一、填空题

1)

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix} = \begin{pmatrix}1 & 4 & 2 & -1\\ 3 & 4 & 5 & 2\\ 2 & 0 & 3 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 4 & 2 & -1\\ 0 & -8 & -1 & 5\\ 0 & -8 & -1 & 5\end{pmatrix} \sim \begin{pmatrix}1 & 4 & 2 & -1\\ 0 & -8 & -1 & 5\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\mathrm{rank}\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix} = 2$$

2)

$$\mathrm{A} = \begin{pmatrix}1\\ -1\\ 2\end{pmatrix}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$$

$$\mathrm{A}^{10} = \mathrm{A} = \begin{pmatrix}1 & 2 & 1\\ -1 & -2 & -1\\ 2 & 4 & 2\end{pmatrix}$$

3)

$$\det(\mathrm{A}^*)=\det(\mathrm{A})^{n-1} = 5^{n-1}$$

4)

$$\text{notice } b_{1n}\mathrm{A}_{1n}+b_{2n}\mathrm{A}_{2n}+...+b_{nn}\mathrm{A}_{nn} = \det\begin{pmatrix}a_{11} & a_{12} & ... & a_{1,n-1} & b_{1n}\\ a_{21} & a_{22} & ... & a_{2,n-1} & b_{2n}\\ ... & ... & & ... & ...\\ a_{n1} & a_{n2} & ... & a_{n,n-1} & b_{nn}\end{pmatrix}$$

$$\text{so } \mathrm{A}_{14} - 3\mathrm{A}_{24} + 2\mathrm{A}_{34} - \mathrm{A}_{44} = \det\begin{pmatrix}1 & 5 & 2 & 1\\ 0 & 3 & 1 & -3\\ 0 & 0 & -1 & 2\\ 1 & 5 & 2 & -1\end{pmatrix} = 6$$

5)

i.e. $\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \beta$ has no solution

$$\begin{pmatrix}1 & 1 & 1 & | & 3\\1 & 2 & -\lambda & | & 9\\2 & -1 & 3 & | & 6\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1 & | & 3\\0 & 1 & -\lambda-1 & | & 6\\0 & -3 & 1 & | & 0\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1 & | & 3\\0 & 1 & -\lambda-1 & | & 6\\0 & 0 & -3\lambda-2 & | & 18\end{pmatrix}$$

$\lambda = -2/3$

6)

$$\mathrm{A}^{-} = \begin{pmatrix}\mathrm{O} & \mathrm{C}^{-}\\ \mathrm{B}^{-} & \mathrm{O}\end{pmatrix},\ (\mathrm{A}^t)^{-} = (\mathrm{A}^{-})^t = \begin{pmatrix}\mathrm{O} & (\mathrm{B}^{-})^t\\ (\mathrm{C}^{-})^t & \mathrm{O}\end{pmatrix}$$

二、判断题

1) $\times$

$$\mathrm{rank(A)} = \mathrm{rank}\begin{pmatrix}2 & 3 & -1\\1 & 5 & -2\\5 & 11 & -4\end{pmatrix} = \mathrm{rank}\begin{pmatrix}0 & -7 & 3\\1 & 5 & -2\\0 & -14 & 6\end{pmatrix} = 2$$

$$\mathrm{rank(B)} = \mathrm{rank}\begin{pmatrix}1 & 0 & -2\\5 & 0 & 4\\3 & 0 & 2\end{pmatrix} = 2$$

« A，B 相抵 » $\Leftrightarrow$ « $\mathrm{rank(A)} = \mathrm{rank(B)}$ »

2) $\times$

$$\alpha_1 = \begin{pmatrix}1\\0\end{pmatrix}, \alpha_2 = \begin{pmatrix}0\\1\end{pmatrix}, \alpha_3 = \begin{pmatrix}1\\1\end{pmatrix}, \mathrm{A} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}$$

« 当 $\ell \geq m$ 时,结论成立,此时 $\mathrm{rank}\begin{pmatrix}\beta_1 & \beta_2 & \dots & \beta_\ell\end{pmatrix} \leq \mathrm{rank}\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_m\end{pmatrix} \leq m-1 \leq \ell-1$ »

3) $\times$

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix},B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\text{rank}(AB)=\text{rank}\begin{pmatrix}0&0\\0&0\end{pmatrix}=0$$

$$\text{rank}(BA)=\text{rank}\begin{pmatrix}0&1\\0&0\end{pmatrix}=1$$

« 若 A，B 中至少有一个可逆，则结论成立，因为可逆矩阵不改变矩阵的秩，因此同学们在找反例时，应避免 A，B 可逆 »

4) $\surd$

$$r=\text{rank}\begin{pmatrix}\alpha_1&\alpha_2&...&\alpha_s\end{pmatrix}=\text{rank}\begin{pmatrix}\alpha_1&\alpha_2&...&\alpha_r\end{pmatrix}$$

因此 $\{\alpha_i\}_{1\le i\le r}$ 线性无关，又因为任意的 $\alpha_i(1\le i\le s)$ 可以被 $\{\alpha_i\}_{1\le i\le r}$ 线性表示，故它是 $\{\alpha_i\}_{1\le i\le s}$ 的极大无关组

三、解方程

$$\left(\begin{array}{cccccc}1&2&-3&4&|&2\\3&8&-1&-2&|&0\\2&5&-2&1&|&a\end{array}\right)\sim\left(\begin{array}{cccccc}1&2&-3&4&|&2\\0&2&8&-14&|&-6\\0&1&4&-7&|&a-4\end{array}\right)\sim\left(\begin{array}{cccccc}1&2&-3&4&|&2\\0&1&4&-7&|&-3\\0&0&0&0&|&a-1\end{array}\right)$$

$\Rightarrow a=1$ 时有解，$x=x_0+t_1\eta_1+t_2\eta_2$，其中 $x_0=\begin{pmatrix}8\\-3\\0\\0\end{pmatrix}$，$\eta_1=\begin{pmatrix}11\\-4\\1\\0\end{pmatrix}$，

$$\eta_2=\begin{pmatrix}-18\\7\\0\\1\end{pmatrix}$$

四、求行列式与逆

求行列式，我们可以计算更一般的情形

$$\det\begin{pmatrix} a & a & ... & a & a \\ b & a & ... & a & a \\ ... & & & & ... \\ b & b & ... & a & a \\ b & b & ... & b & a \end{pmatrix} = \det\begin{pmatrix} a & a & ... & a & a \\ b-a & 0 & ... & 0 & 0 \\ ... & & & & ... \\ b-a & b-a & ... & 0 & 0 \\ b-a & b-a & ... & b-a & 0 \end{pmatrix}$$

$= (-)^{n-1}a(b-a)^{n-1} = a(a-b)^{n-1}$

这里取 $a = 1$，$b = -1$，$\det(\mathrm{A}) = 2^{n-1}$

$\mathrm{A}_{ii} = 2^{n-2}$，$\mathrm{A}_{i,i-1} = -2^{n-2}$，$\mathrm{A}_{1n} = 2^{n-2}$，其余代数余子式均为 0

$$\mathrm{A}^{-} = \frac{1}{2}\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & ... & ... & \\ & & & 1 & -1 \\ 1 & & & & 1 \end{pmatrix}$$

五、基底，类似于书上原题

1) 线性无关

设有线性关系 $a + b(x+1) + c(x+1)^2 + d(x+1)^3 = 0$，左右两边 x^3 的系数必须相等，因此 $d = 0$，类似的可以得到 $c = b = a = 0$，故 S 线性无关

1) 极大性

$\forall f(x) \in \mathbb{P}_3[x]$，$f(x) = a + bx + cx^2 + dx^3$

$= a + b((x+1)-1) + c((x+1)-1)^2 + d((x+1)-1)^3 = (a-b+c-d) + (b-2c+3d)(x+1) + (c-3d)(x+1)^2 + d(x+1)^3$ 可以被 S 线性表示，因此 S 是 $\mathbb{P}_3[x]$ 的一组基

2) 过渡矩阵

$$\begin{pmatrix}1 & x & x^2 & x^3\end{pmatrix} = \begin{pmatrix}1 & x+1 & (x+1)^2 & (x+1)^3\end{pmatrix}\begin{pmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -2 & 3\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 1\end{pmatrix}$$

3) 坐标

$$5+7x-x^2+13x^3 = \begin{pmatrix}1 & x & x^2 & x^3\end{pmatrix}\begin{pmatrix}5\\ 7\\ -1\\ 13\end{pmatrix}$$

$$= \begin{pmatrix}1 & x+1 & (x+1)^2 & (x+1)^3\end{pmatrix}\begin{pmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -2 & 3\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}5\\ 7\\ -1\\ 13\end{pmatrix}$$

$$= \begin{pmatrix}1 & x+1 & (x+1)^2 & (x+1)^3\end{pmatrix}\begin{pmatrix}-16\\ 48\\ -40\\ 13\end{pmatrix} = -16+48(x+1)-40(x+1)^2+13(x+1)^3$$

六、

1)

我们知道，存在可逆矩阵 P, Q s.t. $\mathrm{A} = \mathrm{P}\begin{pmatrix}1 & 0\\ 0 & \mathrm{O}\end{pmatrix}\mathrm{Q} = \begin{pmatrix}p & \mathrm{P}'\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathrm{O}\end{pmatrix}\begin{pmatrix}q\\ \mathrm{Q}'\end{pmatrix} = pq$

$c = \mathrm{tr}(pq) = \mathrm{tr}(qp) = qp$

$\mathrm{A}^2 = qp\mathrm{A} = c\mathrm{A}$

2)

$\det(\mathrm{I} + \mathrm{A}) = \det(\mathrm{I} + pq) = \det(1 + qp) = 1 + c$