Math 522 Homework 2

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Question 3.

(a) We claim (f_n) converges uniformly to $f(x) = 0$. f_n is differentiable on $\mathbf{R}$, so by the Interior Extremum Theorem the maximum and minimum values will appear where $f_n'(x) = 0$. We have

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

which is zero at $x = \pm 1/\sqrt{n}$.

$f_n'(x)$ attains extrema at $\frac{1/\sqrt{n}}{4}$ and $\frac{-1/\sqrt{n}}{4}$
$\implies |f_n(x)| \leq \frac{1}{2\sqrt{n}}$. Since this bound does not depend on x, convergence is uniform.
The zero function is obviously differentiable

(b)

$$\lim_{n\to\infty} f_n'(x) = \frac{1 - nx^2}{1 + 2nx^2 + n^2x^4} = \frac{\frac{1}{n} - x^2}{\frac{1}{n} + 2x^2 + nx^4}$$

which equals to 1 on $x = 0$ and 0 on everywhere else. So the limit of derivative is equal to derivative of limit on everywhere except for $x = 0$

Question 4.

(a)

$$f_k = (-1)^k \frac{x^2}{2^k}$$

,

$$g_k = (-1)^k \frac{1}{k}$$

Let $C = \sup_{x\in I} x^2$. It exists because I is bounded.
$\forall k \in \mathbb{N}$, let $M_k = \frac{C}{2^k}$ be the bound on $|f_k|$. The series

$$\sum_{k=1}^{\infty} M_k = C \sum_{k=1}^{\infty} \frac{1}{2^k}$$

converges because it is geometric series with common ratio $1/2$. Hence (f_k) converges uniformly on I.

(g_n) is a constant function, and the series $\sum g_n(x)$ converges for any x due to alternating series test. Since the difference between the constant function and its limit is also a constant function, the convergence is uniform

The sum of two uniformly convergent series is still uniformly convergent.

$$\implies f_n = (-1)^n(\frac{x^2}{2^k} + \frac{1}{k})$$

converges uniformly on any bounded interval

(b) For any $x \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} |f_n| = x^2 \sum_{n=1}^{\infty} \frac{1}{2^k} = x^2$$

so $\sum f_n$ converges absolutely.

$$\sum_{n=1}^{\infty} |g_n| = \sum_{n=1}^{\infty} \frac{1}{k}$$

which is the divergent harmonic series. Sum of convergent series and divergent series is divergent.

Question 5.

let $y = x^2$

$$\sum_{k=1}^{\infty} k^2 x^{2k+1} = x \sum_{k=1}^{\infty} k^2 y^k = x(y(y(\sum_{k=1}^{\infty} y^k)')')$$

The idea was to generate k by taking derivative, while compensating for the factor y lost during this process.

Since $|x| < 1$, we have $|y| < 1$, so $\sum_{k=1}^{\infty} y^k = \frac{1}{1-y}$

$$\begin{aligned}
\sum_{k=1}^{\infty} k^2 x^{2k+1} &= x(y(y(\frac{1}{1-y})')') \\
&= x(y(\frac{y}{(1-y)^2})') \\
&= x((y\frac{1+y}{(1-y)^3})) \\
&= \frac{x^3(1+x^2)}{(1-x^2)^3}
\end{aligned}$$