Differential Equations Plus (Math 286)

H48 Determine the general solution of the following ODE's (two answers suffice):

a) $(2t+1)y'' + (4t-2)y' - 8y = (6t^2+t-3)\mathrm{e}^t, \quad t > -1/2;$

b) $t^2(1-t)y'' + 2t(2-t)y' + 2(1+t)y = t^2, \quad 0 < t < 1;$

c) $(t^2-4t+4)y'' + (3t-6)y' + 2y = t^2+1, \quad t > 2.$

Hints: The associated homogeneous ODE in a) has a solution of the form $y(t) = \mathrm{e}^{\alpha t}$ and that in b) a solution of the form $y(t) = t^\beta$ with constants α, β. In both cases a particular solution of the inhomogeneous ODE can be determined by reducing it to a first-order system and using variation of parameters (though this may not be the most economic solution). The ODE in c) is an inhomogeneous Euler equation in disguise.

H49 Determine a fundamental system of solutions for Bessel's ODE with $p = \frac{1}{2}$,

$$y'' + \frac{1}{t}y' + \left(1 - \frac{1}{4t^2}\right)y = 0,$$

using the „Ansatz“ $z = \sqrt{t}\,y$.

H50 *On Hermite Polynomials*

In the lecture the Hermite polynomials $\mathrm{H}_n(X) \in \mathbb{R}[X]$ are defined by $\mathrm{H}_n(t) = (-1)^n \mathrm{e}^{t^2} \mathrm{D}^n[\mathrm{e}^{-t^2}]$ for $t \in \mathbb{R}$ $(n = 0, 1, 2, \dots)$.

a) Show that $t \mapsto \mathrm{H}_n(t)$ is a polynomial function, justifying the definition.

b) Show that $\deg \mathrm{H}_n(X) = n$ and the leading coefficient of $\mathrm{H}_n(X)$ is 2^n.

c) Show that $\mathrm{H}_n(X)$ satisfies the recurrence relation $\mathrm{H}_{n+1}(X) = 2X\,\mathrm{H}_n(X) - 2n\,\mathrm{H}_{n-1}(X)$, and compute $\mathrm{H}_n(X)$ for $n \leq 6$.

d) Show that $t \mapsto \mathrm{H}_n(t)$ solves Hermite's differential equation $y'' - 2ty' + 2ny = 0$.
Hint: The equation is equivalent to $Ly = 0$, where $L = \mathrm{D}^2 - 2t\mathrm{D} + 2n\,\mathrm{id}$. Express $L\big[\mathrm{H}_n(t)\big]$ in terms of $\mathrm{D}^n[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}]$, and rewrite the latter using $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] = \mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}]$.

H51 *Optional Exercise*

The function e^t has no zero and satisfies $y' = y$. The function $\sin t$ has no zero in common with its derivative $\cos t$ and satisfies $y'' = -y$. Generalizing this observation, show that a nonzero C^n-function $f\colon I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ of positive length satisfies an explicit (possibly time-dependent) homogeneous linear ODE of order n if and only if $y, y', \dots, y^{(n-1)}$ have no common zero.
Hint: For the if-part work with the function $t \mapsto f(t)^2 + f'(t)^2 + \dots + f^{(n-1)}(t)^2$.

H52 *On Legendre Polynomials (optional exercise)*

In the lecture the Legendre polynomials $\mathrm{P}_n(X) \in \mathbb{R}[X]$ were defined by $\mathrm{P}_n(t) = \frac{1}{2^n n!}\mathrm{D}^n\left[(t^2-1)^n\right]$, $n = 0, 1, 2, \ldots$.

a) Compute $\mathrm{P}_n(X)$ for $n \leq 6$.

b) Show that

$$\int_{-1}^{1} \mathrm{P}_m(t)\mathrm{P}_n(t) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

Hint: Use partial integration and the fact that $(t^2-1)^n$ has a zero of multiplicity n at $t = \pm 1$. For the case $m = n$ it may be helpful to recall from Calculus III that $\int_0^{\pi/2} \sin^{2n+1} t \, \mathrm{d}t = \frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3}$.

c) Show that $\mathrm{P}_n(X)$ has n distinct zeros $\alpha_1^{(n)} < \alpha_2^{(n)} < \cdots < \alpha_n^{(n)}$ in $[-1, 1]$.

d) Suppose $n \in \mathbb{Z}^+$ and $x_1, \ldots, x_n \in \mathbb{R}$ are such that $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$. Show that there are uniquely determined constants ("weights") $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx c_1 f(x_1) + \cdots + c_n f(x_n) \tag{GQ$_n$}$$

is exact for all polynomial functions $f(t)$ of degree $\leq n-1$.

e) Show that for the particular choice $x_i = \alpha_i^{(n)}$, cf. c), Formula (GQ$_n$) is exact for all polynomial functions $f(t)$ of degree $\leq 2n-1$.

Hint: Long division of $f(t)$ by $\mathrm{P}_n(t)$.

f) Determine (GQ$_n$) for $n = 1, 2, 3$ and the special choice $x_i = \alpha_i^{(n)}$.

Due on Thu Nov 18, 7:30 pm

The optional exercises can be handed in one week later.

Solutions (prepared by Li Menglu and TH)

48 a) Let $y_1(t) = e^{\alpha t}$, so that $y_1'(t) = \alpha e^{\alpha t}$, $y''(t) = \alpha^2 e^{\alpha t}$. Substituting these into the associated homogeneous ODE, we get

$$(2t+1)\alpha^2 + (4t-2)\alpha - 8 = 0 \Rightarrow (2\alpha^2 + 4\alpha)t + \alpha^2 - 2\alpha - 8 = 0$$

$$\therefore \alpha = -2 \implies y_1(t) = e^{-2t} \text{ is a solution.}$$

Setting $y_2(t) = u(t)e^{-2t}$ and substituting this into the ODE (cf. "order reduction" in the lecture), we get for $u'(t)$ the 1st-order linear ODE

$$u''(t) + \left[2\frac{-2e^{-2t}}{e^{-2t}} + \frac{4t-2}{2t+1}\right]u'(t) = 0 \iff u''(t) + \left(\frac{4t-2}{2t+1} - 4\right)u'(t) = 0.$$

$$\therefore u'(t) = e^{\int -\frac{4t-2}{2t+1}+4dt} = e^{2t}(2t+1)^2 \implies u(t) = \frac{4t^2+1}{2}e^{2t} \implies y_2(t) = \frac{4t^2+1}{2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & 2t^2+\frac{1}{2} \\ -2e^{-2t} & 4t \end{vmatrix} = (2t+1)^2 e^{-2t} \neq 0$$

$\implies y_1(t)$, $y_2(t)$ form a fundamental system of solutions of the homogeneous ODE.

For the inhomogeneous ODE in explicit form we have $b(t) = \frac{6t^2+t-3}{2t+1}\,e^t$. Using variation of parameters for the order-reduced 2×2 system, we need to extract the first coordinate function of

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \int \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix} \mathrm{d}t = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \int \frac{1}{W}\begin{pmatrix} -y_2 b \\ y_1 b \end{pmatrix} \mathrm{d}t\,.$$

We have

$$\int \frac{1}{W}\begin{pmatrix} -y_2 b \\ y_1 b \end{pmatrix}\mathrm{d}t = \int \begin{pmatrix} -\frac{(2t^2+\frac{1}{2})(6t^2+t-3)}{(2t+1)^3}\,\mathrm{e}^{3t} \\ \frac{(6t^2+t-3)}{(2t+1)^3}\,\mathrm{e}^{t} \end{pmatrix}\mathrm{d}t$$
$$= \begin{pmatrix} \frac{-12t^3+8t^2+5t-4}{6(2t+1)^2}\,\mathrm{e}^{3t} \\ \frac{3t+2}{(2t+1)^2}\,\mathrm{e}^{t} \end{pmatrix}.$$

Remark: For the integration step we have used a computer algebra program. If $r(t)$ is any rational function (quotient of two polynomials) and $a \in \mathbb{C}$ then $r(t)\mathrm{e}^{at}$ can be integrated in finite terms iff there exists a rational function R such that $r(t) = R'(t) + a\,R(t)$; if this is the case then $\int r(t)\mathrm{e}^{at} = R(t)\mathrm{e}^{at}$. (This result is due to Liouville.) Only few rational functions $r(t)$ have this property. In the two cases under consideration one can find $R(t)$ with some effort by using the „Ansatz“ $R = u/v^2$, $v(t) = 2t+1$, which the special form of the integrand suggests. The details are omitted.

$$\implies y_p(t) = \mathrm{e}^{-2t}\,\frac{-12t^3+8t^2+5t-4}{6(2t+1)^2}\,\mathrm{e}^{3t} + \frac{4t^2+1}{2}\,\frac{3t+2}{(2t+1)^2}\,\mathrm{e}^{t}$$
$$= \frac{-12t^3+8t^2+5t-4+36t^3+24t^2+9t+6}{6(2t+1)^2}\,\mathrm{e}^{t}$$
$$= \frac{24t^3+32t^2+14t+2}{6(2t+1)^2}\,\mathrm{e}^{t} = \left(t+\tfrac{1}{3}\right)\mathrm{e}^{t}$$

is a particular solution of the inhomogeneous ODE, and its general solution is

$$y(t) = c_1 e^{-2t} + c_2(4t^2+1) + \left(t+\tfrac{1}{3}\right) \mathrm{e}^t.$$

Remark: A much quicker (but in a way dirty) solution is the following. Using the differential operator

$$L = (2t+1)\mathrm{D}^2 + (4t-2)\mathrm{D} - 8\,\mathrm{id},$$

the inhomogeneous ODE can be written as $L[y] = (6t^2+t-3)\mathrm{e}^t$. It is clear that L maps the space of exponential polynomials of the special form $p(t)\mathrm{e}^t = (p_0 + p_1 t + \cdots + p_d t^d)\mathrm{e}^t$ into itself. Thus we might hope for a particular solution of this form. When determining the images under L of the first few exponential monomials,

$$\begin{aligned} L[\mathrm{e}^t] &= (2t+1)\mathrm{e}^t + (4t-2)\mathrm{e}^t - 8\mathrm{e}^t = (6t-9)\mathrm{e}^t, \\ L[t\,\mathrm{e}^t] &= (2t+1)(t+2)\mathrm{e}^t + (4t-2)(t+1)\mathrm{e}^t - 8t\,\mathrm{e}^t = (6t^2-t)\mathrm{e}^t, \\ &\vdots \end{aligned}$$

we find that

$$6t^2+t-3 = L[t\,\mathrm{e}^t] + \tfrac{1}{3}L[\mathrm{e}^t] = L\left[t\,\mathrm{e}^t + \tfrac{1}{3}\,\mathrm{e}^t\right] = L\left[\left(t+\tfrac{1}{3}\right)\mathrm{e}^t\right].$$

This gives the same particular solution as above. (The general solution is determined in the same way as above.)

b) Let $y_1(t) = t^\beta$, so that $y_1'(t) = \beta t^{\beta-1}$, $y_1''(t) = \beta(\beta-1)t^{\beta-2}$. Substituting these into the associated homogeneous ODE, we get

$$t^2(1-t)\beta(\beta-1)t^{\beta-2} + 2t(2-t)\beta t^{\beta-1} + 2(1+t)t^\beta = 0 \Rightarrow \left[\beta^2+3\beta+2+\left(-\beta^2-\beta+2\right)t\right]t^\beta = 0$$

$$\therefore \beta = -2 \Rightarrow y_1(t) = t^{-2}.$$

Set $y_2(t) = u(t)t^{-2}$ and substitute this into the ODE, we can get

$$u''(t) + \left[2\frac{-2t^{-3}}{t^{-2}} + \frac{2t(2-t)}{t^2(1-t)}\right]u'(t) = 0 \Rightarrow u''(t) + \frac{2}{1-t}u'(t) = 0$$

$$\therefore u'(t) = e^{\int \frac{2}{t-1}dt} = (t-1)^2 \Rightarrow u(t) = \frac{(t-1)^3}{3} \Rightarrow y_2(t) = \frac{(t-1)^3}{3t^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{-2} & \frac{(t-1)^3}{3t^2} \\ -2t^{-3} & (t-1)^2t^{-2} - \frac{2}{3}(t-1)^3t^{-3} \end{vmatrix} = (t-1)^2t^{-4} \neq 0$$

$\Longrightarrow y_1(t)$, $y_2(t)$ form a fundamental system of solutions of the homogeneous ODE.

For the determination of a particular solution of the inhomogeneous ODE we proceed as before, setting $b(t) = \frac{t^2}{t^2(1-t)} = \frac{1}{1-t}$.

$$\begin{aligned} \int \frac{1}{W}\begin{pmatrix} -y_2 b(t) \\ y_1 b(t) \end{pmatrix} \mathrm{d}t &= \int \frac{t^4}{(t-1)^2}\begin{pmatrix} \frac{(t-1)^2}{3t^2} \\ \frac{1}{t^2(1-t)} \end{pmatrix}\mathrm{d}t = \int \begin{pmatrix} \frac{1}{3}t^2 \\ -\frac{t^2}{(t-1)^3} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{1}{9}t^3\right) \\ -\ln(t-1) + \frac{2}{t-1} + \frac{1}{2(t-1)^2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} y_p(t) &= t^{-2}\frac{1}{9}t^3 + \frac{(t-1)^3}{3t^2}\left(-\ln(t-1) + \frac{2}{t-1} + \frac{1}{2(t-1)^2}\right) \\ &= \frac{2t^3+12t^2-21t+9-6(t-1)^3\ln(t-1)}{18t^2} \end{aligned}$$

In the numerator of $y_p(t)$ we can subtract $2(t-1)^3+11$ to change it into $18t^2-27t-6(t-1)^3\ln(t-1)$, since this amounts to adding a linear combination of $y_1(t)$ and $y_2(t)$ to $y_p(t)$. This leaves the simpler function $t\mapsto 1-\frac{3}{2t}-\frac{(t-1)^3}{3t^2}\log(t-1)$. The general solution of the inhomogeneous ODE is then

$$y(t)=c_1t^{-2}+c_2\frac{(t-1)^3}{t^2}+1-\frac{3}{2t}-\frac{(t-1)^3}{3t^2}\log(t-1).$$

c) Writing the associated homogeneous ODE as $(t-2)^2y''+3(t-2)y'+2y=0$ and setting $t-2=x$ we get $x^2y''+3xy'+2y=0$, which is apparently an Euler equation with $\alpha=3$, $\beta=2$. The indicial equation is $r^2+(\alpha-1)r+\beta=r^2+2r+2=0$. It has roots $r_1=-1+\mathrm{i}$, $r_2=-1-\mathrm{i}$, and hence a complex fundamental system of solutions of the (untransformed) homogeneous ODE on $(2,\infty)$ is

$$\begin{aligned}z_1(t)&=(t-2)^{-1+\mathrm{i}}=\mathrm{e}^{\ln(t-2)(-1+\mathrm{i})},\\ z_2(t)&=(t-2)^{-1-\mathrm{i}}=\mathrm{e}^{\ln(t-2)(-1-\mathrm{i})}.\end{aligned}$$

A real fundamental system of solutions—strictly speaking, this is not required—is

$$\begin{aligned}y_1(t)&=\mathrm{Re}z_1(t)=(t-2)^{-1}\cos\ln(t-2),\\ y_2(t)&=\mathrm{Im}z_1(t)=(t-2)^{-1}\sin\ln(t-2).\end{aligned}$$

Since the associated differential operator $L=(t-2)^2\mathrm{D}^2+3(t-2)\mathrm{D}+2\,\mathrm{id}$ maps the space P_2 of (real, say) quadratic polynomials into itself, it is reasonable to guess that there must be a particular solution of the form $y_p(t)=a(t-2)^2+b(t-2)+c=ax^2+bx+c$, $x=t-2$. Substituting $y'=2ax+b$, $y''=2a$ into the inhomogeneous ODE, we obtain

$$\begin{aligned}10ax^2+5bx+2c=x^2+4x+5 &\implies a=\frac{1}{10},\ b=\frac{4}{5},\ c=\frac{5}{2}\\ &\implies y_p(t)=\frac{1}{10}x^2+\frac{4}{5}x+\frac{5}{2}=\frac{1}{10}t^2+\frac{2}{5}t+\frac{13}{10}.\end{aligned}$$

The general real (or complex) solution of $(t-2)^2y''+3(t-2)y'+2y=t^2+1$ on $(2,\infty)$ is therefore

$$y(t)=c_1(t-2)^{-1}\cos\ln(t-2)+c_2(t-2)^{-1}\sin\ln(t-2)+\frac{1}{10}t^2+\frac{2}{5}t+\frac{13}{10}$$

with $c_1,c_2\in\mathbb{R}$ (resp., $c_1,c_2\in\mathbb{C}$). For the complex solution we could have used the complex fundamental system $z_1(t)$, $z_2(t)$ instead.

49 Using the Ansatz $z=\sqrt{t}\,y$, we have

$$\begin{aligned}\frac{dz}{dt}&=\sqrt{t}\,y'+\frac{1}{2\sqrt{t}}y\\ \frac{d^2z}{dt^2}&=\sqrt{t}\,y''+\frac{1}{\sqrt{t}}y'+\left(-\frac{1}{4}\right)t^{-\frac{3}{2}}y\\ \implies\quad 4t^{\frac{3}{2}}\frac{d^2z}{dt^2}&=4t^2y''+4ty'-y\end{aligned}$$

Rewrite the ODE and substituting the above expression, we obtain

$$\begin{aligned} && 4t^2y'' + 4ty' + (4t^2-1)y &= 0 \\ \iff && 4t^{\frac{3}{2}}z'' + 4t^2y &= 0 \\ \iff && 4t^{\frac{3}{2}}(z+z'') &= 0 \\ \iff && z''+z &= 0 \end{aligned}$$

The characteristic equation of $z''+z=0$ is $r^2+1=0$, so that $r_1 = \mathrm{i}$, $r_2 = -\mathrm{i}$.

$$\begin{aligned} \therefore z(t) &= c_1 \cos t + c_2 \sin t, \\ \therefore y(t) &= \frac{c_1 \cos t}{\sqrt{t}} + \frac{c_2 \sin t}{\sqrt{t}}. \end{aligned}$$

Thus a fundamental system of solutions for the ODE is $\frac{\cos t}{\sqrt{t}}$, $\frac{\sin t}{\sqrt{t}}$, the same as obtained in the lecture (except that in the lecture the variable is denoted by x).

50 a) If f is a polynomial function then

$$\mathrm{D}\left[f(t)\mathrm{e}^{-t^2}\right] = f'(t)\mathrm{e}^{-t^2} + f(t)\mathrm{e}^{-t^2}(-2t) = \big(f'(t) - 2t\,f(t)\big)\mathrm{e}^{-t^2} = F(t)\mathrm{e}^{-t^2}, \qquad \text{(H)}$$

where $F(t) = f'(t) - 2t\,f(t)$ is also a polynomial function. Starting with $f(t) = f_0(t) = 1$, it follows by induction that $\mathrm{D}^n[\mathrm{e}^{-t^2}] = f_n(t)\mathrm{e}^{-t^2}$ for some polynomial function f_n. Hence $\mathrm{H}_n(t) = (-1)^n\mathrm{e}^{t^2}[\mathrm{D}^n\mathrm{e}^{-t^2}] = (-1)^n f_n(t)$ is also a polynomial function.

b) From (H) we see that f_n has degree n and leading coefficient $(-2)^n$. Hence $\mathrm{H}_n(t)$ has degree n as well and leading coefficient 2^n.

c) We have

$$\begin{aligned} \mathrm{H}_{n+1}(t) &= (-1)^{n+1}\mathrm{e}^{t^2}\mathrm{D}^n\left[\mathrm{D}\mathrm{e}^{-t^2}\right] = (-1)^{n+1}\mathrm{e}^{t^2}\mathrm{D}^n\left[-2t\,\mathrm{e}^{-t^2}\right] = 2(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n\left[t\,\mathrm{e}^{-t^2}\right] \\ &= 2(-1)^n\mathrm{e}^{t^2}\big(t\mathrm{D}^n[\mathrm{e}^{-t^2}] + n\mathrm{D}^{n-1}[\mathrm{e}^{-t^2}]\big) \qquad \text{(by Leibniz' formula)} \\ &= 2t(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}] + 2n(-1)^n\mathrm{e}^{t^2}\mathrm{D}^{n-1}[\mathrm{e}^{-t^2}] = 2t\,\mathrm{H}_n(t) - 2n\,\mathrm{H}_{n-1}(t). \end{aligned}$$

This proves the recursion formula in view of the 1-1 correspondence between polynomials in $\mathbb{R}[X]$ and polynomial functions on $\mathbb{R}$.

Together with $\mathrm{H}_0(t) = (-1)^0\mathrm{e}^{t^2}(\mathrm{e}^{-t^2}) = 1$, $\mathrm{H}_1(t) = (-1)^1\mathrm{e}^{t^2}(-2t\,\mathrm{e}^{-t^2}) = 2t$ the recursion formula gives

$$\begin{aligned} \mathrm{H}_0(X) &= 1, \\ \mathrm{H}_1(X) &= 2X, \\ \mathrm{H}_2(X) &= 2X(2X) - 2\cdot 1 = 4\,X^2 - 2, \\ \mathrm{H}_3(X) &= 2X(4X^2-2) - 4(2X) = 8\,X^3 - 12\,X, \\ \mathrm{H}_4(X) &= 2X(8X^3 - 12X) - 6(4X^2-2) = 16\,X^4 - 48\,X^2 + 12, \\ \mathrm{H}_5(X) &= 2X(16\,X^4 - 48\,X^2 + 12) - 8(8\,X^3 - 12\,X) = 32\,X^5 - 160\,X^3 + 120\,X, \\ \mathrm{H}_6(X) &= 2X(32X^5 - 160X^3 + 120X) - 10(16X^4 - 48X^2 + 12) = 64X^6 - 480\,X^4 + 720\,X^2 - 120. \end{aligned}$$

d) We have

$$
\begin{aligned}
L\big[\mathrm{H}_n(t)\big] &= (-1)^n\mathrm{D}^2\big[\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\big] - 2t(-1)^n\mathrm{D}\big[\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\big] + 2n(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}],\\
(-1)^n L\big[\mathrm{H}_n(t)\big] &= \mathrm{e}^{t^2}\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] + 2\,2t\mathrm{e}^{t^2}\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] + (2+4t^2)\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\\
&\quad - 2t\mathrm{e}^{t^2}\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] - 2t\,2t\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}] + 2n\,\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}],\\
(-1)^n\mathrm{e}^{-t^2} L\big[\mathrm{H}_n(t)\big] &= \mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] + 2t\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] + 2(n+1)\mathrm{D}^n[\mathrm{e}^{-t^2}].
\end{aligned}
$$

On the other hand, we also have

$$\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] = \mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}] = -2t\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] - 2(n+1)\mathrm{D}^n[\mathrm{e}^{-t^2}].$$

$\Longrightarrow (-1)^n\mathrm{e}^{-t^2}L\big[\mathrm{H}_n(t)\big] = 0 \Longrightarrow L\big[\mathrm{H}_n(t)\big] = 0$, as desired.

51 $\Longrightarrow$: Suppose, by contradiction, that $f^{(n)}(t) = a_0(t)f(t)+a_1(t)f'(t)+\cdots+a_{n-1}(t)f^{(n-1)}(t)$ for all $t\in I$ and $f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0$ for some $t_0\in I$. Then both f and the all-zero function on I solve the IVP $y^{(n)} = a_0(t)y+a_1(t)y'+\cdots+a_{n-1}(t)y^{(n-1)} \wedge y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. The Uniqueness Theorem (for linear ODEs, say) then implies that $y\equiv 0$, which contradicts the assumption.
$\Longleftarrow$: Under the given assumption $g(t) = f(t)^2 + f'(t)^2 + \cdots + f^{(n-1)}(t)^2$ is zero-free on I, i.e., we can write

$$1 = \frac{g(t)}{g(t)} = \frac{f(t)^2}{g(t)} + \frac{f'(t)^2}{g(t)} + \cdots + \frac{f^{(n-1)}(t)^2}{g(t)}.$$

Multiplying this identity by $f^{(n)}(t)$ gives

$$f^{(n)}(t) = \frac{f^{(n)}(t)f(t)^2}{g(t)} + \frac{f^{(n)}(t)f'(t)^2}{g(t)} + \cdots + \frac{f^{(n)}(t)f^{(n-1)}(t)^2}{g(t)},$$

which is an explicit homogeneous linear ODE of order n for f with coefficient functions

$$a_0(t) = \frac{f^{(n)}(t)f(t)}{g(t)},\quad a_1(t) = \frac{f^{(n)}(t)f'(t)}{g(t)},\quad \dots,\quad a_{n-1}(t) = \frac{f^{(n)}(t)f^{(n-1)}(t)}{g(t)}.$$

52 a) Using $(X^2-1)^n = \sum_{k=0}^n \binom{n}{k}(-1)^k X^{n-2k}$, we obtain

$$
\begin{aligned}
&\mathrm{P}_0(X) = 1,\\
&\mathrm{P}_1(X) = \tfrac{1}{2}(X^2-1)' = X,\\
&\mathrm{P}_2(X) = \tfrac{1}{8}(X^4-2\,X^2+1)'' = \tfrac{1}{8}(12\,X^2-4) = \tfrac{1}{2}(3\,X^2-1),\\
&\mathrm{P}_3(X) = \tfrac{1}{48}(X^6-3\,X^4+3\,X^2-1)''' = \tfrac{1}{2}(5\,X^3-3\,X),\\
&\mathrm{P}_4(X) = \cdots = \tfrac{1}{8}(35\,X^4-30\,X^2+3),\\
&\mathrm{P}_5(X) = \cdots = \tfrac{1}{8}(63\,X^5-70\,X^3+15\,X),\\
&\mathrm{P}_6(X) = \cdots = \tfrac{1}{16}(231\,X^6-315\,X^4+105\,X^2-5).
\end{aligned}
$$

Since the rings of polynomials and polynomial functions over an infinite field are isomorphic, it doesn't matter whether we write $\mathrm{P}_n(X)$ or $\mathrm{P}_n(t)$. This applies also to derivatives, which for polynomials are defined formally to resemble the derivative of the corresponding polynomial function, i.e., $\left(\sum_{k=0}^d a_kX^k\right)' := \sum_{k=1}^d ka_kX^{k-1}$.

b) For integers $m, n \geq 0$ we have

$$
\begin{aligned}
2^{n+m} n! m! \int_{-1}^{1} \mathrm{P}_m(t) \mathrm{P}_n(t)\,\mathrm{d}t &= \int_{-1}^{1} \mathrm{D}^m\big[(t^2-1)^m\big]\mathrm{D}^n\big[(t^2-1)^n\big]\,\mathrm{d}t \\
&= \mathrm{D}^{m-1}\big[(t^2-1)^m\big]\mathrm{D}^n\big[(t^2-1)^n\big]\Big|_{-1}^{1} - \int_{-1}^{1} \mathrm{D}^{m-1}\big[(t^2-1)^m\big]\mathrm{D}^{n+1}\big[(t^2-1 \\
&= -\int_{-1}^{1} \mathrm{D}^{m-1}\big[(t^2-1)^m\big]\mathrm{D}^{n+1}\big[(t^2-1)^n\big]\,\mathrm{d}t = \cdots \\
&= (-1)^m \int_{-1}^{1} \mathrm{D}^0\big[(t^2-1)^m\big]\mathrm{D}^{n+m}\big[(t^2-1)^n\big]\,\mathrm{d}t,
\end{aligned}
$$

since for $0 \leq k \leq m-1$ the polynomial $\mathrm{D}^k\big[(t^2-1)^m\big]$ has a zero (in fact an $(m-k)$-fold zero) at $t = \pm 1$. (For $m = 0$ there are no intermediate steps, but the identity holds as well.)

If $m > n$ then $n + m > 2n$ and hence $\mathrm{D}^{n+m}\big[(t^2-1)^n\big] = 0$. It follows that $\int_{-1}^{1} \mathrm{P}_m(t)\mathrm{P}_n(t)\,\mathrm{d}t = 0$ for $m > n$. By symmetry this also holds for $m < n$, showing the claimed formula for $m \neq n$,

In the case $m = n$ we get, using $\mathrm{D}^{2n}\big[(t^2-1)^n\big] = (2n)!$,

$$
\int_{-1}^{1} \mathrm{P}_n(t)^2\,\mathrm{d}t = \frac{(-1)^n(2n)!}{2^{2n}n!^2}\int_{-1}^{1}(t^2-1)^{2n}\,\mathrm{d}t = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\int_{-1}^{1}(1-t^2)^{2n}\,\mathrm{d}t\,.
$$

Moreover, we have

$$
\begin{aligned}
\int_{-1}^{1}(1-t^2)^{2n}\,\mathrm{d}t &= 2\int_{0}^{1}(1-t^2)^{2n}\,\mathrm{d}t \\
&= 2\int_{0}^{\pi/2}\sin^{2n+1}(\theta)\mathrm{d}\theta && \text{(Subst. } t = \cos\theta) \\
&= 2\,\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)} && \text{(cf. Calculus III)}
\end{aligned}
$$

It follows that $\int_{-1}^{1} \mathrm{P}_n(t)^2\,\mathrm{d}t = \frac{2}{2n+1}$, as asserted.

c) In the case $n = 0$ the assertion is trivially true, so we can assume $n \geq 1$. Since $f(t) = (t^2-1)^n$ satisfies $f(-1) = f(1) = 0$, Rolle's Theorem implies that there exists $x \in (-1,1)$ such that $f'(\xi) = 0$. Since $f'(\pm 1) = 0$, we can apply Rolle's Theorem again (provided that $n \geq 2$) and conclude that f'' has zeros $\xi_1 \in (-1,\xi)$ and $\xi_2 \in (\xi,1)$. Continuing in this way, we find that $f^{(n)}(t) = 2^n n!\,\mathrm{P}_n(t)$ has n distinct zeros in $(-1,1)$. The same is then true of $\mathrm{P}_n(t)$ itself, of course.

Remark: In fact we have $f'(t) = 2nt(t^2-1)^{n-1}$ and hence $\xi = 0$. The derivative $f^{(k)}$, $1 \leq k \leq n$, has k simple zeros (and no further zero) in $(-1,1)$, since it has degree $2n-k$ and zeros of multiplicity $n-k$ at $t = \pm 1$.

d) Since both sides of (GQ_n) are linear, (GQ_n) is exact for all polynomials of degree $\leq n-1$ provided it is exact for $1, t, t^2, \ldots, t^{n-1}$. This is the case iff

$$
c_1 x_1^k + \cdots + c_n x_n^k = \int_{-1}^{1} t^k\,\mathrm{d}t = \begin{cases} \frac{2}{k+1} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}
$$

for $k = 0, 1, \ldots, n-1$. The coefficient matrix of this linear system of equations for $c_1, \ldots, c_n$ is Vandermonde, hence invertible, and this shows that their is a unique assignment of weights haing this property.

e) Long division of f by P_n shows that their exist polynomials q, r with

$$f(t) = q(t)\mathrm{P}_n(t) + r(t) = c\,q(t)(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n) + r(t) \qquad \text{(LD)}$$

and $\deg r \leq n-1$ or $r = 0$. The constant c is equal to $2^{-n}\binom{2n}{n}$ (the leading coefficient of P_n), and we have written $\alpha_i = \alpha_i^{(n)}$.

If $\deg f \leq 2n-1$ then $\deg q \leq n-1$, and hence q is a linear combination of the Legendre polynomials $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_{n-1}$. On account of b) this implies

$$\int_{-1}^{1} q(t)\mathrm{P}_n(t)\,\mathrm{d}t = 0,$$

$$\begin{aligned}
\int_{-1}^{1} f(t)\,\mathrm{d}t &= \int_{-1}^{1} q(t)\mathrm{P}_n(t)\,\mathrm{d}t + \int_{-1}^{1} r(t)\text{ţ} = \int_{-1}^{1} r(t)\,\mathrm{d}t \\
&= c_1 r(\alpha_1) + c_2 r(\alpha_2) + \cdots + c_n r(\alpha_n) && \text{(by d))} \\
&= c_1 f(\alpha_1) + c_2 f(\alpha_2) + \cdots + c_n f(\alpha_n). && \text{(cf. (LD))}
\end{aligned}$$

f) For $n = 1$ we have $\mathrm{P}_1(X) = X$.
$\Longrightarrow \alpha_1 = 0$, $c_1 = c_1\alpha_1^0 = \int_{-1}^{1}\mathrm{d}t = 2$, and the approximation is

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx 2\,f(0)$$

For $n = 2$ we have $\mathrm{P}_1(X) = \frac{1}{2}(3X^2 - 1)$.
$\Longrightarrow \alpha_1 = -\frac{1}{3}\sqrt{3}$, $\alpha_2 = \frac{1}{3}\sqrt{3}$, $c_1 + c_2 = \int_{-1}^{1}\mathrm{d}t = 2$, $c_1\alpha_1 + c_2\alpha_2 = \int_{-1}^{1} t\,\mathrm{d}t = 0$
The solution is $c_1 = c_2 = 1$, and hence the approximation is

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx f\left(-\tfrac{1}{3}\sqrt{3}\right) + f\left(\tfrac{1}{3}\sqrt{3}\right).$$

For $n = 3$ we have $\mathrm{P}_1(X) = \frac{1}{2}(5X^3 - 3X)$.
$\Longrightarrow \alpha_1 = -\frac{1}{5}\sqrt{15}$, $\alpha_2 = 0$, $\alpha_3 = \frac{1}{5}\sqrt{15}$. The system in d) is

$$\begin{array}{ccccccc}
c_1 & + & c_2 & + & c_3 & = & 2, \\
\alpha_1 c_1 & + & \alpha_2 c_2 & + & \alpha_3 c_3 & = & 0, \\
\alpha_1^2 c_1 & + & \alpha_2^2 c_2 & + & \alpha_3^2 c_3 & = & \frac{2}{3},
\end{array}$$

which is solved by $c_1 = c_3 = \frac{5}{9}$, $c_2 = \frac{8}{9}$. (Since $\alpha_3 = -\alpha_1$, the 2nd equation gives $c_1 = c_3$, and then the 3rd equation gives $c_1 = 1/3\alpha_1^2 = 5/9$.) Hence the approximation in this case is

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx \frac{5}{9} f\left(-\tfrac{1}{5}\sqrt{15}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\tfrac{1}{5}\sqrt{15}\right).$$