7. Prop: $a_n \geq 0$, $\sum a_n$ converges $\Longrightarrow$ $\sum \frac{\sqrt{a_n}}{n}$ converges

Pf: By AM-GM inequality:

$$\frac{a_n + \frac{1}{n^2}}{2} \geq \left(a_n \cdot \frac{1}{n^2}\right)^{\frac{1}{2}}$$

$$a_n + \frac{1}{n^2} \geq 2 \frac{\sqrt{a_n}}{n}$$

$\sum a_n + \frac{1}{n^2} = \sum a_n + \sum \frac{1}{n^2}$, both of which converges.

so $\sum a_n + \frac{1}{n^2}$ converges.

By comparison test, $\sum \frac{\sqrt{a_n}}{n}$ converges

8. Prop: If $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded, then $\sum a_n b_n$ converges.

Pf: $\{b_n\} \to b$. $C_n = b_n - b$. $\{C_n\} \to 0$.

$A_n = \sum_{k=1}^{n} a_k$, $\{A_n\} \to A$

$$\sum a_n b_n = \sum a_n \cdot (C_n + b)$$

$$= \underbrace{\sum a_n C_n}_{①} + \underbrace{\sum a_n b}_{②}.$$

① is convergent due to Dirichlet's Test.

② $= b \sum a_n = bA$.

Therefore $\sum a_n b_n$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} > \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n+\frac{1}{4}}} = \sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{2}} > \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$$

which diverges.

By comparison test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ diverges

$\sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1}\right)^n$ converges.

Pf. $\limsup_{n\to\infty} \left(\frac{n+1}{2n-1}\right)^{n \cdot \frac{1}{n}} = \limsup_{n\to\infty} \frac{n+1}{2n-1}$

Since $\lim_{n\to\infty} \frac{n+1}{2n-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2}$,

$\limsup_{n\to\infty} \frac{n+1}{2n-1} = \lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} < 1$, by root test,

$\sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1}\right)^n$ converges (absolutely, in fact, which is obvious)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} \text{ is divergent}$$

Pf:

$$\frac{(-1)^n}{\sqrt{n}+(-1)^n} = \frac{(-1)^n(\sqrt{n}-(-1)^n)}{(\sqrt{n}+(-1)^n)(\sqrt{n}-(-1)^n)}$$

$$= \frac{(-1)^n\sqrt{n}-1}{n-1}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} = \sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}-1}{n-1}$$

$$= \underbrace{\sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{n-1}}_{①} - \underbrace{\sum_{n=2}^{\infty} \frac{1}{n-1}}_{②}$$

① converges because $\left(\frac{\sqrt{n}}{n-1}\right) \searrow 0$ as $n \to \infty$, it follows by alternating series test.

② is the harmonic series which is divergent.

Hence ① - ② is divergent. $\square$

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} \text{ Converges non-absolutely.}$$

Pf: $\left(\frac{1}{n}\right) \searrow 0$, so by Dirichlet's Test, it suffices to show

$\sum_{n=1}^{\infty} \sin(n)$ is bounded.

$$\sum_{n=1}^{m} \sin(n) = \sum_{n=1}^{m} \mathrm{Im}(z) \qquad \text{where } z = \cos(n) + i\sin(n)$$

$$= \mathrm{Im} \sum_{n=1}^{m} e^{in}$$

$$= \mathrm{Im}\left(\frac{1-e^{i(m+1)}}{1-e^{i}}\right)$$

$$\left|\sum_{n=1}^{m} \sin(n)\right| = \left|\mathrm{Im}\left(\frac{1-e^{i(m+1)}}{1-ei}\right)\right| \leq \frac{|1-e^{i(m+1)}|}{|1-e^{i}|} \leq \frac{2}{C} \quad C\text{ - some constant}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ is convergent.

We claim $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n}\right|$ is divergent. $\left|\frac{\sin(n)}{n}\right| > \frac{\sin^2(n)}{n} = \frac{1-\cos(2n)}{n}$

$$\sum_{n=1}^{\infty} \frac{1-\cos(2n)}{n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{①} - \underbrace{\sum_{n=1}^{\infty} \frac{\cos(2n)}{n}}_{②}$$

② is convergent, which can be shown similar argument for convergence of $\sum \frac{\sin(n)}{n}$.

So $\sum_{n=1}^{\infty} \frac{1-\cos(2n)}{n}$ is divergent. By comparison test, so is $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n}\right|$.

q.e.d.

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges for $x \in [0, 2)$

$\alpha = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}$

$\ln \alpha = \lim_{n\to\infty} \frac{1}{n} \ln n$ $*$

$\ln \alpha = 0$.

$\alpha = 1$.

Radius of convergence $= \frac{1}{\alpha} = 1$

$*$: $e^x > x$ for all $x > 0$.

$e^{\ln(n)} > \ln(n)$

$n > \ln(n)$

$0 \leq \frac{\ln(n)}{n} = \frac{2\ln(\sqrt{n})}{n} < \frac{2\sqrt{n}}{n}$

$0 \leq \frac{\ln(n)}{n} < \frac{2}{\sqrt{n}}$.

Since $\lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, by squeeze theorem $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

So the original series converges for $x \in \mathbb{R}$ such that $|x-1| < 1$

$\Rightarrow x \in (0, 2)$, and diverges for $x < 0$ and $x > 2$.

Check $x = 0$: $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alternating series test).

Check $x = 2$: $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n}$ diverges.

So $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges for $x \in [0, 2)$ on the reals.

q.e.d.