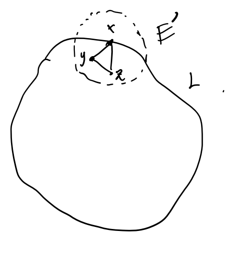
2.6.

E' is the set of limit points of E.

Proposition 1: E' is closed.

Pf. The case where $E' = \emptyset$ is trivial. Assume $E' \neq \emptyset$.

Let L be the set of limit points of E'.



$\forall x \in L$, $\exists y \in E'$ s.t. $\left(d(x,y) < \frac{\varepsilon}{2} \ \forall \varepsilon > 0\right)$

Since $y \in E'$, $\exists z \in E$ s.t. $\left(d(y,z) < \frac{\varepsilon}{2} \ \forall \varepsilon > 0\right)$ and $z \neq x$

By triangle inequality, $d(x,z) \leq d(x,y) + d(y,z)$

$$d(x,z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall \varepsilon > 0$$

$\Rightarrow$ for every open ball centered at x, $\exists z \neq x$ contained in it.

$\Rightarrow x \in E'$.

Since this is true $\forall x \in L$, we have $L \subseteq E'$ $\Rightarrow$ E' is closed. $\square$

Proposition 2: E and $\bar{E}$ have the same limit points

Denote F as the set of limit points of $\bar{E}$

Since $\bar{E} = E \cup E'$, we know that at least $F \subseteq K$. We claim $E' \subseteq F$.

It suffices to show that every limit point of $\bar{E} = E \cup E'$ is contained in E'

$\forall x \in F$, $\exists y \in \bar{E}$ s.t. $0 < d(x,y) < \varepsilon$. If $y \in E'$, then by the same arguments as those in the pf of the previous proposition, we get $x \in E'$. If $y \in E \setminus E'$ then we immediately have $x \in E'$

$\Rightarrow F \subseteq E'$ $\Rightarrow$ $F = E'$ $\square$

Proposition 3: E and E' does not necessarily have the same limit points.

e.g. $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ has limit point $\{0\}$. Limit points of $\{0\}$ is $\emptyset$.

2.7

Let $A_1, A_2, A_3, \dots \subset M$ $\qquad\qquad \overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$

(a) $B_n = \bigcup_{i=1}^{n} A_i$. Proposition : $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$

Pf: We induct on $n \in \mathbb{N}$. The case $n=1$ is trivial.

Base case : $n=2$. Obviously $\overline{A_1} \subseteq \overline{A_1 \cup A_2}$, $\overline{A_2} \subseteq \overline{A_1 \cup A_2}$

$\Rightarrow \overline{A_1} \cup \overline{A_2} \subseteq \overline{A_1 \cup A_2}$

For the other direction of inclusion, it suffices to show ~~that~~

If $x \notin \overline{A_1} \cup \overline{A_2}$, then $x \notin \overline{A_1 \cup A_2}$. $\qquad x \notin \overline{A_1} \cup \overline{A_2} \Rightarrow \exists r>0$ s.t. $\underset{r>0}{B}(x) \cap (A_1 \cup A_2) = \emptyset$

$\Rightarrow \exists r>0$ s.t. $\underset{r>0}{B}(x) \cap A_1 = \emptyset$ and $\underset{r>0}{B}(x) \cap A_2 = \emptyset$.

$\Rightarrow x \notin \overline{A_1}$ and $x \notin \overline{A_2}$ $\Rightarrow x \notin \overline{A_1} \cup \overline{A_2}$ $\Rightarrow \overline{A_1 \cup A_2} \subseteq \overline{A_1} \cup \overline{A_2}$

Hence. $\overline{B_2} = \bigcup_{i=1}^{2} \overline{A_i}$ is true $\therefore$ The rest follows by induction.

$\Rightarrow \overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$

$\square$

(b) $B_n = \bigcup_{i=1}^{\infty} A_i$. Proposition : $\overline{B_n} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$.

It is still true that $\overline{\bigcup_{i=1}^{\infty} A_i} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$. Pf: $\forall x \in \bigcup_{i=1}^{\infty} \overline{A_i}$, $\exists \overline{A_j} \ni x$.

Since $\overline{A_j} \subset \overline{\bigcup_{i=1}^{\infty} A_i}$, we get $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$, hence the inclusion.

But the reverse inclusion may not be true. Consider $A_i = \{ \frac{1}{i} \}$

$\overline{A_i} = A_i = \{ \frac{1}{i} \}$ $\qquad \overline{B_n} = \overline{\{ \frac{1}{n}, n \in \mathbb{N} \}} = 0 \cup \{ \frac{1}{n}, n \in \mathbb{N} \} = 0 \cup (\bigcup_{i=1}^{\infty} \overline{A_i})$

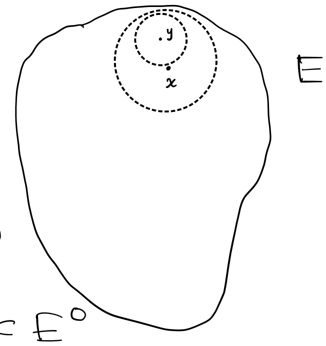
but $0 \notin \bigcup_{i=1}^{\infty} \overline{A_i}$ $\Rightarrow \overline{B_n} \not\subset \bigcup_{i=1}^{\infty} \overline{A_i}$

$\square$

2.9.

(a) Proposition: E° is open.

Pf: By definition, $\forall x \in E^\circ \ \exists \underset{r>0}{B}(x) \subset E$.



Since $\underset{r>0}{B}(x)$ is open, $\exists y \in \underset{r>0}{B}(x)$ such that $\exists \underset{r>0}{B}(y) \subset \underset{r>0}{B}(x)$

$\Rightarrow \underset{r>0}{B}(y) \subset E \Rightarrow y \in E^\circ \ \forall y \in \underset{r>0}{B}(x) \Rightarrow \underset{r>0}{B}(x) \subset E^\circ$

Hence E° is open. $\square$

(b) Proposition: E is open if and only if $E^\circ = E$

Pf: $(\Rightarrow)$

The inclusion $E^\circ \subseteq E$ is trivial.

E is open, so $\forall x \in E \ \exists \underset{r>0}{B}(x) \subset E. \Rightarrow x \in E^\circ \Rightarrow E \subseteq E^\circ$

Hence $E^\circ = E$

$(\Leftarrow)$

In (a) we have shown that E° is open. $E = E^\circ \Rightarrow E$ is open.

$\square$

2.12.

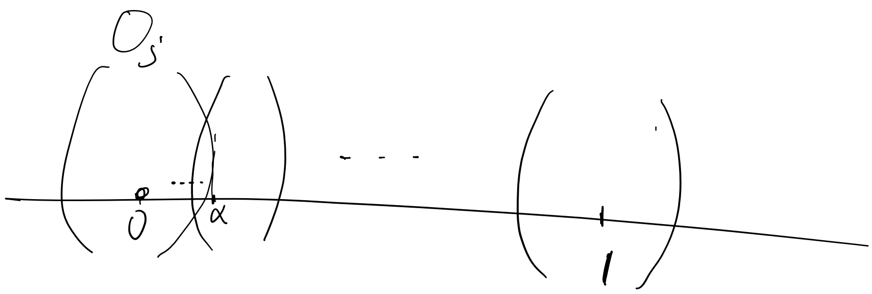
Proposition: $K = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N}\}$ is compact.

Pf. For any open cover $\{O_\lambda : \lambda \in A\}$, $\exists j \in A$ s.t. $0 \in O_j$.

$\exists r > 0$ s.t. $B_r(0) \subset O_j$. By archimedean property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < r$, so O_j covers at least $\{0\} \cup \{\frac{1}{n}, n \geq m\}$

Rest of the elements $\{\frac{1}{n}, n = 1, 2, \dots m-1\} = S$ is a finite set, so $\forall i \in S$, pick an O_i such that $i \in O_i$, hence the collection $\{O_i, i \in S\}$ is a finite cover for S

$\Rightarrow O_j \cup \{O_i, i \in S\}$ is a finite subcover for K



2.19.

(a). Proposition:

A, B are disjoint closed sets in metric space X $\Rightarrow$ A and B are seperated.

Pf:

$A = \bar{A}$. $A \cup B = \emptyset \Rightarrow \bar{A} \cup B = \emptyset$

$B = \bar{B}$ $A \cup B = \emptyset \Rightarrow A \cup B = \emptyset$

Hence A and B are seperated.

2.22 Prop: $\mathbb{R}^k$ contains a countable subset dense in $\mathbb{R}^k$. i.e. $\mathbb{R}^k$ is seperable.

We claim the set of points with rational coordinates $\mathbb{Q}^k = \{(a_1, \ldots, a_k) : a_i \in \mathbb{Q} \ \forall i = 1, \ldots, k\}$ is countable.

Lemma: cartesian product of finitely many countable sets is countable.

Pf: We first show $\mathbb{N} \times \mathbb{N}$ is countable.

By a diagonal emmeration: $\varphi(i,j) = \dfrac{(i+j)(i+j+1)}{2} + j$

which is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

	0	1	2	3
0	0	1	3	6
1	2	4	7	
2	5	8		
3	9			

For any two countable sets, there exist bijections $f: A \to \mathbb{N}$, $g: B \to \mathbb{N}$,

hence $h: \mathbb{N} \times \mathbb{N} \to A \times B$ defined by $h: (m,n) \mapsto (f(m), g(n))$ is bijection.

So $A \times B$ is countable. It follows by induction that $A_1 \times \cdots \times A_n$ is countable if each of $A_1, \ldots, A_n$ is countable.

$\mathbb{Q}$ is countable, so by our lemma $\mathbb{Q}^k$ is countable.

We claim that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. We know $\mathbb{Q}$ is dense in $\mathbb{R}$.

For any $x = \{a_1, \ldots, a_k\} \in \mathbb{R}^k$, there exists $y = \{q_1, \ldots, q_k\} \in \mathbb{Q}^k$ such that $|q_i - a_i| < \frac{r^2}{k}$ $\forall r > 0$, $\forall i = 1, \ldots, k$. So we have:

$$|x - y| = \left(\sum_{1}^{k} |q_i - a_i|\right)^{\frac{1}{2}} < (r^2)^{\frac{1}{2}} = r \quad \forall r > 0$$

$\Rightarrow \mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

$\square$

2-29

Proposition: Every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.

Pf: Let O be any open set in $\mathbb{R}$. We first show that O is the union of a collection of disjoint segments.

$x \in O$. Let $a_x = \inf\{a \in \mathbb{R} : (a,x) \subset O\}$, $b_x = \sup\{b \in \mathbb{R} : (x,b) \subset O\}$

If $\{a \in \mathbb{R} : (a,x) \subset O\}$ is unbounded below, then $a_x = -\infty$. Similarly for $b_x = \infty$.

We claim that $I_x = (a_x, b_x)$ is the maximal segment containing x in a sense that any segment I containing x is a subset of I_x.

It should be noted that $a_x \notin O$ because otherwise there exists a smaller element in O that is also in $\{a \in \mathbb{R} : (a,x) \subset O\}$, contradicting a_x being infimum. Similarly, $b_x \notin O$

For $x, y \in O$, $I_x \cap I_y \neq \emptyset \Rightarrow I_x \cup I_y$ is an open interval containing x

$\Rightarrow I_x \cup I_y \subset I_x \Rightarrow I_x \subseteq I_y$. Similarly it can be shown that $I_y \subseteq I_x$

Hence $I_x \cap I_y \neq \emptyset \Rightarrow I_x = I_y$.

Consider the set $\{I_x : x \in O\}$. Removing duplicates, we arrive at a collection of disjoint segments $\{F_\alpha\}$. It is obvious that $\bigcup F_\alpha \supseteq O$. To see that $\bigcup F_\alpha \subseteq O$, simply notice that any $F_\alpha \in \{F_\alpha\}$ is an open interval and the definition of that open interval implies $F_\alpha \subset O$. We conclude that $O = \bigcup F_\alpha$.

We now show $\{F_\alpha\}$ is at most countable. Suppose $\{F_\alpha\}$ is an uncountable collection of disjoint segments.

Any segment is just an open interval (a,b) where $a \neq b$. Since $a, b \in \mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists\, q \in \mathbb{Q}$ s.t. $a < q < b$. If (a,b) and (c,d) are disjoint, then $q \in (a,b) \Rightarrow q \notin (c,d)$, hence any F_α can be identified uniquely with some rational number q_α.

$\Rightarrow \{q_\alpha\}$ is uncountable. But $\{q_\alpha\} \subset \mathbb{Q}$, so $\mathbb{Q}$ is uncountable, contradiction $\square$