Thesis

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Chapter 1

Special Elements in Semigroups

1.1 Local identities in semigroups

Throughout, let S be a semigroup with associative multiplication, written multiplicatively.

Definition 1 (Left/right/two-sided identities)**.** Let $e \in S$.

- e is a *left identity* if for all $s \in S$, $es = s$.
- e is a *right identity* if for all $s \in S$, $se = s$.
- e is an *identity* (two-sided) if it is both a left and a right identity; equivalently, for all $s \in S$, $es = s = se$.

Lemma 2 (Idempotence of one-sided identities)**.** *Let $e \in S$.*

- *If e is a left identity, then $ee = e$.*
- *If e is a right identity, then $ee = e$.*

Proof. For a left identity, apply the defining property to $s := e$ to get $ee = e$. For a right identity, apply the defining property to $s := e$ to get $ee = e$. □

Lemma 3 (Simplification lemma)**.** *Let $s \in S$ and let $e, f \in S$ be idempotents. If $s = esf$, then $es = s = sf$.*

Proof. Assume $s = esf$. Then

$$es \;=\; e(esf) \;=\; (ee)sf \;=\; esf \;=\; s,$$

using associativity and $e^2 = e$. Similarly,

$$sf \;=\; (esf)f \;=\; es(ff) \;=\; esf \;=\; s,$$

using associativity and $f^2 = f$. □

1.2 Zero elements and null semigroups

Throughout, let S be a semigroup with associative multiplication, written multiplicatively.

Definition 4 (Left/right/two-sided zeros)**.** Let $e \in S$.

- e is a *left zero* if for all $s \in S$, $es = e$.
- e is a *right zero* if for all $s \in S$, $se = e$.
- e is a *zero* (two-sided) if it is both a left and a right zero; equivalently, for all $s \in S$, $es = e = se$.

Lemma 5 (Idempotence of one-sided zeros)**.** *Let $e \in S$.*

- *If e is a left zero, then $ee = e$.*
- *If e is a right zero, then $ee = e$.*

Proof. For a left zero, apply the defining property to $s := e$ to get $ee = e$. For a right zero, apply the defining property to $s := e$ to get $ee = e$. $\square$

Lemma 6 (Uniqueness of zero (at most one zero element))**.** *A semigroup has at most one zero element.*

Proof. Suppose $e, e' \in S$ are both zeros. Then $e = ee'$ since e' is a right zero, and $ee' = e'$ since e is a left zero. Hence $e = e'$. $\square$

Definition 7 (Null semigroup)**.** A semigroup S is *null* if it has a zero element 0_S and for all $x, y \in S$ one has $xy = 0_S$.

1.3 Cancellativity

Throughout, let S be a semigroup with associative multiplication.

Definition 8 (Right/left/two-sided cancellative element)**.** Let $s \in S$.

- s is *right cancellative* if for all $x, y \in S$, $xs = ys \implies x = y$.
- s is *left cancellative* if for all $x, y \in S$, $sx = sy \implies x = y$.
- s is *cancellative* (two-sided) if it is both left and right cancellative.

Definition 9 (Right/left/two-sided cancellative semigroup)**.** A semigroup S is

- *right cancellative* if every $s \in S$ is right cancellative,
- *left cancellative* if every $s \in S$ is left cancellative,
- *cancellative* (two-sided) if every $s \in S$ is cancellative.

1.4 Inverses

Terminology note. The term "inverse" has two distinct usages. In group theory (and, more generally, in monoids), an inverse is defined using a distinguished identity element 1. This notion does not make sense in a bare semigroup that lacks a specified unit. Semigroup theory also uses a different, intrinsic notion of inverse that does not require a unit and is formulated purely in terms of the multiplication.

These notions behave differently:

- In an *infinite* monoid, an element may have several right group inverses and several left group inverses.
- In a *finite* monoid, each element has *at most one* right group inverse and *at most one* left group inverse; if both exist, they coincide (hence give a two-sided group inverse).
- In a semigroup (finite or infinite), an element may have several semigroup inverses, or none at all.

Definition 10 (Semigroup inverse)**.** Let S be a semigroup and $x \in S$. An element $x' \in S$ is a *semigroup inverse* of x if

$$xx'x = x \quad \text{and} \quad x'xx' = x'.$$

Definition 11 (Group inverse (monoid setting))**.** Let M be a monoid with identity 1 and let $x \in M$.

- A *right group inverse* of x is an element $x' \in M$ with $xx' = 1$.
- A *left group inverse* of x is an element $x' \in M$ with $x'x = 1$.
- A *group inverse* of x is an element $x' \in M$ that is both a right and a left group inverse, i.e. $xx' = x'x = 1$.

Lemma 12 (Group inverse $\Rightarrow$ semigroup inverse)**.** *Let M be a monoid and $x, x' \in M$. If x' is a group inverse of x (so $xx' = x'x = 1$), then x' is a semigroup inverse of x in the underlying semigroup:*

$$xx'x = x \quad \textit{and} \quad x'xx' = x'.$$

Proof. Compute $xx'x = (xx')x = 1 \cdot x = x$ and $x'xx' = x'(xx') = x' \cdot 1 = x'$, using associativity and the unit laws. □

Chapter 2

Ordered Semigroups and Monoids

Definition 13 (Ordered semigroup/monoid/group).

- An *ordered semigroup* is a pair $(S, \leq)$ where S is a semigroup and $\leq$ is a partial order on S such that multiplication is monotone in both arguments:

$$\forall a, b, x, y \in S, \quad x \leq y \implies axb \leq ayb.$$

  Equivalently, for all $a, b \in S$, the maps $x \mapsto ax$ and $x \mapsto xb$ are order-preserving.

- An *ordered monoid* is an ordered semigroup $(M, \leq)$ whose underlying semigroup is a monoid $(M, 1)$. We require the same compatibility condition as above (which automatically implies 1 is $\leq$-minimal among right/left translates of any element).

- An *ordered group* is an ordered monoid whose underlying monoid is a group $(G, 1, (\cdot)^{-1})$ and such that the order is bi-invariant in the sense above (equivalently, both left and right multiplication are order embeddings).

Remark. Monotonicity in both coordinates implies: if $x \leq y$ then $ax \leq ay$ and $xb \leq yb$ for all a, b. Conversely, these two conditions together imply $axb \leq ayb$ by associativity.

Chapter 3

Morphisms

Definition 14 (Semigroup morphism)**.** Let S, T be semigroups. A *semigroup morphism* (homomorphism) is a map $\varphi : S \to T$ such that

$$\forall s_1, s_2 \in S, \qquad \varphi(s_1 s_2) = \varphi(s_1)\,\varphi(s_2).$$

Definition 15 (Monoid morphism)**.** Let M, N be monoids with identities $1_M, 1_N$. A *monoid morphism* is a semigroup morphism $\varphi : M \to N$ that also preserves the unit:

$$\varphi(1_M) = 1_N.$$

Definition 16 (Morphism of ordered monoids)**.** Let $(M, \leq_M)$ and $(N, \leq_N)$ be ordered monoids. A *morphism of ordered monoids* is a monoid morphism $\varphi : M \to N$ that is order-preserving:

$$x \leq_M y \implies \varphi(x) \leq_N \varphi(y) \quad \text{for all } x, y \in M.$$

Definition 17 (Group morphism)**.** Let G, H be groups. A *group morphism* is a monoid morphism $\varphi : G \to H$. Equivalently, φ is a semigroup morphism satisfying

$$\varphi(1_G) = 1_H \quad \text{and} \quad \forall g \in G, \;\; \varphi(g^{-1}) = \varphi(g)^{-1}.$$

Lemma 18 (Semigroup morphisms between groups are group morphisms)**.** *Let G, H be groups. Any semigroup morphism $\varphi : G \to H$ is a group morphism.*

Proof. First, $\varphi(1_G) = \varphi(1_G \cdot 1_G) = \varphi(1_G)\varphi(1_G)$, so $\varphi(1_G)$ is idempotent in H, hence $\varphi(1_G) = 1_H$ since 1_H is the unique idempotent in a group. Next, for $g \in G$,

$$\varphi(g^{-1})\varphi(g) = \varphi(g^{-1}g) = \varphi(1_G) = 1_H, \qquad \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1_G) = 1_H,$$

so $\varphi(g^{-1}) = \varphi(g)^{-1}$. $\square$

Definition 19 (Isomorphism)**.** A semigroup (resp. monoid, group) morphism $\varphi : S \to T$ is an *isomorphism* if there exists a morphism $\psi : T \to S$ with $\varphi \circ \psi = \mathrm{id}_T$ and $\psi \circ \varphi = \mathrm{id}_S$.

Lemma 20 (Isomorphism $\Leftrightarrow$ bijective morphism)**.** *A semigroup/monoid/group morphism is an isomorphism if and only if it is bijective.*

Proof. If φ has a two-sided inverse ψ, then φ is bijective. Conversely, if φ is bijective, its set-theoretic inverse φ^{-1} satisfies $\varphi(\varphi^{-1}(x)\varphi^{-1}(y)) = \varphi(\varphi^{-1}(x))\varphi(\varphi^{-1}(y)) = xy$; applying φ^{-1} shows φ^{-1} is a morphism, hence φ is an isomorphism. $\square$

Definition 21 (Isomorphism of ordered monoids)**.** A morphism of ordered monoids $\varphi : (M, \leq_M) \to (N, \leq_N)$ is an *isomorphism of ordered monoids* if it is bijective as a function and reflects the order:

$$\forall x, y \in M, \qquad x \leq_M y \iff \varphi(x) \leq_N \varphi(y).$$

Equivalently, φ is a bijective monoid morphism whose inverse is order-preserving.

Remark. Unlike the unordered case, a bijective morphism of ordered monoids need not be an isomorphism of ordered monoids unless it also reflects the order.

Chapter 4

Algebraic Structures

4.1 Substructures

Definition 22 (Subsemigroup)**.** A *subsemigroup* of a semigroup S is a nonempty subset $T \subseteq S$ such that for all $t_1, t_2 \in T$, one has $t_1 t_2 \in T$.

Definition 23 (Submonoid of a monoid)**.** A *submonoid* of a monoid M is a subsemigroup $T \subseteq M$ containing the identity 1_M.

Definition 24 (Subgroup of a group)**.** A *subgroup* of a group G is a submonoid $H \subseteq G$ that is closed under inversion: $h \in H \Rightarrow h^{-1} \in H$.

Definition 25 (Monoid/group inside a semigroup)**.** Let S be a semigroup.

- A subsemigroup $M \subseteq S$ is a *monoid in* S if there exists an idempotent $e \in M$ such that $(M, \cdot, e)$ is a monoid (with identity e) under the inherited multiplication.
- A subsemigroup $G \subseteq S$ is a *group in* S if there exists an idempotent $e \in G$ such that $(G, \cdot, e, (\cdot)^{-1})$ is a group under the inherited multiplication.

Lemma 26 (Images and preimages preserve substructures)**.** *Let $\varphi : S \to T$ be a semigroup morphism.*

- *If $S' \subseteq S$ is a subsemigroup, then $\varphi(S')$ is a subsemigroup of T.*
- *If $T' \subseteq T$ is a subsemigroup, then $\varphi^{-1}(T')$ is a subsemigroup of S.*

Analogous statements hold for monoid and group morphisms and their corresponding substructures.

Proof. If $t_1 = \varphi(s_1) \in \varphi(S')$ and $t_2 = \varphi(s_2) \in \varphi(S')$ with $s_1, s_2 \in S'$, then $t_1 t_2 = \varphi(s_1)\varphi(s_2) = \varphi(s_1 s_2) \in \varphi(S')$. For preimages: if $s_1, s_2 \in \varphi^{-1}(T')$, then $\varphi(s_i) \in T'$ and $\varphi(s_1 s_2) = \varphi(s_1)\varphi(s_2) \in T'$, hence $s_1 s_2 \in \varphi^{-1}(T')$. □

Lemma 27 (Finite-group test)**.** *A nonempty subsemigroup S' of a finite group G is a subgroup of G.*

Proof. Pick $g \in S'$. The set $\{g^n \mid n \geq 1\} \subseteq S'$ is finite, so $g^i = g^j$ with $1 \leq i < j$. By cancellation in G, $g^{j-i} = 1 \in S'$. For any $h \in S'$, the set $\{h^n \mid n \geq 0\}$ is finite, hence $h^a = h^b$ with $a < b$. Cancelling h^a yields $1 = h^{b-a} \in S'$; then $h^{b-a-1} \in S'$ is an inverse of h. Thus S' is a subgroup. □

4.2 Quotients and Divisions

In this section, "quotient" means "image of a surjective morphism". For finite semigroups, the last two lemmas below show that the *division* relation (defined via quotients of subsemigroups) is a partial order on isomorphism classes.

Definition 28 (Semigroup quotient)**.** A semigroup T is a *quotient* of a semigroup S if there exists a surjective semigroup morphism $\pi : S \twoheadrightarrow T$.

Lemma 29 (Trivial order arises as a quotient)**.** *For any ordered monoid $(M, \leq)$, the identity map $\mathrm{id}_M : (M, =) \to (M, \leq)$ is a surjective morphism of ordered monoids. Hence $(M, \leq)$ is a quotient of $(M, =)$ in the ordered-monoid sense.*

Proof. The identity preserves the monoid structure and is order-preserving from equality to $\leq$ trivially. □

Definition 30 (Division (divisor))**.** A semigroup T *divides* a semigroup S (notation $T \mid S$) if there exist a subsemigroup $U \subseteq S$ and a surjective morphism $\pi : U \twoheadrightarrow T$. Equivalently, T is a quotient of a subsemigroup of S.

Lemma 31 (Transitivity of division)**.** *If $S_1 \mid S_2$ and $S_2 \mid S_3$, then $S_1 \mid S_3$.*

Proof. Let $U_1 \subseteq S_2$ and $\pi_1 : U_1 \twoheadrightarrow S_1$ witness $S_1 \mid S_2$, and $U_2 \subseteq S_3$ and $\pi_2 : U_2 \twoheadrightarrow S_2$ witness $S_2 \mid S_3$. Then $U := \pi_2^{-1}(U_1) \subseteq S_3$ is a subsemigroup and $\pi_1 \circ \pi_2 : U \twoheadrightarrow S_1$ is surjective, so $S_1 \mid S_3$. □

Lemma 32 (Reflexivity and antisymmetry on finite semigroups)**.**

1. *For any semigroup S, $S \mid S$ (take $U = S$ and $\pi = \mathrm{id}$).*
2. *If S_1, S_2 are finite and $S_1 \mid S_2$ and $S_2 \mid S_1$, then $S_1 \cong S_2$.*

Proof. (1) is immediate. For (2), choose $U_1 \subseteq S_2$ with $\pi_1 : U_1 \twoheadrightarrow S_1$ and $U_2 \subseteq S_1$ with $\pi_2 : U_2 \twoheadrightarrow S_2$. Then $|S_1| \leq |U_1| \leq |S_2|$ and $|S_2| \leq |U_2| \leq |S_1|$, hence all are equalities: $|S_1| = |U_1| = |S_2| = |U_2|$. Therefore $U_1 = S_2$, $U_2 = S_1$ and both π_1, π_2 are bijections, i.e. isomorphisms. Thus $S_1 \cong S_2$. □

4.3 Products

Definition 33 (Product of semigroups)**.** Given a family $\{S_i\}_{i \in I}$ of semigroups, the Cartesian product $\prod_{i \in I} S_i$ is a semigroup under coordinatewise multiplication:

$$(s_i)_{i \in I} \cdot (t_i)_{i \in I} \; := \; (\, s_i t_i \,)_{i \in I}.$$

Lemma 34 (Unit object for products)**.** *Let 1 denote the one-element semigroup (resp. monoid, group). Then $S \times 1 \cong S \cong 1 \times S$ via the coordinate projections; these are semigroup (resp. monoid, group) isomorphisms.*

Proof. The maps $(s, *) \mapsto s$ and $s \mapsto (s, *)$ are mutually inverse morphisms. □

Definition 35 (Product of ordered monoids)**.** If $\{(M_i, \leq_i)\}_{i\in I}$ are ordered monoids, their product ordered monoid is $\left(\prod_i M_i, \leq\right)$ where

$$(s_i)_{i\in I} \leq (t_i)_{i\in I} \quad\iff\quad \forall i \in I,\ s_i \leq_i t_i.$$

This order is compatible with the coordinatewise multiplication.

Lemma 36 (Products preserve substructures, quotients, and division)**.** *Let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be families of semigroups.*

- *If each $S_i \subseteq T_i$ is a subsemigroup, then $\prod_i S_i \subseteq \prod_i T_i$ is a subsemigroup.*
- *If each $\pi_i : T_i \twoheadrightarrow S_i$ is a surjective morphism, then $\prod_i \pi_i : \prod_i T_i \twoheadrightarrow \prod_i S_i$ is a surjective morphism; hence products of quotients are quotients of products.*
- *If each S_i divides T_i, then $\prod_i S_i$ divides $\prod_i T_i$.*

Proof. All three items are checked coordinatewise. □

4.4 Ideals

For a semigroup S, write S^1 for S if S already has an identity, and otherwise for the semigroup obtained by adjoining a new identity 1. This allows uniform formulations using S^1.

Definition 37 (Right/left/two-sided ideals)**.** Let S be a semigroup. A *right ideal* is a subset $R \subseteq S$ with $RS \subseteq R$. A *left ideal* is a subset $L \subseteq S$ with $SL \subseteq L$. An *ideal* is a subset $I \subseteq S$ with $SI \subseteq I$ and $IS \subseteq I$.

Lemma 38 (Characterizations via S^1)**.** *Let S be a semigroup and $I \subseteq S$.*

1. *I is a right ideal iff $IS^1 = I$ (equivalently, $IS^1 \subseteq I$).*
2. *I is a left ideal iff $S^1 I = I$ (equivalently, $S^1 I \subseteq I$).*
3. *I is an ideal iff $S^1 I S^1 = I$ (equivalently, $S^1 I S^1 \subseteq I$).*

Proof. If I is a right ideal then $IS \subseteq I$; multiplying by $1 \in S^1$ gives $IS^1 \subseteq I$ and $\supseteq$ is clear. The other items are analogous; for (3) combine the previous two. □

Lemma 39 (Monoid case)**.** *If M is a monoid and $I \subseteq M$, then I is a right ideal iff $IM = I$, a left ideal iff $MI = I$, and an ideal iff $MIM = I$.*

Proof. Same as above with $S^1 = M$. □

Lemma 40 (Intersections)**.** *Arbitrary intersections of (right/left/two-sided) ideals are (right/left/two-sided) ideals.*

Proof. If $\{I_\alpha\}$ are ideals, then $S(\bigcap_\alpha I_\alpha) \subseteq \bigcap_\alpha SI_\alpha \subseteq \bigcap_\alpha I_\alpha$, and similarly on the right. □

Definition 41 (Ideal generated by a set; principal ideals)**.** For $R \subseteq S$, the ideal generated by R is $S^1 R S^1$; the right (resp. left) ideal generated by R is RS^1 (resp. $S^1 R$). An ideal is *principal* if it is generated by a single element.

Lemma 42 (Ideals and morphisms)**.** *Let $\varphi : S \to T$ be a semigroup morphism. If $J \subseteq T$ is an ideal, then $\varphi^{-1}(J)$ is an ideal of S. If φ is surjective and $I \subseteq S$ is an ideal, then $\varphi(I)$ is an ideal of T.*

Proof. For preimages: $S^1\varphi^{-1}(J)S^1 \subseteq \varphi^{-1}(T^1)\varphi^{-1}(J)\varphi^{-1}(T^1) \subseteq \varphi^{-1}(T^1JT^1) = \varphi^{-1}(J)$. For images with φ surjective: $T^1\varphi(I)T^1 = \varphi(S^1)\varphi(I)\varphi(S^1) = \varphi(S^1IS^1) = \varphi(I)$. ☐

Definition 43 (Product of ideals)**.** If $I_1, \dots, I_n$ are ideals of S, their *product* is

$$I_1 \cdots I_n := \{ s_1 \cdots s_n \mid s_k \in I_k \}.$$

Lemma 44 (Product of ideals)**.** *The product $I_1 \cdots I_n$ is an ideal and $I_1 \cdots I_n \subseteq \bigcap_{k=1}^{n} I_k$.*

Proof. Since $S^1I_1 = I_1$ and $I_nS^1 = I_n$, we have $S^1(I_1 \cdots I_n)S^1 = (S^1I_1)\cdots(I_nS^1) = I_1 \cdots I_n$. For inclusion, fix k; then $I_1 \cdots I_n \subseteq S^1I_kS^1 = I_k$. ☐

Definition 45 (Minimal and 0-minimal ideals)**.** A nonempty ideal I of S is *minimal* if $J \subseteq I$ and J an ideal implies $J = I$. If S has a zero 0, a nonempty ideal $I \neq \{0\}$ is 0-*minimal* if every nonempty ideal $J \subseteq I$ satisfies $J = \{0\}$ or $J = I$.

Lemma 46 (Uniqueness of a minimal ideal)**.** *A semigroup has at most one minimal ideal.*

Proof. If I_1, I_2 are minimal ideals, then I_1I_2 is a nonempty ideal contained in $I_1 \cap I_2$. By minimality, $I_1I_2 = I_1 = I_2$. ☐

Lemma 47 (Existence in the finite case)**.** *Every finite semigroup has a minimal ideal.*

Proof. Among the nonempty ideals (nonempty because singletons $\{s\}$ generate ideals), pick one of minimal cardinality; it is minimal by definition. ☐

Lemma 48 (Zero yields a minimal ideal)**.** *If S has a zero 0, then $\{0\}$ is a minimal ideal.*

Proof. $\{0\}$ is an ideal since $S\{0\} = \{0\} = \{0\}S$. If $J \subseteq \{0\}$ is a nonempty ideal, then $J = \{0\}$. ☐

4.5 Simple and 0-Simple semigroups

Definition 49 (Simple and 0-simple)**.** A semigroup S is *simple* if its only ideals are $\emptyset$ and S. If S has a zero 0, then S is 0-*simple* if $S^2 \neq \{0\}$ and the only ideals are $\emptyset, \{0\}, S$. One-sided versions (right/left simple, right/left 0-simple) are defined analogously.

Lemma 50. *If S is 0-simple, then $S^2 = S$.*

Proof. S^2 is a nonempty, nonzero ideal, hence $S^2 = S$. ☐

Lemma 51 (Characterizations via principal two-sided ideals)**.**

1. *S is simple iff $SsS = S$ for every $s \in S$.*
2. *If $S \neq \emptyset$ and has a zero 0, then S is 0-simple iff $SsS = S$ for every $s \in S \ \ \{0\}$.*

Proof. (2) Suppose S is 0-simple. By Lemma 50, $S^2 = S$, so $\bigcup_{s\in S} SsS = S$. The set $I = \{s \in S \mid SsS = \{0\}\}$ is an ideal containing 0 but not equal to S; hence $I = \{0\}$. Thus for $s \neq 0$, $SsS \neq \{0\}$, and being an ideal, it equals S. Conversely, if $S \neq \emptyset$ and $SsS = S$ for all $s \neq 0$, then $S^2 \neq \{0\}$ and any nonzero ideal J contains some $s \neq 0$, hence $S = SsS \subseteq SJS = J$. The proof of (1) is the same without the zero case. ☐

Chapter 5

Semigroup Congruences

Definition 52 (Semigroup congruence)**.** An equivalence relation $\sim$ on a semigroup S is a *congruence* if it is stable under multiplication: $s \sim t$ implies $xsy \sim xty$ for all $x, y \in S^1$.

Lemma 53 (Quotient by a congruence)**.** *If $\sim$ is a congruence on S, the set $S/{\sim}$ of equivalence classes is a semigroup under $[s] \cdot [t] := [st]$. The canonical projection $\pi : S \to S/{\sim}$ is a surjective semigroup morphism.*

Proof. Well-definedness and associativity follow from stability and associativity in S. Surjectivity and multiplicativity of π are immediate. □

Definition 54 (Rees congruence)**.** If I is an ideal of S, the *Rees congruence* $\equiv_I$ identifies all elements of I and keeps distinct elements of $S\ \ I$: $s \equiv_I t \iff (s = t)$ or $(s, t \in I)$. The quotient $S/I := S/{\equiv_I}$ has support $(S\ \ I) \cup \{0\}$ with multiplication

$$s * t = \begin{cases} st, & s, t, st \notin I, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 55 (Syntactic congruence)**.** Given $P \subseteq S$, the *syntactic congruence* $\sim_P$ is

$$s \sim_P t \quad :\iff \quad \forall x, y \in S^1, \;\; xsy \in P \;\Leftrightarrow\; xty \in P.$$

The quotient $S/{\sim_P}$ is the *syntactic semigroup* of P in S.

Definition 56 (Congruence generated by a relation)**.** For a (symmetric) relation $R \subseteq S \times S$, the *congruence generated by* R is the intersection of all congruences containing R.

Lemma 57 (Description of generated congruence)**.** *If $R \subseteq S \times S$ is symmetric, the congruence it generates is the reflexive-transitive closure of*

$$\overline{R} \;:=\; \{(xry,\; xsy) \mid (r, s) \in R,\; x, y \in S^1\}.$$

Proof. Any congruence containing R contains $\overline{R}$ and hence its reflexive-transitive closure $\overline{R}^*$. Conversely, $\overline{R}^*$ is readily checked to be a congruence: if $u \to v$ is a step from $\overline{R}$, then $xuy \to xvy$ is again a step for any $x, y \in S^1$; closures preserve this property. □

Definition 58 (Nuclear congruence)**.** For a morphism $\varphi : S \to T$, the *nuclear congruence* $\sim_\varphi$ on S is defined by $x \sim_\varphi y \iff \varphi(x) = \varphi(y)$.

Theorem 59 (First isomorphism theorem)**.** *Let $\varphi : S \to T$ be a semigroup morphism and $\pi : S \to S/{\sim_\varphi}$ the quotient map. There exists a unique semigroup morphism $\widetilde{\varphi} : S/{\sim_\varphi} \to T$ with $\varphi = \widetilde{\varphi} \circ \pi$. Moreover, $\widetilde{\varphi}$ is an isomorphism $S/{\sim_\varphi} \cong \varphi(S)$.*

Proof. Define $\widetilde{\varphi}([x]) = \varphi(x)$; this is well-defined by definition of $\sim_\varphi$, multiplicative, and has image $\varphi(S)$. It is bijective onto $\varphi(S)$ with inverse given by $[x] \leftarrow \varphi(x)$. $\square$

Theorem 60 (Second isomorphism theorem for congruences)**.** *Let $\sim_1, \sim_2$ be congruences on S with $\sim_2$ coarser than $\sim_1$. Then there is a unique surjective morphism $\Pi : S/{\sim_1} \to S/{\sim_2}$ such that $\Pi \circ \pi_1 = \pi_2$, where $\pi_i : S \to S/{\sim_i}$ are the projections.*

Proof. Define $\Pi([s]_1) := [s]_2$; this is well-defined since $\sim_1 \subseteq \sim_2$, multiplicative, and clearly surjective. $\square$

Lemma 61 (Intersection embeds in a product)**.** *Let $(\sim_i)_{i\in I}$ be congruences on S and $\sim = \bigcap_i \sim_i$. Then $S/\sim$ embeds into $\prod_{i\in I} S/{\sim_i}$ as a subsemigroup.*

Proof. Consider $\pi = (\pi_i)_{i\in I} : S \to \prod_i S/{\sim_i}$. Its nuclear congruence is $\sim$. By Theorem 59, $S/\sim \cong \pi(S)$, a subsemigroup of the product. $\square$

Lemma 62 (Two distinct 0-minimal ideals give a product embedding)**.** *If S has (at least) two distinct 0-minimal ideals I_1, I_2, then S embeds into $S/I_1 \times S/I_2$ as a subsemigroup.*

Proof. Since $I_1 \cap I_2 = \{0\}$, the intersection of the Rees congruences $\equiv_{I_1}$ and $\equiv_{I_2}$ is equality. Apply Lemma 61. $\square$

Definition 63 (Congruences of ordered monoids)**.** Let $(M, \leq)$ be an ordered monoid. A *congruence of ordered monoids* is a stable preorder $\preccurlyeq$ on M that is coarser than $\leq$ and is compatible with multiplication: $x \preccurlyeq y \Rightarrow axb \preccurlyeq ayb$. Writing $x \sim y$ for the induced equivalence $x \preccurlyeq y$ & $y \preccurlyeq x$, the quotient $M/\sim$ carries a well-defined ordered-monoid structure with the order induced by $\preccurlyeq$, and the projection $M \to (M/\sim, \leq)$ is a morphism of ordered monoids.