_\pi(t) - \mathrm{u}_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0;$

b) $y'' + 2y' + y = \begin{cases} \sin(2t) & \text{if } 0 \le t \le \pi/2, \\ 0 & \text{if } t > \pi/2, \end{cases} \quad y(0) = 1,\ y'(0) = 0.$

a) $s^2 Y - s + sY - 1 + Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$

$Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+s+1)} + \frac{s+1}{(s^2+s+1)} = \frac{e^{-\pi s} - e^{-2\pi s}}{s} + (1 - e^{-\pi s} + e^{-2\pi s})\left(\frac{s+1}{s^2+s+1}\right)$

$\Rightarrow \mathcal{L}^{-1}\{Y\} = y_1(t) - u_\pi(t)\, y_1(t-\pi) + u_{2\pi}(t)\, y_1(t-2\pi)$

$y_1(t) = e^{-t/2} \cos \frac{\sqrt{3} t}{2} + \frac{1}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t$

$$y(t) = \begin{cases} y_1(t) & t \in (0,\pi) \\ 1 + y_1(t) - y_1(t-\pi) & t \in [\pi, 2\pi) \\ y_1(t) - y_1(t-\pi) + y_1(t-2\pi) & t \ge 2\pi \end{cases}$$

function $= \sin(2t) - u(t-\frac{\pi}{2}) \cdot \sin(2t$.

b) $s^2 Y - s + 2(sY - 1) + Y = \frac{2}{s^2+4}(1 + e^{-\pi s/2})$

$Y = \frac{2 + 2e^{-\pi s/2}}{(s^2+4)(s^2+1)} + \frac{s+2}{(s+1)^2}$

$\Rightarrow \frac{1}{(s^2+4)(s+1)^2} = -\frac{2s+3}{25(s^2+4)} + \frac{2}{25(s+1)} + \frac{1}{5(s+1)^2}$

$\Rightarrow Y(s) = (2 + 2e^{-\pi s/2})\left(-\frac{2s+3}{25(s^2+4)} + \frac{2}{25(s+1)} + \frac{1}{5(s+1)^2}\right) + \frac{1}{s+1} + \frac{1}{(s+1)^2}$

$\Rightarrow y(t) = \frac{-4}{25}\cos(2t) - \frac{3}{25}\sin(2t) - \frac{4}{25}e^{-t} + \frac{2}{5}te^{-t} - \frac{4}{25}u(t-\frac{\pi}{2})\cos(2t-\pi) - \frac{3}{25}u(t-\frac{\pi}{2})\sin(2t-\pi) + \frac{4}{25}u(t-\frac{\pi}{2})e^{-t+\frac{\pi}{2}} + \frac{2}{5}(t-\frac{\pi}{2})e^{-(t-\pi/2)} + e^{-t} + te^{-t}$

$$= \begin{cases} -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{29}{25}e^{-t} + \frac{7}{5}te^{-t} & t \le \frac{\pi}{2}. \\ \frac{29 + (4 - 5\pi)e^{\pi/2}}{25} e^{-t} + \frac{7 + 2e^{\pi/2}}{5} te^{-t} & t > \frac{\pi}{2} \end{cases}$$

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H73 In each of the following cases, let S be the set of vectors $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ satisfying the given condition. Decide whether S is a subspace of $\mathbb{C}^3/\mathbb{C}$ and, if so, determine the dimension of S.

a) $\alpha = 0$; b) $\alpha\beta = 0$;
c) $\alpha + \beta = 1$; d) $\alpha + \beta = 0$;
e) $\alpha = 3\beta \wedge \beta = (2 - \mathrm{i})\gamma$; f) $\alpha \in \mathbb{R}$.

H74 Let P_3 be the vector space (over $\mathbb{R}$) of polynomials $p(X) \in \mathbb{R}[X]$ of degree at most 3. Repeat the previous exercise for the sets $S \subseteq P_3$ defined by each of the following conditions:

a) $p(X)$ has degree 3; b) $2p(0) = p(1)$;
c) $p(t) \geq 0$ for $0 \leq t \leq 1$; d) $p(t) = p(1 - t)$ for all $t \in \mathbb{R}$.

H75 Do Exercises 11 and 20 in [BDM17], Ch. 7.1.

H76 Find $\mathbf{S}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix for

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Show that $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$ for $k \in \mathbb{N}$, and use this to obtain explicit formulas for the entries of $\mathbf{A}^k$.

H77 *Optional Exercise*

Do Exercise 18 in [BDM17], Ch. 6.5.

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H78 *Optional Exercise*

Repeat Exercises 20, 21 in [BDM17], Ch. 6.6, for the integro-differential equation

$$\phi'(t) = \sin t + \int_0^t \phi(t-\xi)\cos\xi \,\mathrm{d}\xi, \quad \phi(0) = 2.$$

Hint: It may be helpful to use the commutativity of the convolution product.

H79 *Optional Exercise*

Suppose V is a vector space over a field F.

a) Using the vector space axioms, prove the *scalar zero law*

$$0_F\, v = 0_V \quad \text{for all } v \in V.$$

b) Similarly, prove the *vector zero law*

$$a\, 0_V = 0_V \quad \text{for all } a \in F.$$

c) Prove that $(-1)x = -x$ for all $x \in V$.

H80 *Optional Exercise*

a) Write down the linear system of equations satisfied by a classical 3×3 magic square and transform this system into row-echelon form. (What is the magic number in this case?)

b) Use the equations in a) to show that up to obvious symmetries there exists exactly one classical 3×3 magic square.

Due on Fri May 13, 6 pm

The optional exercises can be handed in until Fri May 20, 6 pm.

H76 Find $\mathbf{S}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix for

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Show that $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$ for $k \in \mathbb{N}$, and use this to obtain explicit formulas for the entries of $\mathbf{A}^k$.

As $D = S^{-1}AS \Rightarrow A = SDS^{-1} \Rightarrow A^k = \underbrace{SD\cancel{S^{-1}} \cdot \cancel{S}D\cancel{S^{-1}} \cdots \cancel{S}D \cdot S^{-1}}_{k\text{ terms}} = SD^kS^{-1}$

$\Rightarrow$ Q.E.D.

As $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$

$\Rightarrow (\lambda-2)^2 - 1 = 0 \quad \therefore (\lambda-3)(\lambda-1) = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 3$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{-2} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow D = S^{-1}A \cdot S = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow D^k = \begin{pmatrix} 1^k & \\ & 3^k \end{pmatrix} = \begin{pmatrix} 1 & \\ & 3^k \end{pmatrix}$

$\Rightarrow A^k = SD^kS^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 3^k \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 3^k \\ 1 & -3^k \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{3^k}{2} & \frac{1}{2}-\frac{3^k}{2} \\ \frac{1}{2}-\frac{3^k}{2} & \frac{1}{2}+\frac{3^k}{2} \end{pmatrix}$

H75 Do Exercises 11 and 20 in [BDM17], Ch. 7.1.

Ex.11

For Block 1: $a = \frac{d^2x}{dt^2} \Rightarrow m_1\frac{d^2x_1}{dt^2} = m_1 a = -k_1x_1 + F_1(t) + k_2(x_2 - x_1)$

$= -(k_1+k_2)x_1 + k_2x_2 + F_1(t)$

For Block 2: $\Sigma F_2 = m\frac{d^2x_2}{dt^2} = -(k_2+k_3)x_2 + k_2x_1 + F_2(t)$

Ex.20: $V_L = L\frac{dI}{dt} \quad I_C = C\frac{dV}{dt}$

$V_1 = I_1 \cdot R \quad V_2 = I_2R_2$

$\Rightarrow L\frac{dI}{dt} + I_1R_1 + I_2R_2 = 0 \Rightarrow I_1 = I_2 + I_C = \frac{V_2}{R} + C\frac{dV}{dt}$

H74 Let P_3 be the vector space (over $\mathbb{R}$) of polynomials $p(X) \in \mathbb{R}[X]$ of degree at most 3. Repeat the previous exercise for the sets $S \subseteq P_3$ defined by each of the following conditions:

a) $p(X)$ has degree 3; $\quad$ b) $2p(0) = p(1)$;

c) $p(t) \geq 0$ for $0 \leq t \leq 1$; $\quad$ d) $p(t) = p(1-t)$ for all $t \in \mathbb{R}$.

a) No. For $P_1 = a + bx + cx^2 + x^3$, $P_2 = a + bx + cx^2 - x^3$, $P_1 + P_2 = a + bx + cx^2$ is at most 2 $\Rightarrow$ degree=3 doesnot satisfy.

b) for $p(x) = a_0 + a_1x + a_2x^2 + x^3$

$2a_0 = a_0 + a_1 + a_2 + a_3$ or $a_0 - a_1 - a_2 - a_3 = 0$ $\Rightarrow a_0 = a_1 + a_2 + a_3$

$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1-a_2-a_3 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_3\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\Leftrightarrow p(x) = a_1(x-1) + a_2(x^2-1) + a_3(x^3-1)$ from basis of S

dimension of 3: $(x-1)$, (x^2-1), (x^3-1)

c) $p(x) \in S$: then $(-1)p(t) = -p(t) \leq 0$ for $0 \leq t \leq 1$

$\Rightarrow p(x)$ is not identically zero on $[0,1]$

$\Rightarrow -p(x) \notin X \Rightarrow$ No.

d): $p(t) = p(1-t) \Rightarrow$
$a_0 = a_0 + a_1 + a_2 + a_3$
$a_1 = -a_1 - 2a_2 - 3a_3$
$a_2 = a_2 + 3a_3$
$a_3 = -a_3$

$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ -a_1 \\ a_2 \\ 0 \end{pmatrix} = a_0\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

polynomial 1 & $x^2 - x$ from basis of S $\quad$ dimension 2

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a) $\alpha = 0$;	b) $\alpha\beta = 0$;
c) $\alpha + \beta = 1$;	d) $\alpha + \beta = 0$;
e) $\alpha = 3\beta \wedge \beta = (2 - \mathrm{i})\gamma$;	f) $\alpha \in \mathbb{R}$.

Suppose V is a vector over a field F

A subset $U \subseteq V$ is a subspace of V if U satisfies:

1) U contains 0_V

2) $U + U = \{x+y;\ x, y \in U\} \subseteq U$

3) $FU = \{ax;\ a \in F;\ x \in U\} \subseteq U$

a) $\begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} = \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ dimension = 2

b) $\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \\ \gamma_1 + \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ not satisfy the condition

$\Rightarrow$ not a subspace.

c) $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ doesn't satisfy the condition.

$\Rightarrow$ Not. a subspace.

d) $P_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$, $P_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix}$ $\Rightarrow P_1 + P_2 = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \\ \gamma_1 + \gamma_2 \end{pmatrix}$ $kP_1 = \begin{pmatrix} k\alpha_1 \\ k\beta_1 \\ k\gamma_1 \end{pmatrix}$

$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 0 \\ k\alpha_1 + k\beta_1 = 0 \end{cases}$ $\Rightarrow$ satisfy the condition

$\Rightarrow P_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha_1 \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \gamma_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ dimension = 2

e) $\alpha_1 + \alpha_2 = 3(\beta_1 + \beta_2)$ $\Rightarrow P_1 + P_2, kP_1 \in S$. $P_1 \in S$

$\beta_1 + \beta_2 = (2-i)(\gamma_1 + \gamma_2)$ $P_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} 3(2-i)\gamma_1 \\ (2-i)\gamma_1 \\ \gamma_1 \end{pmatrix} = \gamma_1 \begin{pmatrix} 6-3i \\ 2-i \\ 1 \end{pmatrix}$ $\Rightarrow$ is a subspace & dimension of 1

f) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in S$

but $i \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin S$

$\Rightarrow$ not a subspace.