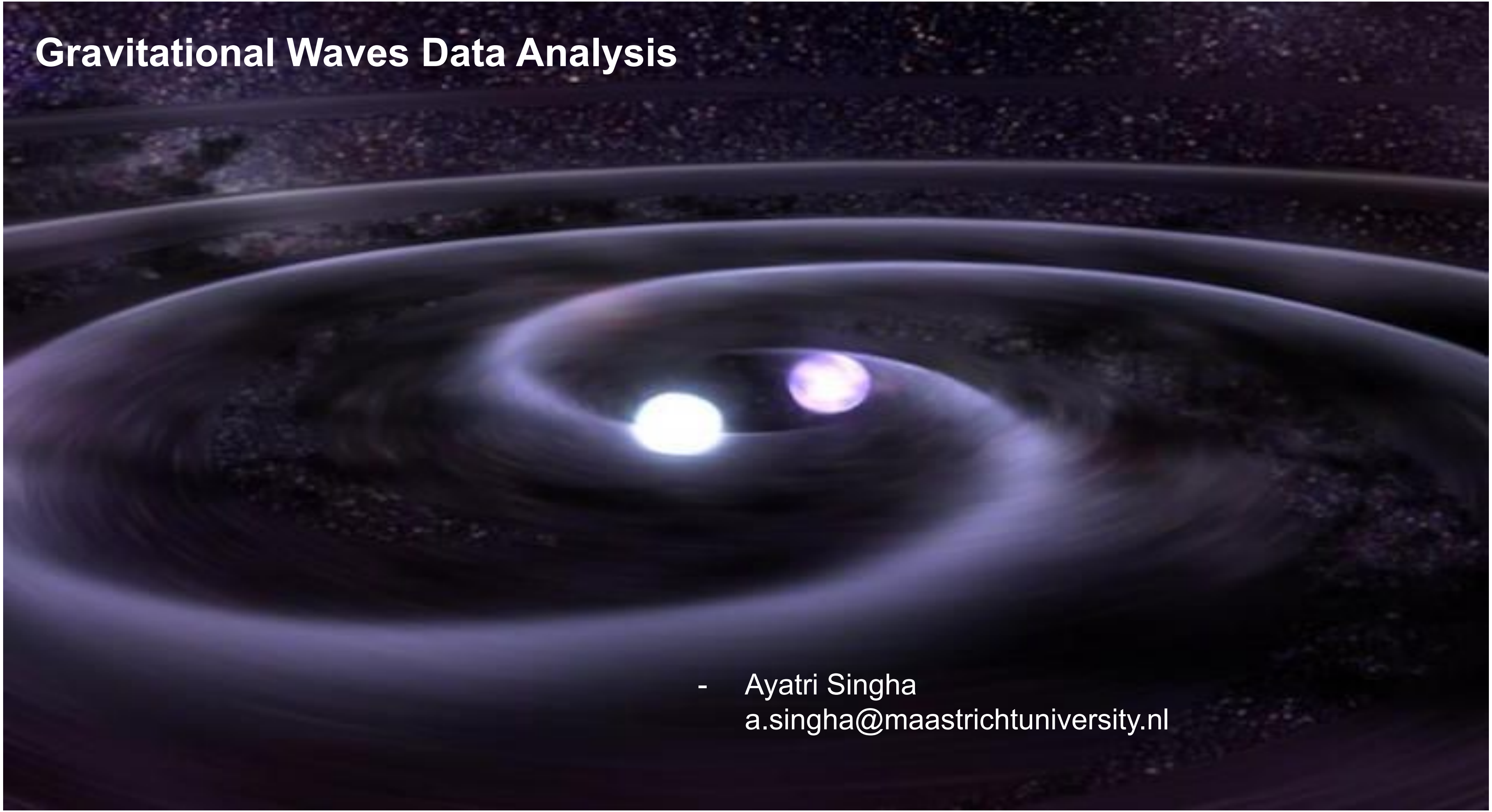
Gravitational Waves Data Analysis



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Gravitational waves

- Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Far away from matter/energy:
  Metric tensor is that of flat spacetime with a small perturbation

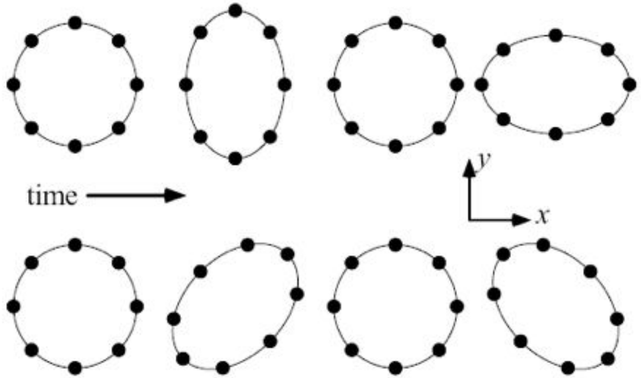
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

- Einstein equations reduce to a wave equation for the perturbation:

$$\left(-\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_{\mu\nu} = 0$$

- In the "transverse-traceless" gauge, wave moving in *z* direction

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$



http://www.ctc.cam.ac.uk/activities/rise/slides/vandenbroeck1.pdf

Interferometric detectors

$$\boxed{h(t) = \frac{1}{2}(h_{xx} - h_{yy})}$$

- Signal from arbitrary direction
  In transverse-traceless frame:

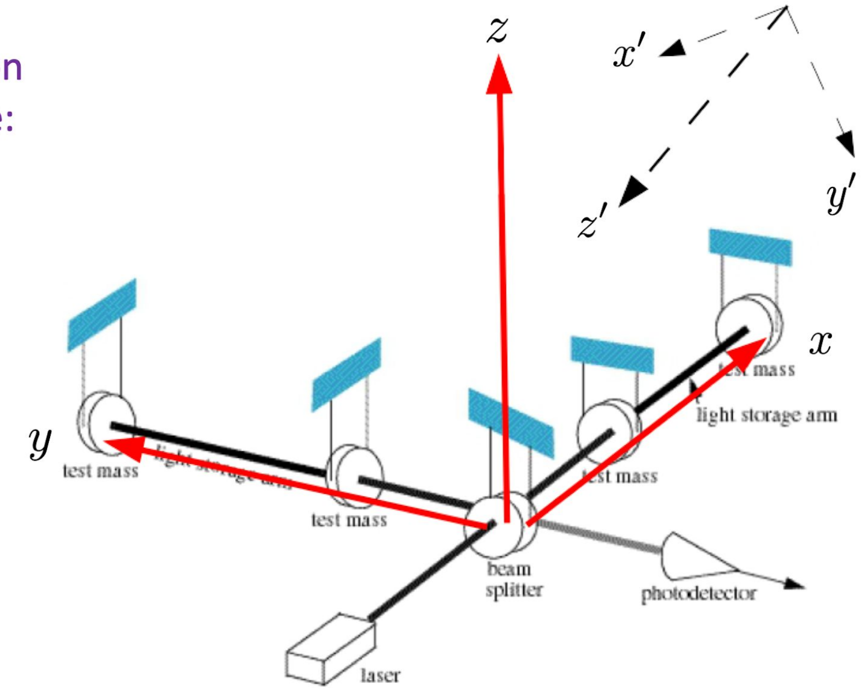
$$h_{ij}^{\hat{\mathbf{n}}=\hat{\mathbf{z}}'} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

- If $x' = x, \;\; y' = y, \;\; z' = z$
  then

$$h(t) = h_+(t)$$

- In general, need to apply linear transformation

$$h_{ij}^{\mathbf{det}} = \mathcal{R}_{ik}\mathcal{R}_{jl}h_{kl}^{\hat{\mathbf{n}}=\hat{\mathbf{z}}'}$$



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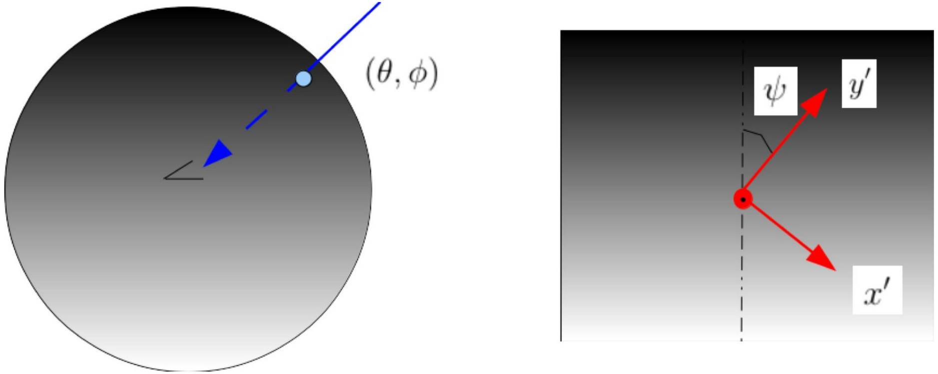
Interferometric detectors

$$\boxed{h(t) = F_+ \, h_+(t) + F_\times \, h_\times(t)}$$

- Signal from an L-shaped interferometer:

$$F_+ = \frac{1}{2}(1+\cos^2(\theta))\,\cos(2\phi)\,\cos(2\psi) - \cos(\theta)\,\sin(2\phi)\,\sin(2\psi)$$
$$F_\times = \frac{1}{2}(1+\cos^2(\theta))\,\cos(2\phi)\,\sin(2\psi) + \cos(\theta)\,\sin(2\phi)\,\cos(2\psi)$$



- (θ, ϕ) sky position, ψ polarization angle

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Maastricht University

Generation of gravitational waves

$$\boxed{\left[h_{ij}^{\mathrm{TT}}(t,\vec{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}\left(t-r/c\right)}$$

- GW radiation is quadrupolar.

No Monopole Radiation

$$\dot{M} = \frac{1}{c}\int_V d^3x\partial_0 T^{00}$$

$$= -\frac{1}{c}\int_V d^3x\partial_i T^{0i}$$

$$= \cancelto{0}{-\frac{1}{c}r^2\int_S d\Omega T^{0i}}$$

No Dipole Radiation

Mass dipole M^i zero (i.e. constant) in center of mass frame

No momentum monopole contribution

$$\dot{P}^i = 0$$

Example I: Quadrupole radiation from a mass in circular orbit

Use Kepler's law, the chirp mass, and the GW frequency to rewrite the solutions.

$$\omega_s^2 = \frac{GM}{R^3} \qquad M_c = \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \qquad \omega_{\mathrm{gw}} = 2\omega_s$$

$$\omega_{\mathrm{gw}} = 2\pi f_{\mathrm{gw}}$$

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{\mathrm{gw}} t_{\mathrm{ret}} + 2\phi)$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}}{c}\right)^{2/3}\cos\theta\sin(2\pi f_{\mathrm{gw}} t_{\mathrm{ret}} + 2\phi)$$

The amplitudes of the GWs emitted depend on the masses m_1 and m_2 only through the combination M_c.

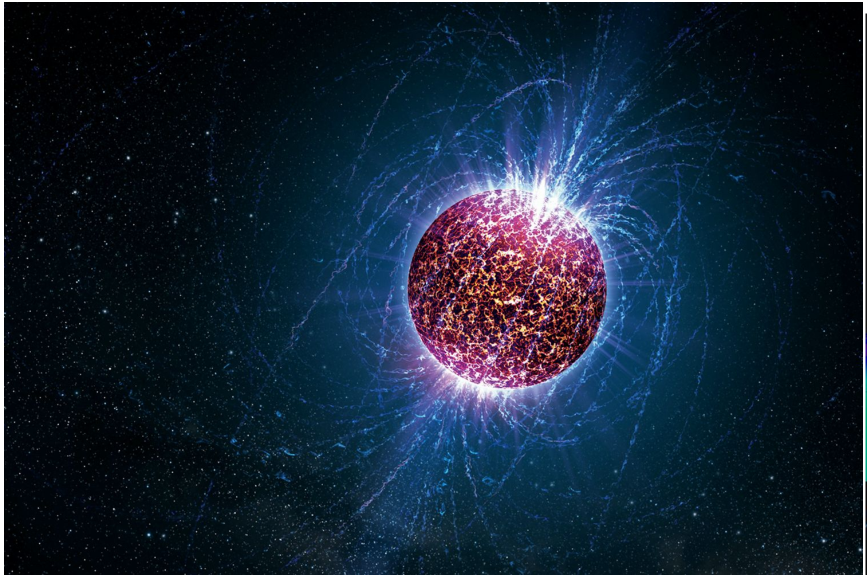
Gravitational wave sources

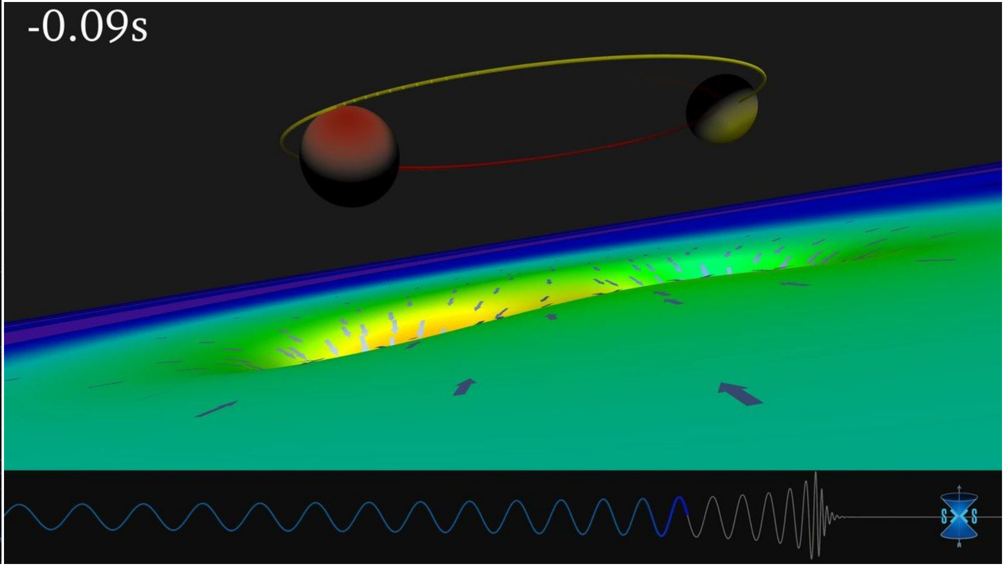
Continuous Gravitational Waves

- Pulsors (isolated massive spinning Neutron star)

Compact Binary Inspiral Gravitational Waves

- Binary Neutron Star (BNS)
- Binary Black Hole (BBH)
- Neutron Star-Black Hole Binary (NSBH)

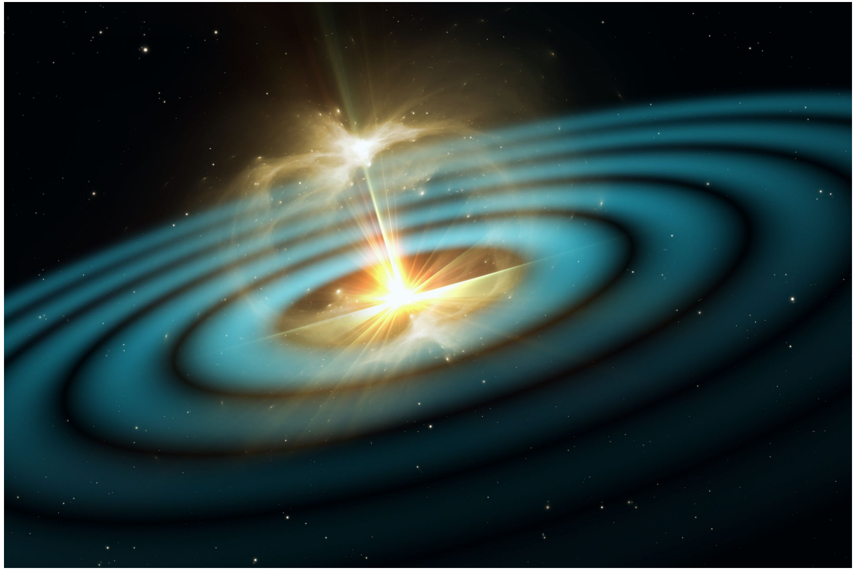




Gravitational wave sources

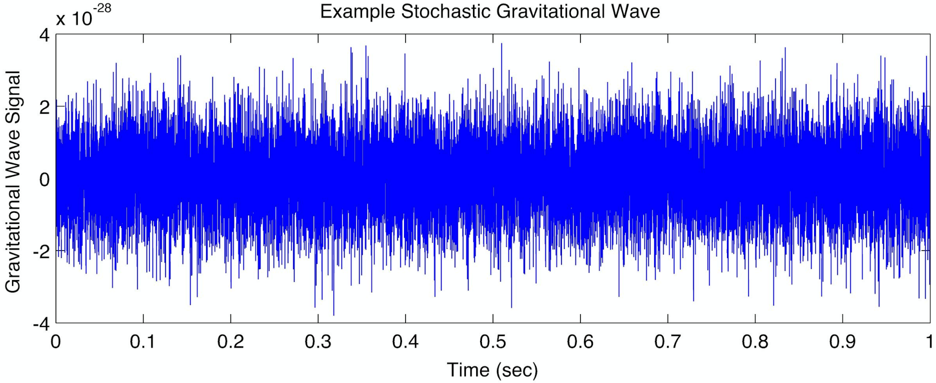
Burst Gravitational Waves

- Core-collapse supernovae
- Gamma-ray bursts

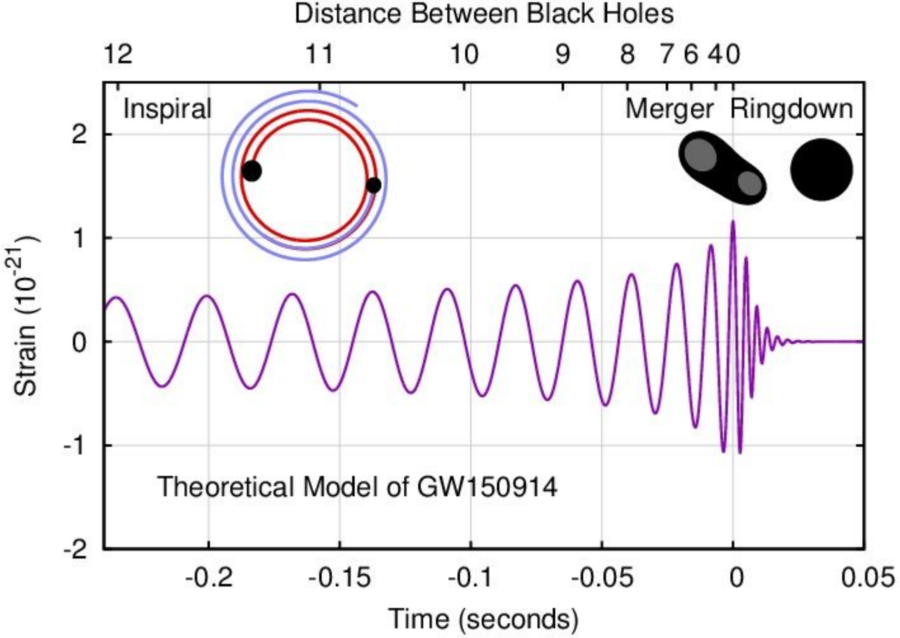


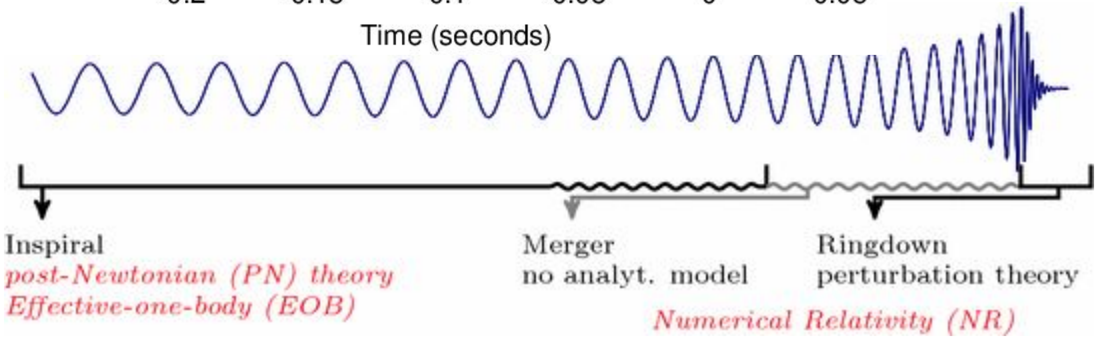
Stochastic Gravitational Waves

- Cosmic Microwave Background (CMB)
- BBH background
- BNS background

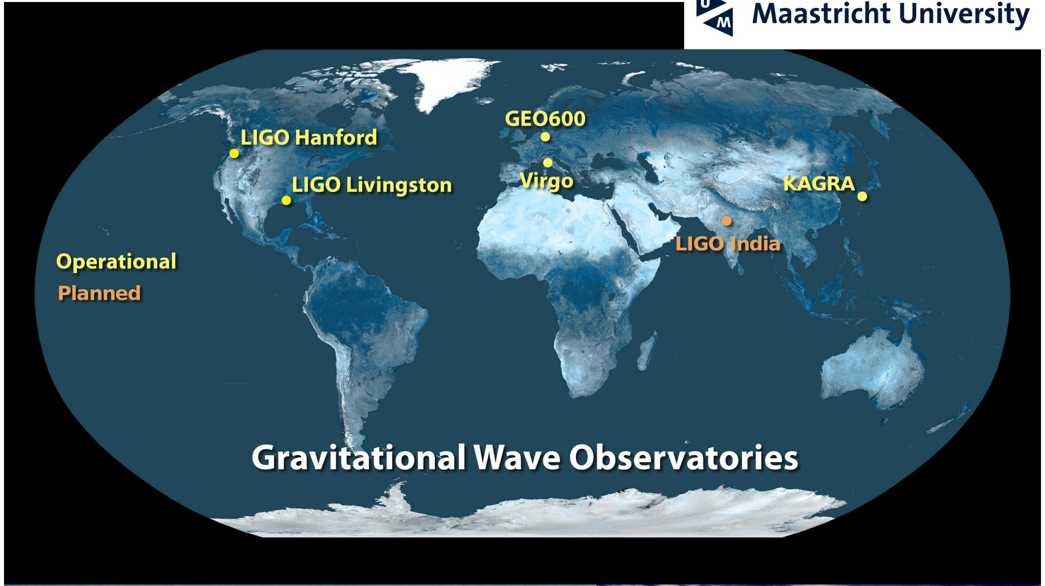


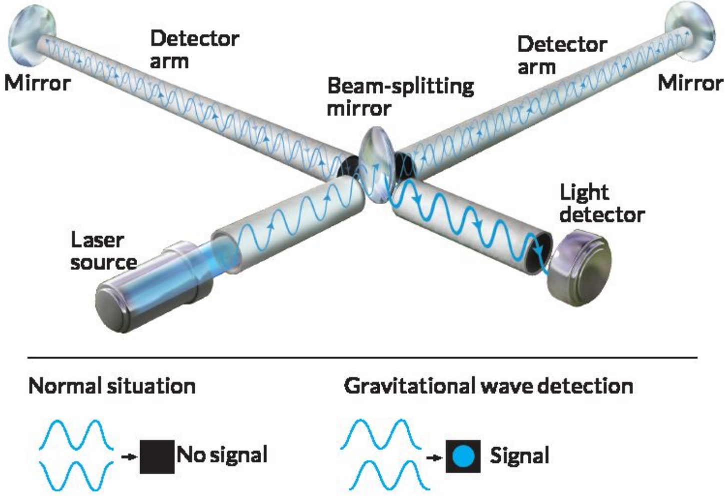
Gravitational waveforms from CBC sources

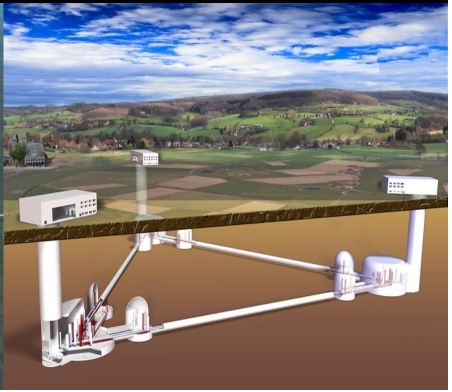




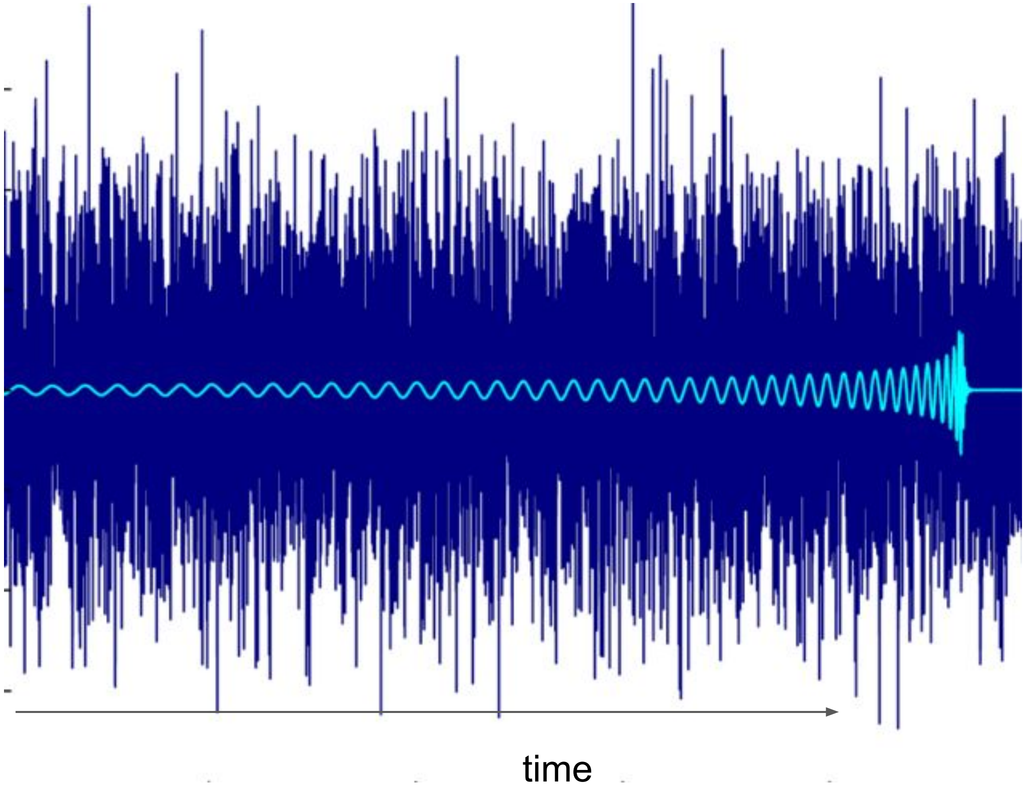
Gravitational wave detectors







What we see in Gravitational wave detectors?
(signal buried into noise)



Digging a signal out of noise

- Detector output can be written as $s(t) = n(t) + h(t)$
- If the shape of signal is known(modeled), then we can integrate the data

$$\frac{1}{T}\int_0^T s(t)\,h(t)\,dt = \frac{1}{T}\int_0^T n(t)\,h(t)\,dt + \frac{1}{T}\int_0^T h(t)^2 dt$$

oscillatory $\sim \left(\frac{\tau_0}{T}\right)^{1/2} n_0 h_0$

positive definite $\sim h_0^2$

To detect a signal, **don't need** $h_0 > n_0$ **but only** $h_0 > (\tau_0/T)^{1/2}\, n_0$

- Binary coalescences:
  $\tau_0 \sim 10^{-2}\,\mathrm{s}, \quad T \sim 100\,\mathrm{s} \quad \rightarrow \quad (\tau_0/T)^{1/2} \sim 10^{-2}$
- Millisecond pulsars:
  $\tau_0 \sim 1\,\mathrm{ms}, \quad T \sim 1\,\mathrm{yr} \quad \rightarrow \quad (\tau_0/T)^{1/2} \sim 10^{-5}$

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Characterizing the noise

- Detector data comes in as a time series
- If only noise: $(n(t_0), n(t_1), \ldots, n(t_N))$ where $t_{i+1} = t_i + t$
- In frequency domain: take discrete Fourier transform-

$$(\tilde{n}(f_0), \tilde{n}(f_1), \ldots, \tilde{n}(f_N)) \quad \text{where} \quad f_{i+1} = f_i + \Delta f$$

  - Notation: $\tilde{n}(f_i) = \tilde{n}_i$

- We will assume noise is stationary and gaussian $p(\tilde{n}_i) \propto e^{-\frac{|\tilde{n}_i|^2}{2\sigma_i^2}}$

Stationarity and Gaussianity: $\langle n_i \rangle = \int \tilde{n}_i \, p(\tilde{n}_i) \, d\tilde{n}_i = 0, \qquad \langle |n_i|^2 \rangle = \int |\tilde{n}_i|^2 \, p(\tilde{n}_i) \, d\tilde{n}_i$

- **noise power spectral density** $\frac{1}{2} S_n(f) \equiv \int_{-\infty}^{\infty} d\tau \, R(\tau) \, e^{2\pi i f \tau}$ where $R(\tau) \equiv \langle n(t+\tau) \, n(t) \rangle$

$$p[n] = \mathcal{N} \, e^{-\int_{-\infty}^{\infty} \frac{|\tilde{n}(f)|^2}{S_n(f)} \, df}$$

$$\langle \tilde{n}^*(f) \, \tilde{n}(f') \rangle = \frac{1}{2} \delta(f - f') \, S_n(f)$$

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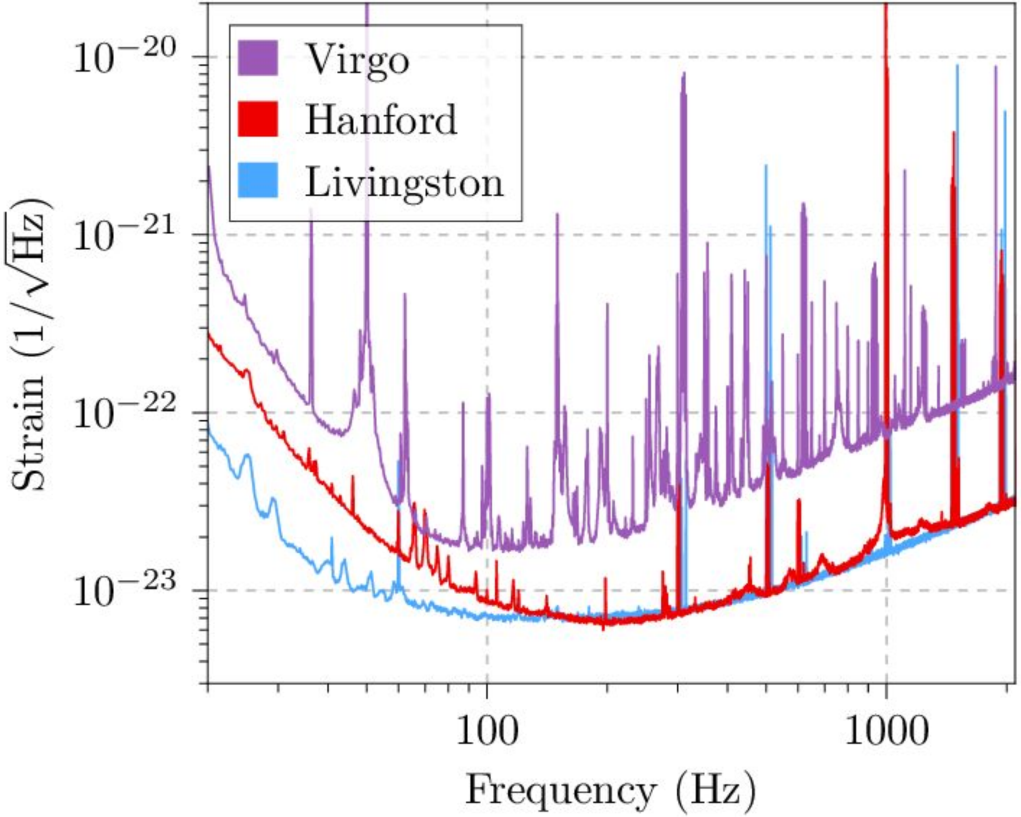
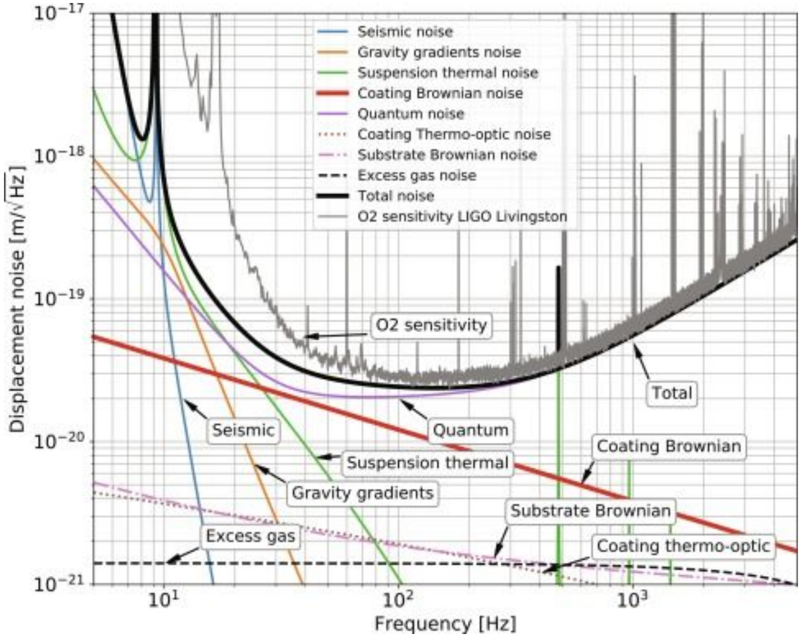
Sensitivity of interferometric GW detectors

(Square root of) noise power spectral density in the first two observing runs of Advanced LIGO and Advanced Virgo: Amplitude spectral density



Various noises limiting sensitivity of GW detectors



Matched filtering

- Instead of integrating the data against waveforms, use more optimal **filter**

$$\hat{s} = \int_{-\infty}^{\infty} dt\, s(t)\, \underbrace{K(t)}_{\text{filter}}$$

- Define S to be the expected value of $\hat{s}$ when a signal $h(t)$ is present, and let N be the root-mean-square value when no signal is present:

$$S = \langle \hat{s} \rangle_h \qquad\qquad N = \left[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2 \right]^{1/2}$$

- Define **signal-to-noise ratio**:

$$S/N = \frac{\langle \hat{s} \rangle_h}{\left[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2 \right]^{1/2}}$$

- Our task: **find out which filter** $K(t)$ **maximizes** S/N

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Matched filtering

- Write S in the frequency domain:

$$\begin{aligned} S &= \langle \hat{s} \rangle_h \\ &= \int_{-\infty}^{\infty} dt\, \langle s(t) \rangle\, K(t) \\ &= \int_{-\infty}^{\infty} dt\, h(t)\, K(t) \\ &= \int_{-\infty}^{\infty} df\, \tilde{h}(f)\, \tilde{K}^*(f) \end{aligned}$$

- Also N :

$$\begin{aligned} N &= \left[\langle \hat{s}^2 \rangle - \langle \hat{s} \rangle^2 \right]_{h=0}^{1/2} \\ &= \left[\langle \hat{s}^2 \rangle \right]_{h=0}^{1/2} \\ &= \left[\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \, \langle n(t)\, n(t') \rangle K(t)\, K(t') \right]^{1/2} \\ &= \left[\int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f)\, |\tilde{K}^*(f)|^2 \right]^{1/2} \end{aligned}$$

- Then we arrive at:

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df\, \tilde{h}(f)\, \tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f)\, |\tilde{K}(f)|^2 \right]^{1/2}}$$

- Now define the **noise-weighted inner product**

$$\begin{aligned} (A|B) &\equiv \Re \int_{-\infty}^{\infty} df\, \frac{\tilde{A}^*(f)\, \tilde{B}(f)}{\frac{1}{2} S_n(f)} \\ &= 4\, \Re \int_{0}^{\infty} df\, \frac{\tilde{A}^*(f)\, \tilde{B}(f)}{S_n(f)} \end{aligned}$$

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Matched filtering

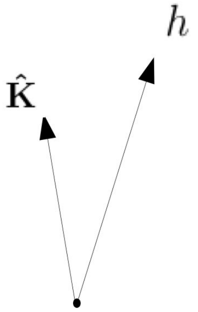
- Now define the **noise-weighted inner product**

$$(A|B) \equiv \Re \int_{-\infty}^{\infty} df \, \frac{\tilde{A}^*(f)\, \tilde{B}(f)}{\frac{1}{2} S_n(f)}$$

$$= 4\, \Re \int_{0}^{\infty} df \, \frac{\tilde{A}^*(f)\, \tilde{B}(f)}{S_n(f)}$$

… and rewrite S/N as

$$\frac{S}{N} = \frac{(\mathbf{K}|h)}{(\mathbf{K}|\mathbf{K})^{1/2}} \qquad \mathbf{K} = \frac{1}{2} S_n(f)\, \tilde{K}(f)$$



… or

$$\frac{S}{N} = (\hat{\mathbf{K}}|h) \qquad \hat{\mathbf{K}} = \frac{\mathbf{K}}{(\mathbf{K}|\mathbf{K})^{1/2}}$$

Maximizing S/N is equivalent to making $\hat{\mathbf{K}}$ point in the same direction as h

http://www.ctc.cam.ac.uk/activities/rise/slides/vandenbroeck1.pdf

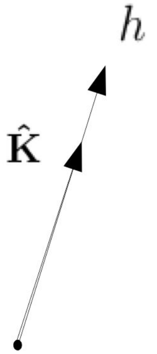
Matched filtering

$$\frac{S}{N} = (\hat{\mathbf{K}}|h) \qquad \hat{\mathbf{K}} = \frac{\mathbf{K}}{(\mathbf{K}|\mathbf{K})^{1/2}} \qquad \mathbf{K} = \frac{1}{2} S_n(f)\, \tilde{K}(f)$$

- **Maximizing S/N is equivalent to making $\hat{\mathbf{K}}$ point in the same direction as h**

$$\hat{\mathbf{K}} \propto h \quad \rightarrow \quad \mathbf{K} \propto h$$

$$\rightarrow \quad \boxed{\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_n(f)}}$$



"Wiener filter"

- **Optimal signal-to-noise ratio**

$$\frac{S}{N} = \frac{(\mathbf{K}|h)}{\sqrt{(\mathbf{K}|\mathbf{K})}} = \frac{(h|h)}{\sqrt{(h|h)}} = \sqrt{(h|h)}$$

$$\boxed{\rho^2 = 4 \int_0^\infty df\, \frac{|\tilde{h}(f)|^2}{S_h(f)}.}$$

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Matched filtering

- So far we assumed that if signal present in the data, it would be of known shape: $\langle s\rangle = h$ and $S/N = (\hat{\mathbf{K}}|\langle s\rangle) = (\hat{\mathbf{K}}|h)$

- In practice we must consider many possible trial waveforms h_i , and apply the optimal filter $\hat{\mathbf{K}}_i \propto h_i$ for each of these to the data:

$$\left(\frac{S}{N}\right)_i = (\hat{\mathbf{K}}_i|\langle s\rangle) = \frac{(\mathbf{K}_i|\langle s\rangle)}{\sqrt{(\mathbf{K}_i|\mathbf{K}_i)}} = \frac{(h_i|\langle s\rangle)}{\sqrt{(h_i|h_i)}}$$

- In practice we only know the detector output s, not its expectation value, so in actual signal processing we approximate

$$\left(\frac{S}{N}\right)_i \simeq \frac{(h_i|s)}{\sqrt{(h_i|h_i)}}$$

Launchpad

http://www.ctc.cam.ac.uk/activities/rise/slides/vandenbroeck1.pdf

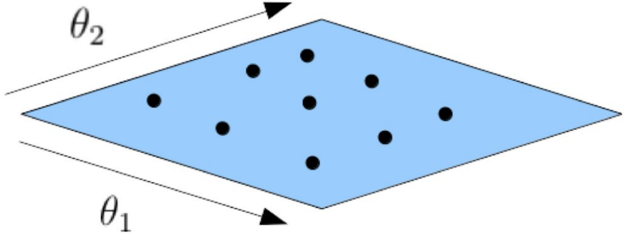
Matched filtering

- For a given trial waveform h_i :

$$\left(\frac{S}{N}\right)_i = \frac{(h_i|s)}{\sqrt{(h_i|h_i)}}$$

- For e.g. compact binary coalescences, waveforms are characterized by source parameters $\bar{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$

- Lay out **template banks** over parameter space: points $\bar{\theta}_i$ give waveforms $h_i = h(\bar{\theta}_i)$



- Find maximum of signal-to-noise ratio over the template bank:

Launchpad

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Amplitude spectral density GW event

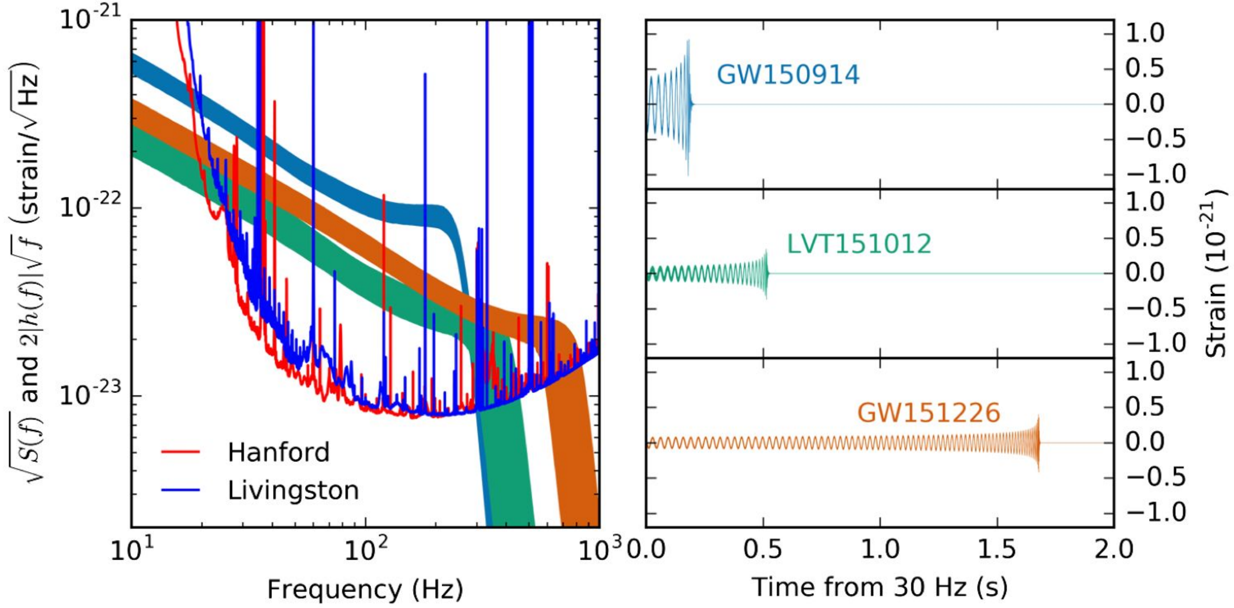
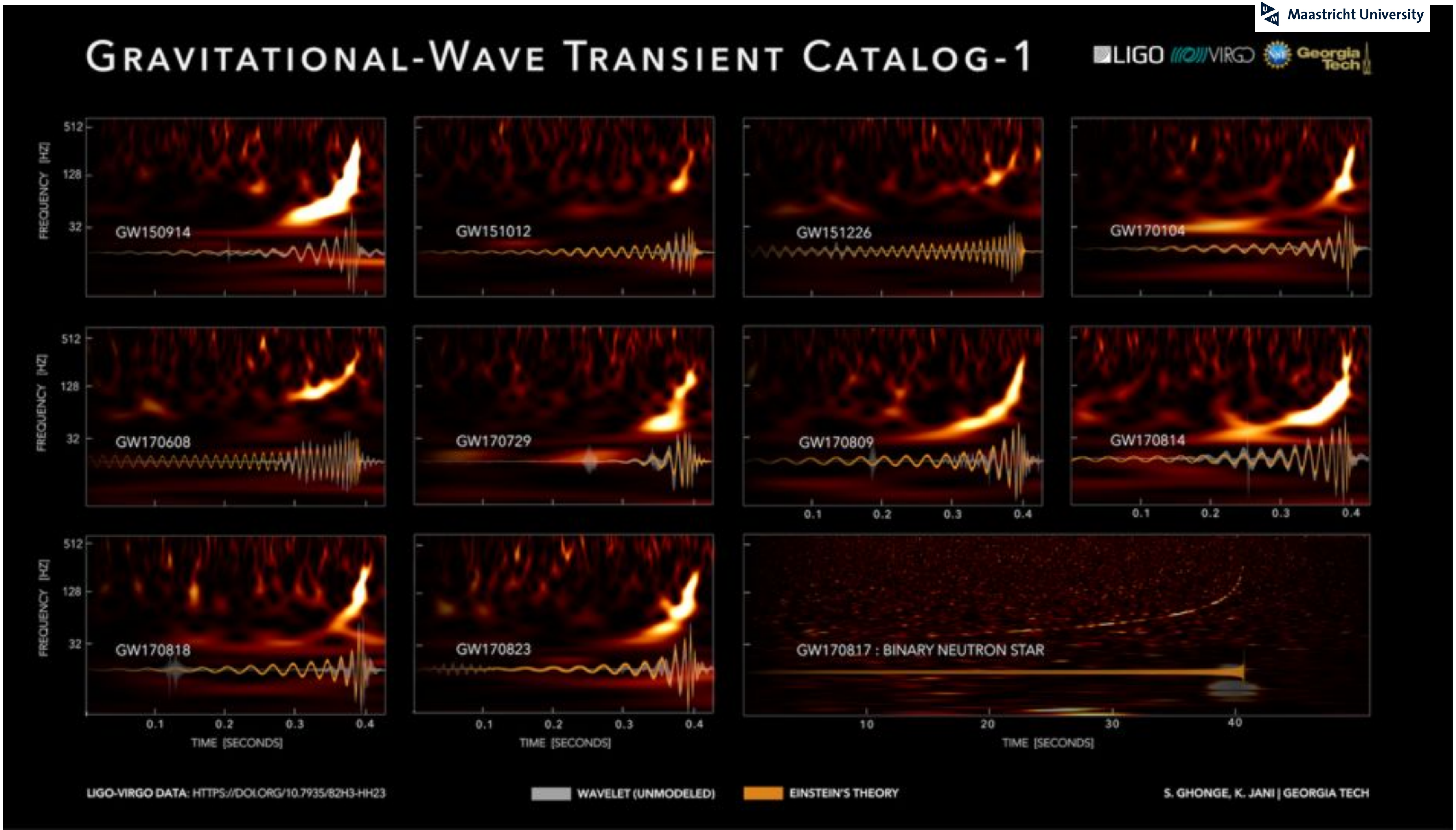


FIG. 1. Left panel: Amplitude spectral density of the total strain noise of the H1 and L1 detectors, $\sqrt{S(f)}$, in units of strain per $\sqrt{\text{Hz}}$, and the recovered signals of GW150914, GW151226, and LVT151012 plotted so that the relative amplitudes can be related to the SNR of the signal (as described in the text). Right panel: Time evolution of the recovered signals from when they enter the detectors' sensitive band at 30 Hz. Both figures show the 90% credible regions of the LIGO Hanford signal reconstructions from a coherent Bayesian

using a nonprecessing spin waveform model [48].

https://arxiv.org/pdf/gr-qc/0411146.pdf


GRAVITATIONAL-WAVE TRANSIENT CATALOG-1
LIGO VIRGO Georgia Tech
FREQUENCY [HZ]
512
128
32
GW150914
GW151012
GW151226
GW170104
GW170608
GW170729
GW170809
GW170814
GW170818
GW170823
GW170817 : BINARY NEUTRON STAR
0.1 0.2 0.3 0.4
10 20 30 40
TIME [SECONDS]
LIGO-VIRGO DATA: HTTPS://DOI.ORG/10.7935/82H3-HH23
WAVELET (UNMODELED)
EINSTEIN'S THEORY
S. GHONGE, K. JANI | GEORGIA TECH

References

Recommended Text Books:

- B. Schutz, 'A first course in general relativity'
- J.B. Hartle, 'Gravity: An introduction to Einstein's general relativity'
- M. Maggiore, 'Gravitational waves volume 1: theory and experiments'

Articles:

- Parameter estimation of inspiralling compact binaries using 3.5 post-Newtonian gravitational wave phasing: The non-spinning case (https://arxiv.org/pdf/gr-qc/0411146.pdf)
- Networks of gravitational wave detectors and three figures of merit - Bernard F Schutz https://hal.archives-ouvertes.fr/hal-00710461/document

Software : PyCBC https://pycbc.org

Tutorial: GWOSC https://www.gw-openscience.org/about/