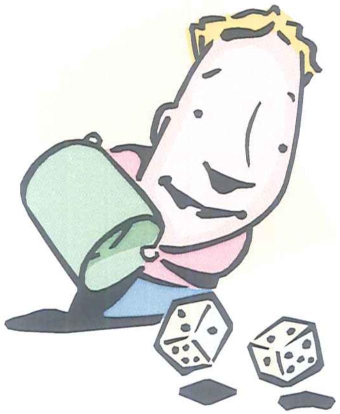
ANNOTATED NOTES FOR

STAT101 Lecture 8

Probability



Today

Learning Objectives:

Calculate probabilities from a contingency table.

Decide whether events are dependent or independent, and explain what this means in a straight-forward example.

Decide whether events are mutually exclusive and collectively exhaustive, and explain what this means in a straight-forward example

Solve simple problems involving probability.

Additive rule

The additive rule is used to calculate compound probabilities for a ***union*** of two events

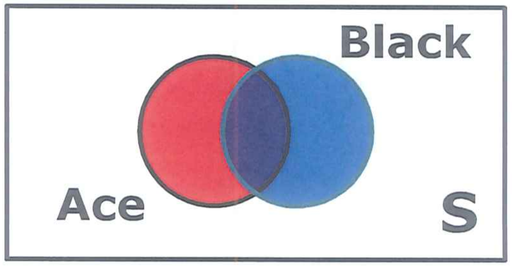
$$pr(A \cup B) = pr(A) + pr(B) - pr(A \cap B)$$

probability of A or B

= pr(A) + pr(B) − pr(A and B)

Additive rule

Experiment: Draw 1 playing card. Note its kind, colour and suit.



$$pr(A \cup B) = pr(A) + pr(B) - pr(A \cap B)$$

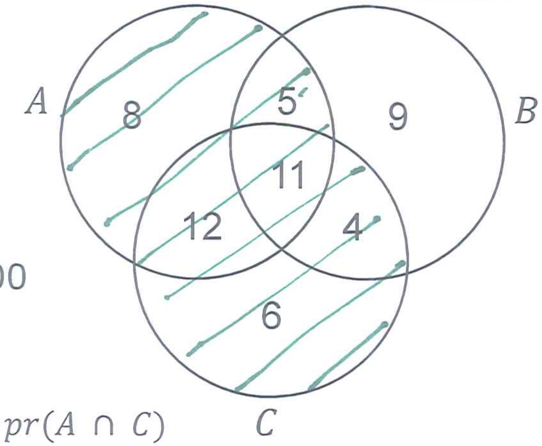
$pr(\text{Ace or black}) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$

Venn diagram example (from lecture 7)

Event A: Easy to deal with. Event B: Longstanding.
Event C: Has over $10,000

Consider the compound event

easy to deal with
OR
has over $10,000



Event $(A \cup C)$
Easy **OR** $10,000

$pr(A \cup C) =$

$= pr(A) + pr(C) - pr(A \cap C)$

$= \frac{36}{55} + \frac{33}{55} - \frac{23}{55} = \frac{46}{55}$

Collectively exhaustive events

Collectively exhaustive:

At least one of the events **must** occur

Examples:

Weather: $A=$ \`it's raining',
$B=$\`it's not raining'

Toss of a die: $A=$ number is 1,2,3,5
$B=$ number is even.

Each sample point is in one of the events, so
$pr(A \cup B) = 1$

Mutually exclusive events

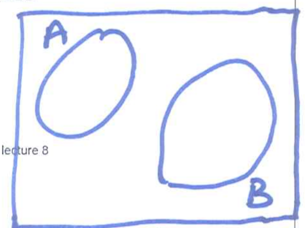
Also called **Disjoint Events** (Triola Pg 168)

The two events **cannot occur at the same time** (i.e., there are no outcomes in common).
Examples: Toss a coin: $A=$ heads, $B=$ tails
Roll two dice: $A=$ the numbers are the same,
$B=$ the sum of the two numbers is odd.

The probability of mutually exclusive events happening together is zero.

$pr(\text{heads} \cap \text{tails}) = 0$

$pr(\text{"3 on die"} \cap \text{"6 on die"}) = 0$



No overlap of A and B in the Venn diagram

Additive rule for mutually exclusive events

Additive rule (slide 3):

$$pr(A \cup B) = pr(A) + pr(B) - pr(A \cap B)$$

But for mutually exclusive events

$$pr(A \cap B) = 0$$

so

$$pr(A \cup B) = pr(A) + pr(B)$$

this is NOT true unless A and B are mutually exclusive!

Conditional probability

Probabilities can change when you gain additional information (this reduces the sample space).

The **conditional probability** is the probability of one event A occurring ***given*** that another event B has occurred.

$pr(A \text{ given } \boldsymbol{B})$ is written $pr(A|B)$.

$$pr(A|B) = \frac{pr(A \cap B)}{pr(B)} \qquad \text{(assuming } pr(B) \neq 0\text{)}$$

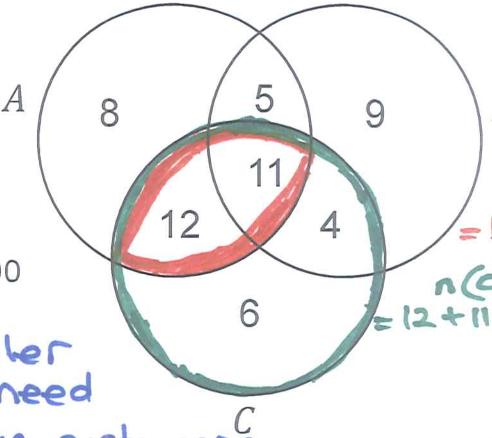
Venn diagram example (from Lecture 7)

Event A: Easy to deal with. Event B: Longstanding.
Event C: Has over $10,000

Consider the event

easy to deal with
GIVEN
they have over $10,000

Event $(A \mid C)$
Easy **GIVEN** $10,000



$n(A \cap C)$
$= 12 + 11 = 23$

$n(C)$
$= 12 + 11 + 4 + 6$
$= 33$

This uses a smaller sample space - we need only to consider the customers with over $10K. (there are 33 of them)

so $pr(A|C) = \frac{pr(A \cap C)}{pr(C)} = \frac{23}{33} = \frac{n(A \cap C)}{n(C)}$

Example 2: Conditional probabilities from table

A sample of 120 cars for sale in last weekend's Press are summarised in the table below:

	Red	White	Blue	TOTALS
Toyota	5	25	10	40
Nissan	10	15	15	40
Subaru	15	20	5	40
TOTALS	30	60	30	120

$pr(\text{Subaru} \mid \text{red}) = 15/30 = \frac{1}{2} = \frac{n(S \cap R)}{n(R)}$

$pr(\text{red} \mid \text{Subaru}) = 15/40$

$= \frac{n(S \cap R)}{n(S)}$

Conditional probability Example 3

Ten mystery envelopes are handed out at a party; three contain a prize and the other seven don't.

The envelopes are put in a hat and handed around. You take one, and your friend does too.

What is the probability your envelope contains a prize? $= 3/10$

Your friend opens her envelope before you and finds she has won a prize. Now what is the probability that your envelope contains a prize?

$2/9$

(2 prizes left out of the remaining 9 envelopes)

Let A be the event 'he has a game of golf'
and B be the event 'he goes fishing'

Conditional probability Example 4

In a weekend the probability that Murray has a game of golf is 0.35, the probability that he goes fishing is 0.65, and the probability that he does both is 0.20.

Find the probability that **if** Murray goes fishing **then** he also has a game of golf.

A B

We need pr (A given B) = pr(A | B)

$$= \frac{pr(A \cap B)}{pr(B)} = \frac{0.2}{0.65} = 0.308 \text{ (3dp)}$$

Statistical independence

Independent events: the occurrence of one event does **not** influence the probability of another event.

Examples:

- The outcomes of tossing a coin twice
- Draw a card, put it back, shuffle the deck, draw another card
- Choose a ball from a bag, replace it, shake the bag, choose another ball

Quiz: statistical independence?

Examples:

1) Finding that your car stereo works *and* finding that your car headlights work. NOT independent
2) Finding that your cellphone works *and* finding that your car starts. Independent
3) Randomly selecting a mall consumer and they tick yes to 'is a student' and yes to `rides a skateboard'. Not independent
4) Wearing plaid shorts with black socks and sandals *and* asking someone on a date and getting a positive response.

Not independent – give it a try!

Multiplicative rule

Recall conditional probability (slide 9):

$$pr(A|B) = \frac{pr(A \cap B)}{pr(B)} \quad \text{(assuming } pr(B) \neq 0)$$

Rearranging gives the **multiplicative rule**

$$pr(A \cap B) = pr(A\,|B)\,pr(B)$$

Multiplicative rule for independent events

General multiplicative rule (slide 16):

$$pr(A \cap B) = pr(A\,|B)\,pr(B)$$

by rearranging the conditional probability formula

For **independent events** we also have:

$$pr(A|B) = pr(A)$$

So $pr(A \cap B) = pr(A) \times pr(B)$

Statistical independence

To test for independence, check if

$$pr(A|B) = pr(A)$$

$$pr(B|A) = pr(B)$$

$$pr(A \cap B) = pr(A) \times pr(B)$$

Any one suffices (if one holds they all do).

This formula is needed for the tutorial questions.

Example from Lecture 7

Are the events 'Hilary is in' and 'Irene is in' independent?

Recall, A was the event 'Irene is in' and
B was the event 'Hilary is in'

We had $pr(A) = 0.6$, $pr(B) = 0.7$

and $pr(A \text{ and } B) = 0.5$

If A and B are independent then

$$pr(A \cap B) = pr(A) \times pr(B)$$

is $0.5 = 0.6 \times 0.7$? No

So the events are NOT independent.

Tree diagrams for probabilities

A probability tree diagram is

- a branched picture of the multiplicative rule used to find joint probabilities, eg, $pr(A \cap B)$
  - each set of branches is an event
  - each set of branches should sum to 1

Multiply along each branch to find the probability of a particular joint event using the **multiplicative rule**:

$$pr(A \cap B) = pr(A) \times pr(B|A)$$

Tree diagram example 1: Ferry travel

A traveller has a choice between the regular ferry and the fast ferry from Picton to Wellington.

pr(late | regular ferry)

The probability she uses the regular ferry is 0.6

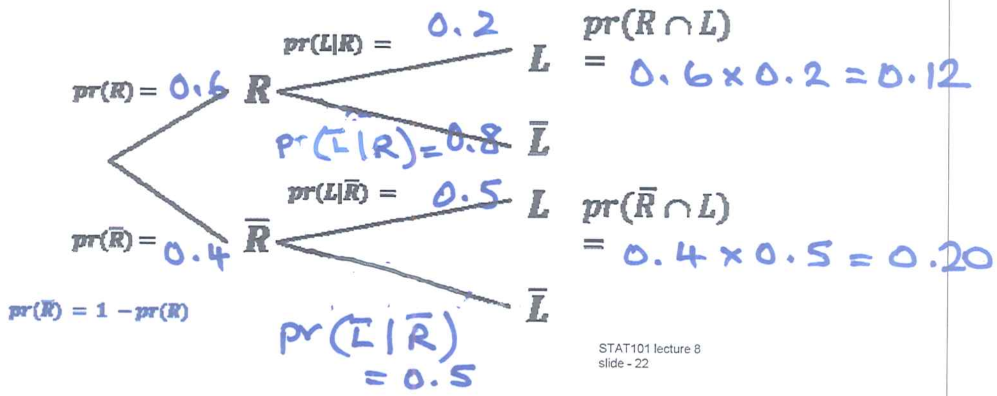
The probability the regular ferry is late is 0.2 and the probability the fast ferry is late is 0.5

Find the probability she is late.

pr(late | fast ferry) = 0.5

Tree diagram example 1: Ferry travel solution

Event R = Traveller chooses regular ferry.
Event L = Ferry is late.



Tree diagram example 1: Ferry travel solution continued

$$pr(L) = pr(R \cap L) + pr(\bar{R} \cap L)$$

multiplying along the branches

$$= 0.6 \times 0.2 = 0.12$$

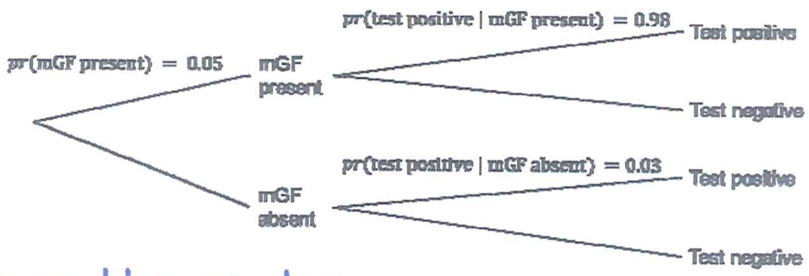
$$= 0.4 \times 0.5 = 0.20$$

So, probability she is late = $0.12 + 0.20$

$= 0.32$

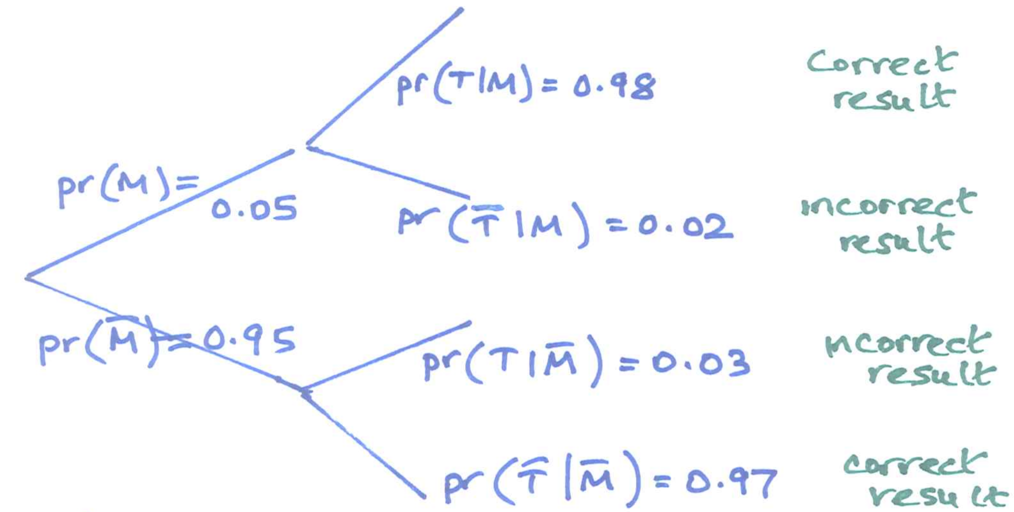
Tree diagram example 2 Testing for an antibody

On average 1 in 20 people have a particular antibody mGF in their blood. The tree diagram sketch shows the results when a machine tests the blood for mGF.
[NB "test positive" means the machine detects the antibodies.]
Find the probability that the machine gives an incorrect result.



Defining the events:
M = mGF present
T = test positive

Tree diagram example 2
Solution



pr(incorrect result) = (0.05 × 0.02) + (0.95 × 0.03) = 0.0295.

Key points from lecture 8

After this lecture you should be able to:

✓ calculate probabilities from a contingency table

✓ decide whether events are dependent or independent, and explain what this means in a straight-forward example

✓ decide whether events are mutually exclusive and collectively exhaustive, and explain what this means in a straight-forward example

✓ solve simple problems involving probability.

Extra Example
Picking from a limited pool (1)

In situations were we pick more than one item from a 'pool' with a limited size the probabilities the second time that we pick depend on (are conditional on) the result of the first pick.

For example:

I buy a new tray of 144 eggs and (unknown to me) there are 6 rotten eggs in the tray. I am going to pick 2 eggs to boil. What is the probability that both eggs I pick are rotten?

The probability that the first egg I pick is rotten is 6/144.

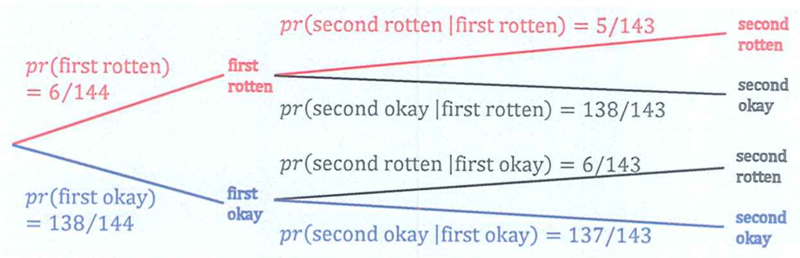
If the first egg is rotten then there are 5 more rotten eggs left out of 143 eggs left in total:

$pr(\text{second rotten} \mid \text{first rotten}) = 5/143$

But if the first egg is okay, there are 6 rotten ones left:

$pr(\text{second rotten} \mid \text{first okay}) = 6/143$

Extra Example
Picking from a limited pool (2)



$$pr(\text{both rotten}) = pr(\text{first rotten}) * pr(\text{second rotten} \mid \text{first rotten})$$
$$= (6/144) * (5/143)$$

A similar approach could be used for the probability that both eggs are okay: $pr(\text{both okay}) = (138/144) * (137/143)$

and the probability that at least one is rotten (complement rule!):

$$pr(\text{at least one okay}) = 1 - pr(\text{both rotten})$$
$$= 1 - (6/144) * (5/143)$$

* you will find questions in Tutorial quiz 4 that use this method *