Algebraic Curves and their Intersections

Yifan Mo

Mentor: Samarth Sheth

December 3, 2025

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Definition

Multiplicity of point $p \in F$: $m_p(F) =$ degree of lowest non-zero term in Taylor expansion of F at p

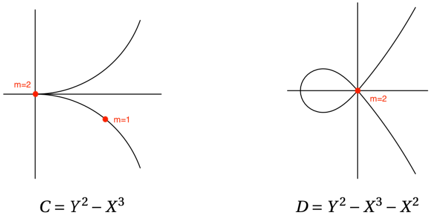

Figure: Multiplicity of points

Important definitions

- (Affine) variety of polynomial function $F \in k[X_1, ..., X_n]$: $V(F) = \{p \in \mathbb{A}^n | F(p) = 0\}$. For simplicity we let F denote $V(F)$
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- Ideal generated by functions F_1, F_2: $(F_1, F_2) = \{AF_1 + BF_2 : A, B \in k[X, Y]\}$
- Quotient ring: $\frac{k[X,Y]}{(F)} = \{\overline{A}\} = \{A + (F) : A \in k[X, Y]\}$

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- Local ring at point p: $\mathcal{O}_p(\mathbb{A}^2) = \{\frac{G}{Q} | G, Q \in k[X, Y], Q(p) \neq 0\}$

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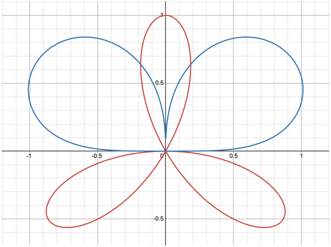
Proposition:

The formula above satisfies the following statements:

- $I(p, F \cap G) \geq m_p(F) \cdot m_p(G)$ with equality iff F and G have no common tangent lines at p
- Distributive: If $F = \prod F_i^{r_i}$, and $G = \prod G_j^{s_j}$, then $I(p, F \cap G) = \sum_{i,j} r_i s_j I(p, F_i \cap G_j)$.
- Invariant under translation
- ...

Example 1

$E = (X^2+Y^2)^2 + 3X^2Y - Y^3$, $F = (X^2+Y^2)^3 - 4X^2Y^2$, and $p = (0,0)$



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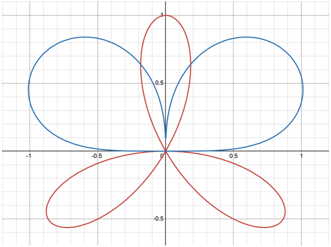

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F and G have no common tangent line at p iff lowest degree terms of two functions have no common factor.

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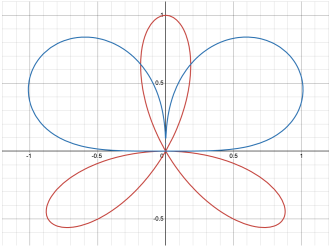
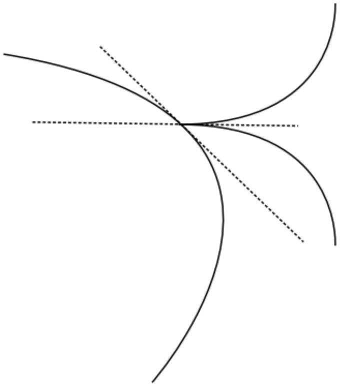


Figure: E, F

F and G have no common tangent line at p iff lowest degree terms of two functions have no common factor.
We can add multiple of one function to the other, because (F, G) and $(F + H \cdot G, G)$ are the same ideal in the local ring!

Example 2

Can you tell the intersection number?



What can we do with non-naive machinery

Count finite-dimensional families of solutions to problems such as:

- How many how many smooth conics are tangent to 5 given conics in $\mathbb{P}^2$
- How many straight lines are there on a cubic surface in $\mathbb{P}^3$

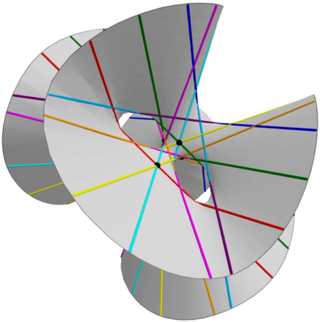


Figure: 27 lines on a Clebsch surface

Thank you!