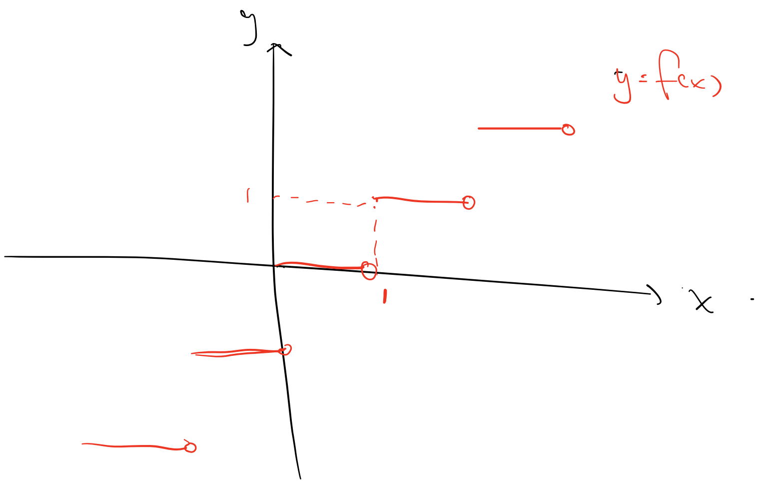
1b.

$f(x) = [x]$ — floor function.

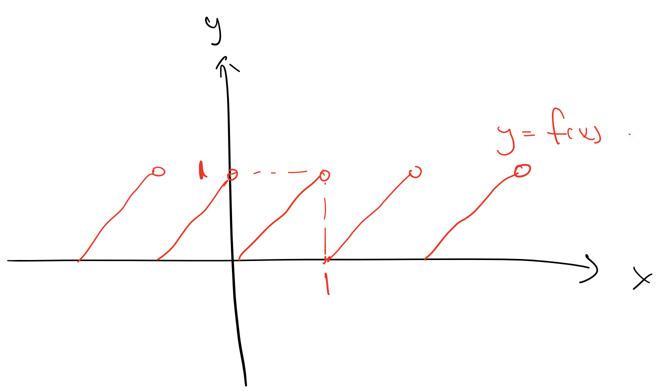


Type-1 discontinuity at $x = \mathbb{Z}$ because

$$\lim_{x \uparrow z} f(x) = z - 1 \neq z = \lim_{x \downarrow z} f(x)$$

for $z \in \mathbb{Z}$

$f(x) = x - [x]$ — fractional function.



Type-1 discontinuity at $x = \mathbb{Z}$ because

$$\lim_{x \uparrow z} f(x) = 1 \neq 0 = \lim_{x \downarrow z} f(x)$$

for $z \in \mathbb{Z}$

20.

20. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(*a*) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

(*b*) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \le d(x, y)$$

for all $x \in X$, $y \in X$.

Hint: $\rho_E(x) \le d(x, z) \le d(x, y) + d(y, z)$, so that

$$\rho_E(x) \le d(x, y) + \rho_E(y).$$

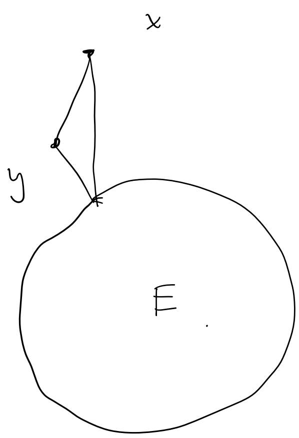
(a). $(\Rightarrow)$: $\rho_E(x) = \inf_{z \in E} d(x, z) = 0 \Rightarrow \forall \varepsilon > 0 \ \exists e \in E$ s.t. $d(x, e) < \varepsilon$

$\Rightarrow x \in \bar{E}$

$(\Leftarrow)$: $x \in \bar{E} \Rightarrow \forall \varepsilon > 0 \ \exists e \in E$ s.t. $d(x, e) - 0 < \varepsilon$. We also know $\inf d(x, z) \geq 0$.

$\Rightarrow \inf d(x, z) = 0$.

(b). $\rho_E(x) \le d(x, z) \le d(x, y) + d(y, z)$, and.



$d(y, z) \geq \rho_E(y)$

$\Rightarrow \rho_E(x) \le d(x, y) + \rho_E(y)$

$\rho_E(x) - \rho_E(y) \le d(x, y)$.

Using the same argument, but swap x and y, we have:

$\rho_E(y) - \rho_E(x) \le d(x, y)$

So $|\rho_E(x) - \rho_E(y)| = d(\rho_E(x), \rho_E(y)) \le d(x, y)$ for all $x, y \in X$

Set $\varepsilon = d(x, y)$, $\delta = \varepsilon$. $d(x, y) < \delta \Rightarrow d(x, y) < \varepsilon \Rightarrow d(\rho_E(x), \rho_E(y)) < \varepsilon$

$\Rightarrow$ For any $\varepsilon > 0$, $\exists \delta = \varepsilon$ such that $d(\rho_E(x), \rho_E(y)) < \varepsilon$ whenever $d(x, y) < \delta$

$\Rightarrow \rho_E$ is uniformly continuous on X.

$$\lim_{x\to\infty} \frac{10\cdot 2^x - x^{10}}{2^{x+1} - \sin(x^2)} = \lim_{x\to\infty} \frac{10 - \frac{x^{10}}{2^x}}{2 - \frac{\sin(x^2)}{2^x}}$$

Claim. $\lim_{x\to\infty} \frac{x^{10}}{2^x} = 0$

Can we deduce this straightly from $\lim_{n\to\infty} \frac{n^{10}}{2^n}$ $(n \in \mathbb{N})$ without the proof below?

Pf: Rudin Thm 3.20 (d): If $p>1$, $\alpha \in \mathbb{R}^+$, then $\lim_{n\to\infty} \frac{n^\alpha}{p^n} = 0$ $(n \in \mathbb{N})$

We now show that it still holds over reals. $n := \lfloor x \rfloor$.

$n \leq x < n+1$. x^α is increasing, so $n^\alpha \leq x^\alpha < (n+1)^\alpha$

$p > 1 \Rightarrow p^x$ is increasing $\Rightarrow p^n \leq p^x < p^{n+1}$

$$\frac{x^\alpha}{p^x} > \frac{n^\alpha}{p^{n+1}} = \frac{1}{p}\cdot\frac{n^\alpha}{p^n} \quad \text{and} \quad \frac{x^\alpha}{p^x} < \frac{(n+1)^\alpha}{p^n} < p\cdot\frac{(n+1)^\alpha}{p^{n+1}}$$

$$\frac{1}{p}\frac{n^\alpha}{p^n} < \frac{x^\alpha}{p^x} < p\,\frac{(n+1)^\alpha}{p^{n+1}}$$

$$\frac{1}{p} b_n < \frac{x^\alpha}{p^x} < p\, b_{n+1} \quad \text{where } b_n = \frac{n^\alpha}{p^n}$$

As $x \to \infty$, $n = \lfloor x \rfloor \to \infty$, so $\lim_{x\to\infty} b_n = \lim_{x\to\infty} b_{n+1} = 0$.

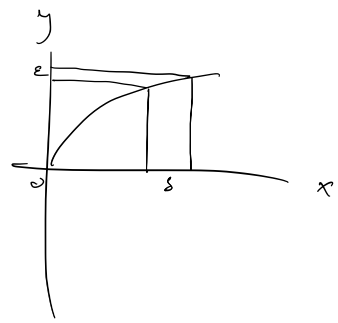
By squeeze theorem, $\lim_{x\to\infty} \frac{x^\alpha}{p^x} = 0$.

Claim: $\lim_{x\to\infty} \frac{\sin(x^2)}{2^x} = 0$. Pf: $\frac{-1}{2^x} \leq \frac{\sin(x^2)}{2^x} \leq \frac{1}{2^x}$,

both sides $\to 0$ as $x \to \infty$. It follows by squeeze theorem

$$\lim_{x\to\infty} \frac{10\cdot 2^x - x^{10}}{2^{x+1} - \sin(x^2)} = \lim_{x\to\infty} \frac{10 - \frac{x^{10}}{2^x}}{2 - \frac{\sin(x^2)}{2^x}} = \frac{10-0}{2-0} = 5$$

$f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$



Pf: $\forall \varepsilon > 0$, Set $\delta = \varepsilon^2$. For all p, q such that $|p-q| < \delta$ and $p, q \in [0, \infty)$:

$$|p-q| < \varepsilon^2$$

$$\Rightarrow \ |\sqrt{p}+\sqrt{q}| \cdot |\sqrt{p}-\sqrt{q}| < \varepsilon^2$$

Since $|\sqrt{p}-\sqrt{q}| \leq |\sqrt{p}+\sqrt{q}|$, we get $|\sqrt{p}+\sqrt{q}| \cdot |\sqrt{p}-\sqrt{q}| \geq |\sqrt{p}-\sqrt{q}|^2$

$$\Rightarrow \ (\sqrt{p}-\sqrt{q})^2 < \varepsilon^2$$

$$\sqrt{p}-\sqrt{q} < \varepsilon$$

$\Rightarrow$ $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$

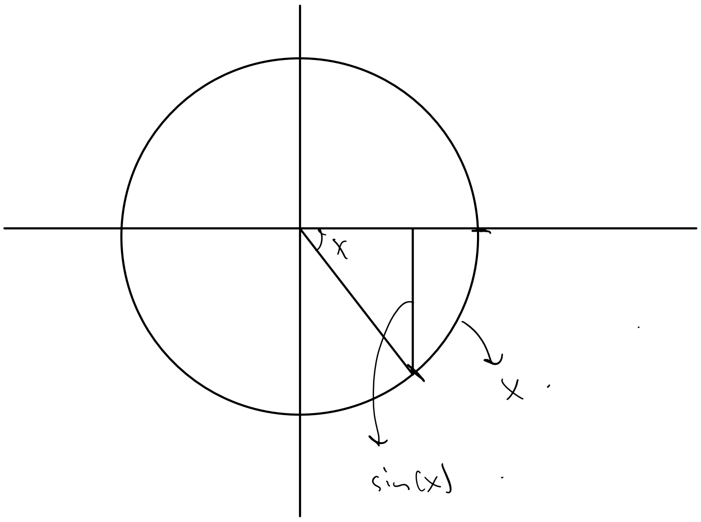
$h(x) = \sin x$ is uniformly continuous on $[0, \infty)$

Pf:

Lemma: $|\sin x| \le |x|$ for $x \in [0, +\infty)$

Pf: When $x = 0$, it is obvious.

When $x \neq 0$, we argue geometrically that the arc length (x) is always larger than $\sin(x)$.



For $p, q \in [0, \infty)$:

$$|p - q| < \delta$$

$$\left|\frac{p-q}{2}\right| < \frac{\varepsilon}{2}$$

Lemma

$$\left|\sin\left(\frac{p-q}{2}\right)\right| < \frac{\varepsilon}{2}$$

because $|\cos(x)| \le 1$

$$\left|\sin \frac{p-q}{2}\right| \left|\cos \frac{p+q}{2}\right| < \frac{\varepsilon}{2}$$

$$\left|2 \sin \frac{p-q}{2} \cos \frac{p+q}{2}\right| < \varepsilon.$$

product to sum formula

$$|\sin(p) - \sin(q)| < \varepsilon.$$

Thus, $\forall \varepsilon > 0$, $\exists \delta$ s.t. $|\sin(p) - \sin(q)| < \varepsilon$ whenever $|p - q| < \delta$. q.e.d.