Thesis

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Chapter 1

Special Elements in Semigroups

1.1 Local identities in semigroups

Throughout, let S be a semigroup with associative multiplication, written multiplicatively.

Definition 1 (Left/right/two-sided identities)**.** Let $e \in S$.

- e is a *left identity* if for all $s \in S$, $es = s$.
- e is a *right identity* if for all $s \in S$, $se = s$.
- e is an *identity* (two-sided) if it is both a left and a right identity; equivalently, for all $s \in S$, $es = s = se$.

Lemma 2 (Idempotence of one-sided identities)**.** *Let $e \in S$.*

- *If e is a left identity, then $ee = e$.*
- *If e is a right identity, then $ee = e$.*

Proof. For a left identity, apply the defining property to $s := e$ to get $ee = e$. For a right identity, apply the defining property to $s := e$ to get $ee = e$. □

Lemma 3 (Simplification lemma)**.** *Let $s \in S$ and let $e, f \in S$ be idempotents. If $s = esf$, then $es = s = sf$.*

Proof. Assume $s = esf$. Then

$$es \;=\; e(esf) \;=\; (ee)sf \;=\; esf \;=\; s,$$

using associativity and $e^2 = e$. Similarly,

$$sf \;=\; (esf)f \;=\; es(ff) \;=\; esf \;=\; s,$$

using associativity and $f^2 = f$. □

1.2 Zero elements and null semigroups

Throughout, let S be a semigroup with associative multiplication, written multiplicatively.

Definition 4 (Left/right/two-sided zeros)**.** Let $e \in S$.

- e is a *left zero* if for all $s \in S$, $es = e$.
- e is a *right zero* if for all $s \in S$, $se = e$.
- e is a *zero* (two-sided) if it is both a left and a right zero; equivalently, for all $s \in S$, $es = e = se$.

Lemma 5 (Idempotence of one-sided zeros)**.** *Let $e \in S$.*

- *If e is a left zero, then $ee = e$.*
- *If e is a right zero, then $ee = e$.*

Proof. For a left zero, apply the defining property to $s := e$ to get $ee = e$. For a right zero, apply the defining property to $s := e$ to get $ee = e$. □

Lemma 6 (Uniqueness of zero (at most one zero element))**.** *A semigroup has at most one zero element.*

Proof. Suppose $e, e' \in S$ are both zeros. Then $e = ee'$ since e' is a right zero, and $ee' = e'$ since e is a left zero. Hence $e = e'$. □

Definition 7 (Null semigroup)**.** A semigroup S is *null* if it has a zero element 0_S and for all $x, y \in S$ one has $xy = 0_S$.

1.3 Cancellativity

Throughout, let S be a semigroup with associative multiplication.

Definition 8 (Right/left/two-sided cancellative element)**.** Let $s \in S$.

- s is *right cancellative* if for all $x, y \in S$, $xs = ys \implies x = y$.
- s is *left cancellative* if for all $x, y \in S$, $sx = sy \implies x = y$.
- s is *cancellative* (two-sided) if it is both left and right cancellative.

Definition 9 (Right/left/two-sided cancellative semigroup)**.** A semigroup S is

- *right cancellative* if every $s \in S$ is right cancellative,
- *left cancellative* if every $s \in S$ is left cancellative,
- *cancellative* (two-sided) if every $s \in S$ is cancellative.

1.4 Inverses

Terminology note. The term “inverse” has two distinct usages. In group theory (and, more generally, in monoids), an inverse is defined using a distinguished identity element 1. This notion does not make sense in a bare semigroup that lacks a specified unit. Semigroup theory also uses a different, intrinsic notion of inverse that does not require a unit and is formulated purely in terms of the multiplication.

These notions behave differently:

- In an *infinite* monoid, an element may have several right group inverses and several left group inverses.
- In a *finite* monoid, each element has *at most one* right group inverse and *at most one* left group inverse; if both exist, they coincide (hence give a two-sided group inverse).
- In a semigroup (finite or infinite), an element may have several semigroup inverses, or none at all.

Definition 10 (Semigroup inverse)**.** Let S be a semigroup and $x \in S$. An element $x' \in S$ is a *semigroup inverse* of x if

$$xx'x = x \quad \text{and} \quad x'xx' = x'.$$

Definition 11 (Group inverse (monoid setting))**.** Let M be a monoid with identity 1 and let $x \in M$.

- A *right group inverse* of x is an element $x' \in M$ with $xx' = 1$.
- A *left group inverse* of x is an element $x' \in M$ with $x'x = 1$.
- A *group inverse* of x is an element $x' \in M$ that is both a right and a left group inverse, i.e. $xx' = x'x = 1$.

Lemma 12 (Group inverse $\Rightarrow$ semigroup inverse)**.** *Let M be a monoid and $x, x' \in M$. If x' is a group inverse of x (so $xx' = x'x = 1$), then x' is a semigroup inverse of x in the underlying semigroup:*

$$xx'x = x \quad \textit{and} \quad x'xx' = x'.$$

Proof. Compute $xx'x = (xx')x = 1 \cdot x = x$ and $x'xx' = x'(xx') = x' \cdot 1 = x'$, using associativity and the unit laws. $\square$