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MATH F651: Homework 1

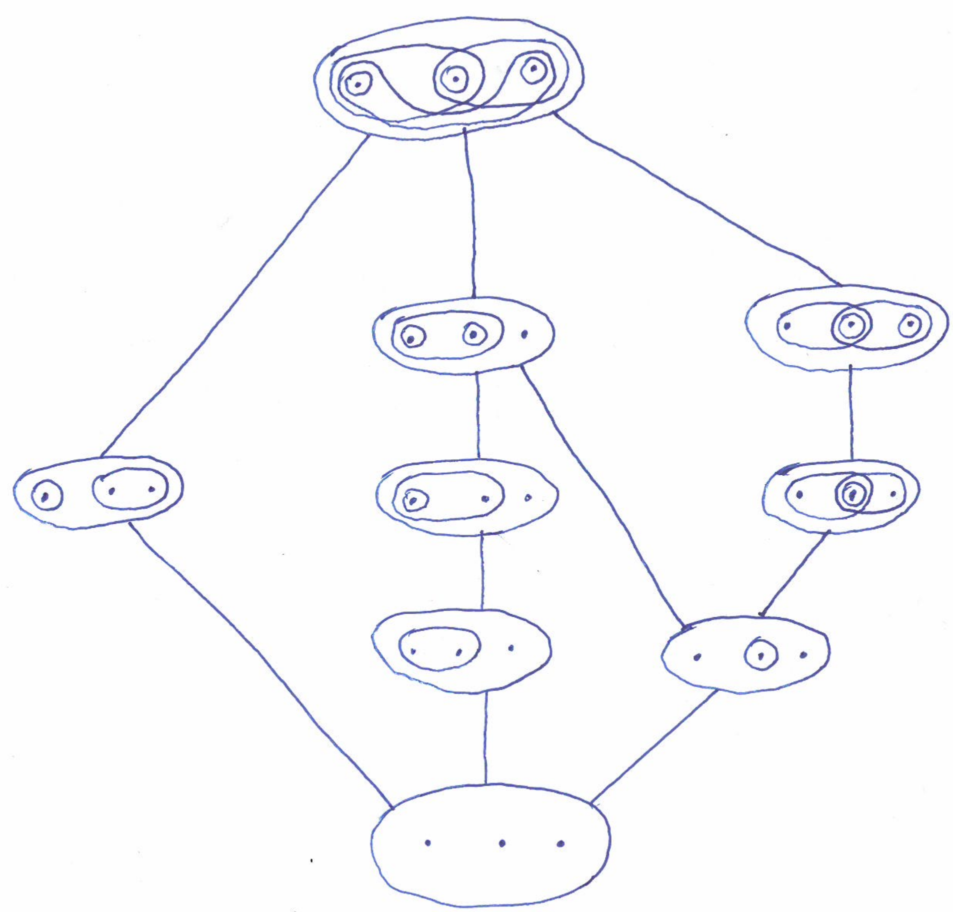
p. 83, # 1

Theorem 1. *Let X be a topological space X, and $A \subseteq X$. Suppose for each $x \in A$ there is an open set U with $x \in U \subseteq A$. Then A is open.*

Proof. For each $x \in A$, it is given that there is an open set U_x such that $x \in U_x \subseteq X$. Note that $A = \bigcup_{x \in A} U_x$ for if $a \in A$, by definition, $a \in U_a \subseteq \bigcup_{x \in A} U_x$ and if $a \in \bigcup_{x \in A} U_x$, then $a \in U_x \subseteq A$ for some $x \in A$. Being the union of open sets, A must hence be open. □

p. 83, # 2 Consider the nine topologies of Example 1 of section 12, and name them T_1 through T_9 so the first row has T_1, T_2, T_3, etc. Then for each of the 36 pairs, determine if they are comparable, and which is finer.

Answer: The comparability structure is presented below as a subset lattice:



p. 83, # 3

Proposition 2. *The collection $\mathcal{T}_c$ in Example 4 of section 12 is a topology on X.*

Proof. Note that $\emptyset = X \setminus X,\ X = X \setminus \emptyset$ so that these sets are, by definition, open in our topology.

Consider, now, any subcollection $\{U_i\}_{i\in I} \subseteq \mathcal{T}_c$ for some arbitrary index set I. Now, each complement $X \setminus U_i$ is countable or X itself. If all such complements are X itself, then necessarily, every U_i must be empty so that the union of this collection will be empty and the result follows. Thus, assume there is some U_j among the collection such that $X \setminus U_j$ is countable. Now, $X \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I}(X \setminus U_i) \subseteq X \setminus U_j$ which implies that $X \setminus \bigcup_{i\in I} U_i$ has to be countable.

Finally, let $U_1, U_2 \in \mathcal{T}_c$. Then, $X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$. Since the union of two countable sets is countable and since if one of the operands of the union is the whole set, the whole union becomes the whole set, we get that $X \setminus (U_1 \cap U_2)$ is countable or the whole set. □

Proposition 3. *The collection $\mathcal{T}_\infty = \{U \subseteq X \mid X \smallsetminus U$ is infinite or empty or $X\}$ is **not necessarily** a topology on X.*

Proof. Indeed, simply consider $X = \mathbb{Z}$ and $U_< = \mathbb{Z}^-,\ U_> = \mathbb{Z}^+$. Then $U_< \cup U_> \subset X$ is not an element of $\mathcal{T}_\infty$ as $X \setminus (U_< \cup U_>) = \{0\}$ is neither infinite, nor empty nor X.

□

p. 83, # 4 (a) Fix a collection $\{\tau_i\}_{i\in I}$ of topologies on X where I is an arbitrary index set. To avoid trivialities, assume that I is non-empty.

Proposition 4. *Then $\bigcap_{i\in I} \tau_i$ is a topology on X.*

Proof. Note that $\emptyset, X$ are in every one of the τ_i's since they each are topologies. So, by definition, they must belong to their combined intersections as well.

Consider, now, any subcollection $\{U_j\}_{j\in J} \subseteq \bigcap_{i\in I} \tau_i$ for some arbitrary index set J. Then note that each U_j belongs to $\bigcap_{i\in I} \tau_i$ so that it must belong to each topology τ_i. Hence, every τ_i, being a topology, must necessarily contain the arbitrary union of opens $\bigcup_{j\in J} U_j$. So, of course, $\bigcup_{j\in J} U_j \in \bigcap_{i\in I} \tau_i$ too.

Finally, let $U_1, U_2 \in \bigcap_{i\in I} \tau_i$. Then both U_1, U_2 must belong to every τ_i. Thus, as every τ_i must be closed under binary intersection, $U_1 \cap U_2$ must also belong to every τ_i i.e. $U_1 \cap U_2 \in \bigcap_{i\in I} \tau_i$. □

Proposition 5. *$\bigcup_{i\in I} \tau_i$ is **not necessarily** a topology on X.*

Proof. For a counterexample, let $X = \{a, b, c\}$ and $\tau_a = \{\emptyset, X, \{a\}\}, \tau_b = \{\emptyset, X, \{b\}\}$. Then τ_a, τ_b are indeed topologies on X. However, $\tau_a \cup \tau_b = \{\emptyset, X, \{a\}, \{b\}\}$ is **not** a topology, for it fails to contain $\{a, b\}$, the union of its two elements $\{a\}, \{b\}$. In fact, working on this theme also reveals that in certain examples, even binary intersection of elements may fail to be contained. □

(b)

Proposition 6. *There is a smallest topology on X containing each of the topologies τ_i.*

Proof. Even though $\bigcup_{i\in I}\tau_i$ is not necessarily a topology on X as demonstrated in (a), we can fix the shortcomings in (a) by letting it be a **subbasis** on X. Indeed, since I is non-empty, there is indeed some topology τ_j in the collection of topologies in discussion. So, for any $x \in X$, we can simply choose $X \in \tau_j \subseteq \bigcup_{i\in I}\tau_i$ to obviously get $x \in X \subseteq X$. So the only condition imposed on a subbasis is satisfied.

Now, it is claimed that the topology generated by this subbasis, $\mathcal{T}$, is the required smallest topology containing all the others. That it contains all the other topologies is clear: $\tau_j \subseteq \bigcup_{i\in I}\tau_i \subseteq \mathcal{T}$. Next, let $\mathcal{T}'$ be another topology on X that contains all the topologies in our collection and let $U \in \mathcal{T}$ be some open set in our candidate topology. Then U is the arbitrary union of finite intersection of elements of $\bigcup_{i\in I}\tau_i$ i.e. it has the form $\bigcup_{j\in J}(U_{j,1}\cap\cdots\cap U_{j,n})$ for some index set J, some $n \in \mathbb{Z}_+$ and where each $U_{j,k} \in \tau_{i(j,k)}$. But then, as $\mathcal{T}'$ contains each $\tau_{i(j,k)}$ and it is a topology, firstly, it must contain the finite intersection $U_{j,1}\cap\cdots\cap U_{j,n}$ (on account of containing each of them) and, as a consequence, it must contain the arbitrary union $\bigcup_{j\in J}(U_{j,1}\cap\cdots\cap U_{j,n})$ which is just U. Hence, $\mathcal{T} \subseteq \mathcal{T}'$. □

Proposition 7. *There is a largest topology on X contained in each of the topologies τ_i.*

Proof. This is precisely $\bigcap_{i\in I}\tau_i$. That it is a topology has already been shown in (a). The fact that it is the largest topology contained in all the others comes from the more general set-theoretic fact that the intersection of a collection of a set-indexed sets is the largest set contained in every set in that collection. □

(c) Let $X = \{a, b, c\}$ and let $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\emptyset, X, \{a\}\{b, c\}\}$ find the smallest topology containing τ_1, τ_2 and the largest topology contained in τ_1, τ_2.

Answer: As per the previously proved propositions, to find the smallest topology containing τ_1, τ_2, one must let $\tau_1 \cup \tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$ be a subbasis. In this case, $\{a, b\} \cap \{b, c\} = \{b\}$ is the only unaccounted for subset in $\tau_1 \cup \tau_2$. So it needs to added to the final topology which is: $\{\emptyset, X, \{a\}, \{b\}\{a, b\}, \{b, c\}\}$. Next, to find the largest topology contained in τ_1, τ_2, one merely needs to intersect the two to find that $\tau_1 \cap \tau_2 = \{\emptyset, X, \{a\}\}$.

p. 83, # 7 Consider the following topologies on $\mathbb{R}$:

$$
\begin{aligned}
\tau_1 &= \text{the standard topology} \\
\tau_2 &= \text{the topology of } \mathbb{R}_K \\
\tau_3 &= \text{the finite complement topology} \\
\tau_4 &= \text{the upper limit topology of having basis elements of the form } (a, b] \\
\tau_5 &= \text{the topology having basis subsets of the form } (-\infty, a) = \{x \in \mathbb{R} \,:\, x < a\}
\end{aligned}
$$

Determine for each of the topologies, which of the others it contains.

Answer: To start with, in the book, it has already been shown that $\mathbb{R}_K$ is strictly finer than the standard topology. It is also been shown that the lower limit topology is strictly finer than the standard topology and an analogous proof can be used to show that the upper limit topology is strictly finer than the standard topology. The question that remains to be settled is how the upper limit topology relates to $\mathbb{R}_K$. Here, it is claimed that the upper limit topology is strictly finer than $\mathbb{R}_K$.

Of course, basis elements of the form (a, b) in $\mathbb{R}_K$ can be generated by basis elements from the upper limit topology for the same reason that upper limit topology is finer than the standard topology. Also, basis elements of the form $(0, b) \setminus K$ can also be so generated; indeed, such basis elements can even be generated by the basis elements of the standard topology:

$$(0, b) \setminus K = \Big(\bigcup_{n \in \mathbb{Z}_+,\ 1/n \leq b} (1/(n+1), 1/n) \Big) \cup (1, b)$$

Consider now a basis element of the form $(a, b) \setminus K$ where $a < 0 < b$. Then, $(a, b) \setminus K = (a, 0] \cup \big((0, b) \setminus K\big)$. The left component $(a, 0]$ is a direct basis element of the upper limit topology while the right component $(0, b) \setminus K$, as already discussed, can be generated by the upper limit topology. Hence, the upper limit topology is finer than $\mathbb{R}_K$. On the other hand, note that the basis element of the upper limit topology, $(2, 3]$, cannot be generated in $\mathbb{R}_K$ for the same reason that it cannot be generated in the standard topology and the fact that the set K only has influence on intervals bounded by 0 and 1. Hence, the upper limit topology is strictly finer than $\mathbb{R}_K$.

For the other relations, note that the finite complement topology is coarser than the standard one because if $\mathbb{R} \setminus U$ is finite ($U \subseteq \mathbb{R}$), U can be decomposed into open rays and open intervals with endpoints chosen from that finite set. Also, the last topology, having rays unbounded below as bases, is coarser than the standard one as rays can be expressed as a union of open intervals: $(-\infty, a) = \bigcup_{n \in \mathbb{Z}_+} (-n, a)$. Finally, the finite complement topology and the last topology are incomparable because, for instance, the open set $(-\infty, 0) \cup (0, \infty)$, of the finite complement topology, cannot be expressed as a union of rays that are only unbounded below while, for instance, the open ray $(-\infty, 0)$, of the last topology, cannot be expressed as an open set in the finite complement topology because its complement is uncountable.

p. 83, # 8 (a)

Proposition 8. *The countable collection of real intervals $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} \ : \ a, b \in \mathbb{Q}\}$ is a basis that generates the standard topology on $\mathbb{R}$.*

Proof. This collection is a basis because, firstly, for any real x, it is possible (using the Archimedean property of reals) to find integers p, q such that $x \in (p, q)$. Next, if a real $x \in (a, b) \cap (c, d)$ is given, then $x \in (\max(a, c), \min(b, d)) \subseteq (a, b) \cap (c, d)$.

To show that this collection generates the same topology on the reals as the standard topology, we apply Lemma 13.2. One direction is clear because obviously this collection is already contained within the usual basis elements of the standard topology. So let the real x be an element of some interval (a, b) where a, b are real. Then by hypothesis, $a < x < b$. From Real Analysis, it is known that the rationals are dense among the reals i.e. between any two unequal real numbers, it is possible to find a rational. Hence, this allows us to find rationals p, q such that $a < p < x < q < b$ so that $x \in (p, q) \subset (a, b)$. $\square$

(b)

Proposition 9. *The collection of real intervals $\mathcal{C} = \{[a, b) \subseteq \mathbb{R} \ : \ a, b \in \mathbb{Q}\}$* ***does not*** *generate the lower limit topology on $\mathbb{R}$.*

Proof. Consider the following basis element in the lower limit topology for $\mathbb{R}$: $[\sqrt{2}, 2)$. Then there cannot exist any element from the above collection that is contained within this interval, while containing $\sqrt{2}$. For suppose there were rationals p, q such that $\sqrt{2} \in [p, q) \subseteq [\sqrt{2}, 2)$. Then certainly, $\sqrt{2} \leq p$ as they are the leftmost endpoints of their respective intervals, one of which is contained in the other. At the same time, since $\sqrt{2} \in [p, q)$, $\sqrt{2} \geq p$. So, the rational p must equal the irrational $\sqrt{2}$, which is absurd. $\square$

p. 91, # 4

Proposition 10. *A map $f : X \to Y$ between topological spaces X, Y is said to be an* ***open map*** *if for every open set U of X, the set $f(U)$ is open in Y. Then, the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.*

Proof. Let $U \subseteq X \times Y$ be an open set of the product topology on $X \times Y$. If U is empty, both projection maps would map U to the empty set, which is surely open in all topological spaces. Otherwise, U is of the form $\bigcup_{i \in I}(V_i \times W_i)$ for an arbitrary non-empty index set I where the V_i, W_i's are basis elements of X, Y respectively (not all of which are empty). But, $\pi_1(U) = \pi_1(\bigcup_{i \in I}(V_i \times W_i)) = \bigcup_{i \in I} V_i \subseteq X$ is open in X as it is an arbitrary union of basis elements of X. Similarly, $\pi_2(U) = \pi_2(\bigcup_{i \in I}(V_i \times W_i)) = \bigcup_{i \in I} W_i \subseteq Y$ is open in Y. $\square$

p. 91, # 9

Proposition 11. *The dictionary order topology on $\mathbb{R}\times\mathbb{R}$ is the same as the product topology on $\mathbb{R}_d\times\mathbb{R}$ where $\mathbb{R}_d$ is the real line with the discrete topology. Furthermore, this topology is strictly finer than the standard topology on $\mathbb{R}^2$.*

Proof. Since $\mathbb{R}\times\mathbb{R}$ with the dictionary order has no largest or smallest element and it obviously has more than one point, the basis for its order topology takes the form of $(a\times b, c\times d)_{\text{dict}}$ for reals a,b,c,d where $a\times b <_{\text{dict}} c\times d$ or $(a\times b, c\times d)_{\text{dict}} = \emptyset$. On the other hand, since the singletons $\{x\}$, for any real x, form a basis for the discrete topology on the reals and intervals of the form (a,b), for reals $a<b$ or $(a,b)=\emptyset$, along with rays of the forms $(a,\infty),\ (-\infty,b),\ (-\infty,\infty)=\mathbb{R}$, consist a basis for the standard topology on the reals, a basis for the product topology on $\mathbb{R}_d\times\mathbb{R}$ consists of subsets of the forms $\{x\}\times(a,b),\ \{x\}\times(a,\infty),\ \{x\}\times(-\infty,b),\ \{x\}\times\mathbb{R}$.

It suffices to show that each basis element of one topology can be generated by basis elements of the other topology. Thus, consider any basis element $(a\times b, c\times d)_{\text{dict}}$ of the dictionary order topology, as described above. If this is the empty set, one is done. Otherwise, $a\times b <_{\text{dict}} c\times d$ which immediately eliminates the case $a>c$. If $a=c$, it must be that $b<d$. Then, clearly, $(a\times b, a\times d)_{\text{dict}}$ is just another notation for $\{a\}\times(b,d)$ which is indeed a basis element of the product topology on $\mathbb{R}_d\times\mathbb{R}$. On the other hand, if $a<c$, then $(a\times b, c\times d)_{\text{dict}}$ consists of the vertical segment above (but not including) (a,b), the vertical segment below (but not including) (c,d) and the entire vertical strip between (but not including) a and c. This basis element can be generated by basis elements from the product topology on $\mathbb{R}_d\times\mathbb{R}$ thus:

$$(a\times b, c\times d)_{\text{dict}} = \Big(\{a\}\times(b,\infty)\Big)\cup\Big(\bigcup_{x\in(a,c)}\{x\}\times\mathbb{R}\Big)\cup\Big(\{c\}\times(-\infty,d)\Big)$$

Conversely, the basis elements of the product topology on $\mathbb{R}_d\times\mathbb{R}$ may be generated by the basis elements of the dictionary order topology on $\mathbb{R}\times\mathbb{R}$ thus:

$$\begin{aligned}
\{x\}\times(a,b) &= (x\times a, x\times b)_{\text{dict}}\\
\{x\}\times(a,\infty) &= \bigcup_{b\in\mathbb{R},\ b>a}(x\times a, x\times b)_{\text{dict}}\\
\{x\}\times(-\infty,b) &= \bigcup_{a\in\mathbb{R},\ a<b}(x\times a, x\times b)_{\text{dict}}\\
\{x\}\times\mathbb{R} &= \bigcup_{R\in\mathbb{R}_+}(x\times -R, x\times R)_{\text{dict}}
\end{aligned}$$

Hence, the two topologies coincide. Finally, to show that this topology is finer than the usual topology on $\mathbb{R}^2$, consider an arbitrary basis element of the standard topology on $\mathbb{R}^2$: $(a,b)\times(c,d)$ where a,b,c,d are reals and either $a<b,\ c<d$ or the set is empty. This is simply an open rectangle in $\mathbb{R}^2$ and it can easily be generated by basis elements of the product topology on $\mathbb{R}_d\times\mathbb{R}$ as follows. If this basis element is empty, we are done. Otherwise, the open rectangle is just a collection of adjacent vertical segments:

$$(a,b)\times(c,d) = \bigcup_{x\in(a,b)}\{x\}\times(c,d)$$

To demonstrate strictness, note, for example, that the vertical segment $\{0\}\times(0,1)$ in the product topology on $\mathbb{R}_d\times\mathbb{R}$ cannot contain any open rectangles inside it (save the trivial empty rectangle) because the segment has no width. Hence, no point in that segment can be covered by any basis element of the standard topology while being contained within it. $\square$