

# Finite Difference Method

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## 1 Calculus of Finite Difference

The motivation behind the calculus of finite differences is basically replacing derivatives with a linear combination of discrete function values.

We can consider the finite differences as a sequence of  $\{z_k\}$ , which can be complex or real. Then we introduce some operators on elements of this sequence called finite different operators.

Consider the element  $z_k$  as a function  $z(kh)$  where the  $k$  is the step, and  $h$  is the length of equidistance discretization. Then, we can define the differentiation operator as

$$D(z_k) = z'(kh) \quad (1)$$

Our purpose is to write  $D$  in terms of the other operators. To achieve that purpose, we first need to define functions of operators as  $g\mathcal{T}$ . We will act such functions as if they are analytic functions, so they will have power series representation.

$$g(\mathcal{T}) = \sum_{i=0}^{\infty} a_i (T^i z)$$

Also, we need to define a half-shift operator to deal with fractional indexes.

$$\mathcal{E}^{\frac{1}{2}} z_k = z_{k+\frac{1}{2}} \quad (2)$$

Now consider  $\mathcal{E}$  acts on  $z(x)$

$$\mathcal{E} z(x) = z(x + h) \quad (3)$$

| Operator                              | Operation   |
|---------------------------------------|---|
| Shift Operator $:= \mathcal{E}$       | $\mathcal{E}(Z_k) = z_{k+1}$  |
| Forward Operator $:= \Delta_+$        | $\Delta_+(z_k) = z_{k+1} - z_k$   |
| Backward Operator $= \Delta_-$        | $\Delta_-(z_k) = z_k - z_{k+1}$   |
| Central Difference $:= \Delta_0$      | $\Delta_0(z_k) = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$                   |
| Averaging Operator $:= \mathcal{T}_0$ | $\mathcal{T}_0(z_k) = \frac{1}{2}(z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}})$ |

Table 1: Table of the Operators

Then expansion of the  $z(x)$  gives

$$\mathcal{E}z(x) = \sum \frac{1}{n!} \frac{d^n z}{dt^n} h^n \quad (4)$$

Replace the derivative with the operator  $D$

$$\mathcal{E}z = \left( \sum \frac{1}{n!} (Dh)^n \right) z \quad (5)$$

Notice that there is the power series expansion of  $e^x$  in the RHS.

$$\mathcal{E}z = e^{(Dh)} z \quad (6)$$

$$\ln(\mathcal{E}) = Dh \quad (7)$$

$$\Rightarrow D = \frac{1}{h} \ln \mathcal{E} \quad (8)$$

Now one should notice that the basic operators can be written as linear combinations of the others. Three important representation of the  $\mathcal{E}$  are

$$\mathcal{E} = \Delta_+ + I \quad (9)$$

$$\mathcal{E} = (\Delta_- + I)^{-1} \quad (10)$$

$$(11)$$

The other one is the result of the following equation

$$\Delta_0 = \mathcal{E}^{\frac{1}{2}} - \mathcal{E}^{\frac{-1}{2}} \quad (12)$$

$$\Rightarrow \mathcal{E}^{\frac{1}{2}} \Delta_0 = (\mathcal{E}^{\frac{1}{2}})^2 - I \quad (13)$$

$$\mathcal{E}^{\frac{1}{2}} = \frac{\Delta_0}{2} + \sqrt{\frac{\Delta_0^2}{4} + I} \quad (14)$$

Then by taking the square of (14)

$$\mathcal{E} = \left( \frac{\Delta_0}{2} + \sqrt{\frac{\Delta_0^2}{4} + I} \right)^2 \quad (15)$$

Finally, we use these definitions of  $\mathcal{E}$  in the equation (8) then Taylor expands the resulting function to obtain the differentiation operator. With definition (9), we get

$$D = \frac{1}{h} [\Delta_+ - \frac{1}{2} \Delta_+^2 + \frac{1}{3} \Delta_+^3] + \mathcal{O}(h^3) \quad (16)$$

and in general

$$D^s = \frac{1}{h^s} [\Delta_+^s - \frac{1}{2} s \Delta_+^{s+1} + \frac{s(3s+5)}{24} \Delta_+^{s+2}] + \mathcal{O}(h^3) \quad (17)$$

With this operator, we can differentiate a function up to an order  $\mathcal{O}(h^3)$ . Also, with the same procedure, one can find a formula with the  $\Delta_-$  operator. The actual problem is to choose one. One should notice that the definition with the  $\Delta_+$  operator approximates the derivation using the elements  $\{z_k, z_{k+1}, \dots, z_{k+s+2}\}$ . The  $\Delta_-$  case on the other hand does the approximation using the points  $\{z_k, z_{k-1}, \dots, z_{k-s-2}\}$ . The problem with such approximations is the fact that each operation uses only one side of the grids. A more proper approximation should have used both sides. This can be achieved by using the definition (15).

It gives the operator as

$$D = \frac{2}{h} \ln \left( \frac{\Delta_0}{2} + \sqrt{I^2 + \left( \frac{\Delta_0}{4} \right)^2} \right) \quad (18)$$

To obtain a useful operation out of (18), let's first consider the function

$$g(\epsilon) = \ln(\epsilon + \sqrt{1 + \epsilon^2}) \quad (19)$$

$$\Rightarrow g'(\epsilon) = \frac{1}{\sqrt{1 + \epsilon^2}} = \sum (-1)^j \binom{2j}{j} \left( \frac{\epsilon}{2} \right)^{2j} \quad (20)$$

$g(0) = \ln(1) = 0$ . By the fundamental identity,

$$g(\epsilon) = g(0) + \int_0^\epsilon g'(t) dt \quad (21)$$

$$= 2 \sum \frac{(-1)^j}{2j+1} \binom{2j}{j} \left( \frac{\epsilon}{2} \right)^{2j+1} \quad (22)$$

$D = \frac{2}{h} g\left(\frac{\Delta_0}{2}\right)$ . Then the operation becomes

$$D = \frac{4}{h} \sum \frac{(-1)^j}{2j+1} \binom{2j}{j} \left( \frac{\Delta_0}{4} \right)^{2j+1} \quad (23)$$

The problem with this operator is all of the terms include odd powers of the central difference operator. Remember that  $\Delta_0$  cannot map  $\mathbb{R}^z \rightarrow \mathbb{R}^z$ . However,  $\Delta_0^2$  does.

$$\Delta_0^2 = \Delta_0(z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}) \quad (24)$$

$$= z_{k+1} - 2z_k + z_{k-1} \quad (25)$$

Hence, we should consider the square of the operator (23). If we take the square and open the series up to some order, we get the following formula for the  $2s$  derivative as

$$D^{2s} = \frac{1}{h^{2s}} \left[ \Delta_0^{2s} - \frac{\Delta_0^{2s+2}}{12} + \frac{\Delta_0^{2s+4}}{45} \right] + \mathcal{O}(h^6) \quad (26)$$

We can obtain odd derivatives by defining an identity using the  $\mathcal{T}_0$  operator.

$$\mathcal{T}_0 = \frac{1}{2}(\mathcal{E}^{\frac{1}{2}} + \mathcal{E}^{-\frac{1}{2}}) \quad (27)$$

$$\Rightarrow \mathcal{T}_0^2 = \frac{1}{4}(\mathcal{E} + \mathcal{E}^{-1} + 2I) \quad (28)$$

$\Delta_0^2 = \mathcal{E} + \mathcal{E}^{-1} - 2I$ . Then some manipulation on (28) and  $\Delta_0^2$  gives following

$$\mathcal{T}_0 = \sqrt{I + \frac{\Delta_0^2}{4}} \Rightarrow I = \mathcal{T}_0 \left( I + \frac{\Delta_0^2}{4} \right)^{-\frac{1}{2}} \quad (29)$$

By the (20),

$$\left( I + \frac{\Delta_0^2}{4} \right)^{-\frac{1}{2}} = \sum (-1)^j \binom{2j}{j} \left( \frac{\Delta_0^2}{16} \right)^j \quad (30)$$

Consequently, we get the following identity

$$I = \mathcal{T}_0 \sum (-1)^j \binom{2j}{j} \left( \frac{\Delta_0^2}{16} \right)^j \quad (31)$$

Multiplying this identity with the differentiation operator (23) gives the following operator

$$D = \mathcal{T}_0 \Delta_0 \left( \sum (-1)^j \binom{2j}{j} \left( \frac{\Delta_0^2}{16} \right)^j \right) \left( \sum (-1)^i \binom{2i}{i} \left( \frac{\Delta_0^2}{16} \right)^i \right) \quad (32)$$

One can expand the series up to some order to get a useful differential operator. However, this is computationally wasteful and time consuming. For example, to get  $\mathcal{O}^4$  order approximation, one will need seven grid points. A better way to get the  $D^{2s+1}$  operator is open the (23) to an odd order and then multiply it with the identity (29). This gives the following order six formula

$$D^{2s+1} = \frac{1}{h^{2s+1}} (\mathcal{T}_0 \Delta_0) [\Delta_0^{2s} - \frac{(s+3)(5s+16)\Delta_0^{2s+2}}{12} + \frac{(s+3)(5s+16)\Delta_0^{2s+4}}{1440}] + \mathcal{O}(h^6) \quad (33)$$

## 2 The Five Point Formula

$$D = \frac{-1}{h} [\Delta_- - \frac{1}{2}\Delta_-^2 + \frac{1}{3}\Delta_-^3] + \mathcal{O}(h^3) \quad (34)$$

$$D^s = \frac{-1}{h^s} [\Delta_-^s - \frac{1}{2}s\Delta_-^{s+1} + \frac{s(3s+5)}{24}\Delta_-^{s+2}] + \mathcal{O}(h^3) \quad (35)$$