Thesis

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September 2, 2025

0.1 Preorders (Test)

0.2 Test

Green's preorders are traditionally defined for semigroups by adjoining a unit element and working in the resulting monoid. Here we start directly with monoids, thereby avoiding the overhead of the ‘WithOne‘ construction and the need to explicitly adjoin an identity. Since a semigroup with an adjoined unit is essentially the same as a monoid, working on monoids simplifies many of the subsequent proofs. In future developments this code might be refactored to operate on semigroups with a unit, but the monoid formulation suffices for our purposes.

Definition 1 (Green's R-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_R y$ if there exists $z \in M$ such that $x = y \cdot z$. Equivalently, x lies in the principal right ideal generated by y.

Lemma 2 (Reflexivity of $\leq_R$)**.** *For every $x \in M$, $x \leq_R x$.*

Lemma 3 (Transitivity of $\leq_R$)**.** *For all $x, y, z \in M$, if $x \leq_R y$ and $y \leq_R z$ then $x \leq_R z$.*

Proof. Suppose $x \leq_R y$ and $y \leq_R z$. By definition there exist $v, u \in M$ with $x = y \cdot v$ and $y = z \cdot u$. Taking the witness $u \cdot v$ we compute $z \cdot (u \cdot v) = (z \cdot u) \cdot v = y \cdot v = x$, so $x \leq_R z$. □