ZJU-UIUC Institute
Prof. Thomas Honold

Fall Semester 2021
Homework 9

Differential Equations Plus (Math 286)

H53 Compute the Taylor series of $z \mapsto 1/(z^2+1)$ at $a=1$ and $a=1+\mathrm{i}$.
Hint: Proceed as for $z \mapsto 1/(1-z)$ in the lecture and then use partial fractions.

H54 Using power series, solve each of the following initial-value problems:

a) $t(2-t)y'' - 6(t-1)y' - 4y = 0, \quad y(1) = 1,\ y'(1) = 0;$

b) $y'' + (t^2+2t+1)y' - (4+4t)y = 0, \quad y(-1) = 0,\ y'(-1) = 1.$

H55 a) Find 2 linearly independent solutions of $\quad y'' + t^3y' + 3t^2y = 0.$

b) Find the first 5 terms in the Taylor series expansion about $t=0$ of the solution $y(t)$ of the initial value problem

$$y'' + t^3y' + 3t^2y = \mathrm{e}^t, \quad y(0) = y'(0) = 0.$$

H56 *A Problem from Friday's Lecture*

Suppose (α_n) and (u_n) are sequences of nonnegative real numbers satisfying

$$\begin{aligned}
\alpha_n &\leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}\alpha_k \quad (n \geq 2),\\
u_n &= \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}u_k \quad (n \geq 2),\\
u_0 &= \alpha_0,\ u_1 = \alpha_1
\end{aligned}$$

for some constant $M > 0$.

a) Show $\alpha_n \leq u_n$ for all n.

b) Show $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$.
Hint: Express u_{n+1} in terms of u_n.

c) Is the sequence (u_n) (and hence (α_n) as well) necessarily bounded from above?

Due on Wed Nov 24, 6 pm

Exercise H56 c) is optional, but should be handed in together with H56 a), b).

Solutions (prepared by Li Menglu and TH)

53 $\underline{a=1:}$

$$\begin{aligned}\frac{1}{z^2+1}&=\frac{1}{(z-\mathrm{i})(z+\mathrm{i})}=\frac{1}{2\mathrm{i}}\left(\frac{1}{z-\mathrm{i}}-\frac{1}{z+\mathrm{i}}\right)\\&=\frac{1}{2\mathrm{i}}\left(\frac{1}{z-1+1-\mathrm{i}}-\frac{1}{z-1+1+\mathrm{i}}\right)\\&=\frac{1}{2\mathrm{i}}\left[\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1-\mathrm{i})^{n+1}}-\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1+\mathrm{i})^{n+1}}\right]\\&=\sum_{n=0}^{\infty}b_n(z-1)^n\end{aligned}$$

with

$$\begin{aligned}b_n&=\frac{(-1)^n}{2^{(n+1)/2}}\frac{(\mathrm{e}^{\mathrm{i}\pi/4})^{n+1}-(\mathrm{e}^{-\mathrm{i}\pi/4})^{n+1}}{2\mathrm{i}}=\frac{(-1)^n}{2^{(n+1)/2}}\sin\frac{(n+1)\pi}{4}\\&=\begin{cases}2^{-n/2-1} & \text{if } n=8k,\ 8k+2,\\ -2^{-(n+1)/2} & \text{if } n=8k+1,\\ 0 & \text{if } n=8k+3,\ 8k+7,\\ -2^{-n/2-1} & \text{if } n=8k+4,\ 8k+6,\\ 2^{-(n+1)/2} & \text{if } n=8k+5.\end{cases}\end{aligned}$$

This can also be written as

$$\frac{1}{z^2+1}=\sum_{k=0}^{\infty}\frac{(z-1)^{8k}}{16^k}\left(\frac{1}{2}-\frac{(z-1)}{2}+\frac{(z-1)^2}{4}-\frac{(z-1)^4}{8}+\frac{(z-1)^5}{8}-\frac{(z-1)^6}{16}\right).$$

and shows the known fact that $\sum_{n=0}^{\infty}b_n(z-1)^n$ has radius of convergence $\sqrt{2}$ (the distance from 1 to the singularities $\pm\mathrm{i}$ of $1/(z^2+1)$).
$\underline{a=1+\mathrm{i}:}$ Since $\frac{1}{(z-a)(z-b)}=\frac{1}{a-b}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)$, we obtain

$$\begin{aligned}\frac{1}{z^2+1}&=\frac{1}{(z-1-\mathrm{i}+1)(z-1-\mathrm{i}+1+2\mathrm{i})}=\frac{\mathrm{i}}{2}\left(\frac{1}{z-1-\mathrm{i}+1}-\frac{1}{z-1-\mathrm{i}+1+2\mathrm{i}}\right)\\&=\frac{\mathrm{i}}{2}\left[\sum_{n=0}^{\infty}(-1)^n(z-1-\mathrm{i})^n-\sum_{n=0}^{\infty}(-1)^n\frac{(z-1-\mathrm{i})^n}{(1+2\mathrm{i})^{n+1}}\right]\\&=\sum_{n=0}^{\infty}c_n(z-1-\mathrm{i})^n\end{aligned}$$

with

$$c_n=\frac{(-1)^n\,\mathrm{i}}{2}\left(1-\frac{(1-2\mathrm{i})^{n+1}}{5^{n+1}}\right)=\frac{(-1)^n\,\mathrm{i}}{2}\left(1-\frac{\left(\frac{1-2\mathrm{i}}{\sqrt{5}}\right)^{n+1}}{5^{(n+1)/2}}\right).$$

Since $\left|\frac{1-2\mathrm{i}}{\sqrt{5}}\right| = 1$, the last representation shows $c_n \simeq (-1)^n \, \mathrm{i}/2$ for $n \to \infty$, implying the known fact that $\sum_{n=0}^{\infty} c_n(z-1-\mathrm{i})^n$ has radius of convergence 1 (the distance from $1+\mathrm{i}$ to the nearest singularity i of $1/(z^2+1)$).

54 a) We look for a solution in the form of a power series about $t_0 = 1$. The series has the form

$$y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n.$$

The point $t_0 = 1$ is an ordinary point of the differential equation, so the power series solution will be analytic at this point. Moreover, since the coefficient functions $p(t) = \frac{-6(t-1)}{t(2-t)}$, $q(t) = \frac{-4}{t(2-t)}$ of the corresponding explicit ODE have their singularities, viz. $t = 0$ and $t = 2$, at distance 1 from t_0, the radius of convergence of the power series will be at least 1, and $y(t)$ will solve the ODE on $(-1, 1)$.

Differentiating the equation term by term, we obtain that

$$y'(t) = \sum_{n=1}^{\infty} a_n n (t-1)^{n-1},$$
$$y''(t) = \sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2}.$$

Substituting the above series into the original equation gives

$$t(2-t)\sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2} - 6(t-1)\sum_{n=1}^{\infty} a_n n(t-1)^{n-1} - 4\sum_{n=0}^{\infty} a_n (t-1)^n = 0.$$

Rewrite the series so that they display the same generic term and using $t(2-t) = 1-(t-1)^2$ gives

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(t-1)^n - \sum_{n=2}^{\infty} a_n n(n-1)(t-1)^n - 6\sum_{n=1}^{\infty} a_n n(t-1)^n - \\ - 4\sum_{n=0}^{\infty} a_n (t-1)^n = 0,$$

which can be simplified to

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2+5n+4)a_n\right](t-1)^n = 0.$$

Hence the coefficients a_n must satisfy the recurrence relation

$$a_{n+2} = \frac{n^2+5n+4)}{(n+2)(n+1)}\, a_n = \frac{n+4}{n+2}\, a_n, \quad n = 0, 1, 2, 3, 4, \ldots$$

According to the initial conditions,

$$a_0 = y(1) = 1, \quad a_1 = y'(1) = 0.$$

The solution is $a_{2k+1} = 0$ for $k = 0, 1, 2, \dots$ and

$$a_{2k} = \frac{2k+2}{2k}\, a_{2k-2} = \dots = \frac{2k+2}{2k} \frac{2k}{2k-2} \cdots \frac{4}{2}\, a_0 = \frac{2k+2}{2} = k+1 \quad \text{for } k = 0, 1, 2, \dots.$$

Substituting these coefficients into the original series, the solution of the IVP is

$$y(t) = \sum_{k=0}^{\infty} (k+1)(t-1)^{2k}, \quad -1 < t < 1.$$

The radius of convergence of this power series is obviously 1.

Remark: Making the variable transformation $x = t - 1$ early on saves some writing (but otherwise leads to the same solution, of course).

b) We look for a solution in the form of a power series about $t_0 = 1$. The series has the form

$$y = \sum_{n=0}^{\infty} a_n (t+1)^n.$$

The point $t_0 = 1$ is an ordinary point of the differential equation, and the coefficient functions $p(t) = (t+1)^2$, $q(t) = -4(t+1)$ are polynomials. Hence the power series will have radius of convergence ∞ and $y(t)$ will be defined and solve the ODE on $\mathbb{R}$. Proceeding as before, we obtain

$$\sum_{n=2}^{\infty} a_n n(n-1)(t+1)^{n-2} + (t+1)^2 \sum_{n=1}^{\infty} a_n n(t+1)^{n-1} - 4(t+1) \sum_{n=0}^{\infty} a_n (t+1)^n = 0,$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(t+1)^{n-2} + \sum_{n=1}^{\infty} a_n n(t+1)^{n+1} - 4 \sum_{n=0}^{\infty} a_n (t+1)^{n+1} = 0,$$

$$2a_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2)a_{n+3} + (n-4)a_n\right] (t+1)^{n+1} = 0.$$

Hence the coefficients a_n must satisfy

$$a_2 = 0, \quad a_{n+3} = -\frac{n-4}{(n+3)(n+2)}\, a_n \quad \text{for } n = 0, 1, 2, 3, \dots.$$

The initial conditions are

$$a_0 = y(-1) = 0, \quad a_1 = y'(-1) = 1.$$

Hence $a_0 = a_3 = a_6 = \dots = 0$, $a_2 = a_5 = a_8 = \dots = 0$,

$$a_4 = -\frac{1-4}{(1+3)(1+2)}\, a_1 = \frac{3}{12} = \frac{1}{4},$$

$$a_7 = -\frac{4-4}{(4+3)(4+2)}\, a_4 = 0,$$

and $a_{10} = a_{13} = \dots = 0$ as well. Substituting these coefficients into the original series, the solution of the IVP is

$$y = (t+1) + \frac{1}{4}(t+1)^4, \quad t \in \mathbb{R}.$$

55 a) As in H54b) solutions at $t_0 = 0$ must be analytic and exist on the whole real line. The power series „Ansatz" $y(t) = \sum_{n=0}^{\infty} a_n t^n$ yields

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^3 \sum_{n=1}^{\infty} a_n n t^{n-1} + 3t^2 \sum_{n=0}^{\infty} a_n t^n = 0,$$

$$\sum_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}t^{n+2} + \sum_{n=0}^{\infty} n a_n t^{n+2} + 3\sum_{n=0}^{\infty} a_n t^{n+2} = 0,$$

$$2a_2 + 6a_3 t + \sum_{n=0}^{\infty} \left[(n+4)(n+3)a_{n+4} + (n+3)a_n\right] t^{n+2} = 0.$$

Hence the coefficients a_n satisfy

$$a_2 = a_3 = 0, \quad a_{n+4} = -\frac{1}{n+4}\, a_n \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Tow linearly independent solutions are obtained by setting $(a_0, a_1) = (1, 0)$ and $(0, 1)$, respectively, i.e.,

$$y_1(t) = 1 - \frac{t^4}{4} + \frac{t^8}{4 \cdot 8} - \frac{t^{12}}{4 \cdot 8 \cdot 12} \pm \cdots,$$
$$y_2(t) = t - \frac{t^5}{5} + \frac{t^9}{5 \cdot 9} - \frac{t^{13}}{5 \cdot 9 \cdot 13} \pm \cdots.$$

b) The right-hand side of the equation can be expressed using Taylor series as

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Inserting this series into the ODE and using the initial conditions $a_0 = y(0) = 0$, $a_1 = y'(0) = 0$, changes the homogeneous recurrence relation in a) to the inhomogeneous recurrence relation $a_0 = a_1 = 0$, $a_2 = \frac{1}{2}\frac{1}{0!} = \frac{1}{2}$, $a_3 = \frac{1}{6}\frac{1}{1!} = \frac{1}{6}$, and

$$a_{n+4} = -\frac{1}{n+4}\, a_n + \frac{1}{(n+4)!} \quad \text{for } n = 0, 1, 2, 3, \ldots$$

The latter is obtained from equating coefficients at t^{n+2}, which gives $(n+4)(n+3)a_{n+4} + (n+3)a_n = \frac{1}{(n+2)!}$. The first few terms in the Taylor series expansion about $t = 0$ of the solution are then

$$y(t) = \frac{1}{2}\,t^2 + \frac{1}{6}\,t^3 + \frac{1}{24}\,t^4 + \frac{1}{120}\,t^5 - \frac{59}{6!}\,t^6 - \frac{119}{7!}\,t^7 - \frac{209}{8!}\,t^8 - \frac{335}{9!}\,t^9 + \frac{29737}{10!}\,t^{10} + \cdots$$

56 a) The assertion is trivially true for $n = 0, 1$. For $n \geq 2$ we may assume by induction that $\alpha_k \leq u_k$ for $0 \leq k < n$.

$$\Longrightarrow \alpha_n \leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}\alpha_k \leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)} u_k = u_n.$$

b) We have

$$u_{n+1} = \frac{1}{(n+1)n}\sum_{k=0}^{n} M(k+1)u_k = \frac{1}{(n+1)n}\left(\sum_{k=0}^{n-1} M(k+1)u_k + M(n+1)u_n\right)$$
$$= \frac{n(n-1)u_n + M(n+1)u_n}{(n+1)n} = \frac{n(n-1)+M(n+1)}{(n+1)n}\,u_n \quad \text{for } n \geq 2.$$

It follows that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n(n-1)+M(n+1)}{(n+1)n} = \lim_{n\to\infty}\frac{n^2+(M-1)n+M}{n^2+n} = 1.$$

c) The answer is "No". For $M \leq 2$ the sequence (u_n) remains bounded, but for $M > 2$ it diverges to $+\infty$ (except in the trivial case $u_0 = u_1 = 0$, in which $u_n = 0$ for all n).

The sum of the coefficients in the definition of u_n is $\frac{Mn(n+1)/2}{n(n-1)} = \frac{M(n+1)}{2(n-1)} \approx M/2$ for large n. For $M < 2$ the coefficient sum is ≤ 1 for large n, and one can prove by induction that (u_n) is bounded. (We had a similar example in the lecture.)

We will now show that if u_0, u_1 are not both zero and $M > 2$ then (u_n) is unbounded. Applying the formula for u_{n+1}/u_n repeatedly, we have

$$u_{n+1} = u_2\prod_{k=2}^{n}\frac{k(k-1)+M(k+1)}{(k+1)k}.$$

This says that the numbers u_n are the partial products of the infinite product

$$\prod_{n=2}^{\infty}\frac{n(n-1)+M(n+1)}{(n+1)n}.$$

It is known that an infinite product $\prod_{n=1}^{\infty}(1+b_n)$ with $b_n \geq 0$ converges (equivalently, is bounded) iff the series $\sum_{n=1}^{\infty} b_n$ converges. (In what follows we need only the implication $\Longrightarrow$, which is clear from $\prod_{k=1}^{n}(1+b_k) \geq 1 + \sum_{k=1}^{n} b_k$.) Since

$$\frac{n(n-1)+M(n+1)}{(n+1)n} = 1 + \frac{(M-2)n+M}{n^2+n} > 1 + \frac{(M-2)n+M-2}{n^2+n} = 1 + \frac{M-2}{n},$$

the divergence of the harmonic series implies for $M > 2$ that $\lim_{n\to\infty} u_n = \infty$ as well. (For $M = 2$ the fact about infinite products quoted above shows that (u_n) converges in $\mathbb{R}$, since this is true of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.)