Rudin.

3.1

Prop: (s_n) converges $\Longrightarrow$ $(|s_n|)$ converges.

Pf: $(s_n) \to s$. $|s_n - s| < \varepsilon \quad \forall n > N$.

$$|s_n - s| \geq \Big| |s_n| - |s| \Big| \implies \Big| |s_n| - |s| \Big| < \varepsilon \quad \forall n > N$$

So $(|s_n|) \longrightarrow |s|$

$\square$

Prop: $(|s_n|)$ converges does not necessarily imply (s_n) converges.

$(s_n) = (1, -1, 1, -1, 1, \dots)$ does not converge but.

$(|s_n|) = (1, 1, 1, 1, \dots) \longrightarrow 1$.

$\square$

3.2

$$\sqrt{n^2+n} - n = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right) = \frac{1}{1+1} = \frac{1}{2}.$$

3-5.

$(a_n), (b_n) \in \mathbb{R}$. Prop: $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

Pf: $A := \limsup_{n\to\infty} a_n$, $B := \limsup_{n\to\infty} b_n$

If $A = +\infty$, $B \neq -\infty$ or $B = +\infty$, $A \neq -\infty$, then. it is obvious that proposition holds.

So we can assume $A, B \neq +\infty$.

For any $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that

$$a_n < A + \frac{\varepsilon}{2} \quad \text{for all } n > N_1$$

$$b_n < B + \frac{\varepsilon}{2} \quad \text{for all } n > N_2$$

Let $N = \max\{N_1, N_2\}$. Then. we have :

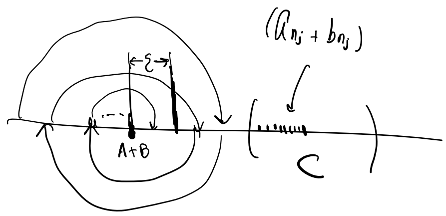
$$\forall \varepsilon > 0, \quad a_n + b_n < A + B + \varepsilon \quad \forall n > N$$

$C := \limsup_{n\to\infty}(a_n+b_n)$ is the supremum of subsequential limits of (a_n+b_n).

Suppose $C > A+B$ Let $\alpha = \frac{C-(A+B)}{4}$, $\varepsilon = \frac{C-(A+B)}{4}$. By definition of C, there exists a subsequence $(a_{n_i}+b_{n_i}) \longrightarrow C_n$ within α-neighborhood of C. Terms $a_{n_j}+b_{n_j}$ such that $n_j > N$ must exist and are larger than $A+B$, This contradicts $a_n+b_n < A+B+\varepsilon$ $\forall n > N$. Hence $C \leq A+B$

$$\Longrightarrow \limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

$\square$



3.20.

Prop: subsequence (p_{n_i}) of Cauchy sequence (p_n) in metrix space. X converges to $p \in X \Rightarrow (p_n) \to p$

p.f: $(p_{n_i}) \to p \Rightarrow \forall \varepsilon > 0$, $\exists N_1$ s.t. $|p_{n_i} - p| < \frac{\varepsilon}{2}$ for all $n_i \geq N_1$.

(p_n) is Cauchy $\Rightarrow \exists N_2$ s.t. $|p_n - p_m| < \frac{\varepsilon}{2}$ for all $n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$

For all $n, n_i > N$, $|p_n - p| \leq |p - p_{n_i}| + |p_{n_i} - p| = \varepsilon$

$\Rightarrow (p_n) \to p$.

$\square$.

1. The set of subsequential limits of $\{x_n\}$ is closed.

Pf: Denote this set as $\tilde{X}$. L is the set of limit-points of $\tilde{X}$.

If L is finite, then $L = \tilde{X}$, L is closed.

If L is infinite, for any $s \in L^*$:

$s = \lim_{n\to\infty} (y_n)$ for some sequence (y_n) consisting of subsequential limits.

There exists y_j in $\frac{3\varepsilon}{4}$ neighborhood of s.

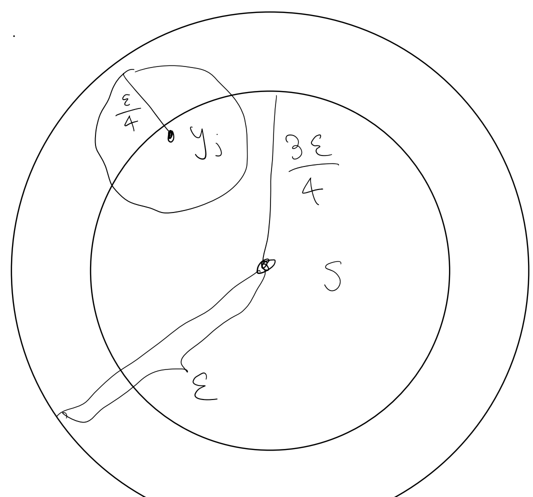
$y_j = \lim_{n\to\infty} x_{n_i}$. There exists an infinite subset of $\{x_{n_i}\}$ that is entirely contained in $\frac{\varepsilon}{4}$ neighborhood of y_j

$\Rightarrow$ There exists an infinite subset of $\{x_n\}$ denoted K, such that for any $x_j \in K$ we have.

$$|x_j - s| \le |x_j - y| + |y - s| < \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} = \varepsilon$$

We can form a subsequence with elements in K, and it is a subsequence of $\{x_n\}$ that converges to s. $\Rightarrow s \in L$

$L^* = L$, L is closed.



$\square$

2.

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n-(-1)^n} = \lim_{n\to\infty} \frac{[n+(-1)^n]^2}{n^2-(-1)^{2n}}$$

$$= \lim_{n\to\infty} \frac{n^2+2n(-1)^n+1}{n^2-1}$$

$$= \lim_{n\to\infty} \frac{n^2-1}{n^2-1} + \frac{2n(-1)^n+2}{n^2-1}$$

$$= 1 + \lim_{n\to\infty} \frac{\frac{2(-1)^n}{n} + \frac{2}{n^2}}{1-\frac{1}{n^2}}$$

$$= 1 + \frac{0+0}{1-0}$$

$$= 1$$

3.

$$\lim_{n\to\infty} \left(\frac{1}{2}+\cdots+\frac{1}{2^n}\right) = \lim_{n\to\infty} \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$$

$$= \lim_{n\to\infty} \left(1-\frac{1}{2^n}\right)$$

$$= 1$$

4 (honors)

$$x_n = \sqrt{\overbrace{7+\sqrt{7+\cdots\sqrt{7}}}^{n \text{ times}}} \qquad x_{n+1} = \sqrt{7+x_n}\ , \qquad x_1 = \sqrt{7}$$

We first show that (x_n) is strictly increasing.

Base case: $x_2 = \sqrt{7+\sqrt{7}} > \sqrt{7} = x_1$

Inductive case: suppose $x_{n+1} > x_n$ is true, then:

$$7 + x_{n+1} > 7 + x_n = x_{n+1}^2$$

$$\sqrt{7+x_{n+1}} > x_{n+1}$$

$$x_{n+2} > x_{n+1}$$

Therefore $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

We now show (x_n) is bounded above by 7

Base case: $x_1 = \sqrt{7} < 7$

Inductive case: suppose $x_n < 7$, $\quad x_{n+1} = \sqrt{7+x_n} < \sqrt{14} < 7$

Therefore $x_n < 7$ for all $n \in \mathbb{N}$.

(x_n) is convergent, hence Cauchy. As $n \to \infty$, $x_{n+1} - x_n < \varepsilon \ \forall \varepsilon > 0$

$$\Rightarrow \lim_{n\to\infty} \sqrt{7+x_n} = \lim_{n\to\infty} x_n. \qquad \text{Let } (x_n) \to L$$

$$\lim_{n\to\infty} (7+x_n) = \lim_{n\to\infty} x_n^2$$

$$7 + \lim_{n\to\infty} x_n = \left(\lim_{n\to\infty} x_n\right)^2$$

$$7 + L = L^2$$

$$L = \frac{1+\sqrt{29}}{2} \quad \text{or} \quad \frac{1-\sqrt{29}}{2} \ (\text{omitted})$$

Hence $\lim_{n\to\infty} x_n = \dfrac{1+\sqrt{29}}{2}$