CS 412 Homework 2

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Question 1

You are part of a research team that focuses on a rare species of butterfly, whose pre-rain flight patterns have been observed to correlate with rainfall amounts. The rate at which the butterfly flaps its wings, *flutterness*, affects the rainfall amounts. You have developed a statistical model, called the *Flutter Distribution*, to encapsulate this phenomenon. This model leverages parameter α, to quantify the observed flutterness, x. The Flutter Distribution is delineated by the probability density function (PDF) where $x \in [0, 1]$ and $\alpha > 0$:

$$p(x|\alpha) = \alpha(1 - x)^{\alpha-1} \tag{1}$$

1.a Maximum Likelihood Estimate (MLE)

Based on a dataset $X = \{x_1, x_2, \ldots, x_n\}$ of flutterness measurements, determine is the Maximum Likelihood Estimate (MLE) the parameter α.

Solution

1. From the given the probability density function:

$$L(\alpha) = \prod_{i=1}^{n} p(x_i|\alpha) = \prod_{i=1}^{n} \alpha(1 - x_i)^{\alpha-1}. \tag{2}$$

2. Clearly, the likelihood function $L(\alpha)$ is not easy to maximize. Therefore, I will take the ln of the likelihood function:

$$\ell(\alpha) = \ln L(\alpha) = \sum_{i=1}^{n} \ln\left[\alpha(1 - x_i)^{\alpha-1}\right]. \tag{3}$$

3. To make it simpler I used some logarithmic rules. The simplified version of the $l(\alpha)$:

$$\ell(\alpha) = \sum_{i=1}^{n} \left[\ln(\alpha) + (\alpha - 1)\ln(1 - x_i)\right]. \tag{4}$$

$$\ell(\alpha) = n\ln(\alpha) + (\alpha - 1)\sum_{i=1}^{n} ln(1 - x_i). \tag{5}$$

4. Let the derivative of $\ell(\alpha)$ with respect to α:

$$\frac{d\ell(\alpha)}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1 - x_i). \tag{6}$$

5. To find the MLE for α, we set the derivative equal to zero and solve for α:

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1 - x_i) = 0. \tag{7}$$

$$\frac{n}{\alpha} = -\sum_{i=1}^{n} \ln(1 - x_i). \tag{8}$$

6. If I get α alone, it gives the MLE:

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \ln(1 - x_i)}. \tag{9}$$

Problem 1.b

Given the prior of α as $p(\alpha) = \lambda\alpha^{\lambda-1}e^{-\lambda\alpha}$ where $\lambda > 0$, find the Maximum a Posteriori (MAP) estimation of the parameter α based on $p(\alpha)$.

Solution

To find the MAP estimation for α, we need to maximize the posterior value which is proportional to multiplication of likelihood and prior value.

1. The likelihood function for α given the dataset is:

$$L(\alpha) = \prod_{i=1}^{n} p(x_i|\alpha) = \prod_{i=1}^{n} \alpha(1 - x_i)^{\alpha-1}. \tag{10}$$

2. The prior distribution for α is:

$$p(\alpha) = \lambda\alpha^{\lambda-1}e^{-\lambda\alpha}. \tag{11}$$

3. The posterior distribution is proportional to the likelihood times the prior:

$$p(\alpha|X) \propto L(\alpha) \cdot p(\alpha) = \left(\prod_{i=1}^{n} \alpha(1 - x_i)^{\alpha-1}\right) \lambda\alpha^{\lambda-1}e^{-\lambda\alpha}. \tag{12}$$

4. We need to take the ln of both sides for making calculation easier:

$$
\begin{aligned}
\ln(p(\alpha|X)) &\propto \ln\left(L(\alpha)\cdot p(\alpha)\right) &(13)\\
&= \sum_{i=1}^{n} \ln(\alpha(1-x_i)^{\alpha-1}) + \ln(\lambda\alpha^{\lambda-1}e^{-\lambda\alpha}) &(14)\\
&= \sum_{i=1}^{n} [\ln(\alpha) + (\alpha-1)\ln(1-x_i)] + (\lambda-1)\ln(\alpha) - \lambda\alpha. &(15)
\end{aligned}
$$

5. We need to take derivative of the both sides:

$$
\frac{d}{d\alpha}\log(p(\alpha|X)) = \sum_{i=1}^{n}\left[\frac{1}{\alpha} + \log(1-x_i)\right] + \frac{\lambda-1}{\alpha} - \lambda. \tag{16}
$$

6. We need to equal the equation to zero:

$$
0 = \sum_{i=1}^{n}\left[\frac{1}{\alpha} + \log(1-x_i)\right] + \frac{\lambda-1}{\alpha} - \lambda. \tag{17}
$$

7. When we leave alone α:

$$
\hat{\alpha}_{MAP} = \frac{n+\lambda-1}{-\sum_{i=1}^{n}\log(1-x_i)+\lambda}. \tag{18}
$$