Finite Difference Method

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1 Calculus of Finite Difference

The motivation behind the calculus of finite differences is basically replacing derivatives with a linear combination of discrete function values.
We can consider the finite differences as a sequence of $\{z_k\}$, which can be complex or real. Then we introduce some operators on elements of this sequence called finite different operators.

Consider the element z_k as a function $z(kh)$ where the k is the step, and h is the length of equidistance discretization. Then, we can define the differentiation operator as

$$D(z_k) = z'(kh) \tag{1}$$

Our purpose is to write D in terms of the other operators. To achieve that purpose, we first need to define functions of operators as $g\mathcal{T}$. We will act such functions as if they are analytic functions, so they will have power series representation.

$$g(\mathcal{T}) = \sum_{i=0}^{\infty} a_i(T^i z)$$

Also, we need to define a half-shift operator to deal with fractional indexes.

$$\mathcal{E}^{\frac{1}{2}} z_k = z_{k+\frac{1}{2}} \tag{2}$$

Now consider $\mathcal{E}$ acts on $z(x)$

$$\mathcal{E} z(x) = z(x+h) \tag{3}$$

Operator	Operation
Shift Operator := $\mathcal{E}$	$\mathcal{E}(Z_k) = z_{k+1}$
Forward Operator := Δ_+	$\Delta_+(z_k) = z_{k+1} - z_k$
Backward Operator = Δ_-	$\Delta_-(z_k) = z_k - z_{k+1}$
Central Difference := Δ_0	$\Delta_0(z_k) = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$
Averaging Operator := $\mathcal{T}_0$	$\mathcal{T}_0(z_k) = \frac{1}{2}(z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}})$

Table 1: Table of the Operators

Then expansion of the $z(x)$ gives

$$\mathcal{E}z(x) = \sum \frac{1}{n!}\frac{d^n z}{dt^n}h^n \tag{4}$$

Replace the derivative with the operator D

$$\mathcal{E}z = \left(\sum \frac{1}{n!}(Dh)^n\right) z \tag{5}$$

Notice that there is the power series expansion of e^x in the RHS.

$$\mathcal{E}z = e^{(Dh)}z \tag{6}$$
$$ln(\mathcal{E}) = Dh \tag{7}$$
$$\Rightarrow D = \frac{1}{h}ln\mathcal{E} \tag{8}$$

Now one should notice that the basic operators can be written as linear combinations of the others. Three important representation of the $\mathcal{E}$ are

$$\mathcal{E} = \Delta_+ + I \tag{9}$$
$$\mathcal{E} = (\Delta_- + I)^{-1} \tag{10}$$
$$\tag{11}$$

The other one is the result of the following equation

$$\Delta_0 = \mathcal{E}^{\frac{1}{2}} - \mathcal{E}^{\frac{-1}{2}} \tag{12}$$
$$\Rightarrow \mathcal{E}^{\frac{1}{2}}\Delta_0 = (\mathcal{E}^{\frac{1}{2}})^2 - I \tag{13}$$
$$\mathcal{E}^{\frac{1}{2}} = \frac{\Delta_0}{2} + \sqrt{\frac{\Delta_0^2}{4} + I} \tag{14}$$

Then by taking the square of (14)

$$\mathcal{E} = \left(\frac{\Delta_0}{2} + \sqrt{\frac{\Delta_0^2}{4} + I}\right)^2 \tag{15}$$

Finally, we use these definitions of $\mathcal{E}$ in the equation (8) then Taylor expands the resulting function to obtain the differentiation operator. With definition (9), we get

$$D = \frac{1}{h}[\Delta_+ - \frac{1}{2}\Delta_+^2 + \frac{1}{3}\Delta_+^3] + \mathcal{O}(h^3) \tag{16}$$

and in general

$$D^s = \frac{1}{h^s}[\Delta_+^s - \frac{1}{2}s\Delta_+^{s+1} + \frac{s(3s+5)}{24}\Delta_+^{s+2}] + \mathcal{O}(h^3) \tag{17}$$

With this operator, we can differentiate a function up to an order $\mathcal{O}(h^3)$. Also, with the same procedure, one can find a formula with the Δ_- operator. The actual problem is to choose one. One should notice that the definition with the Δ_+ operator approximates the derivation using the elements $\{z_k, z_{k+1}, \cdots, z_{k+s+2}\}$. The Δ_- case on the other hand does the approximation using the points $\{z_k, z_{k-1}, \cdots, z_{k-s-2}\}$. The problem with such approximations is the fact that each operation uses only one side of the grids. A more proper approximation should have used both sides. This can be achieved by using the definition (15).
It gives the operator as

$$D = \frac{2}{h} ln \left(\frac{\Delta_0}{2} + \sqrt{I^2 + \left(\frac{\Delta_0}{4}\right)^2} \right) \tag{18}$$

To obtain a useful operation out of (18), lets us first consider the function

$$g(\epsilon) = ln(\epsilon + \sqrt{1+\epsilon^2}) \tag{19}$$

$$\Rightarrow g'(\epsilon) = \frac{1}{\sqrt{1+\epsilon^2}} = \sum (-1)^j \binom{2j}{j} \left(\frac{\epsilon}{2}\right)^{2j} \tag{20}$$

$g(0) = ln(1) = 0$. By the fundamental identity,

$$g(\epsilon) = g(0) + \int_0^\epsilon g'(t)dt \tag{21}$$

$$= 2 \sum \frac{(-1)^j}{2j+1} \binom{2j}{j} \left(\frac{\epsilon}{2}\right)^{2j+1} \tag{22}$$

$D = \frac{2}{h} g(\frac{\Delta_0}{2})$. Then the operation becomes

$$D = \frac{4}{h} \sum \frac{(-1)^j}{2j+1} \binom{2j}{j} \left(\frac{\Delta_0}{4}\right)^{2j+1} \tag{23}$$

The problem with this operator is all of the terms include odd powers of the central difference operator. Remember that Δ_0 cannot map $\mathbb{R}^z \to \mathbb{R}^z$. However, Δ_0^2 does.

$$\Delta_0^2 = \Delta_0 (z_{k+\frac{1}{2}} - z_{k-1\frac{1}{2}}) \tag{24}$$

$$= z_{k+1} - 2z_k + z_{k-1} \tag{25}$$

Hence, we should consider the square of the operator (23). If we take the square and open the series up to some order, we get the following formula for the $2s$ derivative as

$$D^{2s} = \frac{1}{h^{2s}} [\Delta_0^{2s} - \frac{\Delta_0^{2s+2}}{12} + \frac{\Delta_0^{2s+4}}{45}] + \mathcal{O}(h^6) \tag{26}$$

We can obtain odd derivatives by defining an identity using the $\mathcal{T}_0$ operator.

$$\mathcal{T}_0 = \frac{1}{2}(\mathcal{E}^{\frac{1}{2}} + \mathcal{E}^{\frac{-1}{2}}) \tag{27}$$

$$\Rightarrow \mathcal{T}_0^2 = \frac{1}{4}(\mathcal{E} + \mathcal{E}^{-1} + 2I) \tag{28}$$

$\Delta_0^2 = \mathcal{E} + \mathcal{E}^{-1} - 2I$. Then some manipulation on (28) and Δ_0^2 gives following

$$\mathcal{T}_0 = \sqrt{I + \frac{\Delta_0^2}{4}} \Rightarrow I = \mathcal{T}_0 \left(I + \frac{\Delta_0^2}{4}\right)^{\frac{-1}{2}} \tag{29}$$

By the (20),

$$\left(I + \frac{\Delta_0^2}{4}\right)^{\frac{-1}{2}} = \sum (-1)^j \binom{2j}{j} \left(\frac{\Delta_0^2}{16}\right)^j \tag{30}$$

Consequently, we get the following identity

$$I = \mathcal{T}_0 \sum (-1)^j \binom{2j}{j} \left(\frac{\Delta_0^2}{16}\right)^j \tag{31}$$

Multiplying this identity with the differentiation operator (23) gives the following operator

$$D = \mathcal{T}_0 \Delta_0 \left(\sum (-1)^j \binom{2j}{j} \left(\frac{\Delta_0^2}{16}\right)^j \right) \left(\sum (-1)^i \binom{2i}{i} \left(\frac{\Delta_0^2}{16}\right)^i \right) \tag{32}$$

One can expand the series up to some order to get a useful differential operator. However, this is computationally wasteful and time consuming. For example, to get $\mathcal{O}^4$ order approximation, one will need seven grid points. A better way to get the D^{2s+1} operator is open the (23) to an odd order and then multiply it with the identity (29). This gives the following order six formula

$$D^{2s+1} = \frac{1}{h^{2s+1}}(\mathcal{T}_0\Delta_0)[\Delta_0^{2s} - \frac{(s+3)(5s+16)\Delta_0^{2s+2}}{12} + \frac{(s+3)(5s+16)\Delta_0^{2s+4}}{1440}] + \mathcal{O}(h^6) \tag{33}$$

2 The Five Point Formula

$$D = \frac{-1}{h}[\Delta_- - \frac{1}{2}\Delta_-^2 + \frac{1}{3}\Delta_-^3] + \mathcal{O}(h^3) \tag{34}$$

$$D^s = \frac{-1}{h^s}[\Delta_-^s - \frac{1}{2}s\Delta_-^{s+1} + \frac{s(3s+5)}{24}\Delta_-^{s+2}] + \mathcal{O}(h^3) \tag{35}$$

$$D^{2s} = \frac{1}{h^{2s}}[\Delta_0^{2s} - \frac{\Delta_0^{2s+2}}{12} + \frac{\Delta_0^{2s+4}}{45}] + \mathcal{O}(h^6) \tag{36}$$

$$D^{2s} = \frac{1}{h^{2s}}\Delta_0^{2s} + \mathcal{O}(h^2) \tag{37}$$

$$D^{2s} = \frac{1}{h^2}\Delta_0^2 + \mathcal{O}(h^2) \tag{38}$$

$$\frac{1}{(\Delta x)^2}(\Delta_{0,x}^2 + \Delta_{0,y}^2)u_{k,l} = f_{k,l}$$

$$(\Delta x)^2 f_{k,l} = (U_{k+1,l} + U_{k-1,l} - U_{k,l} - U_{k,l}) + (U_{k,l+1} + U_{k,l-1} - U_{k,l} - U_{k,l})$$

$$(\Delta x)^2 f_{k,l} = U_{k+1,l} + U_{k-1,l} + U_{k,l+1} + U_{k,l-1} - 4U_{k,l}$$

$$v_{k,l} = sin\left(\frac{k\alpha\pi}{(m+1)2}\right) sin\left(\frac{l\beta\pi}{(m+1)2}\right)$$

$$\lambda_{\alpha,\beta} = -4[sin^2\left(\frac{k\alpha\pi}{(m+1)2}\right) + sin^2\left(\frac{l\beta\pi}{(m+1)2}\right)]$$

$$\nabla^2 u(x,y) = f(x,y) \quad , u(x,y) = \phi(x,y), \quad (x,y) \in \partial\Omega$$

Theorem 2.1. *The matrix A is symmetric and negative definite. Hence, it is nonsingular.*