ET4386 Estimation and Detection

Accelerometer Calibration

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I. Introduction

This project focuses on calibrating an accelerometer by applying various estimation techniques, such as the Maximum Likelihood Estimator (MLE), and evaluating the performance of these techniques using performance bounds like the Cramer-Rao Lower Bound (CRLB). The calibration process involves comparing the sensor's output with known reference values and adjusting the model parameters to minimize offsets. The accelerometer's output signal is modeled as a combination of a true acceleration signal and additive white Gaussian noise. The challenge is to estimate the unknown parameters, including scale factors, misalignment angles, and bias, using the provided model.

II. System Model

A. Accelerometer Sensor Model

The accelerometer's sensitivity axes are often slightly misaligned from the ideal orthogonal axes due to manufacturing errors. The transformation matrix $\mathbf{T}$ accounts for these small angular misalignments by adjusting the sensor readings to match the platform's coordinate system. The accelerometer also experiences scale factor errors, represented by a matrix $\mathbf{K}$, which describe inaccuracies in how much the sensor output changes in response to acceleration. Additionally, there is a bias $\mathbf{b}$ in the accelerometer readings, which is a constant error present even when no acceleration is applied. Nine calibration parameters consisting of three scale factors, three misalignment angles, and three biases can be collected into one parameter vector as

$$\boldsymbol{\theta} = [k_x, k_y, k_z, \alpha_{yz}, \alpha_{zy}, \alpha_{zx}, b_x, b_y, b_z]^T. \tag{1}$$

The measured output of the accelerometer can be written as

$$\mathbf{y}_k = \boldsymbol{\mu}_k + \mathbf{v}_k \tag{2}$$

where $\mathbf{y}_k$ is the output of the k-th measurement and $\mathbf{v}_k$ is zero-mean white Gaussian noise with variance σ^2. The signal $\boldsymbol{\mu}_k$ is defined as

$$\boldsymbol{\mu}_k = \boldsymbol{\mu}(\boldsymbol{\theta}, \mathbf{u}_k) = \mathbf{K}\mathbf{T}^{-1}\mathbf{u}_k + \mathbf{b} \tag{3}$$

where $\mathbf{u}_k$ is the input force at k-th measurement.

B. Input Force

Gravitational acceleration is used as the reference acceleration since it is easy to relate it to the sensor's basis with a single rotation matrix $\mathbf{R}_n^p$ to the platform's basis. Thus, the input force $\mathbf{u}_k$ to the system is simply

$$\mathbf{u}_k = \mathbf{R}_n^p \mathbf{g} = \mathbf{R}_n^p \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = \begin{bmatrix} -\sin\phi \\ \cos\phi \sin\varphi \\ \cos\phi\cos\varphi \end{bmatrix} \tag{4}$$

where g is the magnitude of the gravity vector, φ is the roll and ϕ is the pitch w.r.t. the navigation frame.

C. Calibration Procedure

The calibration procedure consists of setting the accelerometer's platform to M known orientations represented by

$$\boldsymbol{\eta} = [\varphi_1, \phi_1, \varphi_2, \phi_2, \ldots, \varphi_M, \phi_M]^T \tag{5}$$

and making N observations at each orientation. Note that each of the observations has the same input force $\mathbf{u}_k$ in a single orientation. Since the M orientations are known, the input force $\mathbf{u}_k$ is also known; therefore, the only unknown in $\boldsymbol{\mu}_k$ remaining is the nine calibration parameters $\boldsymbol{\theta}$, which can be estimated for $M \geq 9$. Increasing M and N allows for a more robust estimation.

D. Joint probability density

According to signal model (2), just like equation (13) in [1], the total measurement output $\tilde{\mathbf{y}}$ and the signal model $\tilde{\mu}$ can be defined as

$$\begin{aligned} \tilde{\mathbf{y}} &= \left[\mathbf{y}_1^T, \mathbf{y}_2^T, \ldots, \mathbf{y}_{MN}^T\right]^T, \\ \tilde{\boldsymbol{\mu}} &= \left[\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T, \ldots, \boldsymbol{\mu}_{MN}^T\right]^T. \end{aligned} \tag{6}$$

Thus, joint probability density function of $\tilde{\mathbf{y}}$ is given as

$$p(\tilde{\mathbf{y}}; \boldsymbol{\theta}) = \frac{1}{(2\pi)^{\frac{MN}{2}} |\mathbf{C}|^{\frac{1}{2}}} \exp\left(- \frac{\sum_{k=1}^{MN} ||\mathbf{y}_k - \boldsymbol{\mu}_k(\boldsymbol{\theta})||^2}{2\sigma^2} \right), \tag{7}$$

where

$$\mathbf{C} \triangleq \sigma^2 \mathbf{I}_{3MN}. \tag{8}$$

III. Maximum Likelihood Estimation

To estimate unknown parameters $\boldsymbol{\theta}$, we maximize the log-likehood function defined as

$$L(\boldsymbol{\theta}) = \ln p(\tilde{\mathbf{y}}; \boldsymbol{\theta}) \tag{9}$$

so that according to statistical model assumed, observations $\tilde{\mathbf{y}}$ are the most likely. Maximizing $L(\boldsymbol{\theta})$ is equivalent to minimizing

$$J(\boldsymbol{\theta}) = \sum_{k=1}^{MN} ||\mathbf{y}_k - \boldsymbol{\mu}_k(\boldsymbol{\theta})||^2 \tag{10}$$

It is useful to separate known and unknown variables in the objective function $J(\boldsymbol{\theta})$ before minimization as

$$\boldsymbol{\mu}_k = \mathbf{H}(\boldsymbol{\theta}) \begin{bmatrix} \mathbf{u}_k \\ 1 \end{bmatrix} \tag{11}$$

where $\mathbf{H}(\boldsymbol{\theta})$ is augmented matrix such that $\mathbf{H}(\boldsymbol{\theta}) = \left[\mathbf{K}\mathbf{T}^{-1}|\mathbf{b}\right]$. A simple way to find a minimizer $\hat{\boldsymbol{\theta}}$ can be done by reparametrization of $\mathbf{H}(\boldsymbol{\theta})$ so that the solution can be found by linear least squares estimation.

A. Reparametrization

By extending $\mathbf{H}(\boldsymbol{\theta})$ we can obtain

$$\mathbf{H}(\boldsymbol{\theta}) = \begin{bmatrix} k_x & k_x\alpha_{yz} & k_x(\alpha_{yz}\alpha_{zx} - \alpha_{zy}) & b_x \\ 0 & k_y & k_y\alpha_{zx} & b_y \\ 0 & 0 & k_z & b_z \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ 0 & h_{11} & h_{12} & h_{13} \\ 0 & 0 & h_{22} & h_{23} \end{bmatrix}. \tag{12}$$

Accordingly, $\boldsymbol{\mu}_k$ can be written as

$$\boldsymbol{\mu}_k = \begin{bmatrix} [\mathbf{u}_k]_0 h_{00} + [\mathbf{u}_k]_1 h_{01} + [\mathbf{u}_k]_2 h_{02} + h_{03} \\ [\mathbf{u}_k]_1 h_{11} + [\mathbf{u}_k]_2 h_{12} + h_{13} \\ [\mathbf{u}_k]_2 h_{22} + h_{23} \end{bmatrix} \tag{13}$$

where $[\mathbf{u}_k]_i$ represents the i-th element of vector $\mathbf{u}_k$. Equivalently, (13) can be written as a matrix vector product, i.e.,

$$\boldsymbol{\mu}_k = \begin{bmatrix} [\mathbf{u}_k]_0 & [\mathbf{u}_k]_1 & [\mathbf{u}_k]_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\mathbf{u}_k]_1 & [\mathbf{u}_k]_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\mathbf{u}_k]_2 & 1 \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{03} \\ h_{11} \\ h_{12} \\ h_{13} \\ h_{22} \\ h_{23} \end{bmatrix} = [\mathbf{A}]_{[3(k-1):3k,:]}\mathbf{h} \tag{14}$$

Then, the cost function in (10) can be written as a linear system of equations of $\mathbf{A}\mathbf{h} = \mathbf{b}$ as

$$\begin{bmatrix} [\mathbf{u}_1]_0 & [\mathbf{u}_1]_1 & [\mathbf{u}_1]_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\mathbf{u}_1]_1 & [\mathbf{u}_1]_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\mathbf{u}_1]_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\mathbf{u}_{MN}]_0 & [\mathbf{u}_{MN}]_1 & [\mathbf{u}_{MN}]_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\mathbf{u}_{MN}]_1 & [\mathbf{u}_{MN}]_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\mathbf{u}_{MN}]_2 & 1 \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{03} \\ h_{11} \\ h_{12} \\ h_{13} \\ h_{22} \\ h_{23} \end{bmatrix} = \begin{bmatrix} [\mathbf{y}_1]_0 \\ [\mathbf{y}_1]_1 \\ [\mathbf{y}_1]_2 \\ \vdots \\ [\mathbf{y}_{MN}]_0 \\ [\mathbf{y}_{MN}]_1 \\ [\mathbf{y}_{MN}]_2 \end{bmatrix} \tag{15}$$

whose solution in the least square sense is simply $\hat{\mathbf{h}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$ where $\mathbf{b}$ is the right hand side of the equation. Then the actual parameters θ can be obtained back using equation (12).

IV. Best Linear Unbiased Estimator

Using the reparametrized model, from the Gauss-Markov theorem, the BLUE for the reparametrized parameters $\mathbf{h}$ is obtained by minimizing the following expression:

$$\hat{\mathbf{h}}_{\text{BLUE}} = \arg\min_{\mathbf{h}} (\mathbf{b} - \mathbf{A}\mathbf{h})^T(\mathbf{b} - \mathbf{A}\mathbf{h}) \tag{16}$$

This is the least squares estimator, which solves the normal equation:

$$\mathbf{A}^T\mathbf{A}\hat{\mathbf{h}}_{\text{BLUE}} = \mathbf{A}^T\mathbf{b} \tag{17}$$

The solution to this is:

$$\hat{\mathbf{h}}_{\text{BLUE}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \tag{18}$$

Assuming the errors $\boldsymbol{\epsilon}$ are normally distributed, the likelihood function of the observations $\mathbf{b}$ is given by:

$$L(\mathbf{h}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{b} - \mathbf{A}\mathbf{h})^T(\mathbf{b} - \mathbf{A}\mathbf{h})\right) \tag{19}$$

Taking the logarithm of the likelihood function (log-likelihood):

$$\log L(\mathbf{h}, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{b} - \mathbf{A}\mathbf{h})^T(\mathbf{b} - \mathbf{A}\mathbf{h}) \tag{20}$$

To find the MLE for $\mathbf{h}$, we differentiate the log-likelihood with respect to $\mathbf{h}$ and set it to zero:

$$\frac{\partial \log L(\mathbf{h}, \sigma^2)}{\partial \mathbf{h}} = \frac{1}{\sigma^2}\mathbf{A}^T(\mathbf{b} - \mathbf{A}\mathbf{h}) = 0 \tag{21}$$

This leads to the same normal equation:

$$\mathbf{A}^T\mathbf{A}\hat{\mathbf{h}}_{\text{MLE}} = \mathbf{A}^T\mathbf{b} \tag{22}$$

Thus, the MLE for $\mathbf{h}$ is:

$$\hat{\mathbf{h}}_{\text{MLE}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \tag{23}$$

Since both the BLUE and the MLE give the same estimator for $\mathbf{h}$ in the Linear Gauss-Markov Model, i.e.,

$$\hat{\mathbf{h}}_{\text{BLUE}} = \hat{\mathbf{h}}_{\text{MLE}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \tag{24}$$

we conclude that **the BLUE is the same as the MLE** in this case.

V. Cramér-Rao Lower Bound

Assuming $p(\tilde{\mathbf{y}};\boldsymbol{\theta})$ satisfies the regularity condition, the variance of any unbiased estimator $\hat{\boldsymbol{\theta}}$ satisfies

$$\text{var}(\hat{\boldsymbol{\theta}}(i)) \geq [\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ii} \tag{25}$$

where $\hat{\boldsymbol{\theta}}(i)$ is the i-th element of $\hat{\boldsymbol{\theta}}$ and $[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ij}$ represents the element at the i-th row and j-th column of the inverse Fisher information matrix.

Using the knowledge that $\mathbf{u}_k$ is constant for during each of the M poses and the independence of $\mathbf{C}$ to $\boldsymbol{\theta}$ as shown in (8), the Fisher information matrix for $p(\tilde{\mathbf{y}};\boldsymbol{\theta})$ is

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \frac{N}{\sigma^2}\sum_{m=1}^{M}\left[\frac{\partial\boldsymbol{\mu}_m}{\partial\boldsymbol{\theta}(i)}\right]^T\left[\frac{\partial\boldsymbol{\mu}_m}{\partial\boldsymbol{\theta}(j)}\right]. \tag{26}$$

VI. Numerical Results

We have evaluated the method using Monte Carlo simulations. In the calibration process IMU was rotated to $M = 25$ known orientations, at each orientation $N = 30$ observations were made. Measurement noise variance is $\sigma^2 = 0.024$ $[m/s^2]$ and gravitational acceleration is $g = 9.80665$ $[m/s^2]$. The number of Monte-Carlo simulations is 500.

A. *Estimation of $\hat{\boldsymbol{\theta}}$ Using MLE*

The parameter vector $\hat{\boldsymbol{\theta}}_{\text{MLE}}$ is estimated by solving the system in (15) in the least-squares sense. Using all 500 Monte Carlo datasets, the averages and standard deviations of the estimated parameters, as shown in Table I, are computed. Furthermore, the empirical RMSE and the corresponding square root of the Cramér-Rao Lower Bound are calculated for varying numbers of observations, $N = [1, \ldots, 30]$, at each orientation. The resulting plots are presented in Figure 1.

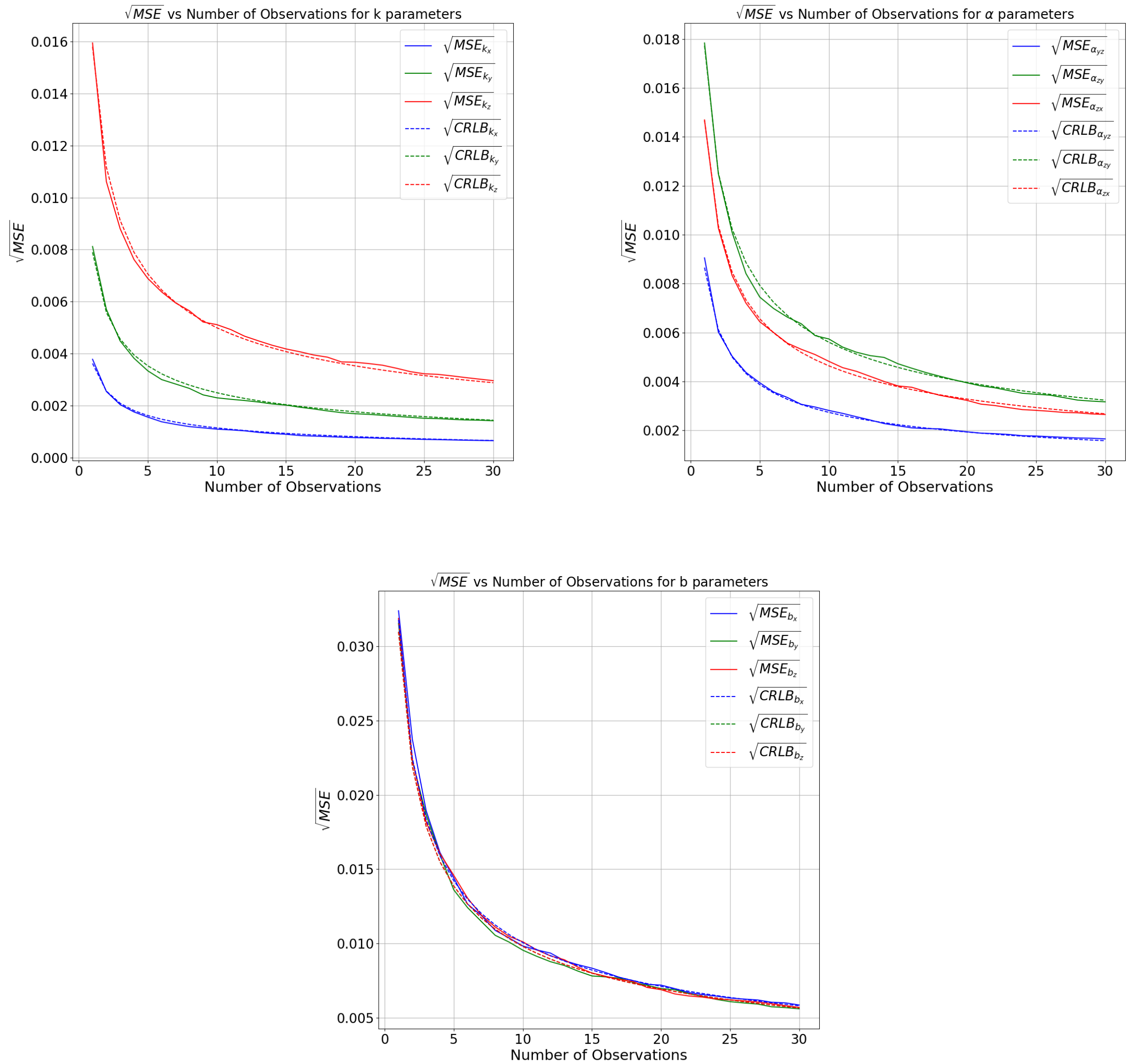

Figure 1: Square root of the (empirical) mean squared errors ($\sqrt{\text{MSE}}$) and square root of the Cramér-Rao Lower Bound ($\sqrt{\text{CRLB}}$) with respect to number of observations used at each orientation.

Calibration Parameter		**True Value**	**Average Value**	**Standard Deviation**
Scale Factors	k_x	0.92	0.919994	0.000655
	k_y	1.08	1.079923	0.001420
	k_z	1.10	1.100030	0.002963
Misalignment [rad]	α_{yz}	0.28	0.280039	0.001653
	α_{zy}	-0.19	-0.189923	0.003167
	α_{zx}	0.16	0.160091	0.002650
Biases $[m/s^2]$	b_x	0.42	0.419692	0.005843
	b_y	-0.67	-0.670073	0.005596
	b_z	0.50	0.500067	0.005689

Table I: The comparison of estimated and true values of calibration parameters and standard deviation of estimated parameters using 500 Monte-Carlo simulations, $N = 30$, $M = 25$, $\sigma^2 = 0.024$ $[m/s^2]$.

B. Cramér-Rao Lower Bound w.r.t. Noise Variance

We have plotted the Cramér-Rao Lower Bound (CRLB) by varying the noise variance σ^2 within the range $[0.01, 0.1]$ for $N = 30$ and $M = 25$, using the true parameter vector $\boldsymbol{\theta}$. The results are presented in Figure 2.

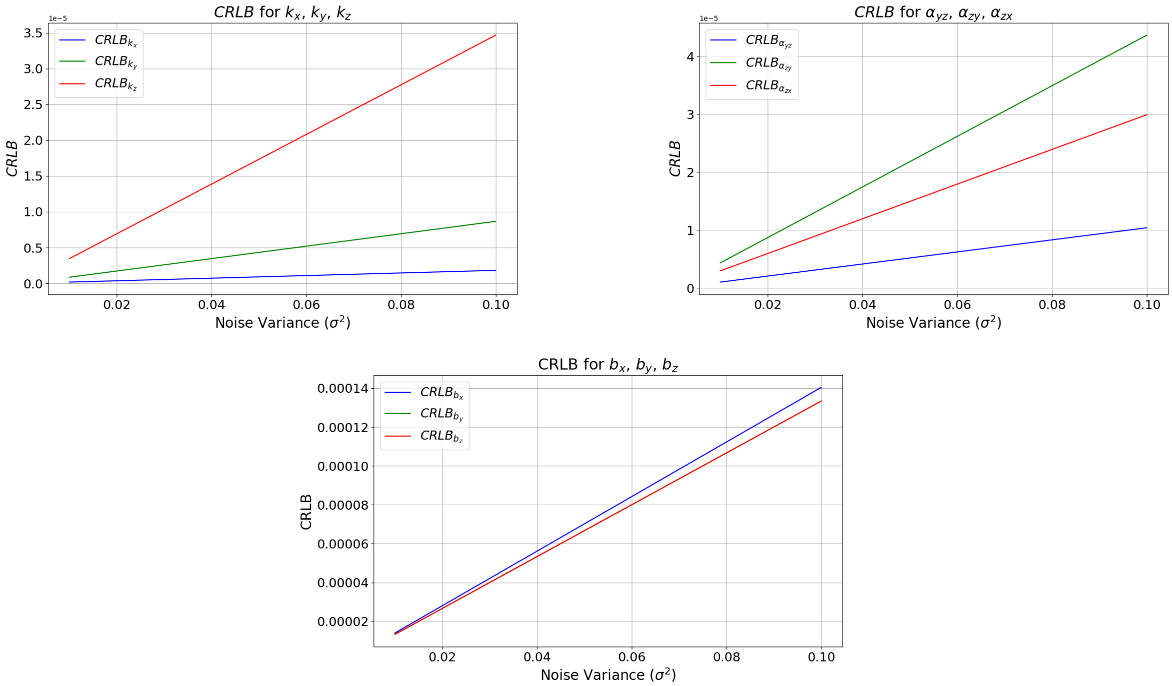

Figure 2: Cramér-Rao Lower Bound (CRLB)) with respect to the noise variance $\sigma^2 \in [0.01, 0.1]$.

VII. Discussion

A. Optimality of MLE

The transformations to obtain the reparametrized model $\mathbf{g}(\boldsymbol{\theta}) = \mathbf{h}$ are one-to-one functions, thus the true parameter estimates are also MLE estimates such that $\mathbf{g}^{-1}(\hat{\mathbf{h}}_{\text{MLE}}) = \hat{\boldsymbol{\theta}}_{\text{MLE}}$. For a linear Gauss-Markov model MLE is optimal because it is unbiased, achieving the lowest possible variance as defined by the CRLB. This optimality arises from the fact that the log-likelihood function of a Gaussian model is convex (10), ensuring the MLE converges to a global minimum. Therefore, by applying the invariance property of the MLE, optimality is attained.

B. Numerical Efficiency Evaluation of MLE

From Figure 1, it is evident that the empirical RMSE converges to the CRLB, indicating the efficiency of the MLE even for small values of N. This behavior is expected, as the system follows a Gauss-Markov model, for which MLE estimates are Minimum Variance Unbiased (MVU).

In Figure 1, the empirical RMSE is observed to occasionally dip below the CRLB. Theoretically, the CRLB provides a lower bound on the true Mean Squared Error (MSE), defined as the expected value of the squared errors, $\mathbb{E}[||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_{\text{true}}||_2^2]$. However, in practice, the MSE is approximated empirically by averaging the squared errors over a finite number of samples. Due to random fluctuations introduced by the finite sample size, the empirical MSE may temporarily fall below the CRLB. As the number of samples increases, the empirical MSE becomes a more accurate approximation of the true MSE and converges to the CRLB, consistent with theoretical expectations.

C. CRLB vs Noise Variance

According to (25) and (26), the CRLB is expected to be directly proportional to the noise variance σ^2, resulting in a linear relationship. This behavior is confirmed by Figure 2, which supports this claim.

Additionally, due to the structure of the Fisher information matrix, the CRLBs for the parameters b_y and b_z are identical.

References

[1] G. Panahandeh, I. Skog, and M. Jansson, "Calibration of the accelerometer triad of an inertial measurement unit, maximum likelihood estimation and cramér-rao bound," in *2010 International Conference on Indoor Positioning and Indoor Navigation*, 2010, pp. 1–6.