Differential Equations Plus (Math 286)

H57 For each of the following ODE's, find two linearly independent real solutions.

a) $4xy'' + 3y' - 3y = 0, \quad x \lessgtr 0;$
b) $x^2y'' - x(1+x)y' + y = 0, \quad x \lessgtr 0;$
c) $x^2y'' + xy' - (1+x)y = 0, \quad x > 0;$
d) $x^2y'' + xy' + (1+x)y = 0, \quad x > 0.$

H58 Consider the ODE

$$xy'' + 3y' - 3y = 0, \quad x > 0.$$

a) Show that the roots of the indicial equation are $r = 0$ and $r = -2$.
b) Find a solution $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$.
c) Find a second solution $y_2(x) = a\, y_1(x) \ln x + x^{-2}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$.

H59 Do Exercises 5, 6, 9 in [BDM17], Ch. 5.7 (Exercises 6, 7, 10 in the 11th edition). Additionally show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$ for $x > 0$; see p. 236 (p. 238 in the 11th edition) for the definition of $\mathrm{Y}_1(x)$. The solution $y_2(x)$ appearing in the definition of $\mathrm{Y}_1(x)$ is the same as that you obtain in Exercise 9 (resp., Exercise 10).

H60 The Γ function is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t$, and for non-integral $x < 0$ by choosing an integer $n > -x$ and setting

$$\Gamma(x) := \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}.$$

a) Show that $\Gamma(x)$ is well-defined for $x < 0$, $x \notin \mathbb{Z}$, and satisfies $\Gamma(x+1) = x\,\Gamma(x)$ for all $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$.
Hint: Recall from Calculus III that $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.

b) Show $\lim_{x \to -n} \frac{1}{\Gamma(x)} = 0$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.
This shows that $1/\Gamma$ can be continuously extended to $\mathbb{R}$ by defining $1/\Gamma(-n) := 0$ for $n \in \mathbb{N}$.

c) The Bessel function of order $\nu \in \mathbb{R}$ is defined as (cf. the lecture)

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\nu+2m} m!\,\Gamma(\nu+m+1)} x^{\nu+2m} \quad \text{for } x \in \mathbb{R},$$

cf. b) for the definition of $1/\Gamma(\nu+m+1)$.
Show $\quad \mathrm{J}_{-\nu} = (-1)^\nu \mathrm{J}_\nu \quad$ for $\nu \in \mathbb{N}$.
Hint: Show first that the coefficients of x^n in the expansion of $\mathrm{J}_{-\nu}(x)$ are zero for $n < \nu$.

H61 *Optional Exercise*

For $x \in \mathbb{R} \setminus \{0\}$, $\nu \in \mathbb{R}$ show:

a) $\mathrm{J}_{\nu+1}(x) = \dfrac{2\nu}{x}\,\mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x)$;

b) $\mathrm{J}_\nu'(x) = -\mathrm{J}_{\nu+1}(x) + \dfrac{\nu}{x}\,\mathrm{J}_\nu(x)$.

Remark: a) Provides a recurrence relation to determine J_ν for $\nu \in \mathbb{N}$ from J_0, J_1. The Neumann functions Y_ν, $\nu \in \mathbb{N}$, satisfy the same recurrence relation and provide a 2nd solution of $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, which is linearly independent of J_ν. Thus in order to determine Y_ν for $\nu \in \mathbb{N}$ (the only case of interest) it suffices to know Y_0 and Y_1.

H62 *Optional Exercise*

Show $\quad \mathrm{J}_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x\sin\theta)\mathrm{d}\theta \quad$ for $x \in \mathbb{R}$.

Due on Thu Dec 2, 7:30 pm

Exercises H57 c), H58, and H59, Ex. 9 are similar, and it suffices to do one of them. (But we recommend to do more, if you have time.)

The optional exercises and H60 can be handed in one week later.

Solutions (prepared by Li Menglu and TH)

57 a) Rewriting the ODE as

$$y'' + \frac{3}{4x}y' - \frac{3}{4x}y = 0,$$

we see that $x = 0$ is a regular singular point and

$$p_0 = \lim_{x\to 0} x\frac{3}{4x} = \frac{3}{4}, \quad q_0 = \lim_{x\to 0} x^2\frac{-3}{4x} = 0$$

. $\implies$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{1}{4}r = 0$$

.

$\implies$ The exponents at the singularity $x = 0$ are $r_1 = 0$, $r_2 = \frac{1}{4}$. Since $r_1 - r_2$ is not an integer, there must be solutions $y_1(x)$, $y_2(x)$ on $(0, \infty)$ of the form

$$y_1(x) = 1 + \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{\frac{1}{4}}\left(1 + \sum_{n=0}^{\infty} a_n x^n\right).$$

In terms of the rational functions $a_n(r)$ defined in the lecture and textbook, the coefficients of $y_1(x)$, $y_2(x)$ are $a_n = a_n(0)$ and $a_n = a_n(1/4)$, respectively. (We use 'a_n' for both, in order to be compatible with the notation used in [BDM17], Theorem 5.6.1.)

i) $r_1 = 0$:

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^n \\
y_1' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\
xy_1'' &= x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^n
\end{aligned}$$

Substituting these into the ODE, we get

$$\begin{aligned}
& 4\sum_{n=1}^{\infty}(n+1)n a_{n+1}x^n + 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 3\sum_{n=0}^{\infty}a_n x^n = 0 \\
\implies \quad & 3a_1 - 3a_0 + \sum_{n=1}^{\infty}\left\{\left[4n(n+1) + 3(n+1)\right]a_{n+1} - 3a_n\right\}x^n = 0 \\
\implies \quad & a_1 = a_0 \quad \text{and} \quad a_{n+1} = \frac{3}{(4n+3)(n+1)}a_n \quad \text{for} \quad n \geq 1.
\end{aligned}$$

Setting $a_0 = 1$, we have

$$\begin{aligned}
y_1(x) &= 1 + x + \frac{3}{7\cdot 2}\,x^2 + \frac{3^2}{7\cdot 2\cdot 11\cdot 3}\,x^3 + \cdots \\
&= 1 + x + \sum_{n=2}^{\infty} \frac{3^{n-1}}{7\cdot 11\cdots(4n-1)\cdot n!}\,x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{3^{n}}{3\cdot 7\cdot 11\cdots(4n-1)\cdot n!}\,x^n \\
&= \sum_{n=0}^{\infty} \frac{3^{n}}{3\cdot 7\cdot 11\cdots(4n-1)\cdot n!}\,x^n,
\end{aligned}$$

using the convention that $\prod_{n=1}^{0}(4n-1) = 1$ (“empty product”).

ii) $r_2 = \frac{1}{4}$:

$$\begin{aligned}
y_2 &= x^{\frac14}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac14} \\
y_2' &= \sum_{n=0}^{\infty}\left(n+\frac14\right)a_n x^{n-\frac34} = \sum_{n=-1}^{\infty}\left(n+\frac54\right)a_{n+1}x^{n+\frac14} \\
xy_2'' &= \sum_{n=0}^{\infty}\left(n+\frac14\right)\left(n-\frac34\right)a_n x^{n-\frac34} = \sum_{n=-1}^{\infty}\left(n+\frac54\right)\left(n+\frac14\right)a_{n+1}x^{n+\frac14}
\end{aligned}$$

Substituting these into the ODE, the coefficient of $x^{-3/4}$ vanishes by construction, and we get

$$\sum_{n=0}^{\infty}\left\{\left[4\left(n+\frac54\right)\left(n+\frac14\right)+3\left(n+\frac54\right)\right]a_{n+1} - 3a_n\right\}x^{n+\frac14} = 0$$

$$\implies \quad a_{n+1} = \frac{3\,a_n}{4\left(n+\frac54\right)\left(n+\frac14\right)+3\left(n+\frac54\right)} = \frac{3\,a_n}{(4n+5)(n+1)} \quad \text{for} \quad n \geq 0.$$

Setting $a_0 = 1$, we obtain

$$y_2(x) = x^{\frac14} + \sum_{n=1}^{\infty}\frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^{n+\frac14} = \sum_{n=0}^{\infty}\frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^{n+\frac14}.$$

As shown in the lecture, $y_1(x)$ and $y_2(x)$ are linearly independent. This is also clear from the fact that $y_1(x)$ is analytic at $x = 0$ and $y_2(x) = x^{1/4} \times$ "nonzero analytic" is not.

As discussed in the lecture (or see Theorem 5.6.1 in [BDM17], p. 227), a fundamental system of solutions on $(-\infty, 0)$ is obtained by replacing the fractional part x^r (if any) in the solutions by $(-x)^r = |x|^r$. This doesn't affect $y_1(x)$ ($y_1(x)$ is analytic on $\mathbb{R}$ and hence solves the ODE on $\mathbb{R}$), but $y_2(x)$ is changed to

$$y_2^-(x) = (-x)^{\frac14}\sum_{n=0}^{\infty}\frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^n, \quad x \in (-\infty, 0).$$

b) Rewriting the ODE as

$$y'' - \left(1 + \frac{1}{x}\right) y' + \frac{1}{x^2}\, y = 0,$$

we see that $x = 0$ is a regular singular point with $p_0 = -1$, $q_0 = 1$.
$\implies$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = (r-1)^2 = 0$$

.

$\implies$ The exponents at the singularity $x = 0$ are $r_1 = r_2 = 1$. Thus there must be solutions $y_1(x)$, $y_2(x)$ on $(0, \infty)$ of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n x^{n+1}, \qquad y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+1}.$$

i) $r_1 = 1$:

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1}, \\
y_1' &= \sum_{n=0}^{\infty} (n+1) a_n x^n, \\
x(1+x)y_1' &= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=1}^{\infty} n a_{n-1} x^{n+1} \\
&= \sum_{n=0}^{\infty} \left[(n+1)a_n + n a_{n-1}\right] x^{n+1}, \qquad (a_{-1} := 0) \\
x^2 y_1'' &= x^2 \sum_{n=1}^{\infty} (n+1) n a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) n a_n x^{n+1} = \sum_{n=0}^{\infty} (n+1) n a_n x^{n+1}.
\end{aligned}$$

Substituting these into the ODE, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1) n a_n x^{n+1} - \sum_{n=0}^{\infty} \left[(n+1)a_n + n a_{n-1}\right] x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=0}^{\infty} (n^2 a_n - n a_{n-1}) x^{n+1} = 0.
\end{aligned}$$

$$\implies \quad a_n = \frac{a_{n-1}}{n} \quad \text{for} \quad n \geq 1$$

Setting $a_0 = 1$, we obtain $a_n = 1/n!$ and

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = x\, \mathrm{e}^x.$$

ii) For the determination of $y_2(x)$ we use the recurrence relation for $a_n(r)$ derived in the lecture; cf. also [BDM17], p. 223, Eq. (8). Since $F(r) = (r-1)^2$, $p_0 = p_1 = -1$, $q_0 = 1$ and all other coefficients p_i, q_i are zero, we have

$$\begin{aligned} a_n(r) &= -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_k(r) \\ &= \frac{-1}{(r+n-1)^2}\,(r+n-1)p_1 a_{n-1}(r) = \frac{a_{n-1}(r)}{r+n-1} \qquad (n \geq 1). \end{aligned}$$

Setting $a_0(r) = 1$, we get

$$\begin{aligned} a_1(r) &= \frac{1}{r}, \\ a_2(r) &= \frac{1}{r(r+1)}, \\ &\vdots \\ a_n(r) &= \frac{1}{r(r+1)(r+2)\cdots(r+n-1)}. \end{aligned}$$

$$\Longrightarrow \quad b_n(r) := a_n'(r) = \frac{a_n'(r)}{a_n(r)}\, a_n(r)$$

$$= -\left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{r+n-1}\right) \frac{1}{r(r+1)(r+2)\cdots(r+n-1)}$$

$$\Longrightarrow \quad b_n = b_n(1) = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\frac{1}{n!} = -\frac{\mathrm{H}_n}{n!}$$

$$\Longrightarrow \quad y_2(x) = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\right) \ln x - \sum_{n=1}^{\infty} \frac{\mathrm{H}_n}{n!}\, x^{n+1} = x\,\mathrm{e}^x \ln x - \sum_{n=1}^{\infty} \frac{\mathrm{H}_n}{n!}\, x^{n+1}$$

The linear independency of $y_1(x)$, $y_2(x)$ was shown in the lecture.
A fundamental system of solutions on $(-\infty, 0)$ is formed by $y_1(x)$ and

$$y_2^-(x) = x\,\mathrm{e}^x \ln(-x) - \sum_{n=1}^{\infty} \frac{\mathrm{H}_n}{n!}\, x^{n+1}, \quad x \in (-\infty, 0).$$

Remark: The coefficients b_n can also be determined by substituting the „Ansatz“

for $y_2(x)$ into the ODE. Writing $L = x^2\mathrm{D}^2 - x(x+1)\mathrm{D} + \mathrm{id}$, we obtain

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln x + \sum_{n\geq 0} b_n x^n, \\
y_2'(x) &= y_1'(x)\ln x + \frac{y_1(x)}{x} + \sum_{n\geq 1} n b_n x^{n-1}, \\
y_2''(x) &= y_1''(x)\ln x + 2\,\frac{y_1(x)}{x} - \frac{y_1(x)}{x^2} + \sum_{n\geq 2} n(n-1) b_n x^{n-2}, \\
L\left[y_2(x)\right] &= L\left[y_1(x)\right]\ln x + 2x\, y_1'(x) - (x+2)y_1(x) + L\left[\sum_{n\geq 0} b_n x^n\right] \\
&= 0 + \underbrace{2x(x+1)\mathrm{e}^x - (x+2)x\mathrm{e}^x}_{=x^2\mathrm{e}^x} + \sum_{n=1}^{\infty}(n^2 b_n - n b_{n-1})x^{n+1} \\
&= \sum_{n=1}^{\infty}\left(n^2 b_n - n b_{n-1} + \frac{1}{(n-1)!}\right) x^{n+1}.
\end{aligned}
$$

$L\left[y_2(x)\right] = 0$ is equivalent to an inhomogeneous linear recurrence relation for b_n, which has the particular solution $b_0 = 0$, $b_n = -\mathrm{H}_n/n!$ for $n \geq 1$ (as can be seen by introducing $B_n = n! b_n$, which satisfies $B_n - B_{n-1} = -1/n$).

c) Rewriting the ODE as

$$
y'' + \frac{1}{x}\,y' - \left(\frac{1}{x^2} + \frac{1}{x}\right) y = 0,
$$

we see that $x = 0$ is a regular singular point with $p_0 = 1$, $q_0 = -1$.
$\Longrightarrow$ The indicial equation is

$$
r^2 + (p_0 - 1)r + q_0 = r^2 - 1 = 0
$$

.

$\Longrightarrow$ The exponents at the singularity $x = 0$ are $r_1 = 1$, $r_2 = -1$.
Since $r_1 - r_2 = 2$ is an integer, there must be solutions $y_1(x)$, $y_2(x)$ on $(0, \infty)$ of the form

$$
y_1(x) = x\left(1 + \sum_{n=1}^{\infty} a_n x^n\right), \qquad y_2(x) = a\, y_1(x)\ln x + x^{-1}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right).
$$

In terms of $a_n(r)$ the coefficients occuring in $y_2(x)$ are given by

$$
a = \lim_{r\to -1}(r+1)a_2(r), \quad c_n = \frac{\mathrm{d}}{\mathrm{d}r}\left[(r+1)a_n(r)\right]\bigg|_{r=-1}.
$$

i) $r_1 = 1$:

$$\begin{aligned}
(1+x)y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+1} \\
x y_1'(x) &= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} \\
x^2 y_1''(x) &= \sum_{n=1}^{\infty} (n+1) n a_n x^{n+1}
\end{aligned}$$

Substituting these into the ODE, we get

$$\sum_{n=1}^{\infty} \left[(n^2+2n) a_n - a_{n-1} \right] x^{n+1} = 0.$$

$$\implies \quad a_n = \frac{1}{n(n+2)} a_{n-1}, \quad \text{for} \quad n \geq 1$$

Setting $a_0 = 1$ gives the solution

$$\begin{aligned}
y_1(x) &= x + \frac{x^2}{1 \cdot 3} + \frac{x^3}{(1 \cdot 3)(2 \cdot 4)} + \frac{x^4}{(1 \cdot 3)(2 \cdot 4)(3 \cdot 5)} + \cdots \\
&= 2 \sum_{n=0}^{\infty} \frac{1}{n!(n+2)!} x^{n+1}.
\end{aligned}$$

Since the solutions form a subspace of $\mathbb{R}^{(0,\infty)}$, the factor 2 could be discarded, but the normalization $a_0 = 1$ is needed for computing $y_2(x)$ with the stated formulas.
Remark: $y_1(x) = 2\sum_{n=0}^{\infty} \frac{1}{n!(n+2)!} x^{n+1}$ is related to the Bessel function $\mathrm{J}_2(x)$ by $\mathrm{J}_2(2\sqrt{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+2)!} x^{m+1} = -\frac{1}{2} y_1(-x)$. The function I_ν with the power series representation $\mathrm{I}_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{2^{2m} m!\,\Gamma(\nu+m+1)} x^{2m}$ (the same as that of J_ν except for the alternating signs of the coefficients) is known as *modified Bessel function*. Thus the above relation can also be written as $y_1(x) = 2\,\mathrm{I}_2(2\sqrt{x})$.

ii) $r_2 = -1$: Since $p_0 = 1$, $q_0 = q_1 = -1$ and all other coefficients p_i, q_i are zero, the recurrence relation for $a_n(r)$ becomes

$$\begin{aligned}
a_n(r) &= -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k) p_{n-k} + q_{n-k} \right] a_k(r) \\
&= \frac{a_{n-1}(r)}{(r+n)^2 - 1} \qquad (n \geq 1).
\end{aligned}$$

Together with $a_0(r) = 1$ this gives

$$\begin{aligned}
a_1(r) &= \frac{1}{(r+2)r}, \\
c_1 &= \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{r+1}{r^2+2r} \right) \bigg|_{r=-1} = \frac{-r^2-2r-2}{(r^2+2r)^2} \bigg|_{r=-1} = -1,
\end{aligned}$$

and for $n \geq 2$

$$
\begin{aligned}
a_n(r) &= \frac{1}{[(r+2)(r+3)\cdots(r+n+1)][r(r+1)\cdots(r+n-1)]},\\
(r+1)a_n(r) &= \frac{1}{r(r+n)(r+n+1)\prod_{k=2}^{n-1}(r+k)^2},\\
a &= \lim_{r\to -1}\frac{1}{r(r+2)(r+3)} = -\frac{1}{2},\\
c_n &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r(r+n)(r+n+1)\prod_{k=2}^{n-1}(r+k)^2}\right)\Bigg|_{r=-1}\\
&= \frac{1}{n!(n-2)!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-1+1+\frac{1}{2}+\cdots+\frac{1}{n-2}\right)\\
&= \frac{1}{n!(n-2)!}\left(\mathrm{H}_n+\mathrm{H}_{n-2}-1\right).
\end{aligned}
$$

(Also valid for $n = 2$ if we set $\mathrm{H}_0 := 0$)

$$
\Longrightarrow \quad y_2(x) = -\left(\sum_{n=0}^{\infty}\frac{1}{n!(n+2)!}x^{n+1}\right)\ln x + x^{-1} - 1 + \sum_{n=2}^{\infty}\frac{\mathrm{H}_n+\mathrm{H}_{n-2}-1}{n!(n-2)!}x^{n-1}
$$

Some further simplifications are possible. For example, we may discard the "-1" in the second numerator, since this amounts to adding a multiple of $y_1(x)$ to $y_2(x)$, and we can make the denominators equal by an index shift. Thus an alternative 2nd solution is

$$
y_2^*(x) = \frac{1}{x} - 1 + \sum_{n=0}^{\infty}\frac{\mathrm{H}_n+\mathrm{H}_{n+2}-\ln x}{n!(n+2)!}\,x^{n+1}.
$$

Remark: As in b) one can determine a 2nd solution by substituting the „Ansatz“ $y_2(x) = a\,y_1(x)\ln x + x^{-1}\sum_{n=0}^{\infty} c_n x^n$ into the ODE. Then the normalization $a_0 = 1$ doesn't matter, so that we can take $y_1(x) = \sum_{n=0}^{\infty}\frac{1}{n!(n+2)!}\,x^{n+1}$ (corresponding to $a_0 = 1/2$). Writing $L = x^2y'' + xy' - (1+y)y$, $c(x) = \sum_{n=0}^{\infty} c_n x^{n-1}$, we obtain

$$
\begin{aligned}
y_2(x) &= a\,y_1(x)\ln x + c(x),\\
y_2'(x) &= a\,y_1'(x)\ln x + a\,y_1(x)\frac{1}{x} + c'(x),\\
y_2''(x) &= a\,y_1''(x)\ln x + 2a\,y_1'(x)\frac{1}{x} + a\,y_1(x)\frac{-1}{x^2} + c''(x),\\
L\left[y_2(x)\right] &= a\,L\left[y_1(x)\right]\ln x + 2ax\,y_1'(x) + L\left[c(x)\right]\\
&= 2ax\,y_1'(x) + L\left[c(x)\right].
\end{aligned}
$$

Further we have

$$
\begin{aligned}
2ax\,y_1'(x) &= \sum_{n=0}^{\infty}\frac{2a(n+1)}{n!(n+2)!}\,x^{n+1} = \sum_{n=2}^{\infty}\frac{2a(n-1)}{n!(n-2)!}\,x^{n-1},\\
L\left[c(x)\right] &= \sum_{n=1}^{\infty}\left[(n^2-2n)c_n - c_{n-1}\right]x^{n-1}.
\end{aligned}
$$

The computation of the latter is omitted. (It should be routine by now.)
Hence $L\left[y_2(x)\right]=0$ is equivalent to $(1^2-2\cdot 1)c_1-c_0=-c_1-c_0=0$ and $(n^2-2n)c_n-c_{n-1}+\frac{2a(n-1)}{n!(n-2)!}=0$ for $n\geq 2$. Setting $c_0=1$, we get $c_1=-c_0=-1$. The equation for $n=2$ leaves c_2 undetermined (corresponding to the fact that constant multiples of $y_1(x)$ can be added to $y_2(x)$ without affecting the property of being a solution) and gives $a=c_1=-1$. (That a has changed is due to the change of the normalization constant; the corresponding relation is $a=-1/(2a_0)$.) The remaining coefficients c_n, $n\geq 3$, are determined by c_2 and the recurrence relation.
The recurrence relation is conveniently solved by introducing $C_n:=n!(n-2)!\,c_n$ and multiplying it by $(n-1)!(n-3)!$. This turns it into

$$C_n-C_{n-1}=-\frac{2a(n-1)}{n(n-2)}=\frac{2n-2}{n(n-2)}=\frac{1}{n}+\frac{1}{n-2}$$

and implies that

$$C_n=C_2+\sum_{k=3}^{n}(C_k-C_{k-1})=2c_2+\sum_{k=3}^{n}\left(\frac{1}{k}+\frac{1}{k-2}\right).$$

Setting $c_2=3/4$ gives $C_n=\mathrm{H}_n+\mathrm{H}_{n-2}$, $c_n=\frac{\mathrm{H}_n+\mathrm{H}_{n-2}}{n!(n-2)!}$, and hence the solution denoted earlier by $y_2^*(x)$.

d) Rewriting the ODE as

$$y''+\frac{1}{x}y'+\left(\frac{1}{x^2}+\frac{1}{x}\right)y=0,$$

we see that $x=0$ is a regular singular point and

$$p_0=\lim_{x\to 0}x\frac{x}{x^2}=1,\quad q_0=\lim_{x\to 0}x^2\frac{1+x}{x^2}=1.$$

$\Longrightarrow$ The indicial equation is

$$r^2+(p_0-1)r+q_0=r^2+1=0.$$

$\Longrightarrow$ The exponents at the singularity $x=0$ are $r_1=\mathrm{i}$, $r_2=-\mathrm{i}$. Thus there must be solutions $y_1(x)$, $y_2(x)$ on $(0,\infty)$ of the form

$$\begin{aligned}
y_1(x)&=x^{\mathrm{i}}\sum_{n=0}^{\infty}a_nx^n=\mathrm{e}^{\mathrm{i}\ln x}\sum_{n=0}^{\infty}a_nx^n,\\
y_2(x)&=x^{-\mathrm{i}}\sum_{n=0}^{\infty}a_nx^n=\mathrm{e}^{-\mathrm{i}\ln x}\sum_{n=0}^{\infty}a_nx^n.
\end{aligned}$$

This time we first determine the functions $a_n(r)$ from the recurrence relation and then substitute $r=\pm\mathrm{i}$. Since $p_1=0$, $q_1=1$, the recurrence relation for $a_n(r)$ is

$$a_n(r)=-\frac{a_{n-1}(r)}{F(r+n)}=-\frac{a_{n-1}(r)}{(r+n)^2+1}.$$

$$\begin{aligned}
\Longrightarrow \quad a_1(r) &= -\frac{a_0(r)}{(r+1)^2+1} = -\frac{1}{(r+1)^2+1},\\
a_2(r) &= \frac{1}{[(r+1)^2+1][(r+2)^2+1]},\\
&\vdots\\
a_n(r) &= \frac{(-1)^n}{[(r+1)^2+1][(r+2)^2+1]\cdots[(r+n)^2+1]},\\
\Longrightarrow y_1(x) &= \mathrm{e}^{\mathrm{i}\ln x}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{[(1+\mathrm{i})^2+1][(2+\mathrm{i})^2+1]\cdots[(n+\mathrm{i})^2+1]}\right),\\
y_2(x) &= \mathrm{e}^{\mathrm{i}\ln x}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{[(1-\mathrm{i})^2+1][(2-\mathrm{i})^2+1]\cdots[(n-\mathrm{i})^2+1]}\right).
\end{aligned}$$

Two linearly independent real solutions $y_1^*(x)$, $y_2^*(x)$ are obtained by extracting real and imaginary part of $y_1(x)$, say.

$$\begin{aligned}
y_1^*(x) &= \cos(\ln x)\left(1-\frac{x}{5}-\frac{x^2}{40}+\frac{3x^3}{520}\mp\cdots\right)-\sin(\ln x)\left(\frac{2x}{5}-\frac{3x^2}{40}+\frac{7x^3}{1560}\mp\cdots\right),\\
y_2^*(x) &= \sin(\ln x)\left(1-\frac{x}{5}-\frac{x^2}{40}+\frac{3x^3}{520}\mp\cdots\right)+\cos(\ln x)\left(\frac{2x}{5}-\frac{3x^2}{40}+\frac{7x^3}{1560}\mp\cdots\right).
\end{aligned}$$

58 a) Rewriting the ODE as

$$y''+\frac{3}{x}y'-\frac{3}{x}=0,$$

we see that $x=0$ is a regular singular point and

$$p_0=\lim_{x\to 0}x\frac{3}{x}=3,\quad q_0=\lim_{x\to 0}x^2\frac{-3}{x}=0$$

.

$\Longrightarrow$ The indicial equation is

$$r^2+(p_0-1)r+q_0=r^2+2r=0$$

.

$\Longrightarrow$ The exponents at the singularity $x=0$ are $r_1=0$, $r_2=-2$.

b) $r_1=0$:

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty}a_n x^n,\\
y_1' &= \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,\\
xy_1'' &= \sum_{n=2}^{\infty}n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n
\end{aligned}$$

Substituting these into the ODE, we get

$$\sum_{n=0}^{\infty}\{[n(n+1)+3(n+1)]\,a_{n+1}-3a_n\}\,x^n=0.$$

$$\Longrightarrow \quad a_{n+1}=\frac{3}{(n+1)(n+3)}a_n, \quad \text{for} \quad n=0,1,2,\ldots$$

Setting $a_0=1$ gives

$$y_1(x)=\sum_{n=0}^{\infty}\frac{2\cdot 3^n}{n!(n+2)!}x^n.$$

c) $r_2=-2$:

$$y_2(x)=a\,y_1(x)\ln x+x^{-2}\left(1+\sum_{n=1}^{\infty}c_nx^n\right)$$

with

$$a=\lim_{r\to-2}(r+2)a_2(r), \quad c_n=\frac{\mathrm{d}}{\mathrm{d}r}\left[(r+2)a_n(r)\right]\Big|_{r=-2}.$$

Since $p_1=0$, $q_1=-3$ (and all other relevant p_i, q_i are zero), the recurrence relation for $a_n(r)$ is

$$a_n(r)=-\frac{-3\,a_{n-1}(r)}{F(r+n)}=\frac{3\,a_{n-1}(r)}{(r+n)(r+n+2)}.$$

Together with $a_0(r)=1$ this leads to

$$\begin{aligned}
a_n(r)&=\frac{3^n}{[(r+1)(r+2)\cdots(r+n)][(r+3)(r+4)\cdots(r+n+2)]},\\
a_N(r)=a_2(r)&=\frac{3^2}{(r+1)(r+2)(r+3)(r+4)},\\
a&=\lim_{r\to-2}\frac{3^2}{(r+1)(r+3)(r+4)}=-\frac{9}{2},\\
c_1&=\left(\frac{3(r+2)}{(r+1)(r+3)}\right)\Bigg|_{r=-2}=-3,\\
c_n&=\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{3^n}{[(r+1)(r+3)(r+4)\cdots(r+n)][(r+3)(r+4)\cdots(r+n+2)]}\right]\Bigg|_{r=-2}\\
&=\frac{3^n}{(n-2)!n!}\left(-1+1+\frac{1}{2}+\cdots+\frac{1}{n-2}+1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\\
&=\frac{3^n}{(n-2)!n!}\left(\mathrm{H}_n+\mathrm{H}_{n-2}-1\right) \qquad \text{for } n\geq 2.
\end{aligned}$$

Thus

$$y_2(x)=-\left(\sum_{n=0}^{\infty}\frac{3^{n+2}}{n!(n+2)!}\,x^n\right)\ln x+x^{-2}-3x^{-1}+\sum_{n=2}^{\infty}\frac{3^n(\mathrm{H}_n+\mathrm{H}_{n-2}-1)}{n!(n-2)!}\,x^{n-2}.$$

Remark: If $y(x)$ solves $xy''+3y'-3y=0$ then $z(x):=x\,y(x/3)$ solves the ODE in H57c). This is suggested by the form of $y_1(x)$ and can be proved easily. Thus we can save the computation in c) and obtain directly $y_2^*(x) = z(3x)/(3x)$, where z denotes the 2nd solution of H57c). The coefficient of $y_2^*(x)$ at x^{-2} is $1/9$, so that $y_2(x) = 9\,y_2^*(x) + c\,y_1(x)$ for some $c \in \mathbb{R}$. Although it is not necessary for the solution, one can check that $c = \frac{9}{2}\ln 3$.

59 a) Exercise 5

Using the ratio test, we get

$$\begin{aligned}
\lim_{m\to+\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to+\infty}\left|\frac{\frac{(-1)^{m+1}x^{2(m+1)}}{2^{2(m+1)}((m+1)!)^2}}{\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}}\right| \\
&= \lim_{m\to+\infty}\left|\frac{-x^2}{4(m+1)^2}\right| \\
&= \lim_{m\to+\infty}\frac{x^2}{4(m+1)^2} \\
&= 0 \\
&< 1
\end{aligned}$$

for all $x \neq 0$.

So, the series for $\mathrm{J}_0(x)$ converges absolutely for all x.

b) Exercise 6

Using the ratio test, we get

$$\begin{aligned}
\lim_{m\to+\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to+\infty}\left|\frac{\frac{(-1)^{m+1}x^{2(m+1)}}{2^{2(m+1)}(m+2)!(m+1)!}}{\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}}\right| \\
&= \lim_{m\to+\infty}\left|\frac{-x^2}{4(m+2)(m+1)}\right| \\
&= \lim_{m\to+\infty}\frac{x^2}{4(m+2)(m+1)} \\
&= 0 \\
&< 1
\end{aligned}$$

for all $x \neq 0$.

So, the series for $\mathrm{J}_1(x)$ converges absolutely for all x.

It follows that we can obtain the derivative of $\mathrm{J}_0(x)$ everywhere by term-wise differentiation:

$$\begin{aligned}
\mathrm{J}_0'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \\
&= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}x^{2m+1}}{2^{2m+1}(m+1)!m!} \\
&= -\frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!} \\
&= -\mathrm{J}_1(x),
\end{aligned}$$

as claimed.

c) Exercise 9

First, we want to show that $a_1(-1) = a_1'(-1) = 0$.

Equation (24) in Ch. 5.7 gives

$$a_1(r)((r+1)^2 - 1)x^{r+1} = 0.$$

Hence $a_1(r) = 0$ for $r \notin \{-2, 0\}$, and in particular $a_1(-1) = a_1'(-1) = 0$. (Alternatively, look at the recurrence relation $a_n(r) = -F(r+n)^{-1} \sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}]\, a_k(r)$, which for $n = 1$ reduces to $a_1(r) = -\frac{1}{r(r+2)} [rp_1 + q_1]\, a_0(r) = \frac{0}{r(r+2)}$, since for the Bessel equation $p_1 = q_1 = 0$.) Next,

$$c_1(-1) = \frac{\mathrm{d}}{\mathrm{d}r}[(r+1)a_1(r)]\bigg|_{r=-1} = 0.$$

Then, from equation (25) in Ch. 5.7 or using the said general recurrence relation for $a_n(r)$, we get

$$a_n(r) = \frac{-a_{n-2}(r)}{(r+n-1)(r+n+1)} \qquad \text{for } n \geq 2.$$

Since $a_1(r) = 0$, this gives $a_n(r) = 0$ for all odd n wherever $a_n(r)$ is defined (i,e., $r \notin \{0, -2, -4, \ldots, -n-1\}$), and hence $c_n(-1) = \frac{\mathrm{d}}{\mathrm{d}r} [(r+1)a_n(r)]\big|_{r=-1} = 0$ for all odd n. For even n the recurrence relation gives by induction

$$\begin{aligned}
a_2(r) &= \frac{-a_0(r)}{(r+1)(r+3)} = -\frac{1}{(r+1)(r+3)}, \\
a_4(r) &= \frac{-a_2(r)}{(r+3)(r+5)} = \frac{1}{(r+1)(r+3)(r+3)(r+5)}, \\
&\vdots \\
a_{2m}(r) &= \frac{-1}{(r+2m-1)(r+2m+1)} \cdot \frac{-1}{(r+2m-3)(r+2m-1)} \cdots \frac{-1}{(r+1)(r+3)} \\
&= \frac{(-1)^m}{(r+1)(r+3)\cdots(r+2m-1)(r+3)(r+5)\cdots(r+2m+1)}.
\end{aligned}$$

So,

$$\begin{aligned}
c_{2m}(-1) &= \frac{\mathrm{d}}{\mathrm{d}r}[(r+1)a_{2m}(r)]\bigg|_{r=-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{(-1)^m}{(r+3)^2(r+5)^2\cdots(r+2m-1)^2(r+2m+1)}\right)\bigg|_{r=-1} \\
&= \left[\left(-\frac{2}{r+3} - \frac{2}{r+5} - \frac{2}{r+2m-1} - \frac{1}{r+2m+1}\right)(r+1)a_{2m}(r)\right]\bigg|_{r=-1} \\
&= \left(-1 - \frac{1}{2} - \cdots - \frac{1}{m-1} - \frac{1}{2m}\right)\frac{(-1)^m}{2^2 4^2 \cdots (2m-2)^2(2m)} \\
&= -\frac{1}{2}(\mathrm{H}_{m-1} + \mathrm{H}_m)\frac{(-1)^m}{2^{2m-1}m!(m-1)!} \\
&= \frac{(-1)^{m+1}(\mathrm{H}_{m-1} + \mathrm{H}_m)}{2^{2m}m!(m-1)!} \qquad \text{for } m = 1, 2, \ldots
\end{aligned}$$

Finally, we need to compute

$$
\begin{aligned}
a &= \lim_{r\to -1}(r+1)a_2(r)\\
&= \lim_{r\to -1}\left(\frac{-1}{r+3}\right)\\
&= -\frac{1}{2}.
\end{aligned}
$$

According to the theory (e.g., Th. 5.6.1 in Ch. 5.6), a 2nd solution of Bessel's equation of order one is

$$
\begin{aligned}
y_2(x) &= -\frac{1}{2}y_1(x)\ln|x| + \frac{1}{|x|}\left(1-\sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m}\right)\\
` &= -\mathrm{J}_1(x)\ln|x| + \frac{1}{|x|}\left(1-\sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m}\right), \quad x\neq 0.
\end{aligned}
$$

For this note that $y_1(x)$ denotes the analytic solution normalized by $a_0 = y_1'(0) = 1$, so that $\mathrm{J}_1(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+1}m!(m+1)!}\,x^{2m+1} = \frac{x}{2}+\cdots = \frac{1}{2}y_1(x)$.
The corresponding Neumann function is then

$$
\begin{aligned}
\mathrm{Y}_1(x) &= \frac{2}{\pi}\left[-y_2(x)+(\gamma-\ln 2)\mathrm{J}_1(x)\right]\\
&= \frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) - \frac{1}{x} + \sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}\,x^{2m-1}\right].
\end{aligned}
$$

Finally we show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$.

$$
\begin{aligned}
\mathrm{Y}_0'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m}(m!)^2}\,x^{2m}\right]\\
&= \frac{2}{\pi}\left[\frac{\mathrm{J}_0(x)}{x} + \left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_0'(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m-1}m!(m-1)!}\,x^{2m-1}\right]\\
&= \frac{2}{\pi}\left[-\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) + \frac{1}{x} + \sum_{m=1}^{\infty}\left(\frac{(-1)^m}{2^{2m}(m!)^2} + \frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m-1}m!(m-1)!}\right)x^{2m-1}\right]\\
&= \frac{2}{\pi}\left[-\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) + \frac{1}{x} - \sum_{m=1}^{\infty}\frac{\frac{(-1)^{m-1}}{m}+(-1)^m2\mathrm{H}_m}{2^{2m}m!(m-1)!}\,x^{2m-1}\right]\\
&= -\mathrm{Y}_1(x),
\end{aligned}
$$

since $2\mathrm{H}_m - \frac{1}{m} = \mathrm{H}_m + \mathrm{H}_{m-1}$.

60 a) To show that $\Gamma(x)$ is well-defined for $x<0$, $x\notin\mathbb{Z}$, we only need to show that different choices of $n > -x$ don't affect the value of $\Gamma(x)$ as specified in the exercise.

$$\begin{aligned}
\Gamma(x) &= \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)} \\
&= \frac{\Gamma(x+n+1)}{x(x+1)\cdots(x+n-1)(x+n)} && \text{(since } \Gamma(x+n+1) = (x+n)\Gamma(x+n)\text{)} \\
&= \frac{\Gamma(x+n+2)}{x(x+1)\cdots(x+n-1)(x+n)(x+n+1)} && \text{(same reasoning)} \\
&= \cdots
\end{aligned}$$

So, as $n > -x$ varies, the result of $\Gamma(x)$ remains the same, which means that $\Gamma(x)$ is well-defined.
Then, we prove that $\Gamma(x+1) = \Gamma(x)$. For $x > 0$ this was shown in Calculus III, so it remains to consider the case $x < 0$, $x \notin \mathbb{Z}$. Choose $n \in \mathbb{N}$ such that $x + n > 0$. Then in the definition of $\Gamma(x+1)$ we can use $n-1$, since $x+1+(n-1) = x+n > 0$.

$$\begin{aligned}
\implies \Gamma(x+1) &= \frac{\Gamma(x+1+n-1)}{(x+1)(x+2)\cdots(x+1+(n-1)-1)} \\
&= \frac{\Gamma(x+n)}{(x+1)(x+2)\cdots(x+n-1)} \\
&= x\,\frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)} \\
&= x\,\Gamma(x)
\end{aligned}$$

For $n = 1$, which is possible only if $-1 < x < 0$, the definition of $\Gamma(x)$ reduces to $\Gamma(x) = \frac{\Gamma(x+1)}{x}$ and the functional equation holds as well. This case is included in the above computation, provided the first denominator is interpreted as 1 (empty product).

b) For x close to $-n$ we have $x + n + 1 > 0$. Hence a) gives

$$\lim_{x\to -n} \frac{1}{\Gamma(x)} = \lim_{x\to -n} \frac{x(x+1)\cdots(x+n)}{\Gamma(x+n+1)}.$$

Since $\Gamma(1) = 1$, the limit evaluates to

$$\lim_{x\to -n} \frac{1}{\Gamma(x)} = \frac{(-n)(-n+1)\cdots(0)}{1} = 0.$$

c) First, we have

$$\mathrm{J}_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{-\nu+2m} m!\,\Gamma(-\nu+m+1)}\, x^{-\nu+2m}.$$

From b), we know that $1/\Gamma(-n) = 0$ for $n \in \mathbb{N}$. So, the coefficients of $x^{-\nu+2m}$ are zero for $m < \nu$. Then

$$\begin{aligned}
\mathrm{J}_{-\nu}(x) &= \sum_{m=\nu}^{\infty} \frac{(-1)^m}{2^{-\nu+2m} m!\,\Gamma(-\nu+m+1)}\, x^{-\nu+2m} \\
&= \sum_{m=\nu}^{\infty} \frac{(-1)^m}{2^{\nu+2(m-\nu)} m!\,\Gamma((m-\nu)+1)}\, x^{\nu+2(m-\nu)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+\nu}}{2^{\nu+2n} (n+\nu)!\,\Gamma(n+1)}\, x^{\nu+2n} && (\text{let } n = m-\nu) \\
&= (-1)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{\nu+2n}(n+\nu)!n!}\, x^{\nu+2n} \\
&= (-1)^{\nu} \mathrm{J}_{\nu}(x).
\end{aligned}$$

61 For $\nu \in \mathbb{N}$ the function $\mathrm{J}_\nu(x)$ was defined in the lecture as the analytic solution of Bessel's equation of order ν normalized by setting the coefficient of x^ν (first nonzero coefficient) equal to $\frac{1}{2^\nu \nu!}$. It can also be derived using Frobenius' method as follows (not part of the exercise):

$$\begin{aligned}
0 &= x^2 y'' + xy' + (x^2 - \nu^2) y \\
&= \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + (x^2 - \nu^2) \cdot \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} a_n [(r+n)^2 - \nu^2] x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} \\
&= a_0 (r^2 - \nu^2) x^r + a_1 [(r+1)^2 - \nu^2] x^{r+1} + \sum_{n=2}^{\infty} \left\{ [(r+n)^2 - \nu^2] a_n + a_{n-2} \right\} x^{r+n}
\end{aligned}$$

For $r = \nu$ there are solutions with arbitrary a_0. These must satisfy $a_n = 0$ for all odd n and $[(\nu+n)^2 - \nu^2] a_n + a_{n-2} = n(n+2\nu) a_n + a_{n-2} 0$ for all even $n \geq 2$. By induction,

$$\begin{aligned}
a_{2m} &= -\frac{a_{2m-2}}{2m(2m+2\nu)} = \cdots = \frac{(-1)^m a_0}{[2m(2m-2)\cdots 2]\,[(2m+2\nu)(2m-2+2\nu)\cdots(2+2\nu)]} \\
&= \frac{(-1)^m a_0}{2^{2m} m! (\nu+1)(\nu+2)\cdots(\nu+m)}.
\end{aligned}$$

Choosing $a_0 = \frac{1}{2^\nu \nu!}$, we get

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m}}{2^{\nu+2m} m! (\nu+m)!}.$$

Then, we solve the exercise:

a)

$$\begin{aligned}
\frac{2\nu}{x}\mathrm{J}_\nu(x)-\mathrm{J}_{\nu-1}(x)&=\sum_{m=0}^{\infty}\frac{(-1)^m x^{\nu+2m-1}\nu}{2^{\nu+2m-1}m!(\nu+m)!}-\sum_{m=0}^{\infty}\frac{(-1)^m x^{\nu+2m-1}}{2^{\nu+2m-1}m!(\nu+m-1)!}\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m x^{\nu+2m-1}}{2^{\nu+2m-1}m!(\nu+m-1)!}\left(\frac{\nu}{\nu+m}-1\right)\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1} x^{\nu+2m-1}}{2^{\nu+2m-1}(m-1)!(\nu+m)!}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^{n+2} x^{\nu+2n+1}}{2^{\nu+2n+1}n!(\nu+n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^{n} x^{\nu+1+2n}}{2^{\nu+1+2n}n!(\nu+1+n)!}\\
&=\mathrm{J}_{\nu+1}(x)
\end{aligned}$$

b) The Bessel functions may be differentiated termwise to yield

$$\begin{aligned}
\mathrm{J}_\nu'(x)&=\frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{\nu+2m}\,m!\,\Gamma(m+\nu+1)}\,x^{\nu+2m}\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m(\nu+2m)}{2^{\nu+2m}\,m!\,\Gamma(m+\nu+1)}\,x^{\nu+2m-1}\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m\nu}{2^{\nu+2m}m!\,\Gamma(m+\nu+1)}\,x^{\nu+2m-1}+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^{\nu+2m-1}(m-1)!\,\Gamma(m+\nu+1)}\,x^{\nu+2m-1}\\
&=\frac{\nu}{x}\,\mathrm{J}_\nu(x)+\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2^{\nu+2m+1}m!\,\Gamma(m+\nu+2)}\,x^{\nu+2m+1}\\
&=\frac{\nu}{x}\,\mathrm{J}_\nu(x)-\mathrm{J}_{\nu+1}(x).
\end{aligned}$$

62 First, $\cos(x\sin\theta)$ can be written as

$$\cos(x\sin(\theta))=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m}\sin^{2m}\theta.$$

If $x\in\mathbb{R}$ is kept fixed, this represents a function series $\sum_{m=0}^{\infty}f_m(\theta)$, which converges uniformly on $[0,\pi]$ by Weierstrass' test. Hence the series can be integrated termwise, and we obtain

$$\frac{1}{\pi}\int_0^\pi\cos(x\sin(\theta))\mathrm{d}\theta=\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m}\int_0^\pi\sin^{2m}\theta\,\mathrm{d}\theta \tag{$\star$}$$

The integral $\int_0^\pi\sin^{2m}\theta\,\mathrm{d}\theta=2\int_0^{\pi/2}\sin^{2m}\theta\,\mathrm{d}\theta$ has been evaluated in Calculus III (or see our Calculus textbook [Ste12/16], Ch. 7.1, Exercise 50):

$$\begin{aligned}
\int_0^\pi\sin^{2m}\theta\,\mathrm{d}\theta&=\frac{(2m-1)(2m-3)\cdots1}{2m(2m-2)\cdots2}\,\pi\\
&=\frac{(2m)!}{(2m)^2(2m-2)^2\cdots2^2}\,\pi=\frac{(2m)!}{2^{2m}(m!)^2}\,\pi.
\end{aligned}$$

Substituting this into ($\star$), we get

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}\cos(x\sin(\theta))\mathrm{d}\theta &= \frac{1}{\pi}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(2m)!}x^{2m}\,\frac{(2m)!}{2^{2m}(m!)^2}\,\pi\right]\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\\ &= \mathrm{J}_0(x)\end{aligned}$$

for $x \in \mathbb{R}$.
Alternative solution: J_0 is the unique solution on $\mathbb{R}$ of the IVP $x^2y'' + xy' + x^2y = 0$, $y(0) = 1$. This follows from the fact that Y_0 is not defined at $x = 0$. The right-hand side $f(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\mathrm{d}\theta$ satisfies $f(0) = \frac{1}{\pi}\int_0^{\pi}\cos(0)\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi}1\,\mathrm{d}\theta = 1$. Since $[0,\pi]$ is compact and the integrand $g(x,\theta) = \cos(x\sin\theta)$ has continuous partial derivatives up to order two (in fact up to any order), we can differentiate twice under the integral sign to obtain

$$\begin{aligned}f'(x) &= \frac{1}{\pi}\int_0^{\pi} -\sin(x\sin\theta)\sin\theta\,\mathrm{d}\theta,\\ f''(x) &= \frac{1}{\pi}\int_0^{\pi} -\cos(x\sin\theta)\sin^2\theta\,\mathrm{d}\theta.\end{aligned}$$

It follows that

$$\begin{aligned}x^2\big(f(\theta) + f''(\theta)\big) &= \frac{1}{\pi}\int_0^{\pi} x^2\cos(x\sin\theta)\cos^2\theta\,\mathrm{d}\theta\\ &= \frac{1}{\pi}\left([(x\cos\theta)\sin(x\sin\theta)]_0^{\pi} + \int_0^{\pi} x\sin\theta\sin(x\sin\theta)\mathrm{d}\theta\right)\\ &= \frac{x}{\pi}\int_0^{\pi}\sin\theta\sin(x\sin\theta)\mathrm{d}\theta = -x\,f'(x).\end{aligned}$$

Thus f solves the same IVP as J_0 and hence must be equal to J_0.
Remark: A different integral representation of $\mathrm{J}_0(x)$ (obtained from the present by an obvious substitution) was the subject of Question 3 in the Calculus III final exam.