CS 173 Study Guide

Aydan Pirani

December 11, 2021

Contents

1 Math Review

1.1 Sets

1.1.1 Important Sets

The following sets are very commonly used when it comes to numerical analysis - make sure to memorize the sets and understand how they work conceptually.

- Integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- Natural Numbers (non-negative integers): $\mathbb{N} = \{0, 1, 2, 3...\}$
- Positive Integers: $\mathbb{Z}^+ = \{1, 2, 3, ...\}$
- Real Numbers (all rationals and irrationals): $\mathbb{R}$
- Rational Numbers (all numbers of the form $\frac{p}{q}$, where p and q are integers): $\mathbb{Q}$
- Complex Numbers (of the form $a + bi$, where a and b are reals and $i = \sqrt{-1}$): $\mathbb{C}$

1.1.2 Things to Know

- Zero is not included in non-positive or non-negative.
- Natural numbers include zero.
- Real numbers include integers, so not necessary for a real to have a decimal.
- We denote that x is an element of the set A with: $x \in A$
- MAKE SURE TO LOOK AT THE TYPES OF YOUR VARIABLES.

1.1.3 Intervals

When working with a lot of numbers in a range, intervals help select a large amount of consecutive elements. They're denoted as such:

- Closed Interval (include both endpoints): $[a, b]$
- Open Interval (include no endpoints): (a, b)
- Half-Open Intervals (include one endpoint): $(a, b]$ or $[a, b)$

1.2 Pairs of Reals

- A pair of reals is a special way of denoting two real numbers that go together.
- A pair of reals is denoted by (x,y).
- The SET of all pairs of reals is denoted by $\mathbb{R}^2$.
- Before working with pairs, you need to understand what each point in the pair represents.

This concept also applies for containers of n dimensions, and the set of all such containers is represented as $\mathbb{R}^n$.

1.3 Exponentials and Logs

1.3.1 Exponents

- An exponent is an abbreviated notation for repeated multiplication. Eg. $a \times a \times a = a^3$.
- Hard to define exponents for fractional or decimal powers, but we assume that it holds.

1.3.2 Special Exponent Cases

The following illustrate some special exponent cases, make sure to memorize these:

- $x^0 = 1$
- $x^0.5 = \sqrt{x}$
- $x^{-n} = \frac{1}{x^n}$

1.3.3 Exponent Manipulation

The following illustrate important exponent rules, make sure to memorize these:

- $x^a x^b = x^{a+b}$
- $x^a y^a = (xy)^a$
- $(x^a)^b = x^a b$
- $x^(a^b) \neq (x^a)^b$

1.3.4 Logarithms

If we have an exponential equation $y = x^a$, then we can invert this to $y = \log_a x$. In most cases, a lack of a base indicates that the default base is 2.

1.3.5 Logarithm Manipulation

The following illustrate some important logarithm rules, make sure to memorize these:

- $b^{\log_b x} = x$
- $\log_b xy = \log_b x + \log_b y$
- $\log_b x^y = y \log_b x$

1.3.6 Change of Base Formula

- Change of Base Formula: $\log_b x = \log_b a \log_a x$
- To apply it correctly, choose a and b such that $\log_b a$ is greater or less than 1.
- Note that both logarithms are merely different by a constant, which gets dropped later on.

1.4 Some Handy Functions

1.4.1 Factorials

- Defined as the product of the first n numbers.
- $n! = 1 \times 2 \times 3 \times ... \times (n-1) \times n$
- Note that 0! is predefined as 1.

1.4.2 Permutations

- Defined as the amount of ways to select n objects from a set in any order.
- Permutations ("n choose k"): $\frac{n!}{k!(n-k)!}$
- Note that this formula requires the set to contain only UNIQUE objects.

1.4.3 Max/Min

- Returns the max and min of their inputs.
- $max(2, 7) = 7$.

1.4.4 Floor/Ceiling

- Floor ($\lfloor x \rfloor$) rounds a real downwards.
- Ceiling ($\lceil x \rceil$) rounds upwards.
- Also applies for negative numbers.

Number	$\lfloor x \rfloor$	$\lceil x \rceil$
-1.5	-2	-1
-1	-1	-1
1	1	1
1.5	2	2

2 Logic

2.1 A Bit About Style

Writing math has two requirements:

- Logical Flow of Ideas
- Express Yourself Fluently

2.2 Propositions

- Proposition: A statement that can be true or false, but not both.
- Doesn't deal with variables or complexity, MUST be predefined.
- Eg. "$1 < 2$" (never changes from true to false)

2.3 Complex Propositions

2.3.1 Chaining Propositions

- Can join propositions to get more complex statements that evaluate to true or false.
- Eg. ”$1 < 2$ and Chicago is in Illinois”.

2.3.2 Mathematical Notation

We can manipulate propositions using operators like ”not”, and we can join propositions using operators like ”and”/”or”. We also have the following mathematical abbreviations for the following operators:

- and: $\wedge$
- or: $\vee$
- not: $\neg$

2.3.3 Truth Tables

We can use truth tables to show the outcome of manipulating propositions. Here’s a truth table for the ”not” operator:

A	$\neg A$
T	F
F	F

Here’s another truth table for the results of using and/or on two propositions.

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Note that the ”or” statement is not exclusive - if both of its inputs are true, then the output is also true. In the context of a single exclusive or, there exists an operator called ”exclusive or”(XOR).

2.4 Implication

- Can join propositions into an ”if A, then B” statement.
- Also verbalized as ”A implies B”.
- Mathematical notation for this: $A \to B$.
- Can also be represented as: $\neg A \vee B$ (not A or B).

Here’s the truth table for implication:

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	F

Note that B's value is only called in if A is true - if A is false, then $A \to B$ automatically is TRUE.

2.5 Converse, Contrapositive, Biconditional

All three of these operate based on the implies statement.

2.5.1 Converse

The converse of $A \to B$ is $B \to A$. To find the converse:

1. Find the proposition chains A and B within the original statement.
2. Flip B and A, and place the "implies" sign in between.

NOTE THAT THE CONVERSE OF A STATEMENT IS NOT EQUAL TO THE ORIGINAL STATEMENT.

2.5.2 Biconditional

If a problem says "A implies B, and conversely", then this means: $A \to B \wedge B \to A$, which is equivalent to $A \leftrightarrow B$. Here's a truth table for the bidirectional operator:

A	B	$A \to B$	$B \to A$	$A \leftrightarrow B$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that in the case of a biconditional relationship, the converse of a statement is equivalent to the original statement.

2.5.3 Contrapositive

The contrapositive of $A \to B$ is $\neg B \to \neg A$. To find the contrapositive:

1. Find the proposition chains A and B within the original statement.
2. Negate both predicates (A and B).
3. Flip the negations of B and A, and place the "implies" sign in between.

Note that the contrapositive of a statement is equal to the original statement.

2.6 Complex Statements

- Can combine longer propositions to achieve chains, works the same way.
- Order of operations: parentheses, not, and/or, implications.
- Can also build truth tables, but lot of work as we need more variables.

2.7 Logical Equivalence

- Logically Equivalent: values of two propositions A and B are equal for all possible input values.
- Denoted mathematically with $\equiv$
- Can get to it via truth tables or simplification.

2.7.1 DeMorgan's Laws

Set of laws regarding boolean algebra:

- $\neg(A \vee B) \equiv \neg A \wedge \neg B$
- $\neg(A \wedge B) \equiv \neg A \vee \neg B$
- $A \wedge \neg A \equiv \text{False}$

2.8 Some Useful Logical Equivalences

2.8.1 Commutative Rules

- $A \wedge B \equiv B \wedge A$
- $A \vee B \equiv B \vee A$

2.8.2 Distributive Rules

- $A \wedge (B \wedge C) \equiv A \wedge B \wedge C$
- $A \vee (B \vee C) \equiv A \vee B \vee C$
- $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
- $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

2.9 Negating Propositions

To negate propositons:

1. Convert English text into mathematical notation.
2. Perform the negation on the implication.
3. Convert the mathematical negation back into plain English.

Note that this is dependent on the negation of the implies: $\neg(A \rightarrow B) \equiv A \wedge \neg B$

2.10 Predicates and Variables

- Predicate: A statement that takes on a T/F value based on the variables passed into it.
- Eg. $x^2 > 10$ is true if x=4, but false if x=3.
- When creating a predicate, need to be explicit about types of variables and assertions.

2.11 Other Quantifiers

2.11.1 There Exists

- Any single element in the set that fulfills the requirements.
- Denoted with $\exists$.

2.11.2 For All

- Every element within the given set.
- Denoted with $\forall$.

2.11.3 Unique Exists

- A single element within the given set, which is the ONLY one that fulfills the requirements.
- Highly unlikely to show up on an examlet.

2.12 Notation

- Mathematical notation has a strict template: quantifier, variable + domain, predicate.
- Can also be expressed in simple English terms.
- Eg. $\forall x \in \mathbb{R}, x^2 + 3 \geq 0$ or "for all values of x in the real numbers, $x^2 + 3 \geq 0$".

2.13 Useful Notation

- Can also tie together multiple claims if two variables have the same type.
- Eg. $\forall x, y \in \mathbb{Z}, x + y \geq x$.
- In this case, x and y don't have to be different - both are independent arbitrary values.
- Can also utilize contrapositive in the case of the predicate.

2.14 Notation for 2D Points

Can do any of the two:

- Create a new 2D pair and refer to it later.
- Create two elements.

2.15 Negating Statements with Quantifiers

To negate a statement:

1. Invert the quantifier (exists becomes for all, and vice versa).
2. Negate the predicate/implies statement.

2.16 Binding and Scope

- A quantifier binds the variable it defines.
- After a variable is not bound, it's considered free.
- If a variable is free, it should be considered invalid.

3 Proofs

NOTE: The approaches defined here change based on universal and existential statements, and based on whether or not the proof is positive or negative. Be sure to take this into account!

3.1 Proving a Universal Statement

1. Define all vocabulary (eg. rationals, integers, ...).
2. Pick a representative value from the set (VARIABLE).
3. Go from the value to the claim (hardest part to prove).

3.3 Direct Proof Outline

- Start with known information, move towards final statement.
- Sometimes need to reason backwards, but ALWAYS write forwards.

3.4 Proving Existential Statements

- Find a value that matches the claim, done.
- Can choose any value, because the claim is existential.

3.5 Disproving a Universal Statement

- Similar process to proving an existential statement.
- Find a value that proves it wrong.

3.6 Disproving an Existential Statement

- Similar process to proving a universal statement.
- Find a representative element, then work from there.

3.7 Recap of Proof Methods

claim	prove	disprove
universal	representative element	counterexample
existential	example	representative element

3.10 Proof by Cases

- If claim is in the form p or q, then break it up, and prove for p and q individually.
- Combine both statements together afterwards.

3.11 Rephrasing Claims

- Depending on the claim, might need to apply negations and DeMorgan's Laws to pull it all together.

3.12 Proof by Contrapositive

- Fairly straightforward - if you can prove the contrapositive, then you have proven the claim.
- No strict rule as for when to use it.

4 Number Theory

4.1 Factors and Multiples

- a divides b if $a = bn$, for any integer n.
- In this case:
  - a is a factor of b
  - b is a multiple of a
- Can express "a divides b" as $a|b$.
- NOTE: the "smaller" number goes on the left, not the right.

NOTE: An integer p is even iff $2|p$.

4.3 Stay in the Set

- Don't introduce rationals if working with ints!

4.4 Prime Numbers

- p ($p \geq 2$)is prime iff the only positive factors of p are p and 1, else it's composite.
- Prime Factorization: Expressing any integer p as the product of only prime numbers.

4.5 GCD and LCM

4.5.1 GCD

- Common Divisor: any value that divides 2 integers
- Greatest Common Divisor: largest common divisor of two integers.
- Can express a GCD as gcd(a,b).
- Can calculate the GCD by extracting common factors from the prime factorization.

4.5.2 LCM

- Smallest value possible such that $a|c$ and $b|c$.
- Can be found with this formula: $lcm(a,b) = \frac{ab}{gcd(a,b)}$.
- Relatively Prime: When two integers have no shared common divisors.

4.6 The Division Algorithm

For any integers a and b, there are unique integers q and r such that $a = bq + r$.

4.7 Euclidean Algorithm

Keep doing this algorithm:

```
gcd(a,b):
    x = a
    y = b
    while y > 0:
        r = remainder(x, y)
        x = y
        y = r
    return x
```

4.10 Congruence mod K

- Two integers are congruent mod k if they differ by a value of k.
- If k is any positive integer, $a \equiv b(modk)$ iff $k|(a-b)$.

4.12 Equivalence Classes

- Congruence Class/Equivalence Class: Set of all integers mod k.
- Eg. in congruence mod 7: [3] = {..., -11, -4, 3, 10, 17, ...}
- Generally only use integers from 0 to k-1 to name these classes.

5 Sets

5.1 Sets

- Sets are unordered collections of objects.
- Items in a set are called elements or members.
- Three ways to define a set:
  - Mathematical English (all integers between 3 and 7, inclusive).
  - List ({3,4,5,6,7})
  - Set Builder Notation ($\{x \in Z, 3 \leq x \leq 7\}$)
- Note that the comma (may be replaced with |) in set builder notation indicates a constant.

5.2 Things to be Careful About

- Set is unordered and unique, so $\{3, 2, 1\} = \{1, 2, 3\}$.
- No duplicates in sets, so $\{3, 2, 1, 2\} = \{1, 2, 3\}$.
- Note that sets aren't tuples (ordered non-unique collections), so use {} and not ().
- Sets can be empty ($\emptyset$) or have a single value.
- Sets can contain objects of multiple types.

5.3 Cardinality, Inclusion

- Cardinality: Amount of (unique) objects in a set.
- Subset: All elements in A are also in B, denoted by $A \subseteq B$.
- Proper Subset: Subset but both sets must be different, denoted by $A \subset B$.

5.4 Vacuous Truth

- Happens when a statement is technically true by definition, but not actually true.
- Eg. is an empty set a subset of A? Yes (all elements in empty set are in A) but also no.

5.5 Set Operations

- Intersection($\cap$): All elements that exist in BOTH sets. If intersection yields empty set, then both sets are disjoint.
- Union($\cup$): Set of all unique elements in both sets combined.
- Set Difference($-$): Set of all elements in the first set but not in the second.
- Cartesian Product ($\times$): Set of all 2D-tuples, containing combinations of elements from both sets.

5.7 Size of Set Union

- Inclusion-Exclusion Principle: $|A \cup B| = |A| + |B| - |A \cap B|$
- Can extend this idea to multiple sets (2+).

5.8 Product Rule

- Very intuitive method to determine cardinality of Cartesian product.
- $|A \times B| = |A| \times |B|$

5.10 Proving Facts About Set Inclusion

- Start with a representative element from A.
- Perform algebra to show that A exists in B.
- Conclude by saying that the claim has been proven.

6 Relations

Relations

7 Functions and Onto

Funcions and Onto

8 Functions and One-To-One

Functions and One-To-One

9 Graphs

Graphs

10 Two-Way Bounding

Two-Way Bounding

11 Induction

Induction

12 Recursive Definition

Recursive Definition

13 Trees

Trees

14 Big-O

Big-O

15 Algorithms

Algorithms

16 NP

NP

17 Proof by Contradiction

Proof by Contradiction

18 Collections of Sets

- Most sets previously contained atomic elements (numbers, strings, tuples).
- Sets can also contain other sets.
- Collection: A set that contains other sets.

18.1 Sets Containing Sets

- Happens when we need to get subsets of another set.
- Can divide a set into an amount of (non)overlapping subsets.
- If a non-overlapping set of subsets is the domain of a function, each subset results in an an output.

18.1.1 Cardinality

- The cardinality of a set is the amount of elements in the set itself, not in its subsets.
- The empty set can be put in another set, and it counts as an element.

18.2 Powersets and Set-Valued Functions

- Powerset of A contains ALL subsets of A (including empty set).
- Denote a powerset with $\mathbb{P}(A)$.
- Powerset of A contains 2^n elements, if A has n elements.
- Use powersets when a function returns MULTIPLE values, and hence consistency is important.
- Are useful for defining the domains of above-mentioned functions.

18.3 Partitions

- Division of a base set A into non-overlapping subsets.
- Corresponds to equivalence reactions (and vice-versa).
- Three requirements for a partition of A:
  1. Elements put together cover all of A.
  2. No empty set in partition.
  3. No overlap within elements of partition sets.

18.4 Combinations

18.4.1 Combinations

- Combinations happen when we have an n-element set, from which we need all subsets of size k.
- K-Combination: Subset of size k.
- Care about order in a permutation, but not in a combination.

18.4.2 Equations

- Choose k elements without an order: $\frac{n!}{(n-k)!}$
- Choose k elements in order (AKA binomial coefficient): $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Note that this is defined if $n \geq k \geq 0$

18.5 Applying the Combinations Formula

- Used to select a set of locations and assign values.
- Might need to apply it multiple times to generate the correct answer.

18.6 Combinations with Repetition

- Need a clever way to count the amount of possibilities with multiple groups.
- Replace each item with a star, and find amount of places you can put the separator.
- To choose k objects from a list of size n: $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$

18.7 Identities for Binomial Coefficients

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

19 State Diagrams

19.1 Introduction

- Directed graph, nodes represent states and edges represent actions.
- Label on an edge indicates what happens as a system moves across states.
- Walks must follow arrows.
- NOTE: INCLUDE STATES AND ACTIONS WHEN WRITING IT OUT.

20 Countability

20.1 The Rationals and the Reals

- Three sets of numbers: integers, rationals, reals.
- Integers are discrete whole numbers
- Majority of real numbers are irrational, and only a few are rational.

20.2 Completeness

- Reals have completeness, rationals don't.
- Completeness: Any subset of reals with upper bound has a SMALLEST upper bound.

20.3 Cardinality

- Two sets have the same cardinality iff there's a bijection from A to B.
- Eg. $f : \mathbb{R} \to \mathbb{Z}, f(n) \begin{cases} \frac{n}{2}, n \equiv 0(mod2) \\ \frac{-(n+1)}{2}, n \equiv 1(mod2) \end{cases}$
- Countably Infinite: Bijection exists from $\mathbb{N}$ or $\mathbb{Z}$ onto an infinite set A.
- Countable: A set is countably infinite or finite. (Includes all subsets of integers).

20.4 Cantor Schroeder Bernstein Theorem

- $|A| \leq |B|$ iff there's a one-to-one function from A to B.
- Can reverse this and do it twice (A to B and then B to A), and show that a bijection exists.