Sample Tests

This section of the *Instructor's Guide* contains sample tests for an introductory course in discrete mathematics. Two tests are included for each chapter of the text. The problems on these tests were used on examinations given in discrete mathematics courses at various schools, or are similar to such questions. The first test contains straightforward problems and is easier than the second test. Some of the problems from these second tests are moderately difficult. I have also included two sample final examinations. The second of these is the more challenging examination.

You may want to use these tests as a source of questions for your own examinations, rather than using them exactly as they are. If you do so, select questions primarily from the first of the two examinations for straightforward questions, and from the second for more challenging questions. Also, for a much richer set of questions, consult the extensive test bank also included in this *Guide*.

These sample tests are an attempt to test students efficiently. Wherever appropriate, problems with numerical or short answers are given. However, there are many places in the course where it is not possible to assess students adequately without requiring longer answers. You will find that there are several problems where I have asked students to prove or disprove a statement. I find that questions of this sort test whether students can think mathematically and write correct mathematical arguments.

On my examinations I give explicit directions to students to provide complete answers, including reasons for the steps of proofs; I advise you to do the same.

Each sample test has been printed on its individual page. Solutions are provided immediately following the test.

Chapter 1 — Test 1

1. What is the truth value of $(p \vee q) \rightarrow (p \wedge q)$ when both p and q are false?

2. What are the converse and contrapositive of the statement "If it is sunny, then I will go swimming"?

3. Show that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are equivalent

(a) using a truth table.

(b) using logical equivalences.

4. Suppose that $Q(x)$ is the statement "$x + 1 = 2x$." What are the truth values of $\forall x\, Q(x)$ and $\exists x\, Q(x)$?

5. Prove each of the following statements.

(a) The sum of two even integers is always even.

(b) The sum of an even integer and an odd integer is always odd.

6. Prove that there are no solutions in positive integers to the equation $x^4 + y^4 = 100$.

Chapter 1—Test 1 Solutions

1. When p and q are both false, so are $(p \vee q)$ and $(p \wedge q)$. Hence $(p \vee q) \to (p \wedge q)$ is true.

2. The converse of the statement is "If I go swimming, then it is sunny." The contrapositive of the statement is "If I do not go swimming, then it is not sunny."

3. (a) We have the following truth table.

p	q	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p$	$q \wedge \neg p$
T	T	F	T	F	F	F
T	F	T	T	F	F	F
F	T	F	F	T	T	T
F	F	T	T	F	T	F

Since the fifth and seventh columns agree, we conclude that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are equivalent.

(b) By De Morgan's law $\neg(p \vee \neg q)$ and $\neg p \wedge \neg\neg q$ are equivalent. By the double negation law, this is equivalent to $\neg p \wedge q$, which is equivalent to $q \wedge \neg p$ by the commutative law. We conclude that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are equivalent.

4. Since $x + 1 = 2x$ is true if and only if $x = 1$, we see that $Q(x)$ is true if and only if $x = 1$. It follows that $\forall x\, Q(x)$ is false and $\exists x\, Q(x)$ is true.

5. (a) Suppose that m and n are even integers. Then there are integers j and k such that $m = 2j$ and $n = 2k$. It follows that $m + n = 2j + 2k = 2(j + k) = 2l$, where $l = j + k$. Hence $m + n$ is even.

(b) Suppose that m is even and n is odd. Then there are integers j and k such that such that $m = 2j$ and $n = 2k + 1$. It follows that $m + n = 2j + (2k + 1) = 2(j + k) + 1 = 2l + 1$, where $l = j + k$. Hence $m + n$ is odd.

6. If $x^4 + y^4 = 100$, then both x and y must be less than 4, since $4^4 = 256$. Therefore the only possible values for x and y are 1, 2, and 3, and the fourth powers of these are 1, 16, and 81. Since none of $1+1$, $1+16$, $1+81$, $16+16$, $16+81$, and $81+81$ is 100, there can be no solution.

Chapter 1—Test 2

1. Prove or disprove that $(p \to q) \to r$ and $p \to (q \to r)$ are equivalent.

2. Let $P(m, n)$ be "n is greater than or equal to m" where the domain (universe of discourse) is the set of nonnegative integers. What are the truth values of $\exists n \forall m\, P(m, n)$ and $\forall m \exists n\, P(m, n)$?

3. Prove that all the solutions to the equation $x^2 = x + 1$ are irrational.

4. (a) Prove or disprove that a 6×6 checkerboard can be covered with straight triominoes.

(b) Prove or disprove that an 8×8 checkerboard can be covered with straight triominoes.

5. A stamp collector wants to include in her collection exactly one stamp from each country of Africa. If $I(s)$ means that she has stamp s in her collection, $F(s, c)$ means that stamp s was issued by country c, the domain for s is all stamps, and the domain for c is all countries of Africa, express the statement that her collection satisfies her requirement. Do not use the $\exists!$ symbol.

Chapter 1—Test 2 Solutions

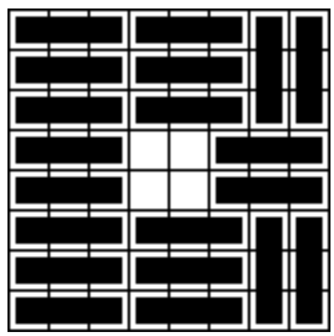
1. Suppose that p is false, q is true, and r is false. Then $(p \to q) \to r$ is false since its premise $p \to q$ is true while its conclusion r is false. On the other hand, $p \to (q \to r)$ is true in this situation since its premise p is false. Therefore $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent.

2. For every positive integer n there is an integer m such that $n < m$ (take $m = n+1$ for instance). Hence $\exists n \forall m\, P(m,n)$ is false. For every integer m there is an integer n such that $n \geq m$ (take $n = m+1$ for instance). Hence $\forall m \exists n\, P(m,n)$ is true.

3. This equation is equivalent to (and therefore has the same solutions as) $x^2 - x - 1 = 0$. By the quadratic formula, the solutions are exactly $(1 \pm \sqrt{5})/2$. If either of these were a rational number r, then we would have $\sqrt{5} = \pm(2r-1)$. Since the rational numbers are closed under the arithmetic operations, this would tell us that $\sqrt{5}$ was rational, which we know from this chapter it is not.

4. (a) The 6×6 board with four squares removed has $36 - 4 = 32$ squares. Since 32 is not a multiple of 3, it cannot be covered by pieces that cover 3 squares each.

(b) The following picture shows that it is possible.



5. The simplest formula is $\forall c \, \exists s \, \forall x((I(x) \land F(x,c)) \leftrightarrow x = s)$.

Chapter 2—Test 1

1. Let $A = \{a, c, e, h, k\}$, $B = \{a, b, d, e, h, i, k, l\}$, and $C = \{a, c, e, i, m\}$. Find each of the following sets.

(a) $A \cap B$

(b) $A \cap B \cap C$

(c) $A \cup C$

(d) $A \cup B \cup C$

(e) $A - B$

(f) $A - (B - C)$

2. Prove or disprove that if A, B, and C are sets then $A - (B \cap C) = (A - B) \cap (A - C)$.

3. Let $f(n) = 2n + 1$. Is f a one-to-one function from the set of integers to the set of integers? Is f an onto function from the set of integers to the set of integers? Explain the reasons behind your answers.

4. Suppose that f is the function from the set $\{a, b, c, d\}$ to itself with $f(a) = d$, $f(b) = a$, $f(c) = b$, $f(d) = c$. Find the inverse of f.

5. Find the values of $\displaystyle\sum_{j=1}^{100} 2$ and $\displaystyle\sum_{j=1}^{100} (-1)^j$.

6. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$. Find $\mathbf{AB}$ and $\mathbf{BA}$. Are they equal?

7. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Find the join, meet, and Boolean product of these two zero-one matrices.

Chapter 2—Test 1 Solutions

1. (a) $\{a, e, h, k\}$

(b) $\{a, e\}$

(c) $\{a, c, e, h, i, k, m\}$

(d) $\{a, b, c, d, e, h, i, k, l, m\}$

(e) $\{c\}$

(f) $A - \{b, d, h, k, l\} = \{a, c, e\}$

2. This is false. For a counterexample take $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$. We have $A - (B \cap C) = \{1, 2\} - \emptyset = \{1, 2\}$, while $(A - B) \cap (A - C) = \{2\} \cap \{1\} = \emptyset$.

3. If $f(n) = f(m)$, then $2n+1 = 2m+1$. It follows that $n = m$. Hence f is one-to-one. Since $f(n) = 2n+1$ is odd for every integer n, it follows that $f(n)$ is not onto; for example, 2 is not in its range.

4. The inverse is $f^{-1}(a) = b$, $f^{-1}(b) = c$, $f^{-1}(c) = d$, $f^{-1}(d) = a$.

5. We have $\sum_{j=1}^{100} 2 = 100 \cdot 2 = 200$ and $\sum_{j=1}^{100} (-1)^j = -1 + 1 - 1 + 1 - \cdots + 1 = 0$.

6. We have $\mathbf{AB} = \begin{bmatrix} 7 & 13 \\ 8 & 13 \end{bmatrix}$ and $\mathbf{BA} = \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 4 \\ 2 & 7 & 18 \end{bmatrix}$. Obviously $\mathbf{AB} \neq \mathbf{BA}$ since they are not even the same size.

7. The join of $\mathbf{A}$ and $\mathbf{B}$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The meet of $\mathbf{A}$ and $\mathbf{B}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. The Boolean product of $\mathbf{A}$ and $\mathbf{B}$ is $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Chapter 2 — Test 2

1. Let A, B, and C be sets. Prove or disprove that $A - (B \cap C) = (A - B) \cup (A - C)$.

2. Consider the function $f(n) = 2\lfloor n/2 \rfloor$ from $\mathbf{Z}$ to $\mathbf{Z}$. Is this function one-to-one? Is this function onto? Justify your answers.

3. Show that the set of odd positive integers greater than 3 is countable.

4. Find $\sum_{j=1}^{100} 2j + 5$ and $\sum_{j=5}^{100} 3^j$.

5. Prove or disprove that $\mathbf{AB} = \mathbf{BA}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices.

Chapter 2—Test 2 Solutions

1. We see that $A - (B \cap C) = A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A - B) \cup (A - C)$. These equalities follow from the definition of the difference of two sets, De Morgan's law, the distributive law for intersection over union, and the definition of the difference of two sets, respectively.

2. Note that $f(0) = 2\lfloor 0/2 \rfloor = 0$ and $f(1) = 2\lfloor 1/2 \rfloor = 0$. Hence f is not one-to-one. Note that $f(n)$ is even for every integer n. Hence f is not onto.

3. The function $f(n) = 2n + 3$ is a one-to-one correspondence from the set of positive integers to the set of odd positive integers greater than 3. Hence this set is countable.

4. We have

$$\sum_{j=1}^{100} 2j + 5 = 2\sum_{j=1}^{100} j + \sum_{j=1}^{100} 5 = 2 \cdot \frac{100 \cdot 101}{2} + 100 \cdot 5 = 10600$$

and

$$\sum_{j=5}^{100} 3^j = \sum_{j=0}^{100} 3^j - \sum_{j=0}^{4} 3^j = \frac{3^{101}}{3-1} - \frac{3^5}{3-1} = \frac{3^{101} - 3^5}{2}.$$

5. This is false. Counterexamples are easy to find. For instance, let $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ while $\mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Chapter 3—Test 1

1. Describe an algorithm for finding the smallest integer in a finite sequence of integers.

2. Determine the worst case complexity in terms of the number of comparisons used for the algorithm you described in problem 1.

3. Let $f(n) = 3n^2 + 8n + 7$. Show that $f(n)$ is $O(n^2)$. Be sure to specify the values of the witnesses C and k.

4. Suppose that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are 3×4, 4×5, and 5×6 matrices of numbers, respectively. Is it more efficient to compute the product $\mathbf{ABC}$ as $(\mathbf{AB})\mathbf{C}$ or as $\mathbf{A}(\mathbf{BC})$? Justify your answer by computing the number of multiplications of numbers needed each way.

Chapter 3—Test 1 Solutions

1. Suppose the terms of the sequence are a_1, a_2, $\ldots$, a_n. First, assign $min := a_1$. Then successively compare a_i with min for $i = 2, 3, \ldots, n$, assigning the value of a_i to min if $a_i < min$. After all the terms have been examined, the value of min will be the smallest integer in the sequence.

2. There are $n - 1$ comparisons used by the algorithm in problem 1, ignoring the bookkeeping. Hence this is a $O(n)$ algorithm in both the worst and average cases.

3. We have $f(n) = 3n^2 + 8n + 7 \le 3n^2 + 8n^2 + 7n^2 = 18n^2$ whenever $n \ge 1$. It follows that $f(n)$ is $O(n^2)$, since we can take $C = 18$ and $k = 1$ in the definition.

4. To multiply $\mathbf{A}$ by $\mathbf{B}$, we will need $3 \cdot 4 \cdot 5 = 60$ multiplications. The result is a 3×5 matrix. To multiply it by $\mathbf{C}$ will require $3 \cdot 5 \cdot 6 = 90$ multiplications. This gives a total of $60 + 90 = 150$ steps. On the other hand, if we multiply $\mathbf{B}$ by $\mathbf{C}$ first, we use $4 \cdot 5 \cdot 6 = 120$ multiplications and then another $3 \cdot 4 \cdot 6 = 72$ to multiply $\mathbf{A}$ by the 4×6 matrix $\mathbf{BC}$. This method uses a total of $120 + 72 = 192$ steps. Therefore the first method is a little faster.

Chapter 3—Test 2

1. (a) Describe an algorithm for finding the second largest integer in a sequence of distinct integers.

(b) Give a big-O estimate of the number of comparison used by your algorithm.

2. Show that $1^3 + 2^3 + 3^3 + \cdots + n^3$ is $O(n^4)$.

3. Show that the function $f(x) = (x+2)\log(x^2+1) + \log(x^3+1)$ is $O(x \log x)$.

4. Describe a brute-force algorithm for determining, given a compound proposition P in n variables, whether P is satisfiable. It is known that this problem is NP-complete. If $\mathrm{P} = \mathrm{NP}$, what conclusion can be drawn about the efficiency of your algorithm compared to the efficiency of the best algorithm for solving this problem?

Chapter 3—Test 2 Solutions

1. (a) We first compare the first and second integers in the sequence a_1, a_2, ..., a_n, setting the value of the variable *firstmax* equal to the larger, and the value of the variable *secondmax* equal to the smaller. For each successive integer a_i in the sequence, $i = 3, 4, \ldots, n$, we first compare it to *firstmax*. If $a_i > \mathit{firstmax}$, then we make the assignments $\mathit{secondmax} := \mathit{firstmax}$ and $\mathit{firstmax} := a_i$. Otherwise, we compare a_i to *secondmax*, and if $a_i > \mathit{secondmax}$, then we make the assignment $\mathit{secondmax} := a_i$. At the end of this procedure the value of *secondmax* will be the second largest integer in the sequence.

(b) We do one comparison at the beginning of the algorithm to determine whether a_1 or a_2 is larger. Then for each successive term, for $i = 3, 4, \ldots, n$, we carry out at most two comparisons. Hence the largest number of comparisons used is $2(n-2)+1 = 2n-3$, ignoring bookkeeping. This is $O(n)$.

2. We have $1^3 + 2^3 + 3^3 + \cdots + n^3 \le n^3 + n^3 + n^3 + \cdots + n^3 = n \cdot n^3 = n^4$ whenever n is a positive integer. It follows that $1^3 + 2^3 + 3^3 + \cdots + n^3$ is $O(n^4)$, with witnesses $C = 1$ and $k = 1$.

3. We have $x+2$ is $O(x)$ since $x + 2 \le 2x$ for all $x \ge 2$; $\log(x^2+1)$ is $O(\log x)$ since $\log(x^2+1) \le \log(2x^2) = \log 2 + 2\log x \le 3\log x$ whenever $x \ge 2$; and similarly $\log(x^3+1)$ is $O(\log x)$. It follows that $(x+2)\log(x^2+1)$ is $O(x \log x)$ and consequently $f(x)$ is $O(x \log x)$.

4. For each string of length n of the letters **T** and **F** (representing true and false), evaluate the given compound proposition when the i^{th} variable is assigned the truth value given by the i^{th} letter in the string. If any of these values of P is **T**, then P is satisfiable; otherwise it is not. This will take at least 2^n steps in the worst case (the case in which P is not satisfiable), once for each such string. If $\mathrm{P} = \mathrm{NP}$, then there is a polynomial worst-case time algorithm, so the brute force algorithm is not the most efficient.

Chapter 4—Test 1

1. Decide whether $175 \equiv 22 \pmod{17}$.

2. Find the prime factorization of 45617.

3. Use the Euclidean algorithm to find

(a) $\gcd(203, 101)$.

(b) $\gcd(34, 21)$.

4. The binary expansion of an integer is $(110101)_2$. What is the base 10 expansion of this integer?

5. Prove or disprove that a positive integer congruent to 1 modulo 4 cannot have a prime factor congruent to 3 modulo 4.

Chapter 4—Test 1 Solutions

1. We have $175 - 22 = 153$ and $17 \mid 153$ since $153 = 17 \cdot 9$. Hence $175 \equiv 22 \pmod{17}$.

2. We see that neither 2, 3, 5, nor 7 divides 45617. Dividing by 11, we find that $45617/11 = 4147$. We divide by 11 again to find that $4147/11 = 377$. We find that 377 is not divisible by 11. We divide by 13 and find that $377 = 13 \cdot 29$. Hence the prime factorization of 45617 is $45617 = 11 \cdot 11 \cdot 13 \cdot 29$.

3. (a) We have $203 = 2 \cdot 101 + 1$ and $101 = 101 \cdot 1$. It follows that $\gcd(203, 101) = 1$.

(b) We have $34 = 1 \cdot 21 + 13$, $21 = 1 \cdot 13 + 8$, $13 = 1 \cdot 8 + 5$, $8 = 1 \cdot 5 + 3$, $5 = 1 \cdot 3 + 2$, $3 = 1 \cdot 2 + 1$, $2 = 2 \cdot 1$ Hence $\gcd(34, 21) = 1$.

4. We have $(110101)_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 32 + 16 + 4 + 1 = 53$.

5. This is false, since $9 = 4 \cdot 2 + 1 = 3 \cdot 3$.

Chapter 4—Test 2

1. Find the prime factorization of 111111.

2. Find each of the following values.

(a) $18 \mathbf{\ mod\ } 7$

(b) $-88 \mathbf{\ mod\ } 13$

(c) $289 \mathbf{\ mod\ } 17$

3. Let m be a positive integer, and let a, b, and c be integers. Show that if $a \equiv b \pmod{m}$, then $a - c \equiv b - c \pmod{m}$.

4. Use the Euclidean algorithm to find

(a) $\gcd(201, 302)$.

(b) $\gcd(144, 233)$.

5. What is the hexadecimal expansion of the $(\mathrm{ABC})_{16} + (\mathrm{2F5})_{16}$?

6. Prove or disprove that there are six consecutive composite integers.

Chapter 4—Test 2 Solutions

1. We see that 2 does not divide 111111, but that 3 does divide 111111, with $111111/3 = 37037$. We see that 3 does not divide 37037, and that 5 does not divide it either. We see that 7 does divide 37037 with $37037/7 = 5291$. We see that 7 does not divide 5291, but 11 does divide 5291 with $5291 = 11 \cdot 481$. We find that 11 does not divide 481, but 13 does, with $481 = 13 \cdot 37$. Since 37 is prime, it follows that the prime factorization of 111111 is $111111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$.

2. (a) We have $18 = 2 \cdot 7 + 4$. Hence $18 \textbf{ mod } 7 = 4$.

(b) We have $-88 = -7 \cdot 13 + 3$. Hence $-88 \textbf{ mod } 13 = 3$.

(c) We have $289 = 17 \cdot 17$. Hence $289 \textbf{ mod } 17 = 0$.

3. Since $a \equiv b \pmod{m}$ we have $m \mid a - b$. Hence there is an integer k such that $a - b = mk$. It follows that $(a-c) - (b-c) = a - b = mk$, so $a - c \equiv b - c \pmod{m}$.

4. (a) We see that $302 = 1 \cdot 201 + 101$, $201 = 1 \cdot 101 + 100$, $101 = 1 \cdot 100 + 1$, and $100 = 100 \cdot 1$. Hence $\gcd(302, 201) = 1$.

(b) We see that $233 = 1 \cdot 144 + 89$, $144 = 1 \cdot 89 + 55$, $89 = 1 \cdot 55 + 34$, $55 = 1 \cdot 34 + 21$, $34 = 1 \cdot 21 + 13$, $21 = 1 \cdot 13 + 8$, $13 = 1 \cdot 8 + 5$, $8 = 1 \cdot 5 + 3$, $5 = 1 \cdot 3 + 2$, $3 = 1 \cdot 2 + 1$, $2 = 2 \cdot 1$. Hence $\gcd(233, 144) = 1$.

5. Working from right to left in base 16, we have $\mathrm{C} + 5 = 11$, so the rightmost digit of the sum is 1 and the carry is 1. We have $\mathrm{B} + \mathrm{F} + 1 = 1\mathrm{B}$, so the second digit from the right is B and the carry is 1. We have $\mathrm{A} + 2 + 1 = \mathrm{D}$. Hence the sum is $(\mathrm{DB1})_{16}$.

6. We can give a constructive proof. The six consecutive integers $7! + 2$, $7! + 3$, $7! + 4$, $7! + 5$, $7! + 6$, and $7! + 7$ are all composite, since $i \mid 7! + i$ for $i = 2, 3, 4, 5, 6, 7$.

Chapter 5—Test 1

1. Use mathematical induction to show that $\sum_{j=0}^{n}(j+1) = (n+1)(n+2)/2$ whenever n is a nonnegative integer.

2. Show that $3^n < n!$ whenever n is an integer with $n \geq 7$.

3. Suppose that the only currency were 3-dollar bills and 10-dollar bills. Show that every amount greater than 17 dollars could be made from a combination of these bills.

4. Suppose that $\{a_n\}$ is defined recursively by $a_n = a_{n-1}^2 - 1$ and that $a_0 = 2$. Find a_3 and a_4.

5. Give a recursive algorithm for computing na using addition, where n is a positive integer and a is a real number.

Chapter 5—Test 1 Solutions

1. The basis step follows since $\sum_{j=0}^{0}(j+1) = 0+1 = (0+1)(0+2)/2$. For the inductive step assume that $\sum_{j=0}^{k}(j+1) = (k+1)(k+2)/2$. It follows that $\sum_{j=0}^{k+1}(j+1) = \sum_{j=0}^{k}(j+1)+[(k+1)+1] = (k+1)(k+2)/2+[(k+1)+1] = (k+1)(k+2)/2+(k+2) = (k+2)[(k+1)/2+1] = (k+2)(k+3)/2 = [(k+1)+1][(k+1)+2]/2$. This completes the proof by mathematical induction.

2. The basis step holds since $3^7 = 2187 < 5040 = 7!$. For the inductive step assume that $3^k < k!$ where k is a positive integer greater than or equal to 7. Using the inductive hypothesis we see that $3^{k+1} = 3 \cdot 3^k < 3 \cdot k! < (k+1)k! = (k+1)!$. This completes the proof.

3. We find that 18 dollars can be made using six 3-dollar bills. Now suppose that n dollars can be formed, where $n \geq 18$. Suppose that at least two 10-dollar bills were used. Then two 10-dollar bills can be replaced by seven 3-dollar bills to form $n+1$ dollars. Otherwise, if zero or one 10-dollar bill were used, then at least three 3-dollar bills were used. Then replace three 3-dollar bills by one 10-dollar bill to form $n+1$ dollars.

4. We have $a_0 = 2$, $a_1 = a_0^2 - 1 = 2^2 - 1 = 3$, $a_2 = a_1^2 - 1 = 3^2 - 1 = 8$, $a_3 = a_2^2 - 1 = 8^2 - 1 = 63$, and $a_4 = a_3^2 - 1 = 63^2 - 1 = 3968$.

5. We can compute na recursively using the following procedure.

procedure *mult*(a : real number, n : positive integer)
if $n = 1$ **then return** a
else return $a + mult(a, n-1)$

Chapter 5—Test 2

2. What is wrong with the following proof that every positive integer equals the next larger positive integer?

> "Proof." Let $P(n)$ be the proposition that $n = n + 1$. Assume that $P(k)$ is true, so that $k = k + 1$. Add 1 to both sides of this equation to obtain $k + 1 = k + 2$. Since this is the statement $P(k + 1)$, it follows that $P(n)$ is true for all positive integers n.

3. Prove that $\sum_{j=n}^{2n-1} (2j + 1) = 3n^2$ whenever n is a positive integer.

4. Use mathematical induction to show that n lines in the plane passing through the same point divide the plane into $2n$ parts.

5. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Show that $a_n \leq 3^n$ for all positive integers n.

6. Describe a recursive algorithm for computing 3^{2^n} where n is a nonnegative integer.

Chapter 5—Test 2 Solutions

2. The error is that no basis step has been done.

3. The basis step holds since $\sum_{j=1}^{1}(2j+1) = 3 = 3\cdot 1^2$. For the inductive step assume that $\sum_{j=k}^{2k-1}(2j+1) = 3k^2$. It follows that $\sum_{j=k+1}^{2(k+1)-1}(2j+1) = \sum_{j=k}^{2k-1}(2j+1) - (2k+1) + (4k+1) + (4k+3) = 3k^2 + 6k + 3 = 3(k+1)^2$. This completes the proof.

4. The basis step follows since one line divides the plane into $2\cdot 1$ parts. For the inductive step assume that k lines passing through a point divide the plane into $2k$ parts. Suppose that we have $k+1$ lines. If we take k of these lines, by the inductive hypothesis they divide the plane into $2k$ parts. Adding the $(k+1)^{\text{st}}$ line splits exactly two of these parts in two. Hence these $k+1$ concurrent lines split the plane into $2k+2 = 2\cdot(k+1)$ parts. This completes the proof.

5. Let $P(n)$ be the proposition that $a_n \le 3^n$. The proof uses strong induction. The basis step follows since $a_1 = 2 \le 3 = 3^1$, and $a_2 = 9 \le 9 = 3^2$. For the inductive step assume that $P(j)$ is true for $1 \le j \le k$. Then $a_j \le 3^j$ for $1 \le j \le k$. Hence $a_{k+1} = 2a_k + 3a_{k-1} \le 2\cdot 3^k + 3\cdot 3^{k-1} = 2\cdot 3^k + 3^k = 3\cdot 3^k = 3^{k+1}$. This shows that $P(k+1)$ is also true, and our proof is complete.

6. We can use the following recursive procedure.

procedure $x(n$: nonnegative integer)
if $n = 0$ **then return** 3
else return $x(n-1)\cdot x(n-1)$

Chapter 6—Test 1

1. Each locker in an airport is labeled with an uppercase letter followed by three digits. How many different labels for lockers are there?

2. There are 805 lockers in the athletic center and 4026 students who need lockers. Therefore, some students must share lockers. What is the largest number of students who must necessarily share a locker?

3. Find the value of each of the following quantities.

(a) $C(5,4)$

(b) $C(5,0)$

(c) $P(5,1)$

(d) $P(5,5)$

4. How many rows are found in a truth table involving nine different propositions?

5. What is the coefficient of x^2y^7 in $(x+y)^9$?

6. How many ways are there to choose five doughnuts if there are eight varieties (and only the type of each doughnut matters)?

7. How many different string can be made using all the letters in the word $GOOGOL$?

Chapter 6—Test 1 Solutions

1. By the product rule for counting there are $26 \cdot 10 \cdot 10 \cdot 10 = 26,000$ different labels for lockers.

2. By the generalized pigeonhole principle there are at least $\lceil 4026/805 \rceil = 6$ students who must share a locker.

3. (a) $C(5,4) = \dfrac{5!}{4!1!} = 5$

(b) $C(5,0) = \dfrac{5!}{0!5!} = 1$

(c) $P(5,1) = 5$

(d) $P(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

4. There are $2^9 = 512$ rows in a truth table involving nine different propositions.

5. By the binomial theorem the coefficient of x^2y^7 in $(x+y)^9$ is $\dbinom{9}{2} = \dfrac{9!}{7!2!} = \dfrac{9 \cdot 8}{2 \cdot 1} = 36$.

6. The number of ways to choose five doughnuts from eight different varieties equals the number of 5-combinations with repetition allowed from a set with 8 elements. This equals $C(5+8-1, 8-1) = C(12,7) = \dfrac{12!}{7!5!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792$.

7. There are three O's, two G's, and one L in $GOOGOL$. The number of different strings that can be made from these letters is $\dfrac{6!}{3!2!1!} = 60$.

Chapter 6—Test 2

1. How many students must be in a class to guarantee that at least five were born on the same day of the week?

2. How many different license plates can be made if each license plate consists of three letters followed by three digits or four letters followed by two digits?

3. (a) How many functions are there from a set with three elements to a set with eight elements?

(b) How many one-to-one functions are there from a set with three elements to a set with eight elements?

(c) How many onto functions are there from a set with three elements to a set with eight elements?

4. What is the coefficient of x^7y^{12} in $(x+y)^{19}$ and in $(2x+3y)^{19}$?

5. Show that $C(n,r) = C(n,n-r)$ using

(a) a combinatorial argument.

(b) algebraic manipulation.

6. (a) How many ways are there to arrange the letters of the word *NONSENSE*?

(b) How many of these ways start or end with the letter O?

7. (a) How many ways are there to choose 12 cookies if there are five varieties of cookies?

(b) How many ways are there to choose 12 cookies if there are five varieties, including chocolate chip, and at least four chocolate chip cookies must be chosen?

Chapter 6—Test 2 Solutions

1. Since there are seven days of the week, to guarantee that at least five students were born on the same day requires at least $7 \cdot 4 + 1 = 29$ students.

2. There are $26^3 10^3 + 26^4 10^2 = 63{,}273{,}600$ different license plates.

3. (a) There are $8^3 = 512$ functions.

(b) There are $8 \cdot 7 \cdot 6 = 336$ one-to-one functions.

(c) There are obviously no onto functions.

4. The coefficient of x^7y^{12} in $(x+y)^{19}$ is $\binom{19}{7} = 50{,}388$. The coefficient of x^7y^{12} in $(2x+3y)^{19}$ is $\binom{19}{7} \cdot 2^7 3^{12} = 3{,}427{,}615{,}885{,}824$.

5. (a) Choosing r elements from a set with n elements is equivalent to picking the $n-r$ elements not to choose.

(b) $C(n,r) = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-r)!(n-(n-r))!} = C(n, n-r)$

6. (a) There are $\dfrac{8!}{3!2!2!1!} = 1680$ ways.

(b) Since the letter O must be in the first position or last position, the number of strings is twice the number of ways to arrange the letters *NNSENSE*. Hence there are $2 \cdot \dfrac{7!}{3!2!2!} = \dfrac{7!}{3!2!} = 420$ ways.

7. (a) The solution is the number of 12-combinations with repetition of five objects. Therefore there are $C(12+5-1, 5-1) = C(16,4) = 1820$ ways.

(b) Since at least four chocolate chip cookies must be chosen, this is equivalent to determining the number of ways to choose eight cookies from five varieties. Consequently the answer is given by the number of 8-combinations with repetition of five objects. Therefore there are $C(8+5-1, 5-1) = C(12,4) = 495$ ways.

Chapter 7 — Test 1

1. What is the probability that a fair coin lands heads four times out of five flips?

2. What is the probability that a positive integer less than 100 picked at random has all distinct digits?

3. Suppose that two cards are drawn without replacement from a well-shuffled deck. What is the probability that both cards have numbers and that the numbers on the cards are the same (note that only the numbers 2 through 10 are shown on cards, since aces, kings, queens, and jacks are represented by letters).

4. A fair red die and a fair blue die are rolled. What is the expected value of the sum of the number on the red die plus three times the number on the blue die?

5. Two identical urns contain balls. One of the urns has 6 red balls and 3 blue balls. The other urn has 5 red balls and 8 blue balls. An urn is chosen at random and a ball is drawn at random from this urn. If the ball turns out to be red, what is the probability that this is the urn with 6 red balls?

Chapter 7—Test 1 Solutions

1. There are 2^5 possible outcomes of five flips. The number of possible ways to have the coin come up heads four times is the number of ways to pick four flips out of five. This can be done in $C(5,4) = 5$ ways. Hence the probability that the coin lands heads four times out of five flips is $5/2^5 = 5/32$.

2. There are 99 choices for the integer. Only 11, 22, 33, $\ldots$, 99 do not have distinct digits. The remaining 90 integers do have distinct digits. Therefore the answer is $90/99$.

3. The probability that the first card has a number on it is $36/52$, since $4 \cdot 9 = 36$ cards have numbers. At that point the deck has 51 remaining cards, and 3 of them have the same number as the first card drawn. Therefore the final answer is $(36/52)(3/51) = 36/884$.

4. Let X_r and X_b be the random variables for the numbers shown on the dice. We are asked for $E(X_r + 3X_b)$. Since these dice are fair, we know that $E(X_r) = E(X_b) = (1+2+3+4+5+6)/6 = 3.5$. By linearity of expectation, we have $E(X_r + 3X_b) = 3.5 + 3 \cdot 3.5 = 14$.

5. Let U be the event that the urn with 6 red balls was chosen. Then $p(U) = p(\overline{U}) = 1/2$. Let R be the event that a red ball was drawn. Because of the contents of the urns, we have $p(R \mid U) = 6/9$ and $p(R \mid \overline{U}) = 5/13$. Therefore by Bayes' theorem

$$p(U \mid R) = \frac{p(R \mid U)p(U)}{p(R \mid U)p(U) + p(R \mid \overline{U})p(\overline{U})} = \frac{(6/9)(1/2)}{(6/9)(1/2) + (5/13)(1/2)} = \frac{26}{41}\,.$$

Chapter 7 — Test 2

1. A computer picks out at random a sequence of six digits.

(a) What is the probability that a person picks all six digits in their correct positions?

(b) What is the probability that a person picks exactly five of the digits, in the correct order?

2. What is the probability that in a group of 200 random people, at least two of them have the same triple of initials (such as RSZ for Ruth Suzanne Zeitman), assuming that each triple of initials is equally likely. Give the answer as a calculation; it is not necessary to evaluate the expression.

3. Suppose that a bag contains six slips of paper: one with the number 1 written on it, two with the number 2, and three with the number 3. What is the expected value and variance of the number drawn if one slip is selected at random from the bag?

4. What is the probability that a random person who tests positive for a certain blood disease actually has the disease, if we know that 1% of the population has the disease, that 95% of those who have the disease test positive for it, and 2% of those who do not have the disease test positive for it.

5. Two identical urns contain balls. One of the urns has 6 red balls and 3 blue balls. The other urn has 5 red balls and 8 blue balls. An urn is chosen at random and two balls are drawn at random from this urn, without replacement.

(a) What is the probability that both balls are red?

(b) What is the probability that the second ball is red, given that the first ball is red?

Chapter 7 — Test 2 Solutions

1. (a) The probability that a person chooses all six digits in the correct order is $1/10^6$.

(b) The number of ways a person can choose exactly five digits correctly is the number of ways to choose one position to be incorrect, namely 6, times the number of ways to choose an incorrect digit for that position, namely 9, times the number of ways to choose the other digits, namely 1. Hence there are 54 ways to choose exactly five digits correctly. The probability is $54/10^6$.

2. This is like the birthday problem, except that there are $26^3 = 17{,}576$ possible triples of initials. Therefore the answer is

$$1 - \frac{17575}{17576} \cdot \frac{17574}{17576} \cdots \frac{17377}{17576}\,.$$

This actually works out to about 68%.

3. The expected value is $\frac{1}{6}\cdot 1 + \frac{2}{6}\cdot 2 + \frac{3}{6}\cdot 3 = \frac{14}{6}$. To compute the variance we compute $E(X^2) - E(X)^2$, where X is the value on the slip. We have $E(X^2) = \frac{1}{6}\cdot 1 + \frac{2}{6}\cdot 4 + \frac{3}{6}\cdot 9 = \frac{36}{6} = 6$, so $V(X) = 6 - \left(\frac{14}{6}\right)^2 = \frac{5}{9}$.

4. Let D be the event that the person has the disease, and let P be the event that the person tests positive. Then we are given $p(D) = 0.01$, $p(P \mid D) = 0.95$, and $p(P \mid \overline{D}) = 0.02$. We are asked for $p(D \mid P)$. We use Bayes' theorem:

$$p(D \mid P) = \frac{p(P \mid D)p(D)}{p(P \mid D)p(D) + p(P \mid \overline{D})p(\overline{D})} = \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.02)(0.99)} \approx 0.324$$

5. (a) Half the time we select the first urn, in which case the probability that the two balls are both red is $(6/9)(5/8) = 5/12$. Half the time we select the second urn, in which case the probability that the two balls are both red is $(5/13)(4/12) = 5/39$. Therefore the answer is

$$\frac{1}{2}\cdot\frac{5}{12} + \frac{1}{2}\cdot\frac{5}{39} = \frac{85}{312}\,.$$

(b) Let F be the event that the first ball is red, and let S be the event that the second ball is red. We are asked for $p(S \mid F)$. By definition, this is $p(S \cap F)/p(F)$. In part (a) we found that $p(S \cap F) = 85/312$. By a simpler calculation, we see that

$$p(F) = \frac{1}{2}\cdot\frac{6}{9} + \frac{1}{2}\cdot\frac{5}{8} = \frac{31}{48}\,.$$

Thus the answer is

$$\frac{85/312}{31/48} = \frac{170}{403}\,.$$

Chapter 8 — Test 1

1. Find a recurrence relation and initial condition for the number of fruit flies in a jar if there are 12 flies initially and every week there are six times as many flies in the jar as there were the previous week.

2. Find the solution of the recurrence relation $a_n = 3a_{n-1}$, with $a_0 = 2$.

3. Find the solution of the linear homogeneous recurrence relation $a_n = 7a_{n-1} - 6a_{n-2}$ with $a_0 = -1$ and $a_1 = 4$.

4. Suppose that $f(n)$ satisfies the divide-and-conquer relation $f(n) = 2f(n/3) + 5$ and $f(1) = 7$. What is $f(81)$?

5. Suppose that $|A| = |B| = |C| = 100$, $|A \cap B| = 60$, $|A \cap C| = 50$, $|B \cap C| = 40$, and $|A \cup B \cup C| = 175$. How many elements are in $A \cap B \cap C$?

6. How many positive integers not exceeding 1000 are not divisible by either 4 or 6?

7. How many onto functions are there from a set with six elements to a set with four elements?

8. List the derangements of the set $\{1, 2, 3, 4\}$.

9. Find a generating function for the sequence $2, 3, 4, 5, \ldots$.

Chapter 8—Test 1 Solutions

1. We have $f(n) = 6f(n-1)$ whenever n is a positive integer where $f(n)$ is the number of fruit flies after n weeks, with $f(0) = 12$.

2. By iteration we find that $a_n = 3a_{n-1} = 3(3a_{n-2}) = 3^2 a_{n-2} = \cdots = 3^n a_0 = 2 \cdot 3^n$. This can be verified using mathematical induction.

3. The characteristic equation is $r^2 - 7r + 6 = (r-1)(r-6) = 0$. The characteristic roots are $r = 1$ and $r = 6$. The solutions are of the form $a_n = c_1 \cdot 1^n + c_2 \cdot 6^n = c_1 + c_2 \cdot 6^n$. Since $a_0 = -1$ and $a_1 = 4$ we have $c_1 + c_2 = -1$ and $c_1 + 6c_2 = 4$. Subtracting the first equation from the second gives $5c_2 = 5$, so $c_2 = 1$. This implies that $c_1 + 1 = -1$, so $c_1 = -2$. Hence the solution is $a_n = -2 + 6^n$.

4. Using the recurrence relation repeatedly, and simplifying when possible, we find that $f(81) = 2 \cdot f(27) + 5 = 2 \cdot (2 \cdot f(9) + 5) + 5 = 4 \cdot f(9) + 15 = 4 \cdot (2 \cdot f(3) + 5) + 15 = 8 \cdot f(3) + 35 = 8 \cdot (2 \cdot f(1) + 5) + 35 = 16 \cdot f(1) + 75 = 16 \cdot 7 + 75 = 187$.

5. By the principle of inclusion–exclusion $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Hence $175 = 100 + 100 + 100 - 60 - 50 - 40 + |A \cap B \cap C|$. Therefore $|A \cap B \cap C| = 175 - 150 = 25$.

6. The number of positive integers not exceeding 1000 that are not divisible by either 4 or 6 equals $1000 - \lfloor 1000/4 \rfloor - \lfloor 1000/6 \rfloor + \lfloor 1000/12 \rfloor = 1000 - 250 - 166 + 83 = 667$. Here we used the fact that the integers divisible by both 4 and 6 are those divisible by 12.

7. There are $4^6 - C(4,3)3^6 + C(4,2)2^6 - C(4,1)1^6 = 4096 - 4 \cdot 729 + 6 \cdot 64 - 4 = 1560$ onto functions.

8. The derangements of $\{1,2,3,4\}$ are the permutations of these four integers that leave no integer in its original position. These are 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, and 4321.

9. We know that the generating function for the sequence $1, 2, 3, 4, \ldots$ is $\dfrac{1}{(1-x)^2}$. Therefore the generating function for the sequence $0, 2, 3, 4, \ldots$ is this function with the constant term omitted, i.e., $\dfrac{1}{(1-x)^2} - 1$. It follows that the generating function for the given sequence is this last function divided by x, namely $\left(\dfrac{1}{(1-x)^2} - 1\right)/x$. This can also be written as $\dfrac{2-x}{(1-x)^2}$.

Chapter 8 — Test 2

1. (a) Find a recurrence relation for the number of ways to climb n stairs if stairs can be climbed two or three at a time.

(b) What are the initial conditions?

(c) How many ways are there to climb eight stairs?

2. What is the solution to the recurrence relation $a_n = 8a_{n-1} + 9a_{n-2}$ if $a_0 = 3$ and $a_1 = 7$?

3. Suppose that $f(n)$ satisfies the divide-and-conquer recurrence relation $f(n) = 3f(n/4) + n^2/8$ with $f(1) = 2$. What is $f(64)$?

4. How many positive integers not exceeding 1000 are not divisible by 4, 6, or 9?

5. How many ways are there to assign six jobs to four employees so that every employee is assigned at least one job?

6. How many permutations are there of the digits in the string 12345 that leave 3 fixed but leave no other integer fixed? (For instance, 24351 is such a permutation.)

7. Use generating functions to solve the recurrence relation $a_k = 5a_{k-1}$ for $k = 1, 2, 3, \ldots$, with initial condition $a_0 = 3$.

Chapter 8—Test 2 Solutions

1. (a) Let a_n be the number of ways to climb n stairs. Suppose that $n \geq 4$. Then $a_n = a_{n-2} + a_{n-3}$, since n stairs can be climbed by going up $n-2$ stairs followed by a step of 2 stairs or by going up $n-3$ stairs followed by a step of 3 stairs.

(b) We see that $a_1 = 0$, $a_2 = 1$, and $a_3 = 1$.

(c) Note that $a_4 = a_2 + a_1 = 1 + 0 = 1$, $a_5 = a_3 + a_2 = 1 + 1 = 2$, $a_6 = a_4 + a_3 = 1 + 1 = 2$, $a_7 = a_5 + a_4 = 2 + 1 = 3$, and $a_8 = a_6 + a_5 = 2 + 2 = 4$.

2. The characteristic equation of this linear homogeneous recurrence relation is $r^2 - 8r - 9 = (r-9)(r+1) = 0$. The characteristic roots are $r = 9$ and $r = -1$. Hence the solutions are of the form $a_n = c_1 \cdot 9^n + c_2(-1)^n$, where c_1 and c_2 are constants. Since $a_0 = 3$ and $a_1 = 7$ we have $3 = c_1 + c_2$ and $7 = 9c_1 - c_2$. Adding these equations gives $10 = 10c_1$, so $c_1 = 1$. Substituting this value of c_1 into the first equation gives $c_2 = 2$. Hence the solution is given by $a_n = 9^n + 2(-1)^n$.

3. We have $f(64) = 3f(16) + 64^2/8 = 3f(16) + 512 = 3(3f(4) + 16^2/8) + 512 = 9f(4) + 608 = 9(3f(1) + 4^2/8) + 608 = 27f(1) + 626 = 27 \cdot 2 + 626 = 680$.

4. By the principle of inclusion–exclusion, the number of positive integers not exceeding 1000 that are divisible by 4, 6, or 9 equals $\lfloor 1000/4 \rfloor + \lfloor 1000/6 \rfloor + \lfloor 1000/9 \rfloor - \lfloor 1000/12 \rfloor - \lfloor 1000/36 \rfloor - \lfloor 1000/18 \rfloor + \lfloor 1000/36 \rfloor = 250 + 166 + 111 - 83 - 27 - 55 + 27 = 389$. Hence there are $1000 - 389 = 611$ positive integers not exceeding 1000 that are not divisible by either 4, 6, or 9.

5. The number of ways to assign six jobs to four employees so that every employee is assigned at least one job equals the number of onto functions from a set with six elements to a set with four elements. This equals $4^6 - C(4,3)3^6 + C(4,2)2^6 - C(4,1)1^6 = 4096 - 2916 + 384 - 4 = 1560$.

6. The number of permutations of 12345 that leave 3 but no other integer fixed equals the number of derangements of 4 integers, namely 1, 2, 4, and 5. This equals $D_4 = 4!(1 - 1/1! + 1/2! - 1/3! + 1/4!) = 24(1 - 1 + 1/2 - 1/6 + 1/24) = 9$.

7. Let $G(x)$ be the generating function for the sequence $\{a_k\}$, i.e., $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$. It follows that $G(x) - 5xG(x) = \sum_{k=0}^{\infty} a_k x^k - 5\sum_{k=1}^{\infty} a_{k-1} x^k = a_0 + \sum_{k=1}^{\infty}(a_k - 5a_{k-1})x^k = 3$, because of the given recurrence relation and initial condition. Thus $G(x) - 5xG(x) = (1-5x)G(x) = 3$, so $G(x) = 3/(1-5x)$. It follows from an identity in Table 1 of Section 8.4 that $G(x) = 3\sum_{k=0}^{\infty} 5^k x^k$. Consequently $a_k = 3 \cdot 5^k$.

Chapter 9—Test 1

1. Which ordered pairs are in the relation $\{(x,y) \mid x > y+1\}$ on the set $\{1,2,3,4\}$?

2. Consider the following relations on $\{1,2,3\}$.

$$R_1 = \{(1,1),(1,3),(2,2),(3,1)\}$$
$$R_2 = \{(1,1),(2,2),(3,1),(3,3)\}$$
$$R_3 = \{(1,2),(2,1),(3,3)\}$$
$$R_4 = \{(1,3),(2,3)\}$$

(a) Which of these relations are reflexive? Justify your answers.

(b) Which of these relations are symmetric? Justify your answers.

(c) Which of these relations are antisymmetric? Justify your answers.

(d) Which of these relations are transitive? Justify your answers.

3. Find the reflexive closure and the symmetric closure of the relation $\{(1,2),(1,4),(2,3),(3,1),(4,2)\}$ on the set $\{1,2,3,4\}$.

4. What is the transitive closure of the relation in problem 3?

5. (a) Show that the relation $R = \{(x,y) \mid x$ and y are bit strings containing the same number of 0s$\}$ is an equivalence relation.

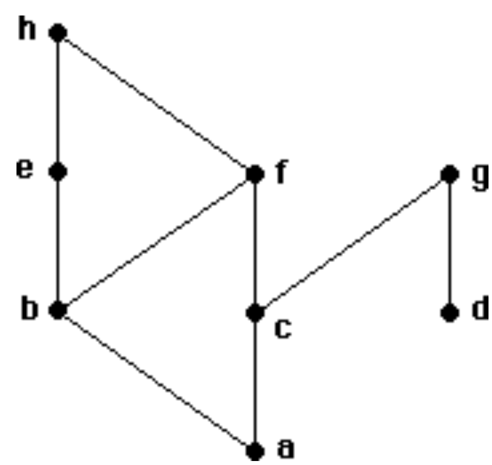
(b) What are the equivalence classes of the bit strings 1, 00, and 101 under the relation R?

6. (a) Are the sets $\{1,3,6\}$, $\{2,4,7\}$, and $\{5\}$ a partition of $\{1,2,3,4,5,6,7\}$?

(b) Are the sets $\{1,2,4,5\}$, $\{3,6,7\}$, and $\{2,3\}$ a partition of $\{1,2,3,4,5,6,7\}$?

7. Show that the inclusion relation, $\{(A,B) \mid A \subseteq B\}$, is a partial ordering on the set of all subsets of $\mathbf{Z}$.

8. What are the minimal and maximal elements in the poset with the following Hasse diagram? Are there least and greatest elements?



Chapter 9—Test 1 Solutions

1. The ordered pairs in this relation are $(3,1)$, $(4,1)$, and $(4,2)$.

2. (a) R_2 is reflexive since it contains $(1,1)$, $(2,2)$, and $(3,3)$. The relations R_1, R_3, and R_4 are not reflexive since they do not contain all three of these ordered pairs.

(b) R_1 and R_3 are symmetric since they contains (i,j) whenever they contain (j,i). To check this for R_1 requires only that we note that both $(1,3)$ and $(3,1)$ are in the relation and to check this for R_3 requires only that we note that both $(1,2)$ and $(2,1)$ are in the relation. R_2 and R_4 are not symmetric since each contains one of $(3,1)$ and $(1,3)$, but not the other.

(c) R_2 and R_4 are antisymmetric since neither contains ordered pairs (i,j) and (j,i) where $i \neq j$. To check this for R_2 requires only that we check that $(1,3)$ is not in R_2, since $(3,1)$ is the only ordered pair in the relation with different first and second elements; to check this for R_4 requires only that we check that neither $(3,1)$ nor $(3,2)$ is in the relation. We see that R_1 is not antisymmetric since both $(1,3)$ and $(3,1)$ are in R_1. We see that R_3 is not antisymmetric since both $(1,2)$ and $(2,1)$ are in R_3.

(d) R_2 and R_4 are transitive. This is easily verified since neither relation has pairs (a,b) and (b,c) with $a \neq b$ and $b \neq c$. R_1 is not transitive since $(3,1)$ and $(1,3)$ belong to R_1 but $(3,3)$ is not in R_1. R_3 is not transitive since $(1,2)$ and $(2,1)$ belong to R_3 but $(1,1)$ does not belong to R_3.

3. The reflexive closure is obtained by adding the pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. The symmetric closure is obtained by adding the pairs $(1,3)$, $(2,1)$, $(2,4)$, $(3,2)$, and $(4,1)$.

4. We add the pairs $(1,3)$, $(2,1)$, $(3,2)$, $(3,4)$, and $(4,3)$ at the first stage; these represent paths of length two. At the second stage we add $(1,1)$, $(2,2)$, $(2,4)$, $(3,3)$, $(4,1)$ and $(4,4)$. We conclude that the transitive closure contains all possible ordered pairs.

5. (a) Let x be a bit string. Then $(x,x) \in R$ since x has the same number of 0's as itself. Hence R is reflexive. Now suppose that $(x,y) \in R$. Then x and y have the same number of 0's. Consequently y and x have the same number of 0's. It follows that $(y,x) \in R$. Next, suppose that $(x,y) \in R$ and $(y,z) \in R$. Then x and y contain the same number of 0's, and y and z contain the same number of 0's. It follows that x and z contain the same number of 0's. Hence $(x,z) \in R$. We conclude that R is transitive.

(b) The equivalence class of 1 is the set of all bit strings that contain no 0's; explicitly, $[1]_R = \{\lambda, 1, 11, 111, 1111, \ldots\}$. The equivalence class of 00 is the set of all bit strings that contain exactly two 0's, that is $[00]_R = \{00, 100, 010, 001, 1100, 1010, 1001, 0110, 0101, 0011, \ldots\}$. The equivalence class of 101 is the set of all bit strings that contain exactly one 0, that is, $[101]_R = \{0, 10, 01, 110, 101, 011, 1110, 1101, 1011, 0111, \ldots\}$.

6. (a) The subsets listed form a partition of $\{1,2,3,4,5,6,7\}$ since they are pairwise disjoint nonempty sets and their union is this set.

(b) These subsets are not pairwise disjoint so they do not form a partition.

7. We see that set inclusion is reflexive since $A \subseteq A$ whenever A is a subset of $\mathbf{Z}$. Since $A \subseteq B$ and $B \subseteq A$ imply that $A = B$ whenever A and B are subsets of $\mathbf{Z}$, we see that set inclusion is antisymmetric. Now suppose that $A \subseteq B$ and $B \subseteq C$ where A, B, and C are subsets of $\mathbf{Z}$. Then $A \subseteq C$, so set inclusion is transitive.

8. The minimal elements are a and d. The maximal elements are h and g. There is no least element and there is no greatest element. If there were a least element then there would be exactly one minimal element, and if there were a greatest element then there would be exactly one maximal element.

Chapter 9—Test 2

1. Consider the following relations on the set of positive integers.

$$\begin{aligned} R_1 &= \{\,(x,y) \mid x+y>10\,\} \\ R_2 &= \{\,(x,y) \mid y \text{ divides } x\,\} \\ R_3 &= \{\,(x,y) \mid \gcd(x,y)=1\,\} \\ R_4 &= \{\,(x,y) \mid x \text{ and } y \text{ have the same prime divisors}\,\} \end{aligned}$$

(a) Which of these relations are reflexive? Justify your answers.

(b) Which of these relations are symmetric? Justify your answers.

(c) Which of these relations are antisymmetric? Justify your answers.

(d) Which of these relations are transitive? Justify your answers.

2. Suppose that R_1 and R_2 are symmetric relations on a set A. Prove or disprove that $R_1 - R_2$ is also symmetric.

3. What is the join of the 3-ary relation

$$\{(\text{Lewis}, \text{MS410}, \text{N507}),\ (\text{Rosen}, \text{CS540}, \text{N525}),\ (\text{Smith}, \text{CS518}, \text{N504}),\ (\text{Smith}, \text{MS410}, \text{N510})\}$$

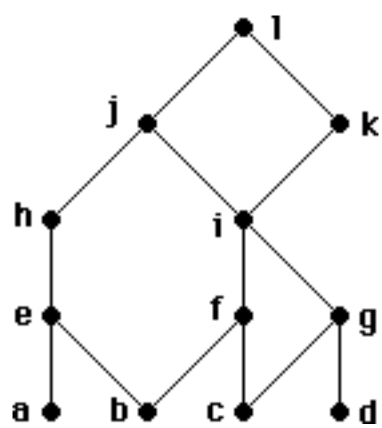
and the 4-ary

$$\{(\text{MS410}, \text{N507}, \text{Monday}, 6{:}00),\ (\text{MS410}, \text{N507}, \text{Wednesday}, 6{:}00),\ (\text{CS540}, \text{N525}, \text{Monday}, 7{:}30),\ (\text{CS518}, \text{N504}, \text{Tuesday}, 6{:}00),\ (\text{CS518}, \text{N504}, \text{Thursday}, 6{:}00)\}$$

with respect to the last two fields of the first relation and the first two fields of the second relation?

4. Show that the relation $R = \{(x,y) \mid x-y \text{ is an integer}\}$ is an equivalence relation on the set of rational numbers. What are the equivalence classes of 0 and $\frac{1}{2}$?

5. Consider the poset with the following Hasse diagram.



(a) Find all maximal elements of the poset.

(b) Find all minimal elements of the poset.

(c) Find the least element of the poset if it exists, or show that it does not exist.

(d) Find the greatest element of the poset if it exists, or show that it does not exist.

(e) What is the greatest lower bound of the set $\{a,b,c\}$?

(f) What is the least upper bound of the set $\{a,b,c\}$?

6. Use a topological sort to order the elements of the poset with the Hasse diagram given in problem 5.

Chapter 9—Test 2 Solutions

1. (a) R_1 is not reflexive since $1+1<10$, so $(1,1)$ is not in R_1. R_2 is reflexive since $x \mid x$ for every positive integer x, so $(x,x)\in R_2$ for all x. R_3 is not reflexive since $\gcd(2,2)=2$, so $(2,2)$ is not in R_3. R_4 is reflexive since x and x have the same prime divisors for every positive integer x, so $(x,x)\in R_4$ for all x.

(b) R_1 is symmetric, since $x+y>10$ implies that $y+x>10$. R_2 is not symmetric since since $1 \mid 2$ but $2 \not\mid 1$. R_3 is symmetric since $\gcd(x,y)=1$ implies that $\gcd(y,x)=1$. R_4 is symmetric since x and y have the same prime divisors if and only if y and x have the same prime divisors.

(c) R_1 is not antisymmetric since $(2,9)$ and $(9,2)$ both belong to R_1. R_2 is antisymmetric since $y \mid x$ and $x \mid y$ imply that $x=y$ if x and y are positive integers. R_3 is not antisymmetric since $\gcd(2,1)=\gcd(1,2)=1$. R_4 is not antisymmetric since 12 and 18 have the same prime divisors, namely 2 and 3, and 18 and 12 have the same prime divisors.

(d) R_1 is not transitive since $(2,9)\in R_1$ and $(9,3)\in R_1$ but $(2,3)\notin R_1$. R_2 is transitive since $y \mid x$ and $z \mid y$ imply that $z \mid x$. R_3 is not transitive since $\gcd(2,3)=1$ and $\gcd(3,2)=1$ but $\gcd(2,2)=2$. R_4 is transitive, for if x and y have the same prime divisors and y and z have the same prime divisors, then x and z have the same prime divisors.

2. Suppose that R_1 and R_2 are symmetric. If $(a,b)\in R_1-R_2$ then $(a,b)\in R_1$ and $(a,b)\notin R_2$. Since R_1 is symmetric it follows that $(b,a)\in R_1$. Since R_2 is symmetric it follows that $(b,a)\notin R_2$, for if $(b,a)\in R_2$ then $(a,b)\in R_2$. Hence $(b,a)\in R_1-R_2$. It follows that R_1-R_2 is symmetric.

3. The join is

$$\begin{aligned}\{&(\text{Lewis}, \text{MS410}, \text{N507}, \text{Monday}, 6{:}00),\ (\text{Lewis}, \text{MS410}, \text{N507}, \text{Wednesday}, 6{:}00),\\ &(\text{Rosen}, \text{CS540}, \text{N525}, \text{Monday}, 7{:}30),\ (\text{Smith}, \text{CS518}, \text{N504}, \text{Tuesday}, 6{:}00),\\ &(\text{Smith}, \text{CS518}, \text{N504}, \text{Thursday}, 6{:}00)\}\end{aligned}$$

4. Since $x-x=0$ is an integer for every rational number x it follows that R is reflexive. Suppose that $(x,y)\in R$. Then $x-y$ is an integer, which implies that $y-x$ is an integer. Hence $(y,x)\in R$. It follows that R is symmetric. Now suppose that $(x,y)\in R$ and $(y,z)\in R$. Then $x-y$ and $y-z$ are integers, so $x-z=(x-y)+(y-z)$ is also an integer. It follows that R is transitive. Hence R is an equivalence relation. We have $[0]_R=\{\, x\in \mathbf{Q} \mid x-0\in \mathbf{Z}\,\}=\mathbf{Z}$ and $[\frac{1}{2}]_R=\{\, x\in\mathbf{Q} \mid x-\frac{1}{2}\in\mathbf{Z}\,\}=\{\, k+\frac{1}{2} \mid k\in \mathbf{Z}\,\}=\{\ldots,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots\}$.

5. (a) The maximal element is l.

(b) The minimal elements are a, b, c, and d.

(c) There is no least element since there is more than one minimal element.

(d) The greatest element is l.

(e) There is no lower bound for the set $\{a,b,c\}$, so there is no greatest lower bound.

(f) The upper bounds for the set $\{a,b,c\}$ are the elements j and l. Since j is less than l, j is the least upper bound.

6. One possible ordering is: a,b,e,h,c,d,g,f,i,k,j,l.

Chapter 10—Test 1

1. How many vertices and how many edges do each of the following graphs have?

(a) K_5

(b) C_4

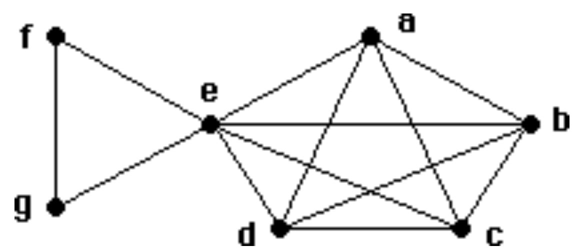
(c) W_5

(d) $K_{2,5}$

2. Does a simple graph that has five vertices each of degree 3 exist? If so, draw such a graph. If not, explain why no such graph exists.

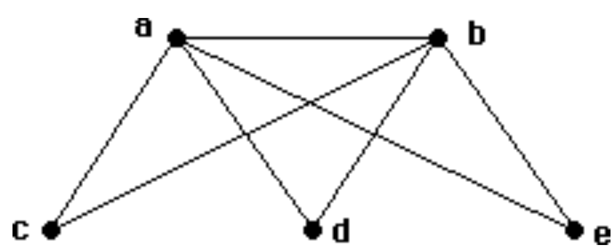
3. How many nonisomorphic simple graphs are there with three vertices? Draw examples of each of these.

4. Is there an Euler circuit in the following graph? If so, find such a circuit. If not, explain why no such circuit exists.



5. Is there a Hamilton circuit in the graph shown in problem 4? If so, find such a circuit. If not, prove why no such circuit exists.

6. Is the following graph planar? If so draw it without any edges crossing. If it is not, prove that it is not planar.



7. What is the chromatic number of each of the graphs in problem 1? Explain your answers.

Chapter 10—Test 1 Solutions

1. (a) K_5 has five vertices and $C(5,2) = 10$ edges.

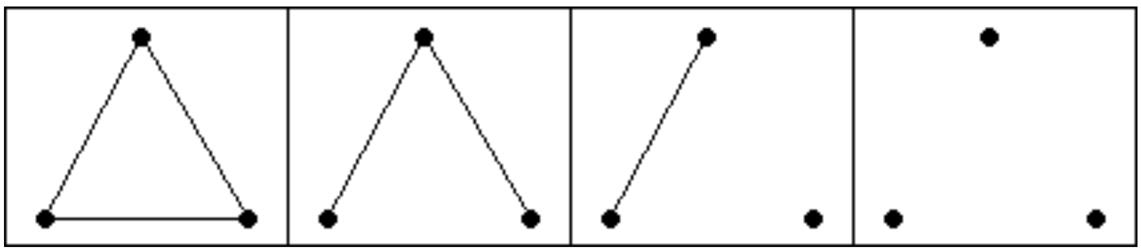
(b) C_4 has four vertices and four edges.

(c) W_5 has $5+1=6$ vertices and $5+5=10$ edges.

(d) $K_{2,5}$ has $2+5=7$ vertices and $2 \cdot 5 = 10$ edges.

2. There is no simple graph with four vertices each of degree 3 since the sum of the degrees of such a graph would be 15, which is odd. This is impossible by the handshaking theorem.

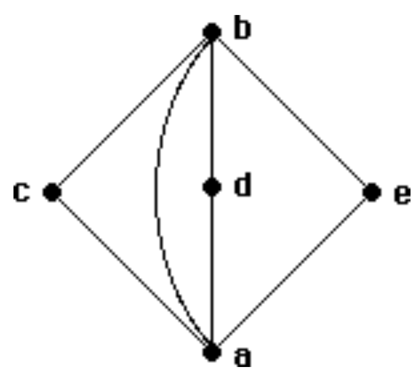
3. There are four nonisomorphic simple graphs with three vertices, as shown.



4. All vertices have even degree (the degree of each of a, b, c, and d is 4, the degree of e is 6, and the degree of each of f and g is 2). Hence the graph has an Euler circuit. One such Euler circuit is $a, b, c, a, d, c, e, d, b, e, g, f, e, a$.

5. There is no Hamilton circuit. Since the degree of each of f and g is 2, any such circuit must contain the edges $\{e, f\}$ and $\{e, g\}$, which implies that the vertex e must be visited twice.

6. The graph is planar, since it can be drawn with no crossing as follows.



7. (a) The chromatic number of K_5 is 5. Each vertex is adjacent to all other vertices in the graph, and so must be assigned its own color.

(b) The chromatic number of C_4 is 2. Suppose that the vertices are v_1, v_2, v_3, and v_4, where the edges are $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, and $\{v_4, v_1\}$. Then we can color v_1 and v_3 red and color v_2 and v_4 blue.

(c) The chromatic number of W_5 is 4. As is easily seen, three colors are needed for the vertices in the cycle with five vertices, and the central vertex must be assigned its own color.

(d) The chromatic number of $K_{2,5}$ is 2. Any bipartite graph can be colored with two colors. In particular, we can color the two vertices in one set in the partition of the vertices red, and the five vertices in the other set in the partition blue.

Chapter 10—Test 2

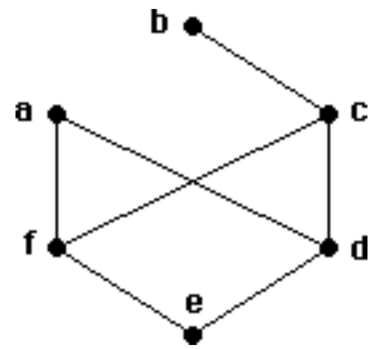
1. For each of the following sequences determine whether there is a simple graph whose vertices have these degrees. Draw such a graph if it exists.

(a) $0, 1, 1, 2$

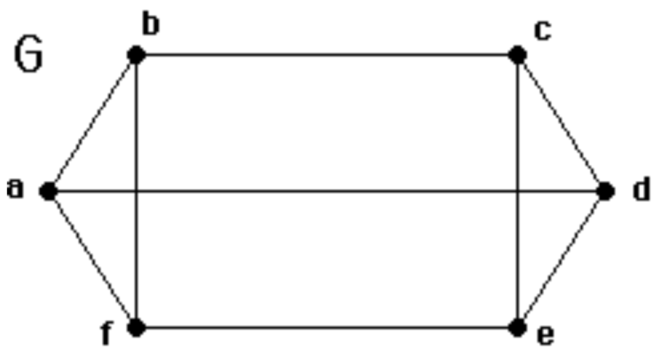
(b) $2, 2, 2, 2$

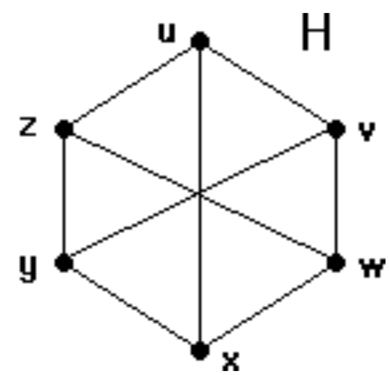
(c) $1, 2, 3, 4, 5$

2. Is the following graph bipartite? Justify your answer.



3. Decide whether the graphs G and H are isomorphic. Prove that your answer is correct.



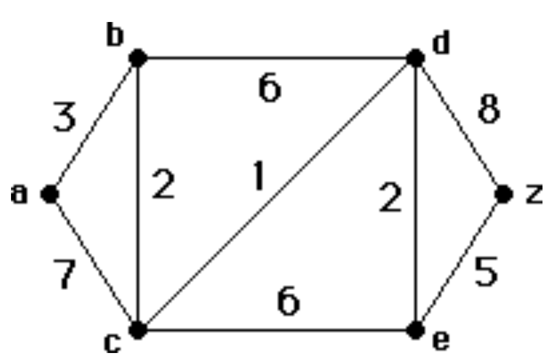


4. Consider the graphs K_5, $K_{2,3}$, and W_5. Which of these graphs have an Euler circuit? Which have an Euler path?

5. Which of the graphs in problem 4 are planar?

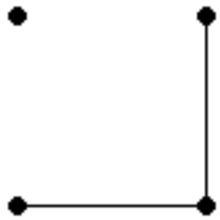
6. What is the chromatic number of each of the graphs in problem 4?

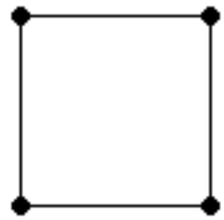
7. Use Dijkstra's algorithm to find the length of the shortest path between the vertices a and z in the following weighted graph.



Chapter 10—Test 2 Solutions

1. (a) Yes (b) Yes





(c) There are at least two reasons why this graph cannot exist. First, the sum of degrees is odd. Second, there can be no vertex of degree 5 in a simple graph with five vertices.

2. The graph is bipartite. The vertex set can be partitioned into $\{a, c, e\}$ and $\{b, d, f\}$. There are no edges connecting a vertex in one set and a vertex in the other set.

3. These graphs are not isomorphic, since G contains a subgraph isomorphic to K_3 but H does not. (In fact, H is bipartite.)

4. K_5 has five vertices each of degree 4, so it has an Euler circuit (and an Euler path) since all its vertices have even degree. $K_{2,3}$ has two vertices of degree 3 and three vertices of degree 2, so it does not have an Euler circuit, but it does have an Euler path since it has exactly two vertices of odd degree. W_5 has five vertices of degree 3 and one vertex of degree 5, so it has neither an Euler circuit nor an Euler path since it has more than two vertices of odd degree.

5. K_5 is nonplanar. $K_{2,3}$ is planar, as can easily be seen by drawing it with no crossings, or since it has no subgraph homeomorphic to $K_{3,3}$ or K_5. W_5 is planar as is seen from the usual way of drawing it.

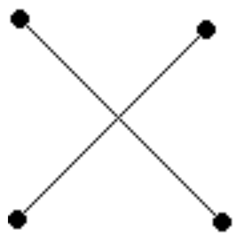
6. The chromatic number of K_5 is 5, since each vertex must be colored differently from all others. $K_{2,3}$ has chromatic number 2, since it is a bipartite graph. W_5 has chromatic number 4, since three colors are required to color C_5 and a fourth color must be used for the hub vertex.

7. First iteration: distinguished vertices: a; labels: $a:0$, $b:3$, $c:7$, $d:\infty$, $e:\infty$, $z:\infty$. Second iteration: distinguished vertices: a, b; labels: $a:0$, $b:3$, $c:5$, $d:9$, $e:\infty$, $z:\infty$. Third iteration: distinguished vertices: a, b, c; labels: $a:0$, $b:3$, $c:5$, $d:6$, $e:11$, $z:\infty$. Fourth iteration: distinguished vertices: a, b, c, d; labels: $a:0$, $b:3$, $c:5$, $d:6$, $e:8$, $z:14$. Fifth iteration: distinguished vertices: a, b, c, d, e; labels: $a:0$, $b:3$, $c:5$, $d:6$, $e:8$, $z:13$. Since at the next iteration z is a distinguished vertex, we conclude that the shortest path has length 13.

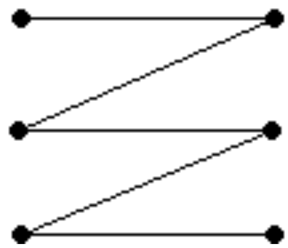
Chapter 11—Test 1

1. Which of the following graphs are trees? Explain your answers.

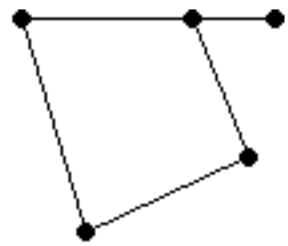
(a)



(b)



(c)



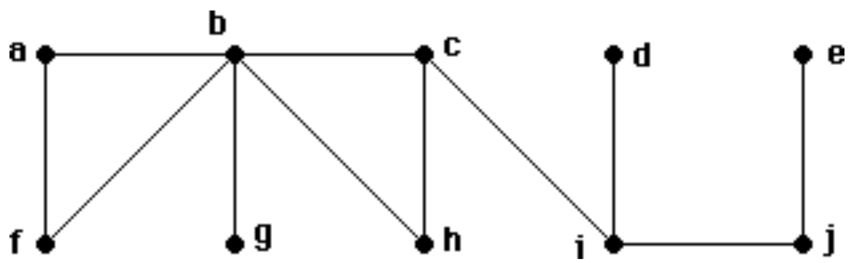
2. A tree has 99 edges. How many vertices does it have?

3. Form a binary search tree from the words of the sentence *This test is not so difficult*, using alphabetical order, inserting words in the order they appear in the sentence.

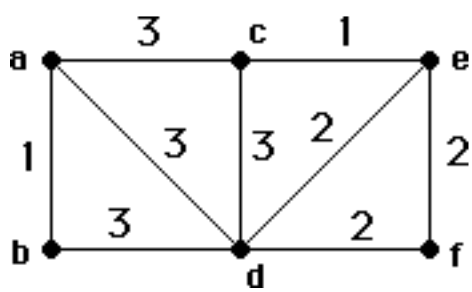
4. Is the code A: 11, B: 10, C: 0 a prefix code?

5. Construct an expression tree for $(3+x)-5\cdot y$ and write this expression is prefix form and postfix form.

6. Use a depth-first search to find a spanning tree of the following graph. Start at the vertex a, and use alphabetical order.



7. Use Prim's algorithm to find a minimum spanning tree for the following weighted graph. Use alphabetical order to break ties.

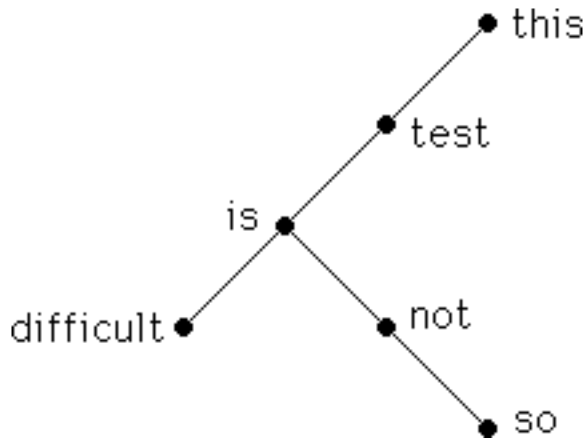


Chapter 11—Test 1 Solutions

1. The graph in part (a) is not connected, so it is not a tree. The graph in part (b) is a tree since it is connected and contains no simple circuits. The graph in part (c) is not a tree since it contains a simple circuit.

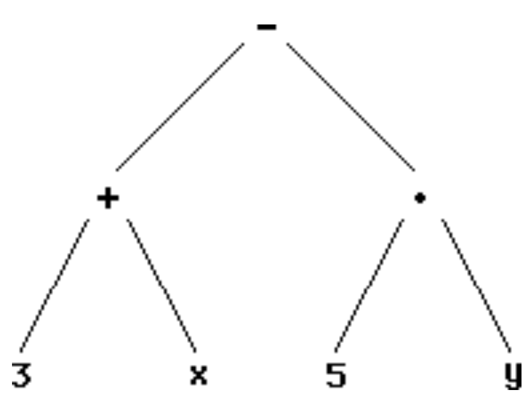
2. If a tree has e edges and n vertices, then $e = n-1$. Hence if a tree has 99 edges, then it has 100 vertices.

3. The following binary search tree is produced.



4. This is a prefix code since the code for A, 11, does not start the code for B or the code for C; the code for B, 10, does not start the code for A or the code for C; and the code for C, 0, does not start the code for A or the code for B.

5. The following tree represents the expression $(3+x) - 5 \cdot y$



The prefix form and postfix form of this expression are obtained by carrying out a preorder and a postorder traversal, respectively. The preorder form is $- + 3\, x \cdot 5\, y$. The postorder form is $3\, x + 5\, y \cdot -$.

6. The edges produced by a depth-first search are $\{a,b\}$, $\{b,c\}$, $\{c,h\}$, $\{c,i\}$, $\{i,d\}$, $\{i,j\}$, $\{j,e\}$, $\{b,f\}$, and $\{b,g\}$.

7. Prim's algorithm adds the edges: $\{a,b\}$ of weight 1, $\{a,c\}$ of weight 3, $\{c,e\}$ of weight 1, $\{d,e\}$ of weight 2, $\{d,f\}$ of weight 2. The weight of the minimum spanning tree is 9.

Chapter 11—Test 2

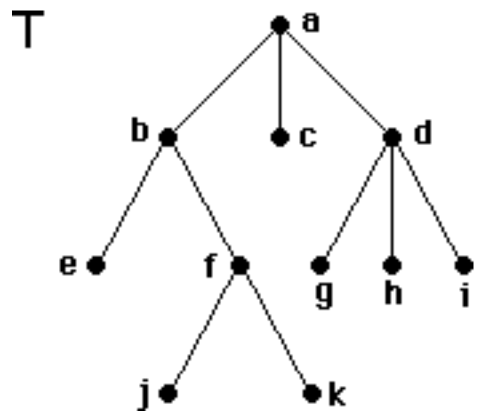
1. (a) Suppose that a full 3-ary tree has 100 internal vertices. How many leaves does it have?

(b) Suppose that a full 4-ary tree has 100 leaves. How many internal vertices does it have?

2. How many nonisomorphic trees are there with four vertices? Draw them.

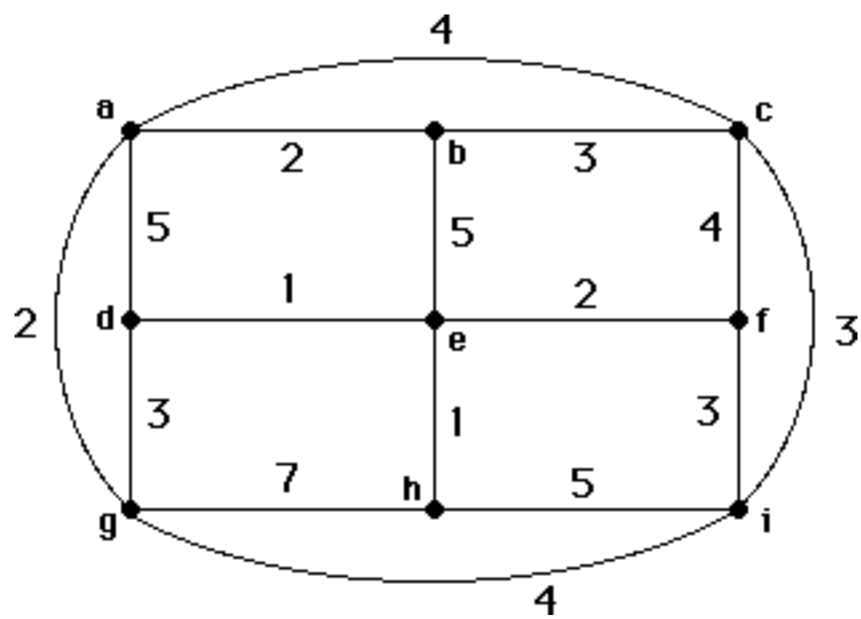
3. Is the code A: 111, B: 101, C: 011, D: 010, E: 10, F: 1101 a prefix code?

4. Perform a preorder, inorder, and postorder traversal of the rooted tree T.



5. Use backtracking to find a sum of integers in the set $\{18, 19, 23, 25, 31\}$ that equals 44.

6. Find a minimum spanning tree in the following weighted graph using Prim's algorithm.

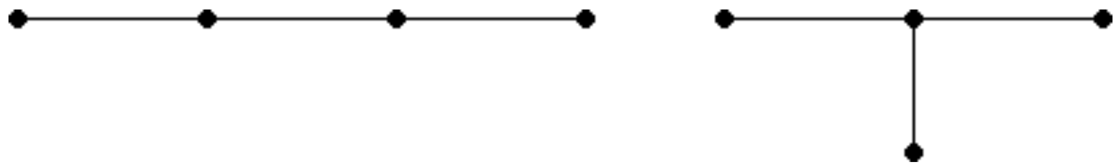


Chapter 11—Test 2 Solutions

1. (a) A full 3-ary tree with 100 internal vertices has $l = (3-1)\cdot 100 + 1 = 201$ leaves.

(b) A full 4-ary tree with 100 leaves has $i = (100-1)/(4-1) = 33$ internal vertices.

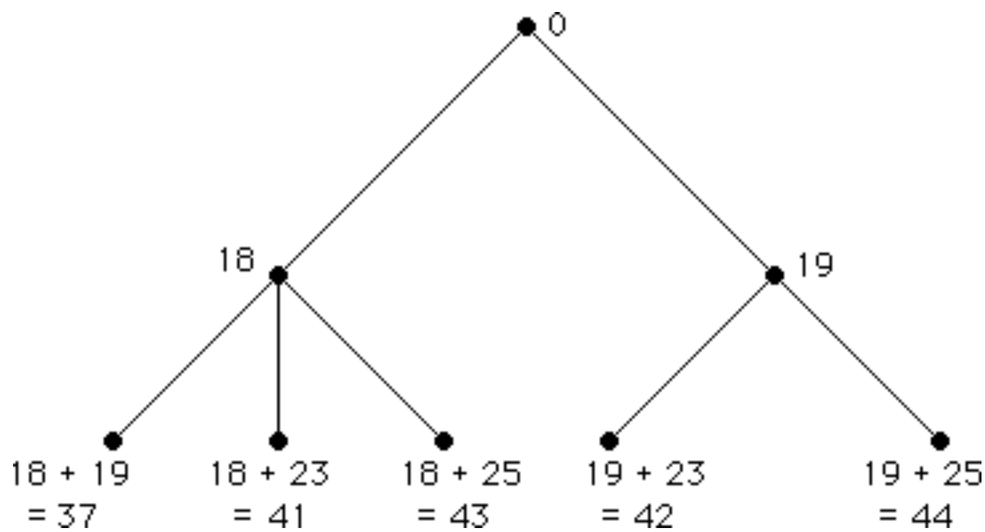
2. There are two nonisomorphic unrooted trees with four vertices, as shown.



3. This is not a prefix code since the code for E, 10, begins the code for B, 101.

4. The preorder traversal is $a, b, e, f, j, k, c, d, g, h, i$; the inorder traversal is $e, b, j, f, k, a, c, g, d, h, i$; and the postorder traversal is $e, j, k, f, b, c, g, h, i, d, a$.

5. We build the following tree using backtracking. We find that $44 = 19 + 25$.



6. Prim's algorithm adds the edges $\{d, e\}$ of weight 1, $\{e, h\}$ of weight 1, $\{e, f\}$ of weight 2, $\{d, g\}$ of weight 3, $\{g, a\}$ of weight 2, $\{a, b\}$ of weight 2, $\{b, c\}$ of weight 3, and $\{c, i\}$ of weight 3. The total weight of the minimum spanning tree is 17.

Chapter 12—Test 1

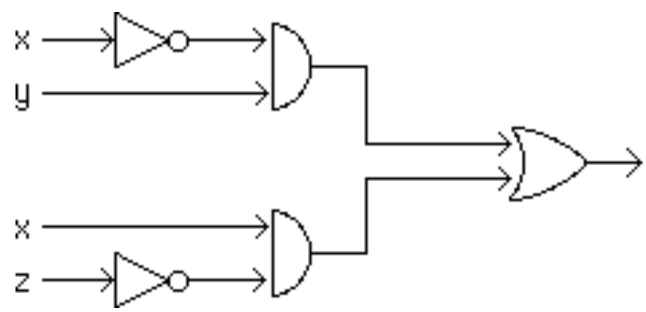
1. What is the value of the Boolean function $f(x, y, z) = (\overline{x} + \overline{y})z + x\,y\,z$ when $x = 1$, $y = 0$ and $z = 1$?

2. Prove or disprove that $x\,y + y = y$ whenever x and y are Boolean variables.

3. How many different Boolean functions are there of degree 3?

4. Find the sum-of-products expansion of a Boolean function $f(x, y, z)$ that is 1 if and only if $x = y = 1$ and $z = 0$, or $x = 0$ and $y = z = 1$, or $x = y = 0$ and $z = 1$.

5. What is the output of the following circuit?



6. Use a K-map to minimize the sum-of-products expansion $x\,y\,z + x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}\,z$.

Chapter 12—Test 1 Solutions

1. We have $f(1,0,1) = (\overline{1}+\overline{0})\cdot 1 + 1\cdot 0\cdot 1 = (0+1)\cdot 1 + 0 = 1\cdot 1 + 0 = 1+0 = 1$.

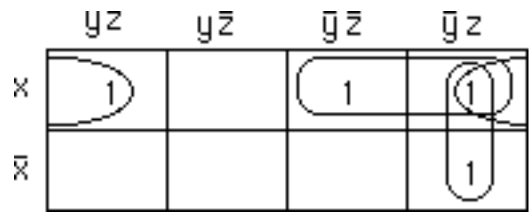
2. When $y=1$ we have $x\,y+y = x+1 = 1 = y$. When $y=0$ we have $x\,y+y = x\cdot 0+0 = 0+0 = 0 = y$. Hence $x\,y+y=y$ for all values of the Boolean variables x and y.

3. There are $2^{2^3} = 2^8 = 256$ Boolean functions of degree 3.

4. The sum-of-products expansion is $x\,y\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,\overline{y}\,z$.

5. The output of the circuit is $\overline{x}\,y + x\,\overline{z}$.

6. We construct the following K-map.



Combining terms gives us the expansion $x\,\overline{y} + x\,z + \overline{y}\,z$.

Chapter 12—Test 2

1. Prove or disprove that $x + x\,y + x\,y\,z = x$ whenever x, y, and z are Boolean variables.

2. Find a Boolean function $f(x, y, z)$ that has the value 1 if and only if exactly two of x, y, and z have the value 1.

3. Is the set of operators $\{+, \cdot\}$ functionally complete? Justify your answer.

4. Construct a circuit using inverters, OR gates, and AND gates that gives an output of 1 if three people on a committee do not all vote the same.

5. Use a K-map to minimize the sum-of-products expansion $x\,y\,z + x\,\overline{y}\,z + \overline{x}\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,\overline{y}\,\overline{z}$.

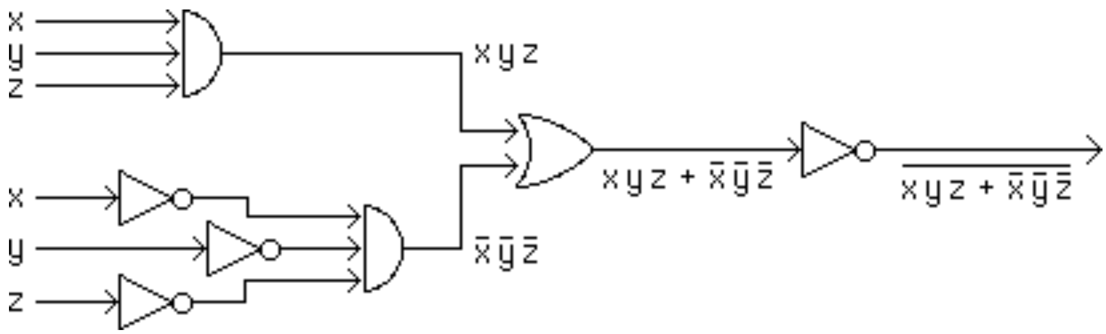
Chapter 12—Test 2 Solutions

1. When x has the value 1, then $x+xy+xyz=1+1\cdot y+1\cdot yz=1=x$. When x has the value 0, then $x+xy+xyz=0+0\cdot y+0\cdot yz=0+0+0=0=x$. Hence this identity holds for all values of the Boolean variables x, y, and z.

2. We want the sum-of-products expansion that has the value 1 if and only if $x=y=1$ and $z=0$, or $x=z=1$ and $y=0$, or $y=z=1$ and $x=0$. Hence, this sum-of-products expansion is $xy\overline{z}+x\overline{y}z+\overline{x}yz$.

3. The set of operators $\{+,\cdot\}$ is not functionally complete. There is no way to obtain $\overline{x}$ from these two operators. To see this, note that $x+x=x$ and $x\cdot x=x$. Therefore every expression involving x and the operators in this set will be equal to x, never $\overline{x}$.

4. Let x, y, and z represent the votes of the three people on the committee with a variable taking the value 1 if the vote is affirmative and the value 0 if the vote is negative. The circuit should produce an output of 1 if and only if not all three of the variables have the same value. The function $f(x,y,z)=\overline{(xyz+\overline{x}\,\overline{y}\,\overline{z})}$ gives this output. Hence we can use the following circuit.



5. We construct the following K-map.

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x	1		1	1
$\overline{x}$	1		1	1

This gives us the expansion $\overline{y}+z$.

Chapter 13—Test 1

1. The productions of a phrase-structure grammar are $S \to S1$, $S \to 0A$, and $A \to 1$. Find a derivation of 0111.

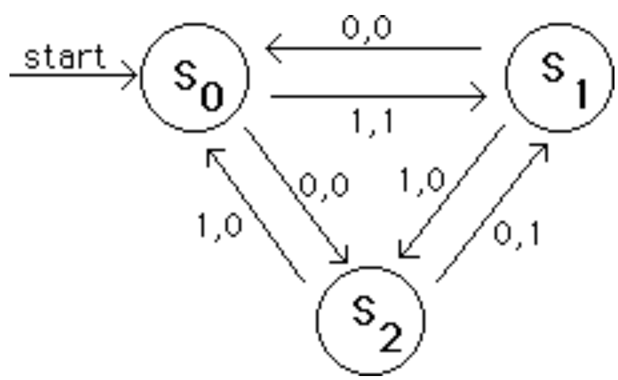
2. What language is generated by the phrase-structure grammar if the productions are $S \to S11$, $S \to \lambda$ where S is the start symbol?

3. Construct a finite-state machine that models a vending machine accepting only quarters that gives a container of orange juice when 50 cents has been deposited, followed by a button being pushed. (The possible inputs are quarters and the button, and the possible outputs are nothing, orange juice, and a quarter. The machine returns any extra quarters.)

4. Suppose that $A = \{1, 11, 01\}$ and $B = \{0, 10\}$. Find AB and BA.

5. Let $A = \{1, 10\}$. Which strings belong to A^*?

6. What is the output produced by the following finite-state automaton when the input string is 11101?



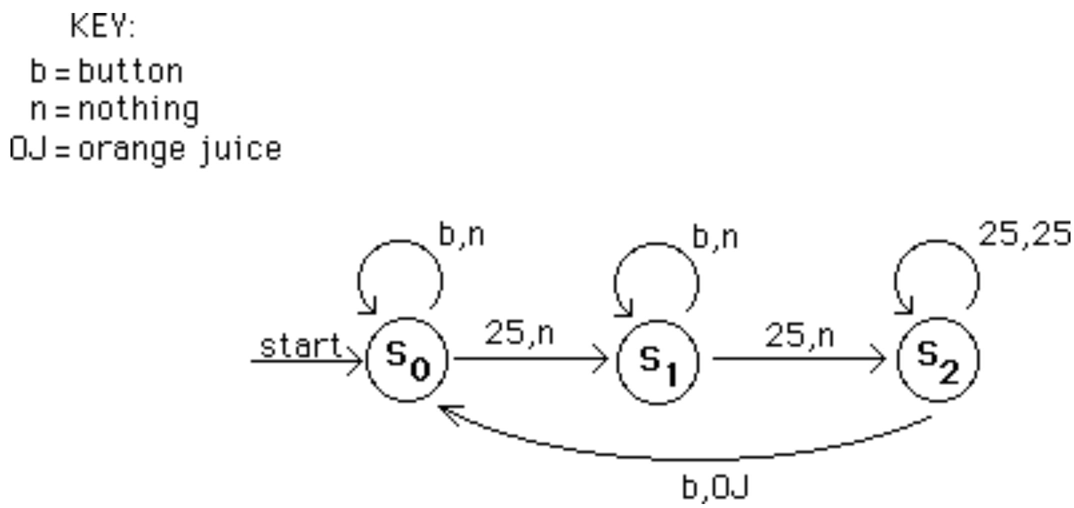
7. Which strings belong to the set represented by the regular expression $\mathbf{0}^* \cup \mathbf{11}$?

Chapter 13—Test 1 Solutions

1. We first apply the production $S \to S1$. Then we apply this production again to obtain $S11$. Next we apply the production $S \to 0A$ to obtain $0A11$. Next we apply the production $A \to 1$ to obtain 0111.

2. The language generated is the set of all strings consisting of an even number of 1's (and no other symbols).

3. The following finite-state machine models the vending machine.



4. We find that $AB = \{10, 110, 1110, 010, 0110\}$ and $BA = \{01, 011, 001, 101, 1011, 1001\}$.

5. The strings in A^* are those for which every 0 is preceded by at least one 1.

6. The output produced is 10000.

7. The set contains strings consisting of all 0s (including the empty string) and the string 11.

Chapter 13—Test 2

1. What is the language generated by the grammar with productions $S \to SA$, $S \to 0$, $A \to 1A$, and $A \to 1$, where S is the start symbol?

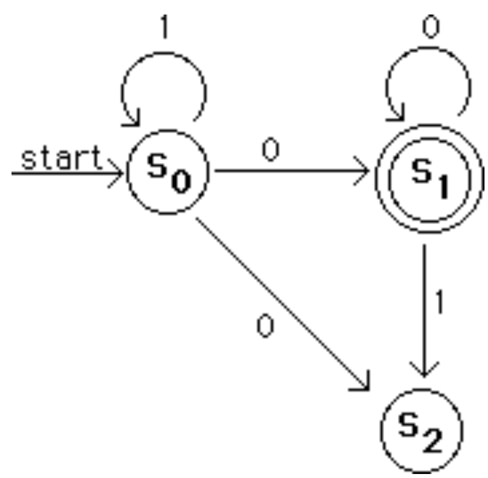
2. Find a grammar for the set $\{\, 0^{2n}1^n \mid n \geq 0 \,\}$.

3. Construct a finite-state machine with output that produces a 1 if and only if the last three input bits read are all 0s.

4. Let $A = \{1, 10\}$. Describe the elements of A^*.

5. Construct a finite-state automaton that recognizes the set represented by the regular expression $\mathbf{10}^*$.

6. Find a deterministic finite-state automaton equivalent to the nondeterministic finite-state machine shown.



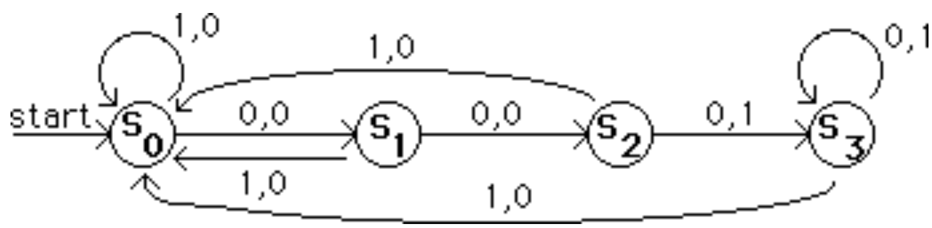
7. Which strings belong to the regular set represented by the regular expression $(\mathbf{1}^*\mathbf{01}^*\mathbf{0})^*$?

Chapter 13—Test 2 Solutions

1. The language is the set of all strings that consist of a 0 followed by an arbitrary number of 1's (possibly none).

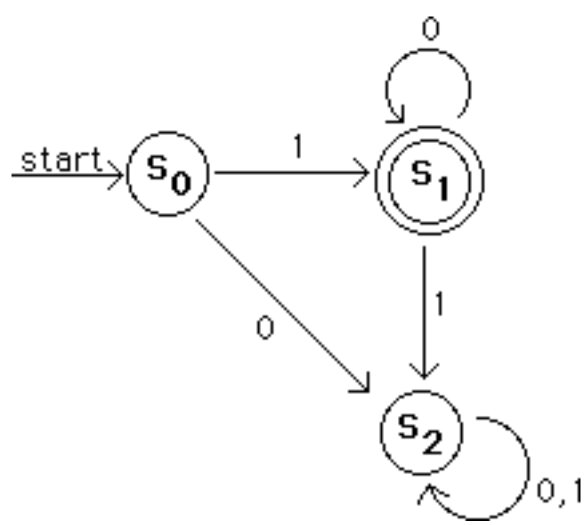
2. We can use the grammar with productions $S \to 00S1$ and $S \to \lambda$ where S is the start symbol.

3. The following finite-state machine produces a 1 if and only if the last three input bits read are all 0's.

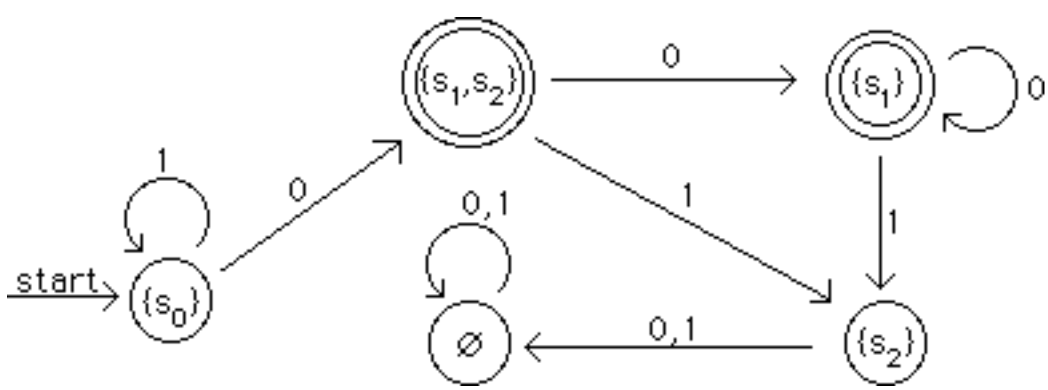


4. The strings in the set A^* are those strings where each 0 is preceded by a 1.

5. The following finite-state automaton recognizes the set represented by the regular expression $\mathbf{10}^*$.



6. The following deterministic finite-state automaton is equivalent.



7. These are the strings containing an even number of 0's and not ending with a 1.

Final Exam 1

1. Prove or disprove that if A and B are sets then $A \cap (A \cup B) = A$.

2. Find the prime factorization of 16575.

3. (a) Prove or disprove: If $a \equiv b \pmod 5$, where a and b are integers, then $a^2 \equiv b^2 \pmod 5$.

(b) Prove or disprove: If $a^2 \equiv b^2 \pmod 5$, where a and b are integers, then $a \equiv b \pmod 5$.

4. Use mathematical induction to prove that $n! \geq 2^{n-1}$ whenever n is a positive integer.

5. Suppose that $a_1 = 10$, $a_2 = 5$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Prove that 5 divides a_n whenever n is a positive integer.

6. How many bit string of length 10 have at least one 0 in them?

7. (a) How many functions are there from a set with three elements to a set with four elements?

(b) How many are one-to-one?

(c) How many are onto?

8. A door lock is opened by pushing a sequence of buttons. Each of the three terms in the combination is entered by pushing either one button or two buttons simultaneously. If there are 5 buttons, how many different combinations are there? (Example: 1-3, 2, 2-4 is a valid combination.)

9. Find a recurrence relation and initial conditions for the number of ways to go up a flight of stairs if stairs can be climbed one, two, or three at a time.

10. How many positive integers not exceeding 1000 are not divisible by either 8 or 12?

11. (a) Show that the relation $R = \{\, (x, y) \mid x - y \text{ is an even integer} \,\}$ is an equivalence relation on the set of real numbers.

(b) What are the equivalence classes of 1 and $\frac{1}{2}$ with respect to R?

12. Answer the following questions about the graph $K_{3,4}$.

(a) How many vertices and how many edges are in this graph?

(b) Is this graph planar? Justify your answer.

(c) Does this graph have an Euler circuit? Does it have an Euler path? Give reasons for your answers.

(d) What is the chromatic number of this graph?

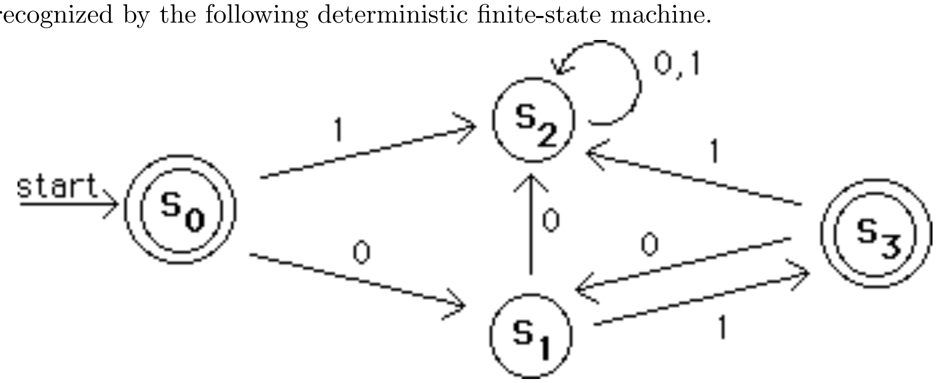
13. Find a spanning tree for the graph $K_{3,4}$ using

(a) a depth-first search.

(b) a breadth-first search.

14. Find the sum-of-products expansion of the Boolean function $f(x, y, z)$ that has the value 1 if and only if an odd number of the variables x, y, and z have the value 1.

15. Find the set recognized by the following deterministic finite-state machine.



16. A fair coin is flipped until a tail first appears, at which time no more flips are made.

(a) What is the probability that exactly five flips are made?

(b) What is the expected number of flips?

Final Exam 1 Solutions

1. Suppose that $x \in A$. Then $x \in A \cup B$, so $x \in A \cap (A \cup B)$. Conversely, suppose that $x \in A \cap (A \cup B)$. Then $x \in A$. Hence $A \cap (A \cup B) = A$.

2. We find that 2 does not divide 16575, but 3 does with $16575/3 = 5525$. We see that 3 does not divide 5525, but 5 does with $5525/5 = 1105$. We see that 5 divides 1105 with $1105/5 = 221$. We see that neither 5, 7, nor 11 divides 221. However, 13 does, with $221/13 = 17$. Hence $16575 = 3 \cdot 5^2 \cdot 13 \cdot 17$.

3. (a) Suppose that $a \equiv b \pmod 5$. Then $5 \mid a - b$, so there is an integer k such that $a - b = 5k$. It follows that $a^2 - b^2 = (a+b)(a-b) = (a+b)5k$. Hence $a^2 - b^2 = 5l$, where $l = (a+b)k$. It follows that $5 \mid a^2 - b^2$, so $a^2 \equiv b^2 \pmod 5$.

(b) We see that $1^2 \equiv 4^2 \pmod 5$, but $1 \not\equiv 4 \pmod 5$.

4. The basis step follows since $1! = 1 = 2^0$. For the inductive hypothesis assume that $n! \geq 2^{n-1}$. Then $(n+1)! = (n+1) \cdot n! \geq (n+1) \cdot 2^{n-1} \geq 2 \cdot 2^{n-1} = 2^n$.

5. The basis step is completed by noting that $a_1 = 10$ and $a_2 = 5$ are both divisible by 5. For the inductive step assume that a_k is divisible by 5 for every positive integer k with $k < n$, where $n \geq 3$. It follows that $a_n = 2a_{n-1} + 3a_{n-2}$ is divisible by 5, since the sum of two integers divisible by 5 is also divisible by 5.

6. The number of bit strings of length 10 with at least one 0 in them is the number of all bits strings of length 10 minus the number of bits strings of length 10 with no 0's in them. This is $2^{10} - 1 = 1024 - 1 = 1023$.

7. (a) There are $4^3 = 64$ functions from a set with three elements to a set with four elements.

(b) There are $4 \cdot 3 \cdot 2 = 24$ one-to-one functions from a set with three elements to a set with four elements.

(c) There are no onto functions from a set with three elements to a set with four elements.

8. A push of buttons in the combinations is either the push of one of the five buttons or the simultaneous push of one of $C(5,2) = 10$ combinations of two of the five buttons. Hence there are $15 \cdot 15 \cdot 15 = 3375$ possible combinations for the door lock.

9. Let a_n denote the number of ways to climb n stairs if stairs can be climbed one, two, or three at a time. Suppose that n is a positive integer, $n \geq 4$. A person can climb n stairs by going up $n-1$ stairs and then climbing one stair, by going up $n-2$ stairs and then climbing two stairs, or by going up $n-3$ stairs and then climbing three stairs. Hence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Note that $a_1 = 1$ since there is only one way to climb one stair, $a_2 = 2$ since two stairs can be climbed one stair at a time or two stairs at once, and $a_3 = 4$ since we can climb three stairs by taking stairs one at a time, by going up two stairs followed by one stair, by going up one stair followed by two stairs, or by taking all three stairs at once.

10. The number of integers not exceeding 1000 that are not divisible by either 8 or 12 is 1000 minus the number of these integers that are divisible by either 8 or 12. Using the principle of inclusion–exclusion, we see that there are $\lfloor 1000/8 \rfloor + \lfloor 1000/12 \rfloor - \lfloor 1000/24 \rfloor = 125 + 83 - 41 = 167$ such integers, where we used the fact that the integers divisible by both 8 and 12 are those divisible by $\operatorname{lcm}(8,12) = 24$. Hence there are $1000 - 167 = 833$ positive integers not exceeding 1000 that are not divisible by either 8 or 12.

11. (a) Since $x - x = 0$ is an even integer for every real number x it follows that R is reflexive. If $(x,y) \in R$ then $x - y$ is an even integer. It follows that $y - x = -(x-y)$ is also an even integer. Hence R is symmetric. Now suppose that $(x,y) \in R$ and $(y,z) \in R$. Then $x - y$ and $y - z$ are even integers. Since $x - z = (x-y) + (y-z)$ and the sum of two even integers is again even, it follows that $x - z$ is also an even integer. This shows that R is transitive. We conclude that R is an equivalence relation.

(b) We have $[1]_R = \{\, x \mid x - 1 \text{ is an even integer} \,\}$. Hence $[1]_R = \{\, x \mid x = 1 + 2k \text{ where } k \text{ is an integer} \,\}$. In other words, $[1]_R$ is the set of odd integers. Similarly, $[\frac{1}{2}]_R = \{\, x \mid x - \frac{1}{2} \text{ is an even integer} \,\}$. Hence $[\frac{1}{2}]_R = \{\, x \mid x = \frac{1}{2} + 2k \text{ where } k \text{ is an integer} \,\}$. This is the set $\{\ldots, -\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\}$.

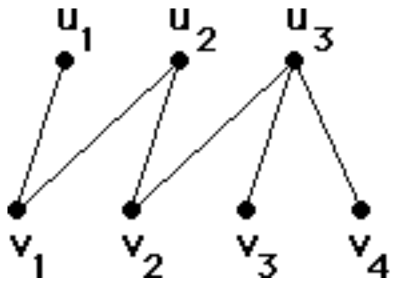
12. (a) The graph $K_{3,4}$ has $3 + 4 = 7$ vertices and $3 \cdot 4 = 12$ edges.

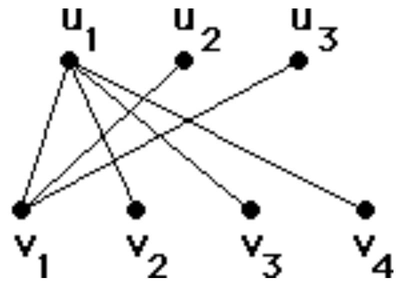
(b) $K_{3,4}$ is not planar since it contains a subgraph isomorphic to $K_{3,3}$, which is not planar.

(c) $K_{3,4}$ has three vertices of degree 4 and four vertices of degree 3. Since there are more than two vertices of odd degree, there is no Euler path in this graph, and therefore also no Euler circuit.

(d) The chromatic number of $K_{3,4}$ is 2 since this graph is bipartite.

13. A depth-first spanning tree of $K_{3,4}$ is shown on the left and a breadth-first spanning tree of $K_{3,4}$ is shown on the right. (Note that the first answer depends on the part of the graph you start in.)





14. The sum-of-products expansion is $f(x, y, z) = x\,y\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z$.

15. The set recognized is the set represented by $(\mathbf{01})^*$.

16. The number of flips follows a geometric distribution with parameter $p = 1/2$.

(a) The coin must land heads four times in a row and then tails, and the probability of this is $(1/2)^5 = 1/32$.

(b) The expected number of flips in a geometric distribution is $1/p = 2$.

Final Exam 2

1. Prove or disprove that $\overline{(A-B)} = \overline{A} \cup B$ whenever A and B are sets.

2. Prove or disprove that the fourth power of an odd positive integer always leaves a remainder of 1 when divided by 16.

3. Use mathematical induction to prove that every postage of greater than 5 cents can be formed from 3-cent and 4-cent stamps.

4. How many bit strings of length 10 have at least eight 1's in them?

5. (a) How many functions are there from a set with four elements to a set with three elements?

(b) How many of these functions are one-to-one?

(c) How many are onto?

6. How many symmetric relations are there on a set with eight elements?

7. (a) Let m be a positive integer greater than 2. Show that the relation R consisting of those ordered pairs of integers (a,b) with $a \equiv \pm b \pmod{m}$ is an equivalence relation.

(b) Describe the equivalence classes of this relation where $m = 4$.

8. (a) Does the graph $K_{2,5}$ have an Euler circuit? If not, does it have an Euler path?

(b) Does the graph $K_{2,5}$ have a Hamilton path?

9. How many nonisomorphic unrooted trees are there with four vertices? Draw these trees.

10. Construct a binary search tree from the words of the sentence *This is your discrete mathematics final*, using alphabetical order, inserting words in the order they appear in the sentence.

11. Find the sum-of-products expansion for the Boolean function $x + y + z$.

12. (a) Describe the bit strings that are in the regular set represented by $\mathbf{0}^*\mathbf{11}(\mathbf{0} \cup \mathbf{1})^*$?

(b) Construct a nondeterministic finite-state automaton that recognizes this set.

13. A thumb tack is tossed until it first lands with its point down, at which time no more tosses are made. On each toss, the probability of the tack's landing point down is $1/3$.

(a) What is the probability that exactly five tosses are made?

(b) What is the expected number of tosses?

Final Exam 2 Solutions

1. This set equality can be proved in several different ways. One method is to use set identities already known to hold. We find that $\overline{(A-B)} = \overline{(A \cap \overline{B})} = \overline{A} \cup \overline{(\overline{B})} = \overline{A} \cup B$, where we have used De Morgan's laws and the double complementation law.

2. Suppose that a is an odd integer. Then $a = 2k+1$. We have $a^4 = (2k+1)^4 = 16k^4+32k^3+24k^2+8k+1 = 16(k^4+2k^3)+8k(3k+1)+1$. If k is even then $8k = 16l$ where $k = 2l$, so $a^4 = 16N+1$, where N is an integer. If k is odd then $3k+1 = 3(2l+1)+1 = 6l+4 = 2m$, where $m = 3l+2$, so again $a^4 = 16N+1$, where N is an integer. It follows that $a \equiv 1 \pmod{16}$ whenever a is an odd integer.

3. The basis step is completed by noting that postage of 6 cents can be formed using two 3-cent stamps. Now assume that postage of n cents can be formed, where n is a positive integer greater than or equal to 6. If a 3-cent stamp was used to form n cents postage, replace this stamp with a 4-cent stamp to obtain postage of $n+1$ cents. Otherwise, if only 4-cent stamps were used, then at least two of them were used, so replace two 4-cent stamps with three 3-cent stamps to obtain postage of $n+1$ cents.

4. The number of bit strings of length 10 with at least eight 1's in them equals the number with exactly eight 1's plus the number with exactly nine 1's plus the number with exactly ten 1's. There are $C(10,8) = 10!/(2!8!) = 45$ such strings with exactly eight 1's, $C(10,9) = 10$ such strings with exactly nine 1's, and $C(10,10) = 1$ such string with exactly ten 1's. Hence there are $45+10+1 = 56$ bit strings of length 10 containing at least eight 1's.

5. (a) There are $3^4 = 81$ functions from a set with four elements to a set with three elements.

(b) There are no one-to-one functions from a set with four elements to a set with three elements since $4 > 3$.

(c) There are $3^4 - C(3,2)2^4 + C(3,1)1^4 = 81-48+3 = 36$ onto functions from a set with four elements to a set with three elements.

6. A symmetric relation is determined by specifying whether (i,j) and (j,i) belong to this relation for the pairs with $i \neq j$, and whether (i,i) belongs to the relation for all elements i in the set. Since there are eight elements in the set, there are $C(8,2) = 28$ pairs (i,j) and (j,i) with $i \neq j$, and eight elements i. Hence there are $2^{28+8} = 2^{36}$ symmetric relations on a set with eight elements.

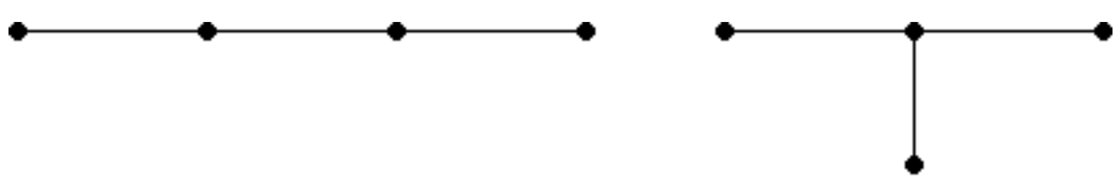
7. (a) Let a be an integer. Then $a \equiv a \pmod{m}$ since $m \mid a-a$. It follows that R is reflexive. Now suppose that $(a,b) \in R$. Then $a \equiv b \pmod{m}$ or $a \equiv -b \pmod{m}$. It is easy to see that $b \equiv a \pmod{m}$ or $b \equiv -a \pmod{m}$. Hence $(b,a) \in R$. It follows that R is symmetric. Now assume that $(a,b) \in R$ and $(b,c) \in R$. Then $a \equiv b \pmod{m}$ or $a \equiv -b \pmod{m}$, and $b \equiv c \pmod{m}$ or $b \equiv -c \pmod{m}$. We can easily see that each of the four combinations leads to $a \equiv c \pmod{m}$ or $a \equiv -c \pmod{m}$. Hence $(a,c) \in R$, and R is transitive.

(b) Let $m = 4$. The equivalence classes of R are $[0]_R = \{\, a \in \mathbf{Z} \mid a \equiv 0 \pmod{4} \,\} = \{\ldots,-8,-4,0,4,8,\ldots\}$, $[1]_R = \{\, a \in \mathbf{Z} \mid a \equiv \pm 1 \pmod{4} \,\} = \{\ldots,-5,-3,-1,1,3,5,\ldots\}$, and $[2]_R = \{\, a \in \mathbf{Z} \mid a \equiv 2 \pmod{4} \,\} = \{\ldots,-6,-2,2,6,\ldots\}$.

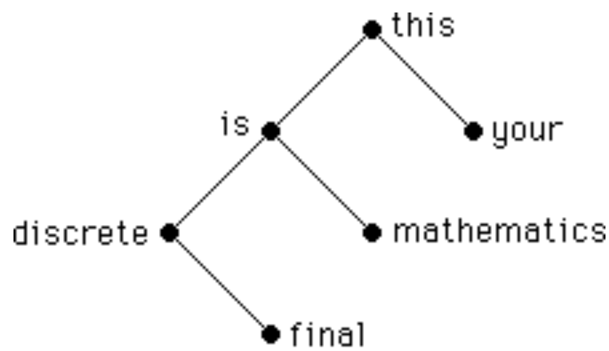
8. (a) The graph $K_{2,5}$ has two vertices of degree 5 and five vertices of degree 2. Hence it has an Euler path and no Euler circuit.

(b) There is no Hamilton path in this graph since any path containing all five vertices of degree 2 must visit some of the vertices of degree 5 more than once.

9. There are two nonisomorphic unrooted trees with four vertices, as shown.



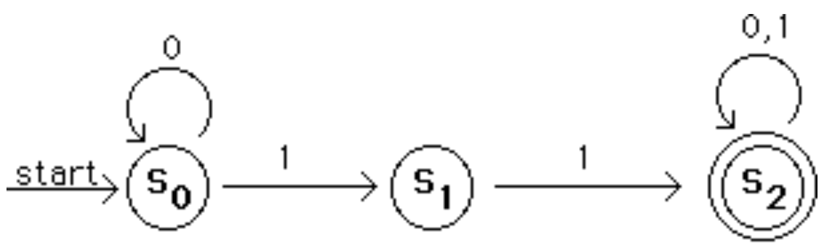
10. We construct the following binary search tree.



11. The Boolean function $x+y+z$ has the value 1 unless $x=y=z=0$, so it has the value 1 for the other seven combinations of the values of these variable. Hence the sum-of-products expansion is $f(x,y,z) = x\,y\,z + x\,y\,\overline{z} + x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z$.

12. (a) The strings in this set are those that begin with an arbitrary number of 0's followed by two consecutive 1's, followed by an arbitrary bit string.

(b) The following nondeterministic finite-state automaton recognizes this set.



13. The number of tosses follows a geometric distribution with parameter $p = 1/3$.

(a) The tack must land point up four times in a row and then point down, and the probability of this is $(2/3)^4(1/3) = 16/243$.

(b) The expected number of tosses in a geometric distribution is $1/p = 3$.