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HW1

- Reading assignment: pp. 1-30, Honors students also need to read Appendix to Chapter I.
- Pages 21–23: 4,5,6.
- 1 Solve equation $z^{101} = i + 1$ over $\mathbb{C}$.
- 2 Simplify expression (try using complex numbers)

$$\sum_{j=0}^{n} \cos(jx)\,, \quad x \in \mathbb{R}\,.$$

- 3 Is it true that $[0, 1) \sim \mathbb{R}$?
- 4 Let S be the set of infinite sequences $(\delta_1, \delta_2, \ldots)$ where $\delta_j \in \{0, 1\}, \forall j$. Is it true that $S \sim \mathbb{R}$?
- 5 (For students not taking the class with honors) Is it true that $\mathbb{R}^2 \sim \mathbb{R}$?
- 6 (For students taking the class with honors) Let S' be the set of infinite sequences $(\delta_1, \delta_2, \ldots)$ where $\delta_j \in \mathbb{R}, \forall j$. Is it true that $S' \sim \mathbb{R}$?

Remark: you are free to use Bernstein-Schroder theorem we proved in class.

Rudin.

1.4.

E is a non-empty subset of an ordered set. $\alpha \leq \inf E$, $\beta \geq \sup E$

Let $x \in E$. $\inf E \leq x$, $\sup E \geq x \Rightarrow \inf E \leq \sup E \Rightarrow \alpha \leq \beta$.

1.5.

$\emptyset \neq A \subset \mathbb{R}$. A is bounded below so $\inf A \in \mathbb{R}$ exists.

Define $-A = \{-y : y \in A\}$. If M is a lower bound for A, then.

$M < x \ \forall x \in A \Rightarrow -M > -x \ \forall x \in A \Rightarrow -M > y \ \forall y \in A \Rightarrow \sup A$ exists.

Let $m = \inf A$. $m \geq M$ for any lower bound M.

$\Rightarrow -m \leq -M$ for any upper bound $(-M)$

$\Rightarrow -\inf A = \sup(-A) \Rightarrow \inf A = -\sup(-A)$

1.6

$b > 1$

(a). $r = m/n = p/q \implies mq = pn$

$$b^{mq} = b^{m \cdot \frac{1}{n} \cdot nq} = ((b^m)^{\frac{1}{n}})^{nq}$$

$$b^{pn} = b^{p \cdot \frac{1}{q} \cdot q \cdot n} = ((b^p)^{\frac{1}{q}})^{qn}$$

According to Theorem 1.21, the positive real solution $y = x^{\frac{1}{n}}$ to $y^n = x$ is unique. $nq, qr \in \mathbb{Z}$, $b^{mq} = b^{pn} > 0$.

$$\Rightarrow (b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$$

(b). $r, s \in \mathbb{Q}$ $r = \frac{a}{b}$, $s = \frac{c}{d}$. Take it for given that law of exponents work on integer exponents. We will extend it to rational exponents.

$$b^{r+s} = b^{\frac{a}{b} + \frac{c}{d}} = b^{\frac{ad+bc}{bd}} = (b^{ad} \cdot b^{bc})^{\frac{1}{bd}} \underset{\uparrow}{=} b^{\frac{ad}{bd}} \cdot b^{\frac{bc}{bd}} = b^r \cdot b^s$$

Corollary of Thm 1.21

(c).

$x \in \mathbb{R}$. $B(x) = \{ b^t : \mathbb{Q} \ni t \leq x \}$

If $r \in \mathbb{Q}$, then $b^r \in B(r)$. $\forall t \leq r$, we have $b^t \leq b^r$, so b^r is an upper bound of $B(r)$;

Suppose y is an upper bound of $B(r)$. $\Rightarrow y \geq b^r \Rightarrow b^r = \sup B(r)$

b^x is certainly an upper bound of $B(x)$. Suppose b^α is also an upper bound and $b^\alpha < b^x$.

$\alpha < x$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$. $\exists q \in \mathbb{Q}$ s.t. $\alpha < q < x$.

$b^\alpha < b^q \in B(x) \Rightarrow b^\alpha$ is not an upper bound of $B(x)$. Contradiction $\Rightarrow b^x = \sup B(x)$.

(d).

$b^{x+y} = \sup B(x+y)$. $B(x+y) = \{ b^{r+s} : r+s \leq x+y \} = \{ b^r \cdot b^s : r \leq x, s \leq y \}$

We claim that $\sup \{ b^r b^s : r \leq x, s \leq y \} = \sup \{ b^r : r \leq x \} \cdot \sup \{ b^s : s \leq y \}$, denoted as $A = B \cdot C$

$B \geq b^r$, $C \geq b^s$, so $B \cdot C \geq b^r b^s \Rightarrow B \cdot C \geq A$. Suppose $B \cdot C > A$, then let $A < \alpha < B \cdot C$.

Let $\alpha = b^{\Omega + \delta}$ where $\Omega \leq x$, $\delta \leq y$. But $b^{\Omega+\delta} > A \Rightarrow \Omega + \delta > x+y$. Contradiction $\Rightarrow A = B \cdot C$.

In (c) we proved: $\sup \{ b^r : r \leq x \} = b^x$, $\sup \{ b^s : s \leq y \} = b^y$. So we have $b^{x+y} = b^x \cdot b^y$

1. $z^{101} = i + 1$

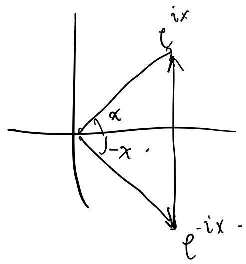
$$(r,\theta)^{101} = \left(\sqrt{1^2+1^2},\ \frac{\pi}{4}\right) \Rightarrow r = 2^{\frac{1}{202}},\quad \theta = \frac{\pi}{4}\cdot\left(\frac{n}{101}\right),\ n \in \{1,2,\dots,100\}$$

2. $$\sum_{j=0}^{n} \cos(jx) = Re\left(\sum_{j=0}^{n} e^{ijx}\right) \qquad \sum_{j=0}^{n} e^{ijx} = \frac{e^{0}(1-e^{inx})}{1-e^{ix}}$$

$$= \frac{e^{\frac{inx}{2}}\cdot e^{\frac{-inx}{2}} - e^{\frac{inx}{2}}\cdot e^{\frac{inx}{2}}}{e^{\frac{ix}{2}}\cdot e^{\frac{-ix}{2}} - e^{\frac{ix}{2}}\cdot e^{\frac{ix}{2}}}$$

Lemma 1: $e^{ix} - e^{-ix} = 2\sin(x)$

Pf:

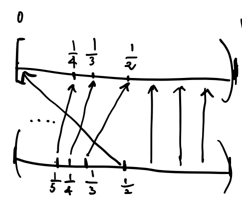


$$= \frac{e^{\frac{inx}{2}}\left(e^{\frac{-inx}{2}} - e^{\frac{inx}{2}}\right)}{e^{\frac{ix}{2}}\left(e^{\frac{-ix}{2}} - e^{\frac{ix}{2}}\right)}$$

$$= e^{\frac{ix(n-1)}{2}} \cdot \frac{2\sin\left(\frac{nx}{2}\right)}{2\sin\left(\frac{x}{2}\right)} \qquad \text{(By Lemma 1)}$$

$$\Rightarrow \sum_{j=0}^{n} \cos(jx) = Re\left(e^{\frac{ix(n-1)}{2}} \cdot \frac{2\sin\left(\frac{nx}{2}\right)}{2\sin\left(\frac{x}{2}\right)}\right) = \cos\left(\frac{(n-1)x}{2}\right) \cdot \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

3. Define $g : [0,1) \longrightarrow (0,1)$ s.t. $g(x) \begin{cases} = 0 & \text{if } x = \frac{1}{2} \\ = \frac{1}{n-1} & \text{if } x = \frac{1}{n} \text{ and } n \in \{3,4,\dots\} \\ = x & \text{otherwise.} \end{cases}$



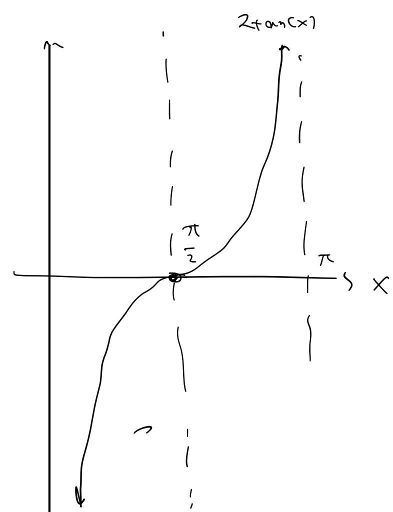
Claim: g is bijective.

Pf: $[0,1) = \{0\} \cup \{\frac{1}{2}, \frac{1}{3}, \dots\} \cup \{x \in [0,1) : x \neq 0, \frac{1}{2}, \frac{1}{3}, \dots\}$

$(0,1) = \{\frac{1}{2}\} \cup \{\frac{1}{3}, \frac{1}{4}, \dots\} \cup \{x \in [0,1) : x \neq 0, \frac{1}{2}, \frac{1}{3}, \dots\}$

Clearly, g, when restricted to each of the three subsets, is bijective.

So $[0,1) \sim (0,1)$



Define $f : (0,1) \longrightarrow \mathbb{R}$ s.t. $f(x) = \tan(\pi x - \frac{\pi}{2})$

f is bijective $\Rightarrow (0,1) \sim \mathbb{R}$

Because $\sim$ is transitive, we have $[0,1) \sim \mathbb{R}$

4. $S = \{$ all possible sequences of 1 and 0 that are at most countable $\}$

Lemma 1: $[0,1] \sim \mathbb{R}$

$$\varphi: (0,1) \to [0,1] \quad \text{s.t.} \quad \varphi(x) \begin{cases} = 1 & \text{if } x = \frac{1}{2} \\ = 0 & \text{if } x = \frac{1}{3} \\ = \frac{1}{n-2} & \text{if } x = \frac{1}{n}, n \in \{4,5,\dots\} \\ = x & \text{otherwise.} \end{cases} \quad \text{is a bijection.}$$

$\Rightarrow (0,1) \sim [0,1]$

In 3. we proved $(0,1) \sim \mathbb{R}$, so $[0,1] \sim \mathbb{R}$.

Define $f: S \to [0,1]$ s.t. $(\delta_1, \delta_2, \dots) \mapsto \sum_{n=1}^{\infty} \frac{\delta_n}{2^n} = 0.\delta_1\delta_2\dots\dots$

The surjectivity of f is obvious.

We delete a subset of S that are 'redundant'. e.g.

$$(\delta_1, \delta_2, \dots, 1, 0, 0, \dots)$$

and

$$(\delta_1, \delta_2, \dots, 0, 1, 1, \dots)$$ map to the same element

So let $(\delta_1, \delta_2, \dots, 0, 1, 1, \dots) \in D$ where D is the set of sequences with trailing 1s. $A := S \setminus D$. D is countably infinite, so $A \sim S$. Now $f|_A$ is injective and (still) surjective. $\Rightarrow f|_A$ is bijective

Hence. $A \sim [0,1]$, $|S| \sim [0,1]$, $S \sim \mathbb{R}$.

6. (Honors)

$\delta_1 = 0.a_1a_2a_3\cdots\cdots$

$\delta_2 = 0.b_1b_2b_3\cdots\cdots$

$\delta_3 = 0.c_1c_2c_3\cdots\cdots$

$\vdots \qquad \vdots$

Because $\mathbb{R} \sim [0,1]$, $S' \sim \mathbb{R}^{\mathbb{N}}$, we have $S' \sim [0,1]^{\mathbb{N}}$

It suffices to show $[0,1]^{\mathbb{N}} \sim [0,1]$

Define. $f: [0,1]^{\mathbb{N}} \longrightarrow [0,1]$ s.t. $f(\delta_1, \delta_2, \delta_3, \cdots) = 0.a_1b_1c_1\cdots a_2b_2c_2\cdots\cdots$

Injectivity of f is obvious.

For any $0.t_1t_2t_3t_4\cdots\cdots \in [0,1]$, let $\delta_1 = 0.t_1t_{k+1}t_{2k+1}\cdots$ where $k = |(\delta_1, \delta_2, \cdots)|$

$\delta_2 = 0.t_2t_{k+2}t_{2k+2}\cdots$

$\vdots \qquad \vdots$

Clearly $\delta_i \in S'$ $\forall i \in \mathbb{N}$.

Hence f is bijective

$\Rightarrow [0,1]^{\mathbb{N}} \sim [0,1]$

$\Rightarrow S' \sim \mathbb{R}$