ZJU-UIUC INSTITUTE

Final Examination

(For Students, please fill in your name and ID number and read any instructions below before starting your exam. Please be aware of your obligation not to receive or give aid to others. Don’t take the test out of the exam room.)

Name:	ID:

(For instructors, please complete the form below)

Course: MATH286	Semester: 2021 Fall Semester	Instructor: Thomas Honold	
Exam Type: Closed-book √ Open-book □ Partly Open-book □ Take Home □			
Exam Date: 2022/01/05	Start Time: 09:00	End Time: 12:00	Duration: 3 hrs
Total number of pages: 1+6+3		Number of questions: 6	

Specific requirements and instructions to students:
Please answer every question and subquestion, and JUSTIFY your answers.
For your answers please use the space provided after each question. If this space is insufficient, continue on the back side, and then on the blank sheets provided.
This is a CLOSED BOOK examination (no books, no lecture notes, no electronic devices of any kind are allowed), except that you may bring 1 sheet of A4 paper (hand-written only) and a Chinese-English dictionary (paper copy only) to the examination.

(For marker, please fill in the score and grade and sign below.)

Score:	Grade:
Signature:	Date:

(Please go on to the next page for questions)

Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = y^4 + y$ satisfying $y(0) = 1$, $y(1) = 0$.

b) There exists a solution $y(t)$ of $y' = ty^2 - t^2 y$ satisfying $\lim_{t\to+\infty} y(t) = 2021$.

c) Every maximal solution of $(x^2+1)y'' + (x+1)y' + y = 1$ has domain $\mathbb{R}$.

d) The initial value problem $(x^2+1)y'' + (x+1)y' + y = 1$, $y(1) = y'(1) = 0$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ which is defined at $x = 3$.

e) Every solution of the system $\mathbf{y}' = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}$ satisfies $\lim\limits_{t\to+\infty} \mathbf{y}(t) = (0,0)^\mathsf{T}$.

f) If $\mathbf{A} \in \mathbb{R}^{3\times 3}$ satisfies $\mathbf{A}^\mathsf{T} = -\mathbf{A}$ then $e^{\mathbf{A}t}$ is an orthogonal matrix for all $t \in \mathbb{R}$.

Question 2 (ca. 9 marks)

Consider the differential equation

$$2x^2y'' + 3x\,y' + (2x-1)y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0,\infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty,0)$ and on $\mathbb{R}$.

Question 3 (ca. 6 marks)

Determine all maximal solutions (including their domains) of

$$y' = \frac{y}{t+1} + y^4, \qquad t > -1. \tag{B}$$

Hint: A substitution of the form $z(t) = y(t)^r$ with $r \in \mathbb{R}$ may help. When translating (B) into an ODE for $z(t)$, you will see how r should be chosen.

Question 4 (ca. 6 marks)

Consider $\mathbf{A} = \begin{pmatrix} 2 & 12 & -32 \\ -4 & -14 & 32 \\ -1 & -3 & 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

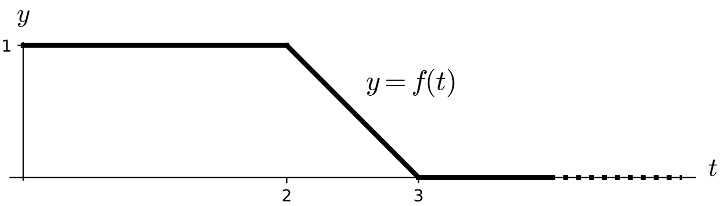
b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{y}(0) = (0, 0, 0)^\mathsf{T}$.

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + y' - 2y = f(t), \quad y(0) = y'(0) = 0$$

with the Laplace transform.



Question 6 (ca. 6 marks)

a) Determine a real fundamental system of solutions of

$$4\,y^{(4)} - 4\,y^{(3)} + 17\,y'' - 16\,y' + 4\,y = 0.$$

b) Determine the general real solution of

$$4\,y^{(4)} - 4\,y^{(3)} + 17\,y'' - 16\,y' + 4\,y = (3 - \cos t)(3 + \sin t).$$