Bounded Ultraspaces

Eigil Fjeldgren Rischel

April 22, 2019

1 Introduction

In [1], Lurie sketches out how certain concepts from classical categorical first-order logic can be generalized to the setting of higher categories. In particular, he defines a generalization of the notion of *ultracategory* to allow for ultrastructures on ∞-categories.

In the classical setting, there is an equivalence of categories between (small) discrete ultracategories, which we may think of as *ultrasets*, and compact Hausdorff spaces. Lurie claims that this can be extended to classify those ultra-∞-categories whose underlying category is a (bounded) Kan complex, in the following way:

Theorem 1.1. Let $\mathcal{U}$ be the category of ultra-∞-categories whose underlying ∞-category is a *bounded* Kan complex, i.e. a Kan complex with only finitely many nonzero homotopy groups. Then we have the following:

(1) $\mathcal{U}$ is an ∞-*pretopos.*

(2) The subcategory of discrete objects, $\mathcal{U}_{\leq 0}$, is equivalent to the (1-) category of ultrasets.

(3) Among ∞-pretopoi, $\mathcal{U}$ enjoys the following universal property: for all ∞-pretopoid $\mathcal{C}$, the natural map

$$\mathrm{Fun}^{\mathrm{pre}}(\mathcal{U}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{pre}}(\mathcal{U}_{\leq 0}, \mathcal{C}_{\leq 0})$$

is an equivalence.

The purpose of this note is to give an explanation of these terms, and prove theorem 1.1. We will prove each statement separately, as theorem 3.9, proposition 3.11, and theorem 3.13.

Remark 1.1. It is more or less clear that theorem 1.1 identifies the category $\mathcal{U}$ up to equivalence among ∞-pretopoi. In [1], Lurie claims that one can replace $\mathcal{U}_{\leq 0}$ with any ordinary pretopos, and form a left adjoint to the functor $(-)_{\leq 0}$ from ∞-pretopoi to pretopoi, which he denotes $(-)^{+}$. In this language, we could state theorem 1.1 as $\mathcal{U} = \mathrm{CHaus}^{+}$ (as Lurie does). We will not make a study of this functor in general, so we stick with the explicit formulation above.

Remark 1.2. Dealing with ultra-∞-categories involves some extra set-theoretic issues. This is essentially analogous to the issues that occur for usual ultracategories, which stem from the fact that the collection of operations involved in an ultra(∞-)category is indexed by the collection of all ultrafilters in all sets. We will not make any particular effort to discuss these issues. See the discussion in [5, Warning 0.0.11]

2 Pretopoi

2.1 Ordinary pretopoi

Definition 2.1. An *equivalence relation* on an object X in a category $\mathcal{C}$ is a subobject $R \hookrightarrow X \times X$ (i.e. a monomorphism) with the property that, for each $A \in \mathcal{C}$, the induced subset

$$\mathrm{Map}(A, R) \subset \mathrm{Map}(A, X \times X) \cong \mathrm{Map}(A, X) \times \mathrm{Map}(A, X)$$

is an equivalence relation on $\mathrm{Map}(A, X)$.

An equivalence relation is *effective* if a coequalizer $R \rightrightarrows X \times X \to C$ exists, and moreover R is the pullback $X \times_C X$.

A morphism $f : X \to C$ is an *effective epimorphism* if

$$X \times_C X \rightrightarrows X \to C$$

is a coequalizer.

Definition 2.2. We say that the coproduct of X and Y, $X \sqcup Y$, is *disjoint* if $X \hookrightarrow X \sqcup Y$ is a monomorphism, and so is $Y \hookrightarrow X \sqcup Y$, and moreover the pullback $X \times_{X \sqcup Y} Y$ is an initial object.

Definition 2.3. A *pretopos* is a 1-category $\mathcal{C}$ with the following properties:

1. $\mathcal{C}$ has finite limits.
2. All equivalence relations in $\mathcal{C}$ are effective.
3. $\mathcal{C}$ has finite coproducts, and coproducts are disjoint.
4. The collection of effective epimorphisms is stable under pullback.
5. The formation of finite coproducts is preserved by pullback. By this we mean that the pullback functor $\mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ preserves finite coproducts for each $f : X \to Y$.

A functor $\mathcal{C} \to \mathcal{D}$ between pretoposes is a *pretopos functor* if it preserves finite coproducts, finite limits, and effective epimorphisms. We denote by $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{D})$ the full subcategory of the functor category spanned by the pretopos functors.

Example 2.4. The categories Fin and Set are both pretopoi.

Definition 2.5. For a pretopos $\mathcal{C}$, the category of pretopos functors $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathrm{Set})$ is denoted by $\mathrm{Mod}(\mathcal{C})$, and referred to as the category of *models* of $\mathcal{C}$.

Pretoposes are the basis for a categorical approach to first-order logic. To each first-order theory T, one can associate a *syntactic pretopos* $\mathrm{Syn}(T)$. From this, one can recover the models of the theory as $\mathrm{Mod}(\mathrm{Syn}(T))$.

2.2 ∞-pretopoi

To define the suitable ∞-categorical version of pretopoi, we will need some preliminary notions. In particular, we will need the notion of *groupoid object* described in [3, Def. 6.1.2.6]. Observe that if $\mathcal{C}$ has finite limits (this is the only case we are interested in), this notion admits the following reformulation:

Definition 2.6. Let $\mathcal{C}$ be an ∞-category. Suppose $\mathcal{C}$ has finite limits. A *groupoid object* in $\mathcal{C}$ is a simplicial object $D : \Delta^{op} \to \mathcal{C}$ with the property that the map $D(\Delta^n) \to D(\Lambda^n_k)$ is an equivalence in $\mathcal{C}$ for each $n \geq 2, 0 \leq k \leq n$.

Here $D(\Lambda^n_k)$ is defined in the obvious way as a limit in $\mathcal{C}$.

We omit a proof of this.

One should think of a groupoid object as an internal Kan complex in $\mathcal{C}$ - this is like a homotopy coherent equivalence relation.

Definition 2.7. We denote by Δ_+ the augmented simplicial category, which is Δ with an extra initial object $[-1]$

Remark 2.8. Note that $\Delta_+ \simeq \Delta^{\triangleleft}$, and $\Delta_+^{op} \simeq (\Delta^{op})^{\triangleright}$. In particular, colimit diagrams of simplicial objects are augmented simplicial objects.

Definition 2.9. Let $\mathcal{C}$ be an ∞-category, and let $D : \Delta_+^{op} \to \mathcal{C}$ be an augmented simplicial object. Then there is always a canonical commutative diagram

$$\begin{tikzcd} D_1 \arrow[r] \arrow[d] & D_0 \arrow[d] \\ D_0 \arrow[r] & D_{-1} \end{tikzcd}$$

We say that D is a *Cech nerve* if $D|_{\Delta^{op}}$ is a groupoid, and this diagram is a pullback.

Proposition 2.10 (HTT 6.1.2.11)**.** Let D be an augmented simplicial object. Then D is a Cech nerve if an only if it is right Kan extended from $(\Delta_+^{\leq 0})^{op}$

Remark 2.11. It follows that a Cech nerve $U : \Delta_+^{op} \to \mathcal{C}$ is uniquely determined by the map $u : U([0]) \to U([-1])$. We also say that it is *the Cech nerve of* u

Definition 2.12. A map $f : X \to Y$ is an *effective epimorphism* if it admits a Cech nerve, which is furthermore a colimit diagram (so that Y is the geometric realization of the underlying simplicial object).

A groupoid object $X_\bullet$ is effective if it admits a geometric realization, and the corresponding colimit diagram is a Cech nerve. Concretely, this means the geometric realization $|X_\bullet|$ exists, and the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & |X_\bullet| \end{array}$$

is a pullback.

Definition 2.13. We call a map $X \to Y$ of in $\mathcal{S}$ a *monomorphism* if it induces an injection on π_0 and an equivalence on all higher homotopy groups, for all basepoints. Equivalently, we may ask that it factors as $X \simeq \tilde{X} \hookrightarrow Y$, where $\tilde{X}$ is a subspace of Y consisting of certain path components.

We call a map $X \to Y$ in a general ∞-category $\mathcal{C}$ a monomorphism if, for each $A \in \mathcal{C}$, we have $\mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ a monomorphism in the above sense.

One easily checks that these definitions agree when $\mathcal{C} = \mathcal{S}$

Remark 2.14. With this definition in hand, the notion of disjoint coproducts makes sense in ∞-categories as well. For brevity we do not write out the definition again.

Definition 2.15. An ∞-pretopos is an ∞-category $\mathcal{C}$ with the following properties:

1. $\mathcal{C}$ has all finite limits.
2. $\mathcal{C}$ has all finite coproducts, and they are preserved by pullback, and disjoint.
3. All groupoid objects in $\mathcal{C}$ are effective.
4. The collection of effective epimorphisms is stable under pullback. Equivalently, taking colimits of groupoid objects is preserved by pullback.

A *functor of ∞-pretopoi*, or just a *pretopos functor* is a functor which preserves finite limits, finite coproducts and effective epimorphisms. We again denote by $\mathrm{Fun}^{\mathrm{pre}}(\mathcal{C}, \mathcal{D})$ the full subcategory of the functor category spanned by the pretopos functors.

A *model* of an ∞-pretopos $\mathcal{C}$ is a functor of ∞-pretopoi $\mathcal{C} \to \mathcal{S}$

Remark 2.16. Our use of the same notation for the categories of pretopos functors in the cases of ∞-pretopoi and ordinary pretopoi is somewhat unfortunate, but should not cause confusion in practice.

Warning 2.17. An 1-category which is an ordinary pretopos is generally *not* an ∞-pretopos. The problem is that groupoid objects in e.g. sets are the same thing as groupoids, *not* equivalence relations. Given a groupoid in sets, the geometric realization is just π_0, and the set of morphisms is not the pullback you want (this pullback is just the equivalence relation on the set of objects).

Definition 2.18. An object X of an ∞-category $\mathcal{C}$ is *n-truncated* if the space $\operatorname{Map}(Y, X)$ is n-truncated for all objects $Y \in \mathcal{C}$ (i.e $\pi_i \operatorname{Map}(Y, X) = 0$ for $i > n$).

The full subcategory of $\mathcal{C}$ spanned by the n-truncated objects is denotes $\mathcal{C}_{\leq n}$ We call an object *discrete* if it is 0-truncated.

Remark 2.19. By [3, Prop. 2.3.4.18], $\mathcal{C}_{\leq 0}$ is (equivalent to the nerve of) an ordinary category.

Example 2.20. $\mathcal{S}_{\leq 0} \simeq \mathrm{Set}$.

We have the following relation between pretopoi and ∞-pretopoi:

Proposition 2.21. Suppose $\mathcal{C}$ is an ∞-pretopos. Then $\mathcal{C}_{\leq 0}$ is an ordinary pretopos. Moreover, if $\mathcal{D}$ is another ∞-pretopos, any pretopos functor $F : \mathcal{C} \to \mathcal{D}$ restricts to a functor $F : \mathcal{C}_{\leq 0} \to \mathcal{D}_{\leq 0}$, which is a pretopos functor.

Proof. It is clear that $\mathcal{C}_{\leq 0}$ is stable under finite limits.

First, we will verify that $\mathcal{C}_{\leq 0}$ is stable under finite coproducts. Let $X, Y \in \mathcal{C}_{\leq 0}$ be given, and let $A \in \mathcal{C}$ be any object. Suppose $f : A \to X + Y$ is a map. We must show that the space of maps homotopic to this map is contractible. To do so, note that since pullbacks preserve coproducts, we may form this diagram:

$$\begin{tikzcd}
A_X \arrow[r, dashed] \arrow[d] & X \arrow[d] \\
A \arrow[r, "f"] & X + Y \\
A_Y \arrow[r, dashed] \arrow[u] & Y \arrow[u]
\end{tikzcd}$$

Here the left column is also a coproduct diagram, and A_X, A_Y are the obvious pullbacks.

Now, A_X and A_Y depend on f, but we may choose them to depend only on its homotopy class. Now giving a map $A \to X + Y$ is equivalent to giving two maps $A_X, A_Y \to X + Y$. But for this map to be homotopic to f, certainly these maps must factor over X and Y. Hence we may equivalently ask for maps $A_X \to X, A_Y \to Y$, since the maps $X, Y \to X + Y$ are monomorphisms.

So we may identify the connected component of f in $\operatorname{Map}(A, X + Y)$ with a subspace of the product $\operatorname{Map}(A_X, X) \times \operatorname{Map}(A_Y, Y)$. Hence this connected component is discrete, hence contractible.

Now let us consider equvalence relations. Given an equivalence relation $X_1 \rightrightarrows X_0$ in $\mathcal{C}_{\leq 0}$, we may extend it to a simplicial diagram with $X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1$ the $n-1$-fold pullback, and the natural face/degeneracy maps coming from the fact that we started with an equivalence relation. Since these objects are discrete, there is no difficulty defining this diagram, and it is clearly a groupoid. Hence is has a geometric realization, which can easily be seen to be discrete. Moreover, this geometric realization is also a quotient for the original diagram.

Hence $\mathcal{C}_{\leq 0}$ has finite products, quotients by equivalence relations and coproducts. The compatibility required to make this a pretopos is implied by the fact that these things are computed in $\mathcal{C}$, as described.

The fact that a pretopos functor preserves discreteness is an immediate consequence of [4, Prop. A.6.7.1]. The fact that the restricted functor is also a pretopos functor is immediate from the above. $\square$

3 Ultracategories

To define the ∞-categorical version of ultracategories, we will need to consider the category of *free stone spaces*. The idea is that maps between free stone spaces describe the various ways of taking ultraproducts.

Definition 3.1. We define two full subcategories of the category of topological spaces:

$$\mathrm{Stone}^{fr} \subseteq \mathrm{Stone} \subseteq \mathrm{Top}$$

Stone is the full subcategory spanned by the *Stone spaces*, those spaces which are compact, locally Hausdorff, and totally disconnected. Recall that an example of a Stone space is βS, the Stone-Cech compactification of a set (i.e. a discrete space) S. Stone^{fr} is the full subcategory of Stone spanned by the spaces βS for each set S

Definition 3.2. An *ultra-∞-category* consists of a locally Cartesian fibration of ∞-categories $\pi : \mathcal{E} \to \mathrm{Stone}^{fr}$ satisfying the following properties.

(a) Let $\mathcal{E}_{\beta I}$ denote the fiber of π over βI. For each set I, and each element $i \in I$, there is an induced functor $\mathcal{E}_{\beta I} \to \mathcal{E}_{\{i\}}$. The induced functor

$$\mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}$$

is an equivalence of ∞-categories.

(b) Suppose we are given composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{E}$, with both f, g locally π-Cartesian. Suppose furthermore $\pi(f)$ preserves isolated points. Then also $g \circ f$ is locally π-Cartesian.

A *functor of ultra-∞-categories* is a functor over Stone^{fr} which preserves locally Cartesian morphisms. This defines the category of ultra-∞-categories as a subcategory of $\widehat{\mathcal{C}\mathrm{at}}_{\infty / \mathrm{Stone}^{fr}}$

Remark 3.3. See [5] for an extended discussion of ultracategories in a 1-categorical setting. We will make use of [5, Thm. 3.1.5], which identifies *ultrasets*, i.e ultracategories with a discrete underlying category, and compact Hausdorff spaces.

We will work with the following class of ultra-∞-categories.

Definition 3.4. An ultra-∞-category is an *ultraspace* if the underlying ∞-category $\mathcal{E}_{\{*\}}$ is a small Kan complex. If this Kan complex is furthermore *bounded*, i.e. has only finitely many nonzero homotopy groups, we say that the ultraspace is bounded.

We define $\mathcal{U}$ to be the ∞-category of bounded ultraspaces.

Lemma 3.5. If $\pi : \mathcal{E} \to \mathrm{Stone}^{fr}$ is a locally Cartesian fibration for which all fibers are Kan complexes, it is a Cartesian fibration.

Proof. By [3, Rmk. 2.4.2.8], the condition that π is Cartesian is the condition that certain natural transformations between functors $\mathcal{E}_X \to \mathcal{E}_Y$ are natural equivalences. But if $\mathcal{E}_Y$ is a Kan complex, all such natural transformations are equivalences, so this is automatic. $\square$

Theorem 3.6. We may identify $\mathcal{U}$ with the full subcategory of $\mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$ spanned by those functors F which map coproducts in Stone^{fr} to products in $\mathcal{S}$, and which furthermore have $F(\{*\})$ a bounded space.

Proof. First suppose A is an ultraspace. Then clearly each $A_{\beta I}$ is also a space, since it is a product of spaces. Hence by the lemma, the category of ultraspaces is a full subcategory of the category of Cartesian fibrations over Stone^{fr}, which is equivalent to $\mathrm{Fun}(\mathrm{Stone}^{fr}, \widehat{\mathcal{C}\mathrm{at}_\infty})$. The conditions of being an ultraspace correspond to mapping coproducts to products, and having image in $\mathcal{S} \subseteq \widehat{\mathcal{C}\mathrm{at}_\infty}$. The further condition of being bounded corresponds to $F(*)$ being bounded. $\square$

Remark 3.7. For a bounded ultraspace F, $F(\beta I) = F(*)^I$ is bounded for all sets I.

Lemma 3.8. The discrete objects of $\mathcal{U}$ are precisely those ultraspaces X where $X(*)$ is a discrete space.

Proof. Under the identification above, we observe that $\mathrm{Map}_{\mathrm{Stone}^{fr}}(-, *)$ acquires the structure of an ultraspace, and in fact is the terminal ultraspace. Let us denote it $*$. By the Yoneda lemma, $\mathrm{Map}(*, X) \simeq X(*)$, so if X is discrete, $X(*)$ is discrete.

For the other direction, suppose $X(*)$ is discrete. Then also $X(A) \simeq X(*)^A$ is discrete for all $A \in \mathrm{Stone}^{fr}$. Let Y be an arbitrary ultraspace, and write $Y = \mathrm{colim}_i \mathrm{Map}(-, A_i)$. (Where this colimit is computed in $\mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$).

Then

$$\mathrm{Map}(Y, X) \simeq \lim_i \mathrm{Map}(\mathrm{Map}(-, A_i), X) \simeq \lim_i X(A_i)$$

it follows that $\mathrm{Map}(Y, X)$ is discrete, so that X is discrete. $\square$

Theorem 3.9. $\mathcal{U}$ is an ∞-pretopos.

Proof. It is clear that $\mathcal{U} \subset \mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$ is stable under finite limits, so $\mathcal{U}$ admits finite limits.

Let us construct finite coproducts in $\mathcal{U}$. We will accomplish this by constructing finite coproducts in the category of functors which preserve finite products, $\mathrm{Fun}^{\times}(\mathrm{Stone}^{fr,op}, \mathcal{S})$, and showing that $\mathcal{U}$ is stable under them.

To do this, observe that we may identify $\mathrm{Fun}^{\times}(\mathrm{Stone}^{fr,op}, \mathcal{S})$ with a full subcategory of the category

$$\mathrm{Fun}^{lax}(\mathrm{Stone}^{fr,op}, \mathcal{S})$$

of lax symmetric-monoidal functors, where we give both $\mathrm{Stone}^{fr,op}$ and $\mathcal{S}$ the Cartesian monoidal structure.

This may further be indentified with $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S}))$, where we give the functor category the Day convolution monoidal structure.

By [2, Prop 3.2.4.7], coproducts in this category exist, and can be identified with tensor products in the underlying category, i.e with Day convolution of the concrete functors in question.

So it only remains to verify that $\mathcal{U}$ is stable under this construction. It is given by the formula

$$(A \circledast B)(\beta I) = \coprod_{J,K:\ \beta J \sqcup \beta K = \beta I} A(\beta J) \times B(\beta K)$$

Suppose that both A and B preserve products. Then the above is equivalent to

$$\coprod_{J,K:\ J \sqcup K = I} A(*)^J \times B(*)^K$$
$$\simeq (A(*) \sqcup B(*))^I$$

Since pullbacks in $\mathcal{U}$ are simply computed pointwise, this formula for the coproduct makes the verification of disjointness and universality of coproducts straightforward.

It remains to see that groupoid objects in $\mathcal{U}$ are effective, and effective epimorphisms are universal. Both of these claims follow directly from the fact that $\mathcal{U}$ is stable under geometric realization of groupoid objects inside $\mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$, which in turn follows from the fact that geometric realization of groupoid objects commutes with products (even infinite ones).

(Here we are also silently using the fact that a diagram in $\mathcal{U}$ is a groupoid if and only if it is a groupoid in $\mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$, if and only if it is levelwise a groupoid).

To prove this, first note that a product of groupoids is again a groupoid. Hence we are essentially asking if the colimit functor $\mathcal{G}\mathrm{pd}(\mathcal{S}) \to \mathcal{S}$ preserves products. We may use the equivalence of [3, Cor. 6.2.3.5] to identify this with the target functor $\mathrm{Fun}(\Delta^1, \mathcal{S})_{eff} \to \mathcal{S}$. Here the subscript denotes the full subcategory of effective epimorphisms. Hence it suffices to show that this subcategory is stable under products. This follows immediately from [3, Cor. 7.2.1.15] $\square$

Construction 3.10. Suppose we have a bounded ultraspace $\mathcal{E}$, and that the underlying space $X = X^{op}$ is discrete. Then for each set I and each ultrafilter μ in I, we obtain a functor

$$X^I \xleftarrow{\simeq} \mathcal{E}_{\beta I} \to \mathcal{E}_{\{\mu\}} = X$$

which is well-defined up to contractible ambiguity. But since X^I and X are sets, this is just a well-defined function. Denote this function $\int_I \bullet d\mu$.

Proposition 3.11. The maps of construction 3.10 determine an ultrastructure on X. Moreover, this assignment determines a functor

$$\mathcal{U}_{\leq 0} \to \mathrm{CHaus}$$

under the identification of compact Hausdorff spaces with ultrasets, which is an equivalence of categories.

Remark 3.12. In fact, one should expect to derive this from a more general proposition, namely that ultra-∞-categories with underlying object a 1-category are equivalent to ordinary ultracategories. But working with just sets brings some simplifications, which we will take advantage of.

Proof. Note that we do not have to define any more data, since X is a set. We just have to check the conditions of [5, Def. 3.1.1]

By construction of the equivalence $X^I \simeq \mathcal{E}_{\beta I}$, if $\mu = \delta_i$ for $i \in I$, the map $\int_I \bullet d\mu$ is equivalent to the ith coordinate projection - in other words, it *equals* it.

Let us suppose we are given an S-indexed family of ultrafilters in T. This is equivalently a function $\nu : S \to \beta T$, which is precisely a map

$$\beta S \to \beta T \text{ in } \mathrm{Stone}^{fr}$$

Furthermore, we are given an ultrafilter μ on S, which we regard as a point

$$\{\mu\} \hookrightarrow \beta S$$

We have a triangle in Stone^{fr}

$$\begin{tikzcd} \{*\} \arrow[r, "\mu"] \arrow[dr, "\int_S \nu_s d\mu"'] & \beta S \arrow[d] \\ & \beta T \end{tikzcd}$$

(The diagonal is simply the composite).

This gives a triangle of functions:

$$\begin{tikzcd} X & X^S \arrow[l, "\int_S \bullet d\nu"'] \\ & X^T \arrow[ul, "\int_T \bullet d(\int_S \nu_s d\mu)"] \arrow[u, "(\int_T \bullet d\nu_s)_{s \in S}"'] \end{tikzcd}$$

Now the Fubini condition is that this diagram commutes. But this follows directly from lemma 3.5

To define the functor, note that being an ultrafunctor(ultrafunction) between ultrasets is a property, not extra structure. So it suffices to show that, given an ultrafunctor $\mathcal{E} \to \mathcal{E}'$ between two 0-truncated ultraspaces, the underlying functor preserves ultraproducts, which is straightforward.

To see that this functor is fully faithful, note that ultrafunctors between 0-truncated ultraspaces are uniquely determined by the underlying function, since one always has commutative diagrams

$$\begin{array}{ccc} \mathcal{E}_{\beta I} & \xrightarrow{\sim} & X^I \\ \downarrow & & \downarrow \\ \mathcal{E}'_{\beta I} & \xrightarrow{\sim} & X'^I \end{array}$$

(and the higher coherence information involved in defining an ultrafunctor is trivial).

Lastly, to see essential surjectivity, let an ultraset X be given. Then we simply define the category $\mathcal{E}$ to have as objects pair $(\beta I, f)$, where βI is a free Stone space and $f : \beta I \to X$ is an ultrafunction (i.e a continuous function). A morphism $(\beta I, f) \to (\beta J, g)$ is a map $\beta I \to \beta J$ rendering the obvious diagram commutative.

Now it is easy to verify that $\mathcal{E} \to \mathrm{Stone}^{fr}$ is an ultra-∞-category which goes to X. $\square$

Theorem 3.13. The restriction functor

$$\mathrm{Fun}^{\mathrm{pre}}(\mathcal{U}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{pre}}(\mathcal{U}_{\leq 0}, \mathcal{C}_{\leq 0})$$

is an equivalence for each ∞-pretopos $\mathcal{C}$.

Remark 3.14. Note that this theorem asserts an equivalence between an ∞-category and a 1-category. So this is a fairly strong statement.

Proof. By theorem 3.6 and lemma 3.8, we may regard $\mathcal{U}_{\leq 0}$ as the full subcategory of $\mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$ spanned by those functors F which preserve products and such that $F(*)$ (and hence $F(\beta I)$ for each I) is a discrete space, i.e a set.

Let $\mathcal{G} \subseteq \mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$ be the category generated by the representables under geometric realizations of groupoid objects. We claim that (under the identification above), $\mathcal{G} = \mathcal{U}$ First, our proof of theorem 3.9 shows that $\mathcal{U}$ is stable under geometric realization of groupoids, and it also contains the representables. This shows $\mathcal{G} \subseteq \mathcal{U}$

To show $\mathcal{U} \subseteq \mathcal{G}$, first consider a 0-bounded (i.e discrete) $X \in \mathcal{U}$. By proposition 3.11, we may view X as an ultraset. Now we may consider the map $\beta X \to X$ which describes the ultrastructure on X. This map is a continous map of compact Hausdorff spaces, so we may view it as a map in $\mathcal{U}$ from the functor represented by βX to X. This is levelwise surjective (on π_0), so by our proof of theorem 3.9, it is an essential epimorphism. Hence X is the geometric realization of the Cech nerve of this map. However, the compact Hausdorff spaces appearing in the Cech nerve are all limits of representable functors, hence representable. Hence X is in $\mathcal{G}$

Now let $X \in \mathcal{U}$ be an arbitary bounded ultraspace. Suppose X is n-truncated, and suppose by induction all $n-1$-truncated ultraspaces are in $\mathcal{G}$ Now $\pi_0 X$ inherits an ultrastructure, since π_0 preserves products. Let $\pi_0 X \to X$ be the inclusion of a point in each connected component, expanded to an ultrafunctor in the natural way. Consider the Cech nerve D of this map. Since $\mathcal{U} \subseteq \mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$ is stable under finite limits, the Cech nerve can be computed levelwise, and since it's also stable under geometric realizations of groupoids, the geometric realization of the Cech nerve may be computed levelwise. Since the map $\pi_0 X \to X$ is levelwise an effective epimorphism in $\mathcal{S}$, we get that the canonical map $|D| \to X$ is levelwise an equivalence, hence an equivalence.

By the limit formula for right Kan extensions, all the spaces in the Cech nerve can be seen to be $n-1$-truncated. Hence X is a geometric realization of a groupoid in $\mathcal{G}$, so $X \in \mathcal{G}$.

Hence $\mathrm{Fun}'(\mathcal{U}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Stone}^{fr}, \mathcal{C})$ is an equivalence, where Fun' denotes the full subcategory of functors preserving geometric realization of groupoids.

Since the representables are clearly discrete, so that we have

$$\mathrm{Stone}^{fr} \subseteq \mathcal{U}_{\leq 0} \subseteq \mathcal{U},$$

we may factor the restriction as

$$\mathrm{Fun}'(\mathcal{U}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{U}_{\leq 0}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Stone}^{fr}, \mathcal{C})$$

By an argument completely analogous to the above (minus the induction), we can show that

$$\mathcal{U}_{\leq 0} \subseteq \mathrm{Fun}(\mathrm{Stone}^{fr,op}, \mathcal{S})$$

is the free completion of Stone^{fr} under quotients by equivalence relations (rather than geometric realizations of groupoids). Hence if we let $\mathrm{Fun}"(\mathcal{U}_{\leq 0}, \mathcal{C})$ denote the full subcategory of those functors which preserve these quotients, we find that the restriction

$$\mathrm{Fun}"(\mathcal{U}_{\leq 0}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Stone}^{fr}, \mathcal{C})$$

is an equivalence, and hence

$$\mathrm{Fun}'(\mathcal{U}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}"(\mathcal{U}_{\leq 0}, \mathcal{C})$$

is an equivalence by 2-of-3.'

It only remains to see that the full subcategories of pretopos functors are identified under this restriction. This follows from the proof of proposition 2.21, as well as the fact that coproducts and finite limits commute with geometric realizations (in $\mathcal{S}$) $\square$

References

[1] Jacob Lurie. *Categorical Logic, Lecture 30X: Higher categorical logic.* http://www.math.harvard.edu/~lurie/278xnotes/Lecture30X-Higher.pdf.

[2] Jacob Lurie. *Higher Algebra.* http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[3] Jacob Lurie. *Higher Topos Theory.* Princeton University Press, 2009.

[4] Jacob Lurie. *Spectral Algebraic Geometry.* http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf.

[5] Jacob Lurie. *Ultracategories.* http://www.math.harvard.edu/~lurie/papers/Conceptual.pdf.