H49 a) $(2t+1)y''+(4t-2)y'-8y=(6t^2+t-3)e^t$

has a sol of $e^{\alpha t}$ in homo case.

$$\therefore (2t+1)\alpha^2 e^{\alpha t}+(4t-2)\alpha e^{\alpha t}-8e^{\alpha t}=0$$
$$\therefore (2\alpha^2 t+\alpha^2+4\alpha t-2\alpha-8)e^{\alpha t}=0$$
$$\therefore (2\alpha^2 t+4\alpha t)+(\alpha^2-2\alpha-8)=0$$
$$\Rightarrow \begin{cases} 2\alpha^2+4\alpha=0 \\ \alpha^2-2\alpha-8=0 \end{cases} \Rightarrow \alpha=-2$$

$\Rightarrow y=e^{-2t}$ is a sol. for homo case.

$$(2t+1)y''+(4t-2)y'-8y=(6t^2+t-3)e^t$$

let solution be $u(t)\cdot e^{-2t}=g(t)$

$$\Rightarrow \frac{dg(t)}{dt}=u'e^{-2t}-2ue^{-2t}=(u'-2u)e^{-2t}$$
$$\frac{d^2g(t)}{dt}=u''e^{-2t}-2u'e^{-2t}-2(u'e^{-2t}-2ue^{-2t})$$
$$=u''e^{-2t}-2u'e^{-2t}-2u'e^{-2t}+4ue^{-2t}$$
$$=(u''-4u'+4u)e^{-2t}$$

$$\Rightarrow (2t+1)y''+(4t-2)y'-8y=(6t^2+t-3)e^t$$
$$(2t+1)(u''-4u'+4u)+(4t-2)(u'-2u)-8u=0$$
$$\Rightarrow (2tu''-8tu'+\cancel{8tu}+u''-4u'+\cancel{4u})$$
$$+\quad 4tu'-\cancel{8tu}\quad -2u'+\cancel{4u}$$
$$+\quad \cancel{8u}=0$$

let $z(t)=u'(t)\Rightarrow z'(t)=u''(t)$

$$\Rightarrow (2t+1)u''-(4t+6)u'=0$$
$$\Rightarrow z'-\left(\frac{4t+6}{2t+1}\right)z=0$$
$$\Rightarrow z=e^{\int\frac{4t+6}{2t+1}dt}=e^{2t+\ln|2t+1|+C}=e^{2t}(2t+1)^2$$
$$\Rightarrow u=\int z\,dt=\int e^{2t}(2t+1)^2dt \Rightarrow u=\tfrac{1}{2}(4t^2+1)e^{2t}$$

$\therefore$ another sol. for homo. case is $(4t^2+1)$

$$\text{Wronskian}=\begin{vmatrix} e^{-2t} & 4t^2+1 \\ -2e^{-2t} & 8t \end{vmatrix}=8te^{-2t}+(8t^2+2)e^{-2t}$$
$$=2(2t+1)^2e^{-2t}\neq 0 \Rightarrow \text{they are basic sols.}$$

$$\Rightarrow Y=c_1\cdot e^{-2t}+c_2\cdot(4t^2+1)$$

For $\tilde{y}$: $(2t+1)y''+(4t-2)y'-8y=(6t^2+t-3)e^t$

guess a form of $\left(\sum_{i=0}^{n}P_ix^i\right)\cdot e^t$

$$\Rightarrow (2t+1)\left[\left(\sum_{i=0}^{n}P_ix^i\right)''+2\left(\sum_{i=0}^{n}P_ix^i\right)'+\sum_{i=0}^{\infty}P_ix^i\right]+(4t-2)\cdot\left[\left(\sum_{i=0}^{n}P_ix^i\right)'+\sum_{i=0}^{\infty}P_ix^i\right]-8\sum_{i=0}^{\infty}P_ix^i=(6t^2+t-3)$$

$\Rightarrow$ easily we can define that the order would be one as max order of $6t^2+t-3$ is two

$$\therefore (2t+1)\cdot[2a+(at+b)]+(4t-2)[a+(at+b)]+8\cdot(at+b)=6t^2+t-3$$
$$4at+2a+2at^2+(a+b)t+b+4at-2a+4at^2+(4b-2a)t-2b+8at+8b=6t^2+t-3$$
$$\Rightarrow a=1\quad b=\tfrac{1}{3}\quad \therefore \tilde{y}=(t+\tfrac{1}{3})e^t$$

$$\boxed{\therefore y=Y+\tilde{y}=c_1\cdot e^{-2t}+c_2\cdot(4t^2+1)+(t+\tfrac{1}{3})e^t \qquad t>-\tfrac{1}{2}}$$

C): $(t-2)^2y''+3(t-2)y'+2y=t^2+1 \qquad t>2$

$$\Rightarrow \text{let } x=t-2 \Rightarrow x^2y''+3xy'+2y=x^2+4x+5 \Rightarrow x>0$$
$$x=e^{\lambda} \Rightarrow y''+3y'+2y=e^{2x}+4e^x+5$$
$$\Rightarrow r^2+2r+2=0$$
$$r_1=-1+i \quad r_2=-1-i$$
$$\Rightarrow Y=e^{-x}\cos x+e^{-x}\sin x$$
$$\Rightarrow Y=\frac{\cos(\ln(t-2))}{t-2}+\frac{\sin(\ln(t-2))}{t-2}$$

$$\tilde{y}_1: e^{2x} \Rightarrow n=0\ \alpha=2 \qquad \tilde{y}_2: 4e^x \Rightarrow n=0\ \alpha=1 \qquad \tilde{y}_3=5\ n=0\ \alpha=0$$
$$\Rightarrow \tilde{y}_1=\frac{e^{2x}}{P(2)}=\frac{e^{2x}}{10} \qquad \tilde{y}_2=\frac{4e^x}{P(1)}=\frac{4e^x}{5} \qquad \tilde{y}_3=\frac{5}{P(0)}=\frac{5}{2}$$
$$\Rightarrow \tilde{y}=\frac{(t-2)^2}{10}+\frac{4(t-2)^2}{5}+\frac{5}{2}$$

$$\boxed{\Rightarrow y=Y+\tilde{y}=\frac{\cos(\ln(t-2))}{t-2}+\frac{\sin(\ln(t-2))}{t-2}+\frac{(t-2)^2}{10}+\frac{4(t-2)^2}{5}+\frac{5}{2}\quad t>2}$$

H50 Proof. by induction

$$n=1: DF=f'g+g'f=\sum_{k=0}^{1}\binom{1}{k}(D^kf)(D^{n-k}g)$$

Suppose D^nF satisfy this rule

For $D^{n+1}F$ case: $\sum_{k=0}^{n}\binom{n}{k}(D^kf)(D^{n-k}g)$

$$=\sum_{k=0}^{n}\binom{n}{k}(D^{k+1}f)(D^{n-k}g)+\binom{n}{k}(D^kf)(D^{n-k+1}g)$$
$$=\sum_{k=0}^{n+1}\binom{n}{k}(D^{k+1}f)(D^{n-k}g)+\binom{n}{k+1}(D^{k+1})(D^{n-k}g)$$
$$=\sum_{k=0}^{n+1}\binom{n+1}{k+1}(D^{k+1}f)(D^{n+1-(k+1)}g)$$

$\Rightarrow$ Induction defined.

$\Rightarrow$ Q.E.D.

H51 a) Proof. by Induction: $n=1 \Rightarrow D^1[e^{-t^2}]=e^{-t^2}\cdot(-2t)=f_1(t)\cdot e^{-t^2}$

suppose $n=n$, the conclusion defined $\Rightarrow n=n+1$

$$D[f_n(t)e^{-t^2}]$$
$$=f_n'(t)e^{-t^2}+f_n(t)\cdot(-2t)\cdot e^{-t^2}$$
$$=e^{-t^2}[f_n'(t)-2tf_n(t)]$$
$$=e^{-t^2}\cdot f_{n+1}(t)$$

$\Rightarrow n+1$ induction defined

$$H_n=e^{t^2}\cdot(-1)^n\cdot D^n[e^{-t^2}]=(-1)^n\cdot f_n(t)$$

$\Rightarrow$ Q.E.D.

b) from Induction:

$$f_{n+1}(t)=[f_n'(t)-2tf_n(t)]$$

$\Rightarrow$ there is leading factor(-2) for the maximum order term.

as $(-1)^n\cdot(-2)^n=2^n$

$\Rightarrow$ leading coeff. is 2^n.

c) $H_{n+1}(t)=(-1)^{n+1}e^{t^2}\cdot D^{n+1}[e^{-t^2}]$

$$=(-1)^{n+1}\cdot e^{t^2}\cdot D^n[-2t\cdot e^{-t^2}]$$
$$=2t\cdot(-1)^ne^{t^2}D^n[e^{-t^2}]+2n(-1)^ne^{t^2}D^{n-1}[e^{-t^2}]$$
$$=2tH_n(t)-2nH_{n-1}(t)$$

$$\Rightarrow \begin{cases} H_0(x)=1 \\ H_1(x)=2x \\ H_2(x)=4x^2-2 \\ H_3(x)=8x^3-12x \\ H_4(x)=16x^4-48x^2+12 \end{cases} \qquad \begin{matrix} H_5(x)=32x^5-160x^3+120x \\ H_6=64x^6-480x^4+720x^2-120. \end{matrix}$$

d) $L[H_n(t)]=(-1)^nD^2[e^{t^2}[D^n[e^{-t^2}]]$

$$-2t(-1)^nD[e^{t^2}D^n[e^{-t^2}]]$$
$$+2n(-1)^ne^{t^2}D[e^{-t^2}]$$

$$e^{-t^2}(-1)^nL[H_n(t)]$$
$$=D^{n+2}[e^{-t^2}]+\cancel{2tD^{n+1}[e^{-t^2}]+2(n+1)D^n[e^{-t^2}]}$$
$$D^{n+2}[e^{-t^2}]=\cancel{-2tD^{n+1}[e^{-t^2}]-2(n+1)D^n[e^{-t^2}]}$$

$\Rightarrow L[H_n(t)]=0 \quad \therefore$ Q.E.D.

H52 $\frac{1}{1+z^2}$ $(a=i)$

$$\frac{1}{(z-i)(z+i)}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)$$
$$=\frac{1}{2i}\left[\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1-i)^{n+1}}-\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1+i)^{n+1}}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{(n+1)/2}}\sin\frac{(n+1)\pi}{4}\cdot(z-1)^n$$

$(a=1+i)$

$$\frac{1}{(z-1-i+1)(z-1-i+1+2i)}$$
$$=\frac{i}{2}\left[\sum_{n=0}^{\infty}(-1)^n(z-1-i)^n-\sum_{n=0}^{\infty}(-1)^n\frac{(z-1-i)^n}{(1+2i)^{n+1}}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot i}{2}\left(1-\frac{(1-2i)^{n+1}}{5^{n+1}}\right)$$