Rudin 1.

For any $x, y \in \mathbb{R}$: $|f(x) - f(y)| \leq (x-y)^2$

$$\Rightarrow \quad 0 \leq \frac{|f(t) - f(x)|}{|t-x|} \leq \frac{(t-x)^2}{|t-x|}$$

$$0 \leq \lim_{t \to x} \frac{|f(t) - f(x)|}{|t-x|} \leq \lim_{t \to x} |t-x| = 0$$

$$\Rightarrow \lim_{t \to x} \frac{|f(t) - f(x)|}{|t-x|} = 0$$

$$\Rightarrow f'(x) = 0 \quad \forall x \in \mathbb{R}$$

By Mean Value Theorem, for any $a, b \in \mathbb{R}$, $\exists x \in (a, b)$ such that

$f(b) - f(a) = f'(x) \cdot (b-a)$. $f'(x) = 0 \Rightarrow f(b) = f(a)$.

$\Rightarrow$ by definition, f is a constant function

2.

Prop: $f'(x) > 0 \ \forall x \in (a,b) \Rightarrow f$ is strictly increasing in (a,b).

Fix any $x \in (a,b)$, let $y > x$. Suppose $f(y) \leq f(x)$.

f is differentiable on $(a,b) \Rightarrow f$ is differentiable on (x,y).

By MVT $\exists t \in (x,y)$ such that $f(y) - f(x) = f'(t) \cdot (y-x)$

$(f(y) - f(x)) \leq 0$, $(y-x) > 0$, so $f'(t) \leq 0$, contradicting

$f'(x) > 0 \ \forall x \in (a,b)$.

Therefore $f(y) > f(x) \ \forall y > x$. q.e.d.

Prop: $g = f^{-1}$. g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$ $(a<x<b)$

$f'(x)$ is defined on $(a,b) \Rightarrow f(x)$ is differentiable on (a,b)

$\Rightarrow f(x)$ is continuous on (a,b).

$f(x)$ is strictly increasing $\Rightarrow$ $(x_1 \neq x_2$ iff $f(x_1) \neq f(x_2))$

$\Rightarrow f: \operatorname{dom} f \to \operatorname{range} f$ is bijection,

$\Rightarrow g = f^{-1}: \operatorname{range} f \to \operatorname{dom} f$ is bijection.

Want to show $g'(\beta) = \lim_{\alpha \to \beta} \frac{g(\alpha) - g(\beta)}{\alpha - \beta}$ exists for every $\beta \in \operatorname{rang} f$.

$\beta = f(s)$, $\alpha = f(t)$.

$$g'(\beta) = \lim_{\alpha \to \beta} \frac{g(f(t)) - g(f(s))}{f(t) - f(s)} = \lim_{\alpha \to \beta} \left(\frac{f(t) - f(s)}{t - s} \right)^{-1}$$

As $\alpha \to \beta$, $g(\alpha) \to g(\beta)$ (because g is injective ?)

We know $s = g(\beta)$, $t = g(\alpha)$, so as $\alpha \to \beta$, $t \to s$.

$$\text{So} \quad \lim_{\alpha \to \beta} \left(\frac{f(t)-f(s)}{t-s} \right)^{-1} = \lim_{t \to s} \left(\frac{f(t)-f(s)}{t-s} \right)^{-1}$$

Lemma 1

$$= \left(\lim_{t \to s} \frac{f(t)-f(s)}{t-s} \right)^{-1}$$

$$= \frac{1}{f'(s)}$$

write $s = x$ ∴

$$\Rightarrow g'(\beta) = g'(f(x)) = \frac{1}{f'(x)} \quad \text{on } (a,b). \quad \text{q.e.d.}$$

Lemma 1: If $\lim_{x \to a} g(x) = b$ and f is continuous at b,

then. $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

Pf. let $\varepsilon > 0$. f is cont at $x = b \Rightarrow \exists \delta_1 > 0$ such that

$f(x)$ and $f(b)$ are ε-close whenever x and b are δ_1-close.

$b = \lim_{x \to a} g(x) \Rightarrow \exists \delta > 0$ so that $g(x)$ and b are δ_1-close

whenever x and a are δ-close.

$\Rightarrow \left| f(g(x)) - f(b) \right| < \varepsilon$ whenever $|x-a| < \delta$

$\Rightarrow \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

4.

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$$

$$f(x) := C_0 + C_1 x + \cdots + C_n x^n$$

$$g(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_n}{n+1} x^{n+1}$$

$$g'(x) = f(x) \quad . \quad g(0) = g(1) = 0$$

$g(x)$ is polynomial $\Rightarrow$ $g(x)$ is differentiable on $\mathbb{R}$

By MVT, $\exists y \in (0,1)$ s.t. $g(1) - g(0) = g'(y) \cdot (1-0)$

$$0 = f(y)$$

$\Rightarrow$, $f(x)$ has at least one root in $(0,1)$

5. f is defined and differentiable on $\mathbb{R}^+$. $f'(x) \to 0$ as $x \to \infty$

$g(x) := f(x+1) - f(x)$. Prop: $g(x) \to 0$ as $x \to \infty$

Pf:

MVT $\Rightarrow$ For any $x>0$, $\exists\, y \in (x, x+1)$ s.t. $f(x+1) - f(x) = f'(y) \cdot (x+1-x)$

i.e. $f'(y) = g(x)$.

As $x \to \infty$, $y \to \infty$, $f'(y) \to 0 \Rightarrow g(x) \to 0$.

q.e.d.

Rmk: Let $a_n = f(n)$, $n \in \mathbb{N}$

$$\sum_{n=1}^{N} (a_{n+1} - a_n) = a_{N+1} - a_1 = f(N+1) - f(1)$$

We also know $a_{n+1} - a_n = g(n)$. So $\sum_{n=1}^{N} g(n) = f(N+1) - f(1)$

which is the discrete analog of integration!

$$\lim_{x\to\infty} x^3\left(\sin\left(\tfrac{1}{x}\right)-\tfrac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)-\frac{1}{x}}{x^{-3}}$$

$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = 0,\ \lim_{x\to\infty} \frac{1}{x^3} = 0$

Apply L' Hospital's Rule.

$$= \lim_{x\to\infty} \frac{-\cos(1/x)\,x^{-2}+x^{-2}}{-3x^{-4}}$$

$$= \lim_{x\to\infty} \frac{1}{3}\left(\cos(1/x)-1\right)x^2$$

$$= \frac{1}{3}\lim_{x\to\infty} \frac{\cos(1/x)-1}{x^{-2}}$$

L' Hospital

$$= \frac{1}{3}\lim_{x\to\infty} \frac{\sin(1/x)\,x^{-2}}{-2x^{-3}}$$

$$= -\frac{1}{6}\lim_{x\to\infty} \sin(1/x)\,x$$

$$= -\frac{1}{6}\lim_{x\to\infty} \frac{\sin(1/x)}{\frac{1}{x}}$$

$$= -\frac{1}{6}\lim_{y\to 0} \frac{\sin(y)}{y}$$

L' Hospital

$$= -\frac{1}{6}\lim_{y\to 0} \frac{\cos(y)}{1}$$

$$= -\frac{1}{6}$$

Method 2: $\lim_{x\to\infty} x^3\left(\sin(1/x)-1/x\right) = \lim_{x\to\infty} x^3\left(\frac{1}{x}-\frac{(1/x)^3}{3!}+\frac{(1/x)^5}{5!}-\cdots-1/x\right)$

let $u=\frac{1}{x}$

$$= \lim_{u\downarrow 0} \frac{1}{u^3}\left(-\frac{u^3}{3!}+\frac{u^5}{5!}-\frac{u^7}{7!}+\cdots\right)$$

$$= \lim_{u\downarrow 0}\left(-\frac{1}{3!}+\frac{u^2}{5!}-\frac{u^4}{7!}+\cdots\right) = -\frac{1}{3!} = -\frac{1}{6}.$$

not sure if we are allowed to use Taylor expansion for $\sin(x)$.

$$\lim_{x \to 1} \frac{1 - \cos(x-1)}{\tan^2(x-1)} = ?$$

Since $\lim_{x\to 1} 1 - \cos(x-1) = 0$ and $\lim_{x\to 1} \tan^2(x-1) = \dfrac{\lim_{x\to 1} \sin^2(x-1)}{\lim_{x\to 1} \cos^2(x-1)}$

$$= \frac{0}{1} = 0,$$

apply L' Hospital Rule to get

$$\lim_{x\to 1} \frac{1-\cos(x-1)}{\tan^2(x-1)} = \lim_{x\to 1} \frac{\sin(x-1)}{2\tan(x-1)\cdot \sec^2(x-1)}$$

Because

$\sin(x-1) \to 0$

$\tan(x-1) \to 0,$

$\sec^2(x-1) \to 1$

$$= \frac{1}{2} \lim_{x\to 1} \frac{\cos(x-1)}{\sec^3(x-1) + \tan^2(x-1)\, 2\sec^2(x-1)} \quad ①$$

As $x \to 1$, $\cos(x-1) \to 1$, $\sec^3(x-1) = \dfrac{1}{\cos^3(x-1)} \to 1$,

$\sec^2(x-1) \to 1$, $\tan^2(x-1) \to 0$.

So $① = \dfrac{1}{2} \cdot \dfrac{1}{1+0} = \dfrac{1}{2}$

- Suppose f is continuously differentiable on $[0,\infty)$ (that is, it is differentiable and the derivative is continuous at every point of $[0,\infty)$) and $|f'(x)| \le 1$ for all $x \ge 0$. Is f bounded on $[0,\infty)$? Is it uniformly continuous on $[0,\infty)$?

Claim: f is not necessarily bounded on $[0,\infty)$.

Pf: $f(x) = \frac{1}{2}x$. $|f'(x)| = \frac{1}{2} \leqslant 1 \quad \forall x \in [0,\infty)$

But for any $M \in \mathbb{R}$, $\exists N = 2M$ s.t. $f(x) > M$ whenever $x > N$

So $f(x)$ is not bounded on $[0,\infty)$.

Claim: f is uniformly continuous on $[0,\infty)$

Pf: For any $\varepsilon > 0$, let $\delta = \varepsilon$, $a, b \in [0,\infty)$ such that $|a-b| < \delta$.

By MVT $\exists c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$

$$|f'(c)| \le 1 \Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| \le 1 \Rightarrow |b-a| > |f(b)-f(a)|$$

$$\Rightarrow |f(b)-f(a)| < \delta = \varepsilon,$$

$\Rightarrow \forall \varepsilon > 0$, $\exists \delta = \varepsilon > 0$ such that $|f(b)-f(a)| < \varepsilon$ whenever $|b-a| < \delta$ for any $a, b \in [0,\infty)$. q.e.d.