$\mathbb{E}_k$-indecomposables and $\mathbb{E}_k$-homology

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Abstract

In [1], Galatius–Kupers–Randall-Williams describe a set of foundations for $\mathbb{E}_k$-*homology*. This is an invariant for non-unital $\mathbb{E}_k$-algebras, which contains rich information about the multiplicative structure. They apply this theory to homological stability problems. In this note, we make some initial progress on a rephrasing of this theory in the language of ∞-categories and ∞-operads, as developed by Lurie in [2]. We describe some preliminary results in this direction, as well as the chief difficulties encountered.

1 Introduction

In [1], Galatius–Kupers–Randall-Williams describe an approach to homological stability problems by means of $\mathbb{E}_k$-homology. The basic idea is that, for instance, the general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$ can be extracted as the automorphism groups in the groupoid $\mathrm{Vect}^{\simeq}_{\mathbb{F}_q}$. However, this groupoid carries addition structure, namely a *symmetric monoidal structure* of direct sum. Moreover, the stabilization maps $\mathrm{GL}_n(\mathbb{F}_q) \hookrightarrow \mathrm{GL}_{n+1}\,\mathbb{F}_q$ correspond to the map

$$\mathrm{Aut}(\mathbb{F}_q^n) \overset{-\oplus 1_{\mathbb{F}_q}}{\to} \mathrm{Aut}(\mathbb{F}_q^{n+1})$$

Thus, the monoidal structure is intimately connected to questions of homological stability. The basic idea of [1] is to view the symmetric monoidal groupoid $\mathrm{Vect}^{\simeq}_{\mathbb{F}_q}$ as an $\mathbb{E}_\infty$-algebra, and extract homological stability data from a study of this $\mathbb{E}_\infty$-algebra (or, in general, an $\mathbb{E}_k$-algebra). The basic ingredient for this is a theory of $\mathbb{E}_k$-*homology.*

In translating the machinery [1] into the ∞-categorical setting, many of the translations have already been defined, mostly by Lurie in [2]. Thus there is already a very complete account of (nonunital) $\mathbb{E}_k$-algebras, their homotopy theory, the extraction of spectral sequences from a filtration, and so on. The main difficulty seems to be defining the $\mathbb{E}_k$-homology, which in turn is defined in terms of the functor of $\mathbb{E}_k$-*indecomposables*, $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}\,\mathcal{C} \to \mathcal{C}_*$.

In the classical setting, one can define a left Quillen functor $Q^{\mathbb{E}_k} \operatorname{Alg}^{\mathrm{nu}}_{\mathbb{E}_k} \mathcal{C} \to \mathcal{C}_*$ by sending an algebra A to the quotient A/A^2 (pointed in the obvious way). The left derived functor $\mathbb{L}Q^{\mathbb{E}_k}$ is then called the *derived* $\mathbb{E}_k$ *indecomposables*, and $\mathbb{E}_k$-homology is defined in terms of this object.

However, in the setting of ∞-categories, writing down a formula on the level of objects (and morphisms) is not generally sufficient to define a functor. In certain good settings, it is possible to define this functor by passing to a certain model category; this approach is taken in this note. However, this construction makes it difficult to prove theorems about $Q^{\mathbb{E}_k}$.

1.1 Conventions

When there is no chance of confusion, we ignore the distinction between an ordinary category and its nerve. In a similar vein, we will sometimes refer to an invertible map in an ∞-category as an isomorphism, when there is no stronger notion of isomorphism around. We adopt Lurie's abuse of notation in not distinguishing, when $\mathcal{O}^\otimes \to \mathrm{Fin}_*$ is an ∞-operad, between $\mathcal{O}$, $\mathcal{O}^\otimes$ and $\mathcal{O}^\otimes \to \mathrm{Fin}_*$.

We denote the little k-cubes ∞-operad by $\mathbb{E}_k$, and conventionally let $\mathbb{E}_\infty = \mathrm{Comm}$, the commutative ∞-operad (where $\mathrm{Comm}^\otimes = \mathrm{Fin}_*$)

2 Preliminaries on ∞-operads and monoidal ∞-categories

We will be relying heavily on the contents of [2]. A certain degree of familiarity with the contents of that volume is assumed, but we will go through some of results that we use in this section. We will phrase these results only in the cases we need, rather than the extreme generality in which Lurie phrases them. Hopefully this will enhance comprehension. Mostly, the "proofs" of this section just describe how to derive our special cases from Lurie's general cases. As such, they can safely be skipped.

2.1 Nonunital ∞-operads

Definition 2.1. An ∞-operad $\mathcal{O}^\otimes \xrightarrow{p} \mathrm{Fin}_*$ is *non-unital* if $p(\alpha)$ is a surjection for each $\alpha : X \to Y$ in $\mathcal{O}^\otimes$.

Remark 2.2. Suppose $\alpha : X \to Y$ is a map in $\mathcal{O}^\otimes$ so that $p(\alpha) : \langle n \rangle \to \langle m \rangle$ is not a surjection. Suppose $i \in \langle m \rangle^\circ$ is not in the image. Then we can choose a coCartesian lift $\bar{\rho}^i : Y \to Z$ of $\rho^i : \langle m \rangle \to \langle 1 \rangle$, since $\mathcal{O}$ an ∞-operad. We have this commutative diagram in Fin_*:

$$\begin{tikzcd}
\langle n \rangle \arrow[r, "\rho^i \circ p(\alpha)"] \arrow[d] & \langle 1 \rangle \\
\langle 0 \rangle \arrow[ur] &
\end{tikzcd}$$

By choosing a coCartesian lift $X \to W$ of $\langle n \rangle \to \langle 0 \rangle$, we obtain a map $\beta : W \to Z$ so that $p(\beta)$ goes from $\langle 0 \rangle \to \langle 1 \rangle$.

We thus see that being non-unital is equivalent to the statement that there are no maps from $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$ to $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ - in other words, no "null-ary" operations. This is also the intuitive idea of non-unital operads - their algebras have no specified maps $\mathbb{1} \to X$, no "units".

Definition 2.3. $\mathrm{Fin}_*^{\mathrm{surj}}$ is the subcategory of Fin_* spanned by the surjective maps.

Remark 2.4. An ∞-operad $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ is a non-unital operad if and only if it factors over $\mathrm{Fin}_*^{\mathrm{surj}} \hookrightarrow \mathrm{Fin}_*$.

Definition 2.5. Let $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad. Then $\mathcal{O}^{\mathrm{nu},\otimes}$ is defined as the pullback $\mathcal{O}^{\otimes} \times_{\mathrm{Fin}_*} \mathrm{Fin}_*^{\mathrm{surj}}$. Just as we abuse notation and denote $\mathcal{O}^{\otimes}$ just by the symbol $\mathcal{O}$, we will also denote $\mathcal{O}^{\mathrm{nu},\otimes}$ by $\mathcal{O}^{\mathrm{nu}}$.

Note that there is a canonical map $\mathcal{O}^{\mathrm{nu},\otimes} \to \mathcal{O}^{\otimes}$, which is clearly a map of ∞-operads.

Remark 2.6. $\mathbb{E}_0^{\mathrm{nu}} \simeq \mathrm{Triv}$

Example 2.7. $\mathbb{E}_k^{\mathrm{nu}}$ is the operad of *non-unital* $\mathbb{E}_k$*-algebras*. It plays a big role in this project.

Notation 2.8. Let $\mathcal{C}$ be an $\mathbb{E}_n$-monoidal ∞-category, for $0 \leq n \leq \infty$. Let $k \leq n$. We will abuse notation and denote the categories $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_n}(\mathcal{C})$ by $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$. We will also denote $\mathrm{Alg}_{\mathbb{E}_k^{\mathrm{nu}}/\mathbb{E}_n}(\mathcal{C})$ by $\mathrm{Alg}_{\mathbb{E}_k}^{\mathrm{nu}}(\mathcal{C})$.

2.2 Day Convolution

Throughout this section, let $\mathcal{O}^{\otimes}$ be a fixed ∞-operad. Suppose that $\mathcal{O}_{\langle 1 \rangle}$ has a single object, so that there is a natural notion of *underlying* ∞*-category* for $\mathcal{O}$-monoidal ∞-categories. In our applications, $\mathcal{O}^{\otimes} = \mathbb{E}_k^{\otimes}$ or $\mathbb{E}_k^{\mathrm{nu},\otimes}$.

Theorem 2.9. Suppose $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ are two $\mathcal{O}$-monoidal ∞-categorical. Then there exists an $\mathcal{O}$-monoidal category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$, equipped with a lax $\mathcal{O}$-monoidal functor

$$\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{D}^{\otimes}$$

which is the universal such thing - in other words, defining a lax $\mathcal{O}$-monoidal functor $\mathcal{E}^{\otimes} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ is equivalent to defining a lax $\mathcal{O}$-monoidal functor

$$\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes} \to \mathcal{D}^{\otimes}$$

Moreover, the underlying ∞-category of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ is $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (hence the notation).

Remark 2.10. We will generally denote the tensor product in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ by $\circledast$

Corollary 2.11. $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$ equivalent to the category of lax $\mathcal{O}$-monoidal functors $\mathcal{C}\to\mathcal{D}$.

[2, Rem. 2.2.6.8]

Corollary 2.12. Given a (lax) $\mathcal{O}$-monoidal functor $\mathcal{A}^{\otimes}\to\mathcal{B}^{\otimes}$, and a third $\mathcal{O}$-monoidal ∞-category $\mathcal{C}$, there's an induced (lax) $\mathcal{O}$-monoidal structure on the natural functor

$$\mathrm{Fun}(\mathcal{B},\mathcal{C})\to\mathrm{Fun}(\mathcal{A},\mathcal{C})$$

Given a (lax) $\mathcal{O}$-monoidal functor $\mathcal{B}^{\otimes}\to\mathcal{C}^{\otimes}$, and a third $\mathcal{O}$-monoidal ∞-category $\mathcal{A}$, there's an induced (lax) $\mathcal{O}$-monoidal functor

$$\mathrm{Fun}(\mathcal{A},\mathcal{B})\to\mathrm{Fun}(\mathcal{A},\mathcal{C})$$

Proof. The first one is induced by the composite

$$\mathcal{A}^{\otimes}\times_{\mathcal{O}^{\otimes}}\mathrm{Fun}(\mathcal{B},\mathcal{C})^{\otimes}\to\mathcal{B}^{\otimes}\times_{\mathcal{O}^{\otimes}}\mathrm{Fun}(\mathcal{B},\mathcal{C})^{\otimes}\to\mathcal{C}^{\otimes}$$

The second one is induced by the composite

$$\mathrm{Fun}(\mathcal{A},\mathcal{B})^{\otimes}\times_{\mathcal{O}^{\otimes}}\mathcal{A}^{\otimes}\to\mathcal{B}^{\otimes}\to\mathcal{C}^{\otimes}$$

□

2.3 Monoidal structures on adjoints

Proposition 2.13. Let $\mathcal{O}$ be an ∞-operad, and suppose we have an $\mathcal{O}$-monoidal functor $F:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$, with the property that for each $X\in\mathcal{O}$, the underlying functor $F_X:\mathcal{C}_X\to\mathcal{D}_X$ admits a right adjoint. Then F itself admits a right adjoint $G:\mathcal{D}^{\otimes}\to\mathcal{C}^{\otimes}$. Moreover this G is a lax $\mathcal{O}$-monoidal functor.

This proposition is just a slight reformulation of [2, Cor. 7.3.2.7]

Theorem 2.14. Let $\mathcal{C}$ be an ∞-category, and $\mathcal{O}$ an ∞-operad with a single object in $\mathcal{O}^{\otimes}_{\langle 1\rangle}$. Then there is an equivalence between $\mathcal{O}$-monoidal structures on $\mathcal{C}$ and $\mathcal{O}$-monoidal structures on the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$.

Proof. Follows from the fact that $\mathcal{C}\mathrm{at}_\infty\xrightarrow{\mathrm{op}}\mathcal{C}\mathrm{at}_\infty$ preserves products (hence is symmetric monoidal), and the equivalence between $\mathcal{O}$-monoidal ∞-categories and $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$ □

Definition 2.15. Let $\mathcal{C},\mathcal{D}$ be $\mathcal{O}$-monoidal ∞-categories. An *oplax* monoidal functor $\mathcal{C}\to\mathcal{D}$ is a lax monoidal functor $\mathcal{C}^{\mathrm{op}}\to\mathcal{D}^{\mathrm{op}}$.

Remark 2.16. Let $\mathcal{C}\to\mathcal{D}$ be a (strong) $\mathcal{O}$-monoidal functor, and suppose the underlying functor admits a left adjoint. Then there is a canonical oplax monoidal structure on this right adjoint, by theorem 2.13.

Lemma 2.17. Let $\mathcal{C} \to \mathcal{D}$ be a lax $\mathbb{E}_k$-monoidal functor of $\mathbb{E}_k$-monoidal categories, for $1 \leq k \leq \infty$. Let $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$ be the binary tensor products of $\mathcal{C}$ and $\mathcal{D}$. Suppose the induced map $F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$ is an equivalence for each pair of objects $X, Y \in \mathcal{C}$. Suppose further that the map $1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ is an equivalence. Then F is a (strong) $\mathbb{E}_k$-monoidal functor.

Proof. We must verify that F carries coCartesian morphisms to coCartesian morphisms. By factoring morphisms in Fin_* by inert maps followed by active maps, and lifting this factorization to $\mathcal{C}$, it suffices to show that F carries coCartesian lifts of active maps to coCartesian morphisms. By postcomposing with (lifts of) ρ^i, it suffices to carry out this check for coCartesian maps lying over the active maps $\mu : \langle n \rangle \to \langle 1 \rangle$.

This is equivalent to the statement that $\otimes_\mu F(X_i) \to F(\otimes_\mu X_i)$ is always an equivalence. For $n = 1$, this is clear. For $n = 0, 2$, this is precisely the assumption of the theorem. The cases for $n > 2$ follows by the fact that $\otimes_\mu X_i \simeq X_1 \otimes \cdots \otimes X_n$. $\square$

Remark 2.18. This theorem holds for $k = 0$ as well, but in this case it is trivial.

Theorem 2.19. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{O}$-monoidal categories, where $\mathcal{O}$ again has a single object, and suppose F admits a right adjoint G

Let $\mathcal{O}' \to \mathcal{O}$ be a map of ∞-operads. Then the induced functor $F : \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$ is left adjoint to the induced functor $G : \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$.

Proof. Observe that the induced functors F and G are given by postcomposing with the adjoint functors $F^\otimes : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ and $G^\otimes : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$. Since these are adjoint by construction, there is a natural transformation $1_{\mathcal{C}^\otimes} \to F^\otimes G^\otimes$. This induces a natural transformation $1_{\mathcal{C}^\otimes} \circ - \to F^\otimes G^\otimes \circ -$, which then restricts to a natural transformation

$$\eta : 1_{\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})} \to FG$$

The counit

$$\varepsilon : GF \to 1_{\mathrm{Alg}\,\mathcal{O}'/\mathcal{O}(\mathcal{D})}$$

is constructed in an analogous way. The triangle identities are easily verified, and this suffices to construct the adjunction, by [3, Prop. 5.2.2.12] $\square$

3 $\mathbb{E}_k$-indecomposables

Consider the "free nonunital $\mathbb{E}_k$-algebra"-monad $T : \mathrm{sSet}_* \to \mathrm{sSet}_*$. There is an obvious map of monads $T \to 1$, which collapses all the non-identity operations to the basepoint. This gives a functor $Q : \mathrm{Alg}_T(\mathrm{sSet}_*) \to \mathrm{sSet}_*$, which is left adjoint to the "trivial algebra" functor. One can show that this is a Quillen adjunction, and so define the *derived indecomposables* as the left derived functor $\mathbb{L}Q : \mathrm{Alg}_T\, \mathrm{sSet}_* \to \mathrm{sSet}_*$. (Here one can replace sSet_* with a quite general class of model categories). This is the basis for a useful theory of $\mathbb{E}_k$-homology, which is developed in [1].

The topic of this section is the development of this construction in the ∞-categorical setting. However, the approach taken above is much more difficult, since equipping a natural transformation with the structure of a monad map is not easily done in this setting. Instead, our approach is to give a description of the right adjoint Triv. To do this, we employ a rectification result of Nikolaus–Sagave, reducing the problem to the situation of a pointed model category, where defining trivial algebras is of course easy.

This approach has some deficiencies: It works only for presentably symmetric monoidal ∞-categories, and it poses some difficulties in proving theorems about the resulting functor. For instance, an important result is that the derived indecomposables can be computed by a bar construction - it's unclear how to get such a theorem in this setting.

Nevertheless, some basic results can be proved. We give some examples of these.

For the following, $N(\mathcal{M})$ refers to the ordinary nerve of $\mathcal{M}$, with no maps inverted, *not* the homotopy coherent nerve.

Theorem 3.1. Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal ∞-category. Suppose further that $\mathcal{C}$ is pointed. Then there exists a symmetric monoidal simplicial left proper combinatorial model category $(\mathcal{M}, \otimes)$, so that $N^{\otimes}(\mathcal{M}^{\circ})[W^{-1}] \simeq \mathcal{C}^{\otimes}$, and so that $\mathcal{M}$ is pointed (as a 1-category). Moreover, we can choose this model in such a way that the functor

$$\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^{\circ}))[W^{-1}] \to \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^{\circ})[W^{-1}])$$

is an equivalence.

Proof. By [4], we can find a model $(\mathcal{M}, \otimes)$ for $\mathcal{C}^{\otimes}$, although $\mathcal{M}$ may not be pointed. However, the unique map $\emptyset \to *$ is a weak equivalence. We may also choose it so that it satisfies conditions (i-ii) of [4, Thm 2.5].

Now we consider the undercategory $\mathcal{M}_{*/}$. This inherits the structure of a simplicial left proper combinatorial model category. We equip it with the symmetric monoidal structure given by "smash product", i.e given $* \to X, * \to Y$, we form a functorial pushout

$$\begin{tikzcd} X \otimes * \sqcup * \otimes Y \arrow[r] \arrow[d] & * \arrow[d] \\ X \otimes Y \arrow[r] & X \wedge Y \end{tikzcd}$$

And define the smash product to be the right vertical morphism.

The structure isomorphisms for the symmetric monoidal structure are constructed in an obvious way from the ones in $\mathcal{M}$. We only need to verify that this smash product is compatible with the model structure.

For brevity, we denote the object $X \otimes * \sqcup * \otimes Y$ as $X \vee Y$. This is extremely misleading, since for the prototypical example of spaces and pointed spaces,

$X \vee Y$ is a quotient of this space. However, it is helpful in remembering that $X \wedge Y = X \otimes Y / X \vee Y$.

The only nontrivial part of the verification is checking that, for every pair of cofibrations $X \to Y, X' \to Y'$, the induced map

$$X \wedge Y' \coprod_{X \wedge X'} Y \wedge X' \to Y \wedge Y'$$

is a cofibration, and moreover it is trivial if either of the cofibrations given is trivial. To see this, consider the following commutative diagram, where the bottom square is a given lifting problem, with $A \to B$ a trivial fibration.

$$\begin{tikzcd}
X \vee Y' \coprod_{X \vee X'} Y \vee X' \arrow[rr] \arrow[d] \arrow[ddr] & & Y \vee Y' \arrow[d] \arrow[ddl] \\
X \otimes Y' \coprod_{X \otimes X'} Y \otimes X \arrow[rr] \arrow[dd] & & Y \otimes Y' \arrow[dd] \arrow[ddll, dashed] \\
 & * \arrow[dl] \arrow[dr] & \\
X \wedge Y' \coprod_{X \wedge X'} Y \wedge X' \arrow[rr] \arrow[d] & & Y \wedge Y' \arrow[d] \\
A \arrow[rr] & & B
\end{tikzcd}$$

By assumption, the monoidal structure on $\mathcal{M}$ is compatible with the model structure, and so the horizontal map with target $Y \otimes Y'$ is a cofibration, trivial if either of the given ones are. Hence we can find a lift as the dashed arrow. It suffices to show that this arrow extends uniquely over $Y \wedge Y'$. By the universal property of pushouts, it suffices that the precomposition with $Y \vee Y' \to Y \otimes Y'$ factors uniquely over $*$. Since any map is required to be compatible with the pointing $* \to A$, it's clear that the only possible factorization is the given one $* \to X \wedge Y' \wedge_{X \wedge X'} Y \wedge X' \to A$ - we merely have to show that the two maps from $Y \vee Y'$ agree.

Since the top horizontal map is an epimorphism, it suffices to verify this after composing with it - but this now follows from a routine diagram chase.

Moreover, one can show that the functor $X \mapsto X \sqcup *$ is a symmetric monoidal left Quillen functor. Hence it induces a symmetric monoidal functor $\mathcal{C}^{\otimes} \simeq N^{\otimes}(\mathcal{M})[W^{-1}] \to N^{\otimes}(\mathcal{M}_{*/})[W^{-1}]$ Since the underlying functor $\mathcal{M} \to \mathcal{M}_{*/}$ is a model for the equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{C}_*$, it is an equivalence, hence $\mathcal{C}^{\otimes} \simeq N^{\otimes}(\mathcal{M}_{*/})[W^{-1}]$ as desired.

For the statement on algebras, consider the diagram

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ))[W^{-1}] \arrow[r] \arrow[d] & \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ_{*/}))[W^{-1}] \arrow[d] \\
\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ)[W^{-1}]) \arrow[r, "\sim"] & \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ_{*/})[W^{-1}])
\end{tikzcd}$$

Here the bottom map is an equivalence because it's induced by a symmetric monoidal equivalence.

Moreover, the left vertical map is an equivalence, since the arguments of [4, Thm 2.5] apply without modification to the case of non-unital monoids. It remains to show that the top map is an equivalence. Note that, on the 1-categorical level, we have a monoidal adjunction given by $(-)\coprod *$ and the forgetful functor, which induces an adjunction between $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{M}^\circ)$ and $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{M}^\circ_{*/})$. Since the forgetful functor also preserves weak equivalences, it induces a right adjoint $\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ_{*/}))[W^{-1}] \to \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ))[W^{-1}]$. Now it just remains to observe that both the unit and counit transformations are weak equivalences, so this is really an inverse functor. □

Definition 3.2. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, which is pointed. Let $(\mathcal{M}, \otimes)$ be a pointed model for $\mathcal{C}$ theorem 3.1. We thus have a symmetric monoidal functor $N(\mathcal{M}^\circ) \to \mathcal{C}$, which induces a symmetric monoidal equivalence $N_\Delta(\mathcal{M}^\circ) \simeq N(\mathcal{M}^\circ)[W^{-1}] \to \mathcal{C}$.

Note that this gives us a map $\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$. Note also that the former category can be identified with the (nerve of the) category of nonunital algebras of $\mathcal{M}^\circ$. There is thus an obvious functor $N(\mathcal{M}^\circ) \to \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ))$ which takes every object $X \in \mathcal{M}^\circ$ to the non-unital commutative algebra with $\mu : X \otimes X \to X$ given by the (unique) zero map. The composite

$$\mathcal{C} \simeq N(\mathcal{M}^\circ) \to \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$$

will be denotes Triv.

Proposition 3.3. The functor $\mathrm{Triv} : \mathcal{C} \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$ is well-defined up to equivalence, in the sense that it does not depend on the choice of $\mathcal{M}$.

Proof. Given two choices of strictly pointed model, say $\mathcal{M}, \mathcal{M}'$, we get functors

$$N^{\otimes}(\mathcal{M}^\circ)[W^{-1}] \xrightarrow{\sim} \mathcal{C}^{\otimes} \xleftarrow{\sim} N^{\otimes}(\mathcal{M}'^\circ)[W^{-1}]$$

We may restrict these equivalences to obtain

$$N(\mathcal{M}^\circ)[W^{-1}] \xrightarrow{\sim} \mathcal{C} \xleftarrow{\sim} N(\mathcal{M}'^\circ)[W'^\circ]$$

Our goal is now to show that the following diagram

$$\begin{tikzcd}
N(\mathcal{M}^\circ)[W^{-1}] \arrow[r, "\sim"] \arrow[d] & \mathcal{C} & N(\mathcal{M}'^\circ)[W'^{-1}] \arrow[l, "\sim"'] \arrow[d] \\
\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ)[W^{-1}]) \arrow[r, "\sim"] & \mathrm{CAlg}(\mathcal{C}) & \mathrm{CAlg}(N(\mathcal{M}'^\circ)[W'^{-1}]) \arrow[l, "\sim"']
\end{tikzcd}$$

commutes, in the natural sense. We may let a composite $F : N^{\otimes}(\mathcal{M}^\circ)[W^{-1}] \xrightarrow{\sim} N^{\otimes}(\mathcal{M}'^\circ)[W'^{-1}]$ be given, and it then suffices to show that the following diagram

$$\begin{array}{ccc}
N(\mathcal{M}^\circ)[W^{-1}] & \xrightarrow{\quad\sim\quad} & N(\mathcal{M}'^\circ)[W'^{-1}] \\
\downarrow & & \downarrow \\
\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ))[W^{-1}] & & \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}'^\circ))[W'^{-1}] \\
\downarrow\sim & & \downarrow\sim \\
\mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}^\circ)[W^{-1}]) & \xrightarrow{\sim} & \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}'^\circ)[W'^{-1}])
\end{array}$$

commutes. Here we define the top functor as the underlying functor of F, and the bottom functor as the obvious thing induced from F. Note that the bottom vertical maps are equivalences by construction.

It now suffices to prove that the two possible functors $N(\mathcal{M})[W^{-1}] \to \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}))[W^{-1}]$ are equivalent. To do this, it's enough to observe that both functors take $X \in \mathcal{M}$ to a nonunital commutative algebra with underlying object X and zero multiplications. $\square$

Remark 3.4. Observe that, by construction, $U\,\mathrm{Triv} \simeq 1_{\mathcal{C}}$.

Proposition 3.5. For each $0 \leq k \leq \infty$, the functor $\mathrm{Triv} : \mathcal{C} \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C})$ preserves limits and sifted colimits.

Proof. The case of limits follows from [2, Prop. 3.2.2.1] and the fact that Triv is a section of U. The case of colimits follows analogously by [2, Prop. 3.2.3.1]. $\square$

Definition 3.6. By [2, Cor. 3.2.3.5], $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C})$ is presentable. By the preceding proposition, this implies that $\mathrm{Triv} : \mathcal{C} \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C})$ admits an essentially unique left adjoint. We denote this functor by

$$Q^{\mathbb{E}_k} : \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$$

In the event that $\mathcal{C}$ is not pointed, we define $Q^{\mathbb{E}_k}$ as the composite

$$\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \overset{(-)_+}{\to} \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}_*) \overset{Q^{\mathbb{E}_k}}{\to} \mathcal{C}_*$$

(There is no risk of confusion, since the two possible definitions when $\mathcal{C}$ is pointed correspond to each other under the canonical identification $\mathcal{C} \simeq \mathcal{C}_*$.

We will also abuse notation and denote the composites

$$\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \overset{Q^{\mathbb{E}_k}}{\to} \mathcal{C}$$

for $n \geq k$, and

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \overset{Q^{\mathbb{E}_k}}{\to} \mathcal{C}$$

by $Q^{\mathbb{E}_k}$.

$Q^{\mathbb{E}_k}$ is called the $\mathbb{E}_k$-indecomposables.

Proposition 3.7. Let $\mathcal{C} \to \mathcal{D}$ be a symmetric monoidal left adjoint functor between pointed, presentably symmetric monoidal ∞-categories. Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \arrow[r, "Q^{E_k}"] \arrow[d, "F"] & \mathcal{C} \arrow[d, "F"] \\
\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{D}) \arrow[r, "Q^{\mathbb{E}_k}"] & \mathcal{D}
\end{tikzcd}$$

Proof. Let R be a right adjoint to L, equipped with the canonical lax structure. It suffices to verify that the corresponding diagram of right adjoints commute. It also suffices to verify this for the case $k = \infty$, so that we may consider

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) & \mathcal{C} \arrow[l, "\mathrm{Triv}"'] \\
\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D}) \arrow[u, "R"] & \mathcal{D} \arrow[l, "\mathrm{Triv}"] \arrow[u, "R"']
\end{tikzcd}$$

Now, by [4, Thm 2.8], we may model the adjunction $L \dashv R$ by a symmetric monoidal Quillen adjunction between models $\mathcal{M}$ and $\mathcal{N}$. It's clear that this descends to aa symmetric monoidal adjunction between the strictly pointed models $\mathcal{M}_{*/}$ and $\mathcal{N}_{*/}$.

Now, by definition, Triv is the composite

$$\mathcal{C} \simeq N(\mathcal{M}_{*/})[W^{-1}] \to \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}_{*/})[W^{-1}]) \simeq \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$$

Hence we may express the above diagram as follows:

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) & \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{M}_{*/})[W^{-1}]) \arrow[l] & N(\mathcal{M}_{*/})[W^{-1}] \arrow[l] & \mathcal{C} \arrow[l] \\
\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D}) \arrow[u] & \mathrm{CAlg}^{\mathrm{nu}}(N(\mathcal{N}_{*/})[W^{-1}]) \arrow[l] \arrow[u] & N(\mathcal{N}_{*/})[W^{-1}] \arrow[l] \arrow[u] & \mathcal{D} \arrow[l] \arrow[u]
\end{tikzcd}$$

Here the rightmost square and the leftmost square commute by construction, whereas the middle square commutes by definition of Triv. $\square$

Corollary 3.8. Suppose $\mathcal{C}$ is a pointed presentably symmetric monoidal ∞-category. Then $\mathrm{Sp}(\mathcal{C})$ inherits this structure, and $\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ is a symmetric monoidal functor. hence

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Sp}(\mathcal{C})) \arrow[r] & \mathrm{Sp}(\mathcal{C})
\end{tikzcd}$$

commutes.

Definition 3.9. We define the $\mathbb{E}_k$-homology $H_n^{\mathbb{E}_k}(R; A)$ of a non-unital algebra $R \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{S})$ as the singular homology groups of $\mathbb{Q}^{\mathbb{E}_k}(R)$, with coefficients in A.

Remark 3.10. Even though the range of results that can be proved about $\mathbb{E}_k$-homology in this setup is somewhat limited, it does allow us to develop some tools for calculation.

We give some examples below.

Proposition 3.11. Let $F \dashv U$ denote the free-forgetful adjunction between $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C})$ and $\mathcal{C}$. The natural isomorphism $1_{\mathcal{C}} \xrightarrow{\sim} U\,\mathrm{Triv}$ induces a natural transformation $QF \to 1_{\mathcal{C}}$ via the two adjunctions. This is a natural equivalence.

Proof. It suffices to observe that QF is left adjoint to $U\,\mathrm{Triv}$, since adjoints compose. □

Theorem 3.12. Equip the poset $\mathbb{Z}_{\leq}$ with the symmetric monoidal structure of $+$. Then the colimit functor $\mathcal{S}^{\mathbb{Z}_{\leq}} \to \mathcal{S}$ acquires a symmetric monoidal structure, inducing a functor

$$\mathrm{colim} : \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{S}^{\mathbb{Z}_{\leq}}) \to \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{S})$$

Let $A_\bullet \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{S}^{\mathbb{Z}_{\leq}})$, and let $A := \mathrm{colim}\, A_\bullet$.

Then there exists a spectral sequence $E^r_{p,q}$, converging to $H_{p+q}(Q^{\mathbb{E}_k} A)$. Moreover,

$$E^1_{p,q} = H_p(Q^{\mathbb{E}_k}(A_\bullet)_{q+1}, Q^{\mathbb{E}_k}(A_\bullet)_q)$$

Proof. Since colim and $Q^{\mathbb{E}_k}$ commutes, we see that

$$\mathrm{colim}\, Q^{\mathbb{E}_k}(A_\bullet) = Q^{\mathbb{E}_k} A$$

Moreover, the functor $\Sigma^\infty(-) \otimes H\mathbb{Z}$ commutes with colimits. The desired spectral sequence is now the one constructed in [2, Section 1.2.2]. □

Remark 3.13. We have shown that $Q^{\mathbb{E}_k} F X \simeq X$. Since free algebras generate $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C})$ under colimits, this is quite useful when it comes to calculating $Q^{\mathbb{E}_k}$.

The map $\mathrm{Map}(FX, FY) \to \mathrm{Map}(QFX, QFY) = \mathrm{Map}(X, Y)$ is given by

$$\mathrm{Map}(FX, FY) \xrightarrow{\sim} \mathrm{Map}(X, UFY) \xrightarrow{f_*} \mathrm{Map}(X, Y)$$

where f_* is postcomposition with $f : UFY \to U\,\mathrm{Triv}\,Y \simeq Y$ Given by $FY \to \mathrm{Triv}\,Y$ classified by $QFY \xrightarrow{\sim} Y$.

The map $UFY \to Y$ should be understandable in terms of formula for FY.

The universal property of the map $Y \to UFY$ is spelled out in HA 3.1.3.13. Is says that the induced maps $\mathrm{Sym}^n_{\mathcal{O}}(Y) \to UF(Y)$ should present $UF(Y)$ as a coproduct $\coprod_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O}}(Y)$.

We show below that $\mathrm{Sym}^1_{\mathbb{E}^{\mathrm{nu}}_k}(X) = X$, and $\mathrm{Sym}^0_{\mathbb{E}^{\mathrm{nu}}_k}(X) = \emptyset$. The conjecture is then that the map $UFY \to Y$ is given by collapsing everything except the 1st summand to 0.

This should be enough to calculate homotopy colimits, at least of diagrams with ordinary index categories (which is sufficient to compute all homotopy colimits.)

Proposition 3.14. Recall the definition of $\mathrm{Sym}^n_{\mathcal{O},Y}(C)$ from HA 3.1.3.1.9. Consider the special case of $\mathrm{Sym}^n_{\mathbb{E}_k^{\mathrm{nu}}}(X)$, for $X \in \mathcal{C}$, where $\mathcal{C}$ is an $\mathbb{E}_k^{\mathrm{nu}}$-monoidal category. Suppose $\mathcal{C}$ is cocomplete. Then $\mathrm{Sym}^0_{\mathbb{E}_k^{\mathrm{nu}}}(X) = \emptyset$, and $\mathrm{Sym}^1_{\mathbb{E}_k^{\mathrm{nu}}}(X) = X$.

Proof. By construction, we have a homotopy fiber sequence

$$\mathrm{Mul}_{\mathbb{E}_k^{\mathrm{nu}}}(\langle n \rangle, \langle 1 \rangle) \to \mathcal{P}(n) \to B\Sigma_n$$

In the case $n = 0$, the first space is empty, which implies the second space must be empty as well (since the latter space is connected). So Sym^0 is a colimit of an empty diagram, hence initial. In the case $n = 1$, the first space is contractible, which implies that $\mathcal{P}(1) \simeq B\Sigma_1$, which is also contractible. It is clear that the value of the functor $\bar{h_1} : * \simeq \mathcal{P}(1) \to \mathcal{C}$ must be C. Hence also

$$\mathrm{Sym}^1_{\mathbb{E}_k^{\mathrm{nu}}}(X) = \operatorname{colim} \bar{h_1} = C$$

$\square$

Remark 3.15. For each $n \geq 0$, the assigment $X \mapsto \mathrm{Sym}^n_{\mathbb{E}_k^{\mathrm{nu}}}(X)$ may be upgraded to a functor $\mathcal{C} \to \mathcal{C}$ (this is not really a special feature of $\mathbb{E}_k^{\mathrm{nu}}$)

We end this section with a few desirable results on the $\mathbb{E}_k$-indecomposables.

Definition 3.16. Let $T : \mathcal{C} \to \mathcal{C}$ be the monad corresponding to the free-forgetful adjunction

$$\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k} \mathcal{C} \to \mathcal{C}$$

Then $T \simeq \coprod_{n \geq 0} \mathrm{Sym}^n_{\mathbb{E}_k^{\mathrm{nu}}}(-)$, by HA 3.1.3.13. Hence we may define a natural transformation $T \to 1_{\mathcal{C}}$ by choosing a zero map $\mathrm{Sym}^n_{\mathbb{E}_k^{\mathrm{nu}}}(-) \to 1_{\mathcal{C}}$ for each $n \neq 1$, and letting the map $\mathrm{Sym}^1_{\mathbb{E}_k^{\mathrm{nu}}}(-) \to 1_{\mathcal{C}}$ be the equivalence constructed above. (This works because limits and colimits in functor categories are pointwise.)

Conjecture 3.17. This map $T \to 1_{\mathcal{C}}$ can be upgraded to a map of monads, so that the induced change-of-monad functor $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$ is equivalent to $Q^{\mathbb{E}_k}$.

Remark 3.18. This should be a viable definition of $Q^{\mathbb{E}_k}$ even outside the presentable setting.

Conjecture 3.19. Suppose $\mathcal{C}$ is pointed, admits suspension (more assumptions here)? Then there is a natural equivalence $\tilde{B}^{\mathbb{E}_k} \simeq \Sigma^k Q^{\mathbb{E}_k}$. Here $\tilde{B}^{\mathbb{E}_k}$: $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$ is defined as the composition

$$\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathcal{C}) \overset{\oplus \mathbb{1}}{\to} \mathrm{Alg}^{\mathrm{aug}}_{\mathbb{E}_k}(\mathcal{C}) \overset{\tilde{B}^{\mathbb{E}_k}}{\to} \mathcal{C}$$

Where this version of $\tilde{B}^{\mathbb{E}_k}$ the cofiber of

$$B(\mathbb{1}, 1_{\mathbb{1}}) \to B(-)$$

where $B : \mathrm{Alg}^{\mathrm{aug}}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$ is the bar construction functor.

Remark 3.20. This is at least true for free algebras. Since free algebras generate the relevant categories under colimits, and the two functors in question are left adjoints, it suffices to exhibit a natural transformation which restricts to an isomorphism on the full subcategories of free algebras.

4 $\mathbb{E}_k$-algebras from groupoids

The aim of [1] is to study homological stability problems by relating them to $\mathbb{E}_k$-homology. This section describes this procedure in the ∞-categorical setting.

Definition 4.1. Consider $\mathbb{N}_0$, regarded as a discrete space. Equip it with the $\mathbb{E}_\infty$-structure of addition. Note that this induces an $\mathbb{E}_k$ structure for each $0 \leq k \leq \infty$. A *graded* $\mathbb{E}_k$*-space* is an $\mathbb{E}_k$-space X, equipped with a map $r : X \to \mathbb{N}_0$ in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})$.

If the space X is a groupoid (i.e $\pi_i(X)$ vanishes for $i \geq 2$), we call it a *graded* E_k*-groupoid.*

Proposition 4.2. Let $r : G \to \mathbb{N}_0$ be a graded $\mathbb{E}_k$-space. Then the subspace $r^{-1}(\mathbb{N}) =: G_\mathbb{N}$ has a canonical $\mathbb{E}_k^{\mathrm{nu}}$-algebra structure.

Proof. We may form the following pullback in the category of nonunital $\mathbb{E}_k$-algebras:

$$\begin{tikzcd} G_\mathbb{N} \arrow[r] \arrow[d] & G \arrow[d] \\ \mathbb{N} \arrow[r] & \mathbb{N}_0 \end{tikzcd}$$

By [2, Prop. 3.2.2.1], the corresponding diagram in $\mathcal{S}$ is also a pullback diagram, hence the underlying object of $G_\mathbb{N}$ is $r^{-1}(\mathbb{N})$, as desired. □

Proposition 4.3. Let $r : G \to \mathbb{N}_0$ be a map of $\mathbb{E}_k$-spaces Then pullback along r induces an $\mathbb{E}_k$-monoidal functor

$$r^* : \mathrm{Fun}(\mathbb{N}_0, \mathcal{S}) \to \mathrm{Fun}(G, \mathcal{S})$$

Moreover, this functor admits a left adjoint, $r_!$. It comes equipped with a canonical monoidal structure. It also admits a right adjoint, r_*, which has a canonical lax monoidal structure.

Proof. The left and right adjoints are simply given by left Kan extension, so they exist because $\mathcal{S}$ admits all colimits and limits. The lax monoidal structure on r_* comes from theorem 2.13.

Now consider $r^!$. By theorem 2.16, it admits a canonical *oplax* monoidal structure. It suffices to show that this structure is strong. In this situation, it suffices to show that the maps $r_!(F \circledast G) \to r_!(F) \circledast r_!(G)$ are equivalences, by theorem 2.17. This can be seen by examining the colimit formulas for $\circledast$ and $r_!$. □

Warning 4.4. In [1], the functor which we name $r_!$ is named r_*.

Definition 4.5. Let $G \xrightarrow{r} \mathbb{N}_0$ be a graded $\mathbb{E}_k$-space, for $0 \leq k \leq \infty$

Consider the functor $* : G \to \mathcal{S}$, constant at $*$. As the terminal object, it acquires a unique $\mathbb{E}_k$ algebra structure in $\mathrm{Fun}(\mathbb{N}_0, \mathcal{S})$.

Similarly, consider the functor $*: G_{\mathbb{N}} \to \mathcal{S}$. It acquires a unique $\mathbb{E}_k^{\mathrm{nu}}$ algebra structure. Hence the right Kan extension along $G_{\mathbb{N}} \to G$, $*_{>0}: G \to \mathcal{S}$, acquires a canonical $\mathbb{E}_k^{\mathrm{nu}}$-algebra structure. By the standard formula for Kan extensions, this is given by

$$*_{>0}(x) = \begin{cases} \emptyset & r(x) = 0 \\ * & \text{else} \end{cases}$$

Now by theorem 4.3, we have a monoidal functor $\mathrm{Fun}(G, \mathcal{S}) \to \mathrm{Fun}(\mathbb{N}_0, \mathcal{S})$, given by left Kan extension. Hence we obtain an $\mathbb{E}_k$-algebra $\bar{R}(G) := r_!(*)$, and an $\mathbb{E}_k^{\mathrm{nu}}$-algebra $R(G) := r_!(*_{>0})$.

Remark 4.6. By inserting the formula for Kan extensions, we see that $\bar{R}(G)(n) = r^{-1}(\{n\})$ Moreover, $R(G)(n) = r^{-1}(\{n\})$ for $n \geq 1$, and $\emptyset$ otherwise. $\bar{R}(G) = \mathbb{1} \oplus R(G)$, the unitalization of $R(G)$.

We will frequently omit the G and simply write R when the space in question is understood, just as we suppress the grading.

Definition 4.7. Let $r: G \to \mathbb{N}$ be a graded E_k-space. Let A be a commutative ring.

Recall that $\mathcal{D}(A)$ is the derived ∞-category of A, which is equivalent to Mod_A, the ∞-category of HA-module spectra, by ([2, Cor. 4.5.1.5] and [2, Rmk. 7.1.1.16]).

There is a functor $(-) \wedge HA : \mathrm{Sp} \to \mathcal{D}(A)$, which is symmetric monoidal. It is the left adjoint of the forgetful functor $\mathcal{D}(A) \to \mathrm{Sp}$ (see [2, Thm 4.5.3.1]).

Recall also that the pointed suspension spectrum functor $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ is symmetric monoidal.

Hence the postcomposition functor

$$\mathrm{Fun}(\mathbb{N}, \mathcal{S}) \to \mathrm{Fun}(\mathbb{N}, \mathrm{Sp}) \to \mathrm{Fun}(\mathbb{N}, \mathcal{D}(A))$$

acquires a (lax) symmetric monoidal structure, and we define $\bar{R}(G)_A$ to be the image of $\bar{R}(G)$ in $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Fun}(\mathbb{N}, \mathcal{D}(A)))$, and similarly define $R(G)_A \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_k}(\mathrm{Fun}(\mathbb{N}, \mathcal{D}(A)))$.

Once again, $\bar{R}(G)_A$ is the unitalization of $R(G)_A$.

Remark 4.8. In this definition, there is no need for A to be a discrete ring - one could replace it with any $\mathbb{E}_{k+1}$-ring spectrum, if $\mathcal{D}(A)$ is also replaced by Mod_A.

In particular, we may write $R(G)_{\mathbb{S}}$ for the algebra in $\mathrm{Fun}(\mathbb{N}, \mathrm{Sp})$. The discreteness assumption is needed for the following proposition to make sense.

Remark 4.9. Recall that $\mathcal{D}^+(A)$, the ∞-category of left-bounded chain complexes, can be identified with the full subcategory of $\mathrm{Mod}_A = \mathcal{D}(A)$ given by the left bounded objects, i.e by those with no negative homotopy groups. Since the homotopy groups of $R(G)_A(n) \in \mathcal{D}(A)$ are precisely the (singular) homology groups of $R(G)_n$ with coefficients in A, they are certainly concentrated in non-negative degrees. Hence $R(G)_A$ is in $\mathrm{Fun}(\mathbb{N}, \mathcal{D}^+(A))$, and since this subcategory is stable under the tensor product in $\mathrm{Fun}(\mathbb{N}, \mathcal{D}(A))$, it has a unique structure of an E_k-algebra in this category.

Conjecture 4.10. Let G be a graded $\mathbb{E}_k$-space as above. Suppose furthermore that there is a unique object σ of degree 1. Suppose $*_{>0}$ has $\mathbb{E}_1$-homology concentrated in degree $r(x)+1$ for each $x \in G$.

Then the objects of G are precisely the powers of σ, and $H_d(G_{\sigma^n}, G_{\sigma^{n-1}}; \mathbb{Z}) = 0$ for $2d \leq n-1$. Here G_a denotes the connected component of $a \in G$.

This conjecture was proved for braided (i.e. $\mathbb{E}_2$) groupoids using the classical formulation of $\mathbb{E}_k$-homology in [1].

References

[1] Sren Galatius, Alexander Kupers, and Oscar Randall-Williams. "Cellular $\mathbb{E}_k$-algebras". In: *ArXiv e-prints* (2018). arXiv: `1805.07184`.

[2] J. Lurie. *Higher Algebra*. Available at `http://www.math.harvard.edu/~lurie/papers/HA.pdf`. 2017.

[3] J. Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[4] Thomas Nikolaus and Steffen Sagave. "Presentably symmetric monoidal infinity-categories are represented by symmetric monoidal model categories". In: *Algebr. Geom. Topol. 17 (2017) 3189-3212* (2017). arXiv: `1506.01475`.