第 4 章 不定积分习题课

本章主要内容

1. 原函数的概念.

2. 不定积分的概念与性质

3. 求不定积分的方法.

4.特殊函数的不定积分求法

一.原函数概念

1. 定义 设$f(x)$在区间I有定义,如果区间上可微函数$F(x)$满足$F'(x) = f(x)$ $(x \in$ I),则称$F(x)$是$f(x)$在I上的一个原函数.

注意定义中:$F(x)$**是**$f(x)$**: (1)在I上的(2)一个原函数.**

2原函数的性质

(i) 如果$F(x)$是$f(x)$的原函数,则对任意常数c, $F(x) + c$也是$f(x)$的原函数.

(ii) $f(x)$的任意两个原函数之差为常数,所以

$$\{F(x) + c \mid c \in R\}$$

就是$f(x)$的全体原函数组成的集合.

(iii) $f(x)$在区间I上的原函数$F(x)$在I上一定是连续函数.

3原函数存在定理 设$f(x)$在区间I上连续,则在I上一定有原函数.

注: (1) 区间I上的连续函数一定有原函数,但原函数未必能用初等函数表示.例如

$$\frac{\sin x}{x}, \quad \frac{1}{\ln x}, \quad \frac{e^x}{x}, \quad e^{-x^2}, \quad \sqrt{1 - k^2 \sin^2 x} \qquad (0 < k < 1)$$

(2) 函数若在区间I上有第一类间断点,则一定在I上没有原函数.

三.不定积分

1.定义 设$F(x)$是$f(x)$的一个原函数,则称集合$\{F(x)+c \mid c \in R\}$为$f(x)$的不定积分,记成$\int f(x)dx = F(x) + c$.

2.性质

$$\frac{d}{dx}\int f(x)dx = f(x), \qquad d\left(\int f(x)dx\right) = f(x)dx.$$

$$\int F'(x)dx = F(x) + c, \qquad \int dF(x) = F(x) + c.$$

3.不定积分的几何意义

$\{F(x)+c \mid c\in R\}$是一族积分曲线,它们是$y=F(x)$沿Oy轴上下平移而得的.在这些曲线上,对应着x 那一点的切线都有相同的斜率$F'(x)=f(x)$.

4.不定积分的运算法则

如果$F(x)$和$G(x)$分别是$f(x)$和$g(x)$的原函数, 则$aF(x)+bG(x)$是$af(x)+bg(x)$的原函数(其中a,b是常数). $\int[af(x)+bg(x)]dx=a\int f(x)dx+b\int g(x)dx$.

5.不定积分的公式

$$\int 0dx=c;\qquad \int \sec x\tan xdx=\sec x+c;$$
$$\int x^{\mu}dx=\frac{x^{\mu+1}}{\mu+1}+c,\ \mu\neq-1;\ \int\csc x\cot xdx=-\csc x+c;$$
$$\int\frac{dx}{x}=\ln|x|+c;\quad \int\frac{dx}{\sqrt{1-x^2}}=\arcsin x+c;$$
$$\int a^xdx=\frac{a^x}{\ln a}+c;\quad \int\frac{dx}{1+x^2}=\arctan x+c;$$
$$\int\cos xdx=\sin x+c;\quad \int \mathrm{ch}xdx=\mathrm{sh}x+c;$$
$$\int\sin xdx=-\cos x+c;\quad \int\mathrm{sh}xdx=\mathrm{ch}x+c;$$
$$\int\csc^2xdx=-\cot x+c;\quad \int\sec^2xdx=\tan x+c;$$
$$\int\frac{1}{\sqrt{x^2-a^2}}dx=\ln|x+\sqrt{x^2-a^2}|+c.$$
$$\int\frac{1}{\sqrt{x^2+a^2}}dx=\ln(x+\sqrt{x^2+a^2})+c;.$$

6.求不定积分的方法

(i) 直接利用公式及运算法则

(ii) 第一类换元法(凑微分法)

(iii) 第二类换元法

(iv) 分部积分法

7. 常见凑微分

$dx=\frac{1}{a}d(ax+b);$ $x^n dx=\frac{1}{n+1}d(x^{n+1});$

$\frac{1}{x}dx=d(\ln|x|);$ $e^x dx=d(e^x);$

$\cos x dx=d(\sin x);$ $\sin x=-d(\cos x);$

$\sec^2 x dx=d(\tan x);$ $\csc^2 x dx=-d(\cot x);$

$\frac{1}{1+x^2}dx=d(\arctan x);$ $\mathrm{ch}x dx=d\mathrm{sh}x;$

$\frac{1}{1+x^2}dx=-d(\mathrm{arccot}x);$ $\mathrm{sh}x dx=d\mathrm{ch}x;$

$\frac{1}{2\sqrt{x}}dx=d(\sqrt{x});$ $\frac{1}{\sqrt{1-x^2}}dx=d(\arcsin x).$

8.常用换元法

(1.)三角代换 (2.) 双曲代换 (3.)倒代换

一. 求不定积分

要求熟练掌握凑微分法、第二换元法、分部积分法、有理函数及三角有理函数的积分法

1. (17) (6分) 求不定积分$\int \max\{1,x^2\}\mathrm{d}x.$

2. (17) (6分）求不定积分$I=\int \sqrt{a^2+x^2}\mathrm{d}x.$

3. (17) (6分）求不定积分$I=\int \frac{\mathrm{d}x}{1+x^3}.$

4. (16) (6分) 求不定积分$\int \min\{x^2,x^5\}\,dx \qquad (-\infty<x<+\infty).$

5. (15,16) (5分,6分)求不定积分$\int x^2\arctan x dx.$

6. (15) (5分)求不定积分$\int \frac{1}{x(1+x^4)}dx.$

7. (14) (10分)已知$f''(x)$连续,$f'(x)\neq 0$,求$\int\left[\frac{f(x)}{f'(x)}-\frac{f^2(x)f''(x)}{(f'(x))^3}\right]dx.$

8. (14) (6分)$\int|\ln x|dx.$

9. (14) (6分)$\int\frac{x^3-x}{1+x^4}dx.$

10. (13) (6分) $\int \frac{1}{1-x^4}dx.$

11. (13) (6分) $\int \max\{x^2, x^4\}dx.$

12. (13) (10分) 设$f(x)$可微,且$\int x^3 f'(x)dx = x^2\cos x - 4x\sin x - 6\cos x + C$,求$f(x)$.

13. (13) (8分) 设$f(x)$是$(-\infty, +\infty)$上的可微函数且有反函数,已知$F(x)$是$f(x)$的一个原函数,求$\int f^{-1}(x)dx.$

14. (12) (5分) $\int x(x-1)^n dx\ (n>0)$

15. (12) (5分) $\int \sin(2x)\cos^2 x dx$

16. (12) (5分) $\int \sin\sqrt{x}dx$

17. (12) (5分) $\int \ln(x+\sqrt{x^2+1})dx$

18. (11)(8分) $\int \frac{xe^x}{\sqrt{e^x-1}}dx$

19. (10) (5分) $\int x^2 e^x dx$

20. (10) (5分) $\int \frac{1}{\sqrt{e^x+1}}dx$

21. (09) (8分) $\int x^2\ln^2 x dx$

22. (08) (5分) $\int \frac{\arctan x}{x^3}dx$

23. (06) (9分) $\int \frac{\cos x}{\sin x(\sin^2 x+1)}dx$

24. (05) (9分) $\int \frac{\ln x}{\sqrt{x-1}}dx$

25. (04) (6分) $\int \frac{\sin 2x}{\sqrt{1+\cos^2 x}}dx$

26. (03) (5分) $\int x\tan x\sec^2 x dx$

27. (02) (8分) $\int e^{\sqrt{x}}dx$,

28. (02) (8分) $\int \frac{dx}{\sin^2 x - 3\sin x\cos x + 2\cos^2 x}$

例题：

1. $\int x\tan x\sec^4 xdx$.

解：

$$\begin{aligned}
&\int x\tan x\sec^4 xdx = \int x\sec^3 xd\sec x = \frac{1}{4}\int xd(\sec^4 x)\\
= &\frac{1}{4}(x\sec^4 x - \int sec^4 xdx) = \frac{1}{4}x\sec^4 x - \frac{1}{4}\int(1+\tan^2 x)d(\tan x)\\
= &\frac{1}{4}x\sec^4 x - \frac{1}{4}\tan x - \frac{1}{12}\tan^3 x + c
\end{aligned}$$

2. $\int \frac{xe^x}{\sqrt{e^x-2}}dx$.

解：

$$\int \frac{xe^x}{\sqrt{e^x-2}}dx = \int x\frac{d(e^x-2)}{\sqrt{e^x-2}} = 2\int xd\sqrt{e^x-2} = 2x\sqrt{e^x-2} - 2\int\sqrt{e^x-2}dx$$

= (以下自己完成)

3. $\int \frac{x(\arcsin x)^2}{\sqrt{1-x^2}}dx$.

解：

$$\begin{aligned}
&\int \frac{x(\arcsin x)^2}{\sqrt{1-x^2}}dx = -\int(\arcsin x)^2 d\sqrt{1-x^2}\\
= &-(\arcsin x)^2\sqrt{1-x^2} + \int\frac{\sqrt{1-x^2}2\arcsin x}{\sqrt{1-x^2}}dx\\
= &-(\arcsin x)^2\sqrt{1-x^2} + 2x\arcsin x - 2\int\frac{x}{\sqrt{1-x^2}}dx\\
= &-(\arcsin x)^2\sqrt{1-x^2} + 2x\arcsin x + 2\sqrt{1-x^2} + c
\end{aligned}$$

4. $\int \frac{\sin 2x}{\sqrt{3-\cos^4 x}}dx$.

解：

$$\int \frac{\sin 2x}{\sqrt{3-\cos^4 x}}dx = -\int\frac{d(\cos^2 x)}{\sqrt{3-\cos^4 x}} = -\arcsin\frac{\cos^2 x}{\sqrt{3}} + c.$$

5. $\int \frac{x\ln x}{\sqrt{(x^2-1)^3}}dx$.

解：

$$\int \frac{x\ln x}{\sqrt{(x^2-1)^3}}dx = -\int \ln x d(\frac{1}{\sqrt{x^2-1}}) = -\frac{\ln x}{\sqrt{x^2-1}} + \int \frac{1}{x\sqrt{x^2-1}}dx$$
$$= \quad -\frac{\ln x}{\sqrt{x^2-1}} + \int \frac{dx}{x^2\sqrt{1-\frac{1}{x^2}}} = -\frac{\ln x}{\sqrt{x^2-1}} - \int \frac{d(\frac{1}{x})}{\sqrt{1-(\frac{1}{x})^2}}$$
$$= \quad -\frac{\ln x}{\sqrt{x^2-1}} - \arcsin\frac{1}{x} + c.$$

6. $\int \frac{x^2}{1+x^2}\arctan x dx.$

解：

$$\int \frac{x^2}{1+x^2}\arctan x dx = \int \frac{x^2-1+1}{1+x^2}\arctan x dx = \int \arctan x dx - \int \frac{\arctan x}{1+x^2}dx$$
$$= \quad x\arctan x - \int \frac{x}{1+x^2}dx - \frac{1}{2}\int d(\arctan x)^2$$
$$= \quad x\arctan x - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + c.$$

7. $\int \frac{xe^x}{(1+x)^2}dx.$

解：

$$\int \frac{xe^x}{(1+x)^2}dx = \int \frac{(1+x-1)e^x}{(1+x)^2}dx = \int \frac{e^x}{1+x}dx - \int \frac{e^x}{(1+x)^2}dx$$
$$= \quad \int \frac{d(e^x)}{1+x} - \int \frac{e^x}{(1+x)^2}dx = \frac{e^x}{1+x} + \int \frac{e^x}{(1+x)^2}dx - \int \frac{e^x}{(1+x)^2}dx = \frac{e^x}{1+x} + c.$$

8. $\int \frac{\ln x - 1}{(\ln x)^2}dx.$

解法一：

$$\int \frac{\ln x-1}{(\ln x)^2}dx = \int (\frac{1}{\ln x} - \frac{1}{\ln^2 x})dx = \int \frac{dx}{\ln x} - \int \frac{1}{\ln^2 x}dx$$
$$= \quad \frac{x}{\ln x} + \int \frac{x}{\ln^2 x}\cdot\frac{1}{x}dx - \int \frac{1}{\ln^2 x}dx = \frac{x}{\ln x} + c.$$

解法二：

$$\int \frac{\ln x-1}{(\ln x)^2}dx \xlongequal{\ln x = t} \int \frac{t-1}{t^2}\cdot e^t dt.$$

9. $\int \frac{e^x(1+\sin x)}{1+\cos x}dx.$

解：

$$\int \frac{e^x(1+\sin x)}{1+\cos x}dx = \int \frac{e^x}{1+\cos x}dx + \int \frac{e^x \sin x}{1+\cos x}dx$$

$$= \int \frac{e^x}{2\cos^2 \frac{x}{2}}dx + \int e^x \tan \frac{x}{2}dx = \int e^x d(\tan \frac{x}{2}) + \int e^x \tan \frac{x}{2}dx = e^x \tan \frac{x}{2} + c.$$

10. $\int (1+x-\frac{1}{x})e^{x+\frac{1}{x}}dx.$

解：

$$\int (1+x-\frac{1}{x})e^{x+\frac{1}{x}}dx = \int e^{x+\frac{1}{x}}dx + \int (x-\frac{1}{x})e^{x+\frac{1}{x}}dx$$

$$= xe^{x+\frac{1}{x}} - \int xe^{x+\frac{1}{x}}(1-\frac{1}{x^2})dx + \int (x-\frac{1}{x})e^{x+\frac{1}{x}}dx = xe^{x+\frac{1}{x}} + c.$$

11. $\int \frac{x}{x^8-1}dx.$

解：

$$\int \frac{x}{x^8-1}dx \xlongequal{x^2=t} \frac{1}{2}\int \frac{dt}{t^4-1} = \frac{1}{4}\int \left(\frac{-1}{2(t+1)} + \frac{1}{2(t-1)} - \frac{1}{t^2+1}\right)dt.$$

12. $\int \frac{dx}{x^8(1+x^2)}.$

解：

$$\int \frac{dx}{x^8(1+x^2)} = \int \frac{1-x^8+x^8}{x^8(1+x^2)}dx = \int \frac{(1+x^4)(1+x^2)(1-x^2)}{x^8(1+x^2)}dx + \int \frac{dx}{1+x^2}dx$$

$$= \int \left(\frac{1}{x^8} - \frac{1}{x^6} + \frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{1+x^2}\right)dx = -\frac{1}{7x^7} + \frac{1}{5x^5} - \frac{1}{3x^3} + \frac{1}{x} + \arctan x + c.$$

13. $\int \frac{x^9-8}{x^{10}+8x}dx.$

解：

$$\int \frac{x^9-8}{x^{10}+8x}dx = \int \frac{x^8(x^9-8)}{x^9(x^8+8)}dx = \frac{1}{9}\int \frac{x^9-8}{x^9(x^9+8)}d(x^9) \xlongequal{x^9=t} \frac{1}{9}\int \frac{t-8}{t(t+8)}dt$$

$$= \frac{1}{9}\int (\frac{2}{t+8} - \frac{1}{t})dt = \frac{1}{9}\ln \frac{(t+8)^2}{t} + c = \frac{1}{9}\ln \frac{(x^9+8)^2}{x^9} + c.$$

14. $\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x}dx.$

解：

$$\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x}dx = \int \frac{4\sin x + 2\cos x + 2\cos x - \sin x}{2\sin x + \cos x}dx = 2\int dx + \int \frac{d(2\sin x + \cos x)}{2\sin x + \cos x}$$

$$= 2x + \ln|2\sin x + \cos x| + c.$$

15. $\int \frac{\ln(1+x^2)}{x^3} dx.$

解:

$$\int \frac{\ln(1+x^2)}{x^3} dx = -\frac{1}{2}\int \ln(1+x^2) d(\frac{1}{x^2}) = -\frac{1}{2}\left[\frac{\ln(1+x^2)}{x^2} - \int \frac{1}{x^2}\cdot\frac{2x}{1+x^2}dx\right]$$

$$= \quad -\frac{\ln(1+x^2)}{2x^2} + \int (\frac{1}{x} - \frac{x}{1+x^2})dx = -\frac{\ln(1+x^2)}{2x^2} + \ln|x| - \frac{1}{2}\ln(1+x^2) + c.$$

16. $\int \frac{1}{\sin^3 x \cos^5 x} dx.$

解:

$$\int \frac{1}{\sin^3 x \cos^5 x} dx = \int \frac{1}{\tan^3 x \cos^8 x} dx = \int \frac{(1+\tan^2 x)^3}{tan^3 x} d(\tan x)$$

$$= \quad \frac{-1}{2\tan x^2} + \frac{3}{2}\tan^2 x + 3\ln|\tan x| + \frac{1}{4}\tan^4 x + c.$$

17. $\int \frac{x^5}{\sqrt[4]{x^3+1}} dx.$

解:

$$\int \frac{x^5}{\sqrt[4]{x^3+1}} dx = \frac{1}{3}\int \frac{x^3}{\sqrt[4]{x^3+1}} d(x^3) \xlongequal{t=x^3} \frac{1}{3}\int \frac{tdt}{\sqrt[4]{t+1}} = \frac{1}{3}\int \left[(t+1)^{\frac{3}{4}} - (t+1)^{\frac{-1}{4}}\right] dt$$

$$= \quad \frac{4}{21}(1+t)^{\frac{7}{4}} - \frac{4}{9}(1+t)^{\frac{3}{4}} + c = \frac{4}{21}(1+x^3)^{\frac{7}{4}} - \frac{4}{9}(1+x^3)^{\frac{3}{4}} + c$$

18. $\int e^x \left(3^x + \frac{e^{-x}}{x\ln x}\right) dx.$

解:

$$\int e^x \left(3^x + \frac{e^{-x}}{x\ln x}\right) dx = \int (3e)^x dx + \int \frac{1}{\ln x} d(\ln x) = \frac{(3e)^x}{\ln(3e)} + \ln|\ln x| + c.$$

19. $\int \frac{e^{-\frac{1}{x}}}{x^4} dx.$

解:

$$\int \frac{e^{-\frac{1}{x}}}{x^4} dx \xlongequal{\frac{1}{x}=t} \int \frac{t^4}{e^t}\cdot(-\frac{1}{t^2})dt = \int t^2 d(e^{-t}) = t^2 e^{-t} + 2te^{-t} + 2e^{-t} + c$$

$$= \quad (\frac{1}{x})^2 e^{-\frac{1}{x}} + 2te^{-\frac{1}{x}} + 2e^{-\frac{1}{x}} + c$$

20. $\int \frac{xe^{\arctan x}}{(1+x^2)^2} dx.$

解:

$$\int \frac{xe^{\arctan x}}{(1+x^2)^2}dx \xlongequal{\arctan x=t} \int \frac{\tan t e^t \sec^2 t}{(1+\tan^2 t)^2}dt = \frac{1}{2}\int e^t \sin 2t dt$$

$$\int e^t \sin 2t dt = \int \sin 2t de^t = e^t \sin 2t - 2\int e^t \cos 2t dt = e^t \sin 2t - 2\int \cos 2t de^t$$

$$= \quad e^t \sin 2t - 2e^t \cos 2t - 4\int e^t \sin 2t dt$$

$$\int e^t \sin 2t dt = \frac{1}{5}(e^t \sin 2t - 2e^t \cos 2t).$$

$$\int \frac{xe^{\arctan x}}{(1+x^2)^2}dx = \frac{1}{10}(e^t \sin 2t - 2e^t \cos 2t) = \frac{1}{10}(e^{\arctan x}\sin 2\arctan x - 2e^{\arctan x}\cos 2\arctan x) + c.$$

21. 设

$f'(\ln x) = \begin{cases} 1, & 0 < x \leq 1 \\ x, & 1 < x < +\infty \end{cases}$,且$f(0)=0$,求$f(x)$.

解: 令$\ln x = t,\ x = e^t$

$$f'(t) = \begin{cases} 1, & 0 < e^t \leq 1 \Longleftrightarrow -\infty < t \leq 0 \\ e^t, & 1 < e^t < +\infty \Longleftrightarrow 0 < t < +\infty \end{cases}$$

$$f(t) = \begin{cases} t + c_1, & -\infty < t \leq 0 \\ e^t + c_2, & 0 < t < +\infty \end{cases}$$

$f(0-0) = f(0+) = f(0)$,得$c_1 = 1 + c_2$

$$f(x) = \begin{cases} x + 1 + c, & -\infty < t \leq 0 \\ e^x + c, & 0 < t < +\infty \end{cases}$$

因为$f(0)=0$,可得$c=-1$ 所以

$$f(x) = \begin{cases} x, & -\infty < t \leq 0 \\ e^x - 1, & 0 < t < +\infty \end{cases}$$

22. 已知$f(x) = \dfrac{e^x}{x}$,求$\int xf''(x)dx$.

解:

$$\int xf''(x)dx = \int xdf'(x) = xf'(x) - \int f'(x)dx$$

$$= \quad xf'(x) - \int df(x) = xf'(x) - f(x) + c = (\frac{e^x}{x})'x - \frac{e^x}{x} + c.$$

练习题

1. $\int \frac{\ln x}{(1-x)^2}dx$

2. $\int \frac{dx}{\sin 2x + 2\sin x}$

3. $\int \frac{\arctan x}{x^2(1+x^2)}dx$

4. $\int e^{2x}(\tan x+1)^2dx$

5. $\int \frac{dx}{x\sqrt{1+x^2}}$

6. $\int \frac{dx}{x+\sqrt{x^2-1}}$

7. $\int x^2\sin^2 xdx$

8. $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$

9. 设$f(x^2-1)=\ln\frac{x^2}{x^2-2}$,且$f(\varphi(x))=\ln x$,求$\int\varphi(x)dx$.

10. 设$f(\ln x)=\frac{\ln(1+x)}{x}$,计算$\int f(x)dx$.