ODEs (Chapter 1)

Lipschitz Condition

- We want to approximate solutions of problems in the form of

$$y' = f(t, y) \quad , \; t \geq t_0 \;\; \& \;\; y(t_0) = y_0$$

when we mean solving an ODE numerically.

Here, we need $f(t,y)$ be a sufficiently well-behaved function maps the space $[t_0, \infty) \times \mathbb{R}^D \to \mathbb{R}^D$. By the mean of well-behaved, we want to be ensure that at least, the f obeys given vector norm $\| \, \|$, and Lipschitz Condition which is

$$\|f(t,y) - f(t,x)\| < \lambda \|x-y\| \quad \forall x, y \in \mathbb{R}^D \;\&\; t \geq t_0 \;\&\; \lambda > 0.$$

The λ is called lipschitz constant & it is independent of the choice of x & y.

NOTE Check the proof that an ODE under the above conditions has a unique solution.

As a stronger condition, we can also assume that f is analytic. (The book will assume). In that case, there is a theory (check) that says the solution is also analytic.

NOTE

- Lipschitz condition is a strong form of uniform continuity.
- $\{\text{Continuously diff. functions}\} \subseteq \{\text{Functions obey Lipschitz Condition}\}$ (?)
- $f'(y) = \lim\limits_{x \to y} \dfrac{\|f(y) - f(x)\|}{\|x-y\|} < \lambda$, so $f'(y) < \lambda \quad \forall y \in \mathbb{R}^D$ (Every function with bounded first derivative obeys Lipschitz condition)

(The rate of change is bounded)

Euler's Method

Consider the ODE,

$$y' = f(t,y), \quad y(t_0) = y_0 \qquad t_0 > t \qquad (*)$$

We basically now that one can write $y(t)$ as

$$y(t) = y(t_0) + \int_{t_0}^{t} \frac{dy}{d\tau} \, d\tau \qquad \text{by the F.T.C}$$

By the given ODE,

$$y(t) = y(t_0) + \int_{t_0}^{t} f(\tau, y(\tau)) \, d\tau$$

Now, if we consider our integration region $[t_0, t]$ small enough, we can use the approximation

$$f(t, y(t)) \approx f(t_0, y(t_0)) \qquad \text{w/ } t \in [t_0, t_0+h] \text{ \& } h \text{ is small.}$$

Then we can approximate $y(t)$ as

$$y(t) = y(t_0) + (t - t_0) f(t_0, y_0)$$

Via this method, we can create a sequence of time as $t_0, t_0+h = t_1, t_1+h = t_2, \ldots$, and use the time sequence to calculate a sequence of $\{y_n\}$ which is an estimation of the exact solution $y(t_n)$

$$y_1 = y_0 + h f(t_0, y_0)$$

$$\vdots$$

$$y_{n+1} = y_n + h f(t_n, y_n), \quad n = 0, 1, 2, \ldots$$

Convergence of the solution is another must requirement for a numerical solution. Euler Method converges always but we need to know what convergence of a solution means actually.

Suppose we want to solve an ODE (*) with a Lipschitz function f in the compact interval $[t_0, t_0+t^*]$. We want to be sure that our numerical solution tends to exact solution as $h \to 0$.

Proposition: A method is said to be convergent if, for every ODE in the form (*) with a lipschitz function f and every $t^*>0$ it is true that

$$\lim_{h\to 0^+} \max_{n=0,1,\dots,\lfloor t^*/h \rfloor} \|y_{n,h} - y(t_n)\| = 0$$ w/ $\lfloor t^*/h \rfloor$ is the integer part of t^*/h.

NOTE

(?)

(1) $y_n = y_{n,h}$, $n = 0, 1, \dots, \lfloor t^*/h \rfloor$ (I think this is to say that I should choose the length $\{n\}$ at least as long as $\lfloor t^*/h \rfloor$)

(2) $$\lim_{h\to 0^+} \max_{n=0,1,\dots,\lfloor t^*/h \rfloor} \|y_{n,h} - y(t_n)\| = 0$$ (What does the max mean here? Check this with python)

Theorem: Euler's Method is convergent.

Proof: Firstly, for the given $h>0$, lets define $e_{n,h} = y_{n,h} - y(t_n)$ $= y_n, n=0,\dots,\lfloor t^*/h \rfloor$. Let say f is analytic, so does the solution $y(t)$. Then, by the Taylor Exp., we have

$$y(t_{n+1}) = y(t_n) + y'(t_n)h + O(h^2)$$

$$y(t_{n+1}) = y(t_n) + f(t_n, y(t_n)) + O(h^2)$$

Then substract this from the E.M solution in $[t_0, t_0+t^*]$

$$y_{n+1} - y(t_{n+1}) = y_n - y(t_n) + h\left(f(t_n, y_n) - f(t_n, y(t_n))\right) + O(h^2)$$

$y_n = y(t_n) + e_{n,h}$

$$\Rightarrow \quad e_{n+1,h} = e_{n,h} + h\left[f(t_n, y(t_n)+e_{n,h}) - f(t_n, y(t_n))\right] + O(h^2)$$

Now, consider $O(h^2)$. Because y is analytic, so continuously diff, $O(h^2)$ can be bounded for all $h>0$ & $n < \lfloor t^*/h \rfloor$ by a term ch^2, $c>0$.

$\mathcal{O}(h^2) \leq ch^2$

$$\|e_{n+1,h}\| \leq \|e_{n,h}\| - h\|f(t_n, y(t_n)+e_{n,h}) - f(t_n, y(t_n))\| + ch^2$$

By the Lipschitz condition, $\exists \lambda > 0$ s.t

$$\|e_{n+1,h}\| \leq \|e_{n,h}\| + h\lambda \|y_n - y(t_n)\| + ch^2$$

$$\|e_{n+1,h}\| \leq (1+h\lambda)\|e_{n,h}\| + ch^2 \quad , n = 0, 1, \dots, \lfloor t^*/h \rfloor - 1$$

Claim

$$\|e_{n,h}\| \leq (c/\lambda) h [(1+h\lambda)^n - 1]$$ (proved by induction)

, so

$$\|e_{n+1,h}\| \leq (1+h\lambda) \frac{c}{\lambda} h[(1+h\lambda)^n - 1] + ch^2$$

$$\begin{aligned}
(1+h\lambda)\frac{c}{\lambda}h[(1+h\lambda)^n - 1] + ch^2 &= \frac{ch}{\lambda}[(1+h\lambda)^{n+1} - (1+h\lambda)] + \frac{ch}{\lambda}\lambda h \\
&= \frac{ch}{\lambda}(1+h\lambda)^{n+1} - \frac{ch}{\lambda} - ch^2 + ch^2 \\
&= \frac{ch}{\lambda}[(1+h\lambda)^{n+1} - 1]
\end{aligned}$$

$$\Rightarrow \|e_{n+1,h}\| \leq \frac{c}{\lambda}[(1+h\lambda)^{n+1} - 1] \cdot h$$

$h\lambda > 0 \Rightarrow 1 + h\lambda \leq e^{h\lambda}$

$$\Rightarrow (1+h\lambda)^n \leq e^{nh\lambda} \quad \text{w/ } n \in \{0, 1, \dots, \lfloor t^*/h \rfloor\}$$

so $e^{nh\lambda} \leq e^{\lfloor t^*/h \rfloor h\lambda} \leq e^{t^*\lambda}$

$$\Rightarrow \|e_{n+1,h}\| \leq \frac{c}{\lambda}(e^{t^*\lambda} - 1)h$$

These steps are done to reduced the dependences on the h.

Consequently,

$$\lim_{h \to 0^+} \frac{c}{\lambda}(e^{t^*\lambda} - 1)h = 0$$

$$\Rightarrow \lim_{\substack{h \to 0^+ \\ 0 \leq nh \leq t^*}} \|e_{n+1,h}\| = 0$$

$\blacksquare$

α The important result from the proof is that the error is actually bounded.

$$\| e_{n+1,h} \| \leq \frac{c}{\lambda}\left(e^{t^* \lambda} - 1\right) h \quad \forall n \in \{0, 1, \ldots, \lfloor t^*/h \rfloor\}$$

However, this boundry is much bigger than the actual boundry of the error.

If we write the exact solution in the E.M solution, we get

$$y(t_{n+1}) - \left[y(t_n) + h f(t_n, y(t_n)) \right] = 0$$

By the Taylor Exp.

$$\left[y(t_n) + h f(t_n, y(t_n)) + \mathcal{O}(h^2) \right] - \left[y(t_n) + h f(t_n, y(t_n)) \right] = \mathcal{O}(h^2)$$

This is to say that the Euler-Method is order of one. In general, for a given numerical method,

$$y_{n+1} = \mathcal{Y}(f, h, y(t_0), y(t_1), \ldots, y(t_n))$$

The method is called order of p if

$$y_{n+1} - \mathcal{Y}(f, h, \ldots, y(t_n)) = \mathcal{O}(h^{p+1})$$

For a order of p method, the local error (error between t_{n+1} & t_n) decays as $\mathcal{O}(h^{p+1})$ (for sufficiently small h, $\| e_n \| \sim \mathcal{O}(h^{p+1})$). The # of steps increases as $\mathcal{O}(h^{-1})$. However, our main interest is not the local but global behaviour of the method. The naive expectation is that the global error decreases as $\mathcal{O}(h^p)$ but it is not true for all methods & shouldn't be taken granted. However, the global error of the Euler's method decreases as $\mathcal{O}(h)$.

NOTE This error discussion is very important but I couldn't understand it well. I also review the $\mathcal{O}$ thing.

Trapezoidal Rule

- A better approximation for the solution

$$y(t) = y(t_n) + \int_{t_n}^{t} f(\tau, y(\tau))\, d\tau$$

is to taking average of the function values at the integral limits

$$y(t) \approx y(t_n) + \frac{1}{2}(t - t_n)\left[f(t, y(t)) + f(t_n, y(t_n)) \right]$$

This approximation is called trapezoidal rule

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]$$

- Order of the trapezoidal Rule is

$$y(t_{n+1}) - \left[y(t_n) + \frac{1}{2} h \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right] \right] = 0$$

$$y(t_n) + h f(t_n, y_n) + \frac{1}{2} h^2 y''(t_n) + \mathcal{O}(h^3) - \left[y(t_n) + \frac{1}{2} h \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right] \right]$$

$$y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1})) = y'(t_n) + h y''(t_n) + \mathcal{O}(h^2)$$

$$\Rightarrow \cancel{y(t_n)} + \cancel{h y'} + \cancel{\frac{1}{2} h^2 y''} + \mathcal{O}(h^3) - \left[\cancel{y(t_n)} + \cancel{\frac{1}{2} h y'} + \frac{1}{2} h \left[\cancel{y'} + \cancel{h y''} + \mathcal{O}(h^2) \right] \right]$$

$$= \mathcal{O}(h^3)$$

, so order of trapezoid rule is two.

Theorem: Trapezoidal Rule is Convergent.

Proof:

$$y_{n+1} = y_n + \tfrac{1}{2} h \left(f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right)$$

$$y(t_{n+1}) = y(t_n) + \tfrac{1}{2} h \left(f(t_n, y(t_n)) - f(t_{n+1}, y(t_{n+1})) \right) + \mathcal{O}(h^3)$$

$$\Rightarrow e_{n+1} = e_n + \frac{1}{2} h \left[f(t_n, y_n) - f(t_n, y(t_n)) \right] + \frac{1}{2} h \left[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y(t_{n+1})) \right] + \mathcal{O}(h^3)$$

by the Lipschitz Condition,

$$\|e_{n+1}\| \leq \|e_n\| + \frac{1}{2} h \lambda \|e_n\| + \frac{1}{2} h \lambda \|e_{n+1}\| + \mathcal{O}(h^3)$$

$O(h^3) \le ch^3$ w/ $c > 0$

$$\Rightarrow \left(1 - \tfrac{1}{2}h\lambda\right)\|e_{n+1,h}\| \le \left(1 + \tfrac{1}{2}h\lambda\right)\|e_{n,h}\| + Ch^3$$

$$\Rightarrow \|e_{n+1,h}\| \le \left(\frac{1 + 1/2\, h\lambda}{1 - 1/2\, h\lambda}\right)\|e_{n,h}\| + \frac{c}{(1 - \frac{1}{2}h\lambda)} h^3$$

?? Somehow $\|e_{n,h}\| \le \frac{c}{\lambda}h^2\left[\left(\frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda}\right)^n - 1\right] + \frac{Ch^2}{\lambda}$

$$\Rightarrow \|e_{n+1,h}\| \le \frac{ch^2}{\lambda}\left(\frac{1 + 1/2\, h\lambda}{1 - 1/2\, h\lambda}\right)^{n+1} + \frac{1}{1 - 1/2\, h\lambda}\left(C\cancel{h^3} - \frac{ch^2}{\lambda} - \frac{1}{2}\cancel{ch^3}\right)$$

$$\le \frac{Ch^2}{\lambda}\left(\frac{1 + 1/2\, h\lambda}{1 - 1/2\, h\lambda}\right)^{n+1} + \frac{1}{1 - 1/2\, h\lambda}\left(1 - 1/2\, h\lambda\right)\left(\frac{Ch^2}{\lambda}\right)$$

or Choice of this.

?? (?) There is sth. wrong with the algebra. I think it is wrong & it is left as an exercise. I couldn't do it!!!

$$\|e_{n+1,h}\| \le \frac{Ch^2}{\lambda}\left(\frac{1 + \frac{1}{2}h\lambda}{1 - 1/2\, h\lambda}\right)^{n+1}$$

$$\Rightarrow \|e_{n,h}\| \le \frac{Ch^2}{\lambda}\left(\frac{1 + 1/2\, h\lambda}{1 - 1/2\, h\lambda}\right)^{n}$$

Because we will look the inequality as $h \to 0$, we can say $0 < nh < 2$ by the archimedian property. This restriction on "nh" allows us to say

$$\left(\frac{1 + 1/2\, h\lambda}{1 - 1/2\, h\lambda}\right) = 1 + \frac{h\lambda}{1 - 1/2\, h\lambda} \le \sum \frac{1}{l!}\left(\frac{h\lambda}{1 - 1/2\, h\lambda}\right)^n = \exp\left(\frac{h\lambda}{1 - 1/2\, h\lambda}\right)$$

, so we have

$$\|e_{n,h}\| \le \frac{ch^2}{\lambda}\exp\left(\frac{nh\lambda}{1 - 1/2\, h\lambda}\right)$$

we have the condition $n \in \{0, 1, \dots, \lfloor t^*/h \rfloor\}$, so $nh \le t^*$

$$\Rightarrow \|e_{n,h}\| \le \frac{Ch^2}{\lambda}\exp\left(\frac{t^*\lambda}{1 - \frac{1}{2}h\lambda}\right)$$

$$\Rightarrow \lim_{\substack{h \to 0 \\ 0 \le nh \le t^*}} \|e_{n,h}\| = 0 \qquad \boxtimes$$

Because the trapezoid method converges, we can say that the global error of the method is decreasing as $O(h^2)$.

* Another important difference between E.M & T.R. is that T.R is an implicit method whereas the E.M is explicit. This means that we can directly calculate "y_{n+1}" using the data & f but this is not possible for the T.R because the right-hand side also depends on y_{n+1}.

$$y_{n+1} - \frac{1}{2} h f(t_{n+1}, y_{n+1}) = \underbrace{y_n + \frac{1}{2} h f(t_n, y_n)}_{\vec{v}}$$

Hence, to approximate the solution, we should solve the algebraic equation

$$y_{n+1} - \tfrac{1}{2} h f(t_{n+1}, y_{n+1}) = \vec{v} \quad \text{in each step.}$$

NOTE ⁉

I couldn't understand the error discussion in the first chapter. What does it mean to say that error decreases with h or $O(h)$. However, the discussion tells that a convergent method of order p, we have $\|e\| = ch^p$, so $\ln|e| = \ln c + p \ln(h)$. For two different step sizes, say h_1 & h_2. The corresponding errors e_1, e_2 are related as

$$\|e_2\| \approx \frac{\|e_1\|}{(h_2/h_1)^p}, \quad \text{so} \quad \ln\|e_2\| \approx \ln\|e_1\| - p \ln(h_2/h_1)$$

NOTE

In the discussion about the figure, $h_1 = 10^{-2}$ & $h_2 = 10^{-3}$. Then, for the last error, $e_1 = 5 \times 10^{-2}$ & $e_3 = 5 \times 10^{-3}$. The book says error decays as $O(h)$.

Then, I think by the saying error decreases as $\mathcal{O}(h^p)$, we mean that if $\left(\frac{h_2}{h_1}\right) \approx 10^{-n}$ then $\left(\frac{e_2}{e_1}\right) \approx 10^{-p}$

Above, we bounded $\mathcal{O}(h^p)$ with ch^p. $(\mathcal{O}(h^p) \leq ch^p)$. If we define $\|e\| \approx ch^p$ then what I say above make sense. However, it is said in the book that the boundary ch^p is a very long bound which makes it useless. However, in the examples the error seems pretty much like that. I didn't understand this.

$$\|e\| \approx ch^2$$ (See the results in the Jupyter Notebook)

Implicit Midpoint Method

$$y_{n+1} = y_n + h f\left(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y_n + y_{n+1})\right)$$

This is a second order convergent Method. (Prove This). This is also special form of the Runge-Kutta Method.

The Theta Method

- The Euler M. & Trapezoid Rule are special form of the theta method.

$$y_{n+1} = y_n + h\left(\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right), \quad \theta \in [0,1]$$

↳ $\theta = 1 \Rightarrow$ E.M & only explicit method

↳ $\theta = 1/2 \Rightarrow$ T.R

∝ Order of the the theta method:

$$y(t_{n+1}) - y(t_n) - h\left[\theta y'(t_n) + (1-\theta) y'(t_{n+1})\right] = 0$$

$$\left[\cancel{y(t_n)} + h y'(t_n) + \tfrac{1}{2}h^2 y''(t_n) + \tfrac{1}{6}h^3 y'''(t_n)\right] - \cancel{y(t_n)} - h\left[\theta y'(t_n) + (1-\theta)\left[y'(t_n) + h y''(t_n) + \tfrac{1}{2}h^2 y^{(3)}\right]\right] + \mathcal{O}(h^4)$$

$$= h y'(t_n)(1-\theta+1-\theta) + h^2\left(\tfrac{1}{2} - (1-\theta)\right) y'' + y''' h^3\left(\tfrac{1}{6} - (1-\theta)\tfrac{1}{2}\right) + \mathcal{O}(h^4)$$

$$= h^2\left(\tfrac{1}{2} - \theta\right) y'' + h^3\left(\tfrac{\theta}{2} - \tfrac{1}{3}\right) y''' + \mathcal{O}(h^4) \quad (*)$$

, so the method is order of 2 if $\theta = 1/2$. Otherwise, it is always order of one.

∝ The θ method always convergent for $\theta \in [0,1]$. (Check the error for a first order implicit method.)

∝ Error Analysis of Theta Method:

$$y_{n+1} - y_n - h\left[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right] = 0$$

by Extracting the $y(t_{n+1}) - \mathcal{Y}(t_n, y(t_n), f) = (*)$, we get

$$e_{n+1} - e_n - h\theta\left[f(t_n, y_n) - f(t_n, y(t_n))\right] - h(1-\theta)\left[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y(t_{n+1}))\right]$$

$$e_{n+1} - e_n - h\theta\left[f(t_n, y(t_n)+e_n) - f(t_n, y(t_n))\right] - h(1-\theta)\left[f(t_{n+1}, y(t_{n+1})+e_{n+1}) - f(t_{n+1}, y(t_{n+1}))\right]$$

$$= \begin{cases} -\tfrac{1}{12} h^3 y'''(t_n) + \mathcal{O}(h^4) & , \theta = 1/2 \\ (\theta - 1/2) h^2 y''(t_n) + \mathcal{O}(h^3) & , \theta \neq 1/2 \end{cases}$$

For sufficiently small "h", by the implicit function theorem, we get

$$e_{n+1} - e_n = \begin{cases} -\tfrac{1}{12} h^3 y'''(t_n) + \mathcal{O}(h^4) & , \theta = 1/2 \\ (\theta - 1/2) h^2 y''(t_n) + \mathcal{O}(h^3) & , \theta \neq 1/2 \end{cases}$$

NOTE $\theta = 0 \Rightarrow y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$ is called backward euler's method. This method specifically useful for solving stiff ODE's.