Ch. 8 Indeterminate Forms and Improper Integrals

8.1 Indeterminate Forms of Type 0/0

1 Use L'Hopital's Rule To Find Limit I

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Use L'Hôpital's rule to find the limit.

1) $\lim\limits_{x \to 9} \dfrac{x^2 - 81}{x - 9}$

A) 18 B) –18 C) 9 D) –9

2) $\lim\limits_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5}$

A) 4 B) 16 C) –1 D) 10

3) $\lim\limits_{x \to 1} \dfrac{x^3 - 7x^2 + 6}{x - 1}$

A) –11 B) 17 C) 14 D) 10

4) $\lim\limits_{x \to 0} \dfrac{\cos 9x - 1}{x^2}$

A) $-\dfrac{81}{2}$ B) $\dfrac{81}{2}$ C) $\dfrac{9}{2}$ D) 0

5) $\lim\limits_{x \to \frac{\pi}{3}} \dfrac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}}$

A) $-\dfrac{\sqrt{3}}{2}$ B) $\dfrac{\sqrt{2}}{2}$ C) $\dfrac{\sqrt{3}}{2}$ D) $-\sqrt{3}$

6) $\lim\limits_{x \to 0} \dfrac{x}{\sin x}$

A) 1 B) 0 C) –1 D) $\dfrac{1}{2}$

7) $\lim\limits_{x \to 0} \dfrac{\sin (5x)}{\sin x}$

A) 5 B) 1 C) 0 D) –5

8) $\lim\limits_{x \to 0} \dfrac{\sin 8x}{\tan 7x}$

A) $\dfrac{8}{7}$ B) $\dfrac{7}{8}$ C) $-\dfrac{8}{7}$ D) 0

9) $\lim_{x \to 0} \frac{\sin x^4}{x}$

A) 0 B) 1 C) ∞ D) $-\infty$

10) $\lim_{x \to 1} \frac{x^3 - 8x^2 + 7}{x - 1}$

A) -13 B) 19 C) 16 D) 11

2 Use L'Hopital's Rule To Find Limit II

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Use L'Hôpital's rule to find the limit.

1) $\lim_{x \to 0} \frac{5 - 5 \cos x}{\sin 4x}$

A) 0 B) 1 C) ∞ D) $\frac{5}{4}$

2) $\lim_{x \to 0} \frac{e^x - 1}{\sin 6x}$

A) $\frac{1}{6}$ B) 1 C) 0 D) $-\frac{1}{6}$

3) $\lim_{x \to 0} \frac{\sin x}{2x + x^4}$

A) $\frac{1}{2}$ B) 1 C) 0 D) $\frac{1}{4}$

4) $\lim_{x \to 0} \frac{7^x - 1}{9^x - 1}$

A) $\frac{\ln 7}{\ln 9}$ B) 0 C) 1 D) $\frac{7}{9}$

5) $\lim_{x \to \pi/2} \frac{\ln (\sin x)^{17}}{\pi/2 - x}$

A) 0 B) 1 C) -1 D) $\frac{1}{17}$

6) $\lim_{x \to 1} \frac{\sqrt{x} - x^5}{\ln x}$

A) $-\frac{9}{2}$ B) -1 C) -5 D) 0

7) $\lim_{x \to 0} \frac{\tan 2x}{\ln (1 + x)}$

A) 2 B) 1 C) 4 D) $\frac{7}{2}$

8) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{4x^3}$

A) $-\dfrac{1}{24}$ B) $\dfrac{1}{8}$ C) $-\dfrac{1}{12}$ D) -4

9) $\lim\limits_{x\to 17} \dfrac{e^x - e^{17}}{x - 17}$

A) e^{17} B) e^{16} C) 0 D) e^{20}

10) $\lim\limits_{x\to 0} \dfrac{\int_0^x \sqrt{1 + \sin t}\, dt}{8x}$

A) $\dfrac{1}{8}$ B) 1 C) 0 D) $\dfrac{1}{7}$

3 Find Continuous Extension

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Solve the problem.

1) $f(x) = \begin{cases} \dfrac{2x - 8}{x - 4}, & x \neq 4 \\ c, & x = 4 \end{cases}$

What value of c makes f(x) continuous at $x = 4$?

A) 2 B) -8 C) -2 D) 4

2) $f(x) = \begin{cases} x^2 + 5, & x < 0 \\ c, & x = 0; \quad c = 0 \\ -3(x - 5) - 10, & x > 0 \end{cases}$

What value of c makes f(x) continuous at $x = 0$?

A) 5 B) -3 C) -5 D) 0

3) $f(x) = \begin{cases} \dfrac{5e^x - 5}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$

What value of c makes f(x) continuous at $x = 0$?

A) 5 B) 4 C) 7 D) $\dfrac{11}{2}$

4 *Know Concepts: Indeterminate Forms

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

Provide an appropriate response.

1) Which one is correct, and which one is wrong? Give reasons for your answers.

(a) $\lim_{x \to 4} \frac{x - 4}{x^2 - 4} = \lim_{x \to 4} \frac{1}{2x} = \frac{1}{8}$

(b) $\lim_{x \to 4} \frac{x - 4}{x^2 - 4} = \frac{0}{12} = 0$

2) Which one is correct, and which one is wrong? Give reasons for your answers.

(a) $\lim_{x \to -2} \frac{x + 2}{x^2 - 4} = \lim_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$

(b) $\lim_{x \to -2} \frac{x + 2}{x^2 - 4} = \frac{0}{-8} = 0$

3) Let $f(x) = \begin{cases} 2x + 4 & x \neq 0 \\ 0 & x = 0 \end{cases}$ and $g(x) = \begin{cases} x + 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 2$ but $\lim_{x \to 0} \frac{f(x)}{g(x)} = 4$. Explain why this does not contradict l'Hopital's Rule.

4) Find the error in the following incorrect application of L'Hopital's Rule.

$$\lim_{x \to -2} \frac{x^3 - 2x^2 + 1}{3x^2 - 6x} = \lim_{x \to -2} \frac{3x^2 - 4x}{6x - 6} = \lim_{x \to -2} \frac{6x - 4}{6} = \frac{-16}{6}.$$

8.2 Other Indeterminate Forms

1 Use L'Hopital's Rule To Find Limit I

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Use L'Hôpital's rule to find the limit.

1) $\lim_{x \to \infty} \frac{x^2 - 4x + 11}{x^3 - 5x^2 + 12}$

A) 0 B) 1 C) -1 D) ∞

2) $\lim_{x \to \infty} \frac{9x^2 - 3x + 6}{8x^2 + 6x + 10}$

A) $\frac{9}{8}$ B) $\frac{8}{9}$ C) $-\frac{9}{8}$ D) 1

3) $\lim_{x \to -\infty} \frac{11 + 9x - 16x^2}{9 - 7x - 14x^2}$

A) $\frac{8}{7}$ B) $\frac{11}{9}$ C) ∞ D) 1

4) $\lim_{x \to \infty} \frac{3x + 6}{6x^2 + 4x - 4}$

A) 0 B) 1 C) $\frac{1}{2}$ D) $\frac{1}{4}$

5) $\lim_{x \to \infty} x \sin \frac{12}{x}$

A) 12 B) $\frac{1}{12}$ C) 0 D) 1

6) $\lim_{x \to \infty} (\sqrt{x^2 + 7x} - x)$

A) $\frac{7}{2}$ B) 7 C) 0 D) $-\frac{7}{2}$

7) $\lim_{x \to \infty} \frac{\ln x}{x^{1/7}}$

A) 0 B) ∞ C) $\frac{1}{7}$ D) 7

8) $\lim_{x \to \infty} \left(1 + \frac{4}{x^2}\right)^x$

A) 1 B) 0 C) ∞ D) 4

9) $\lim_{x \to 0^+} x^{\sin x}$

A) 1 B) 0 C) -1 D) ∞

10) $\lim_{x \to \infty} \frac{e^x}{3x^2 + 6}$

A) ∞ B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) 0

2 Use L'Hopital's Rule To Find Limit II

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Use L'Hôpital's rule to find the limit.

1) $\lim_{x \to \infty} \frac{8x^{507}}{e^x}$

A) 0 B) 8 C) 4056 D) 1

2) $\lim_{x \to 0^+} (x^6 \ln x)$

A) 0 B) 1 C) 6 D) -1

3) $\lim\limits_{x \to 1} x^{1/(4x - 4)}$

A) $e^{1/4}$ B) $e^{2/4}$ C) e D) e^5

4) $\lim\limits_{x \to 0} 3x \csc x$

A) 3 B) 0 C) 1 D) ∞

5) $\lim\limits_{x \to \pi/2} \dfrac{\csc 10x}{\csc 8x}$

A) $-\dfrac{4}{5}$ B) $\dfrac{5}{4}$ C) $\dfrac{4}{5}$ D) $-\dfrac{5}{4}$

6) $\lim\limits_{x \to \infty} \dfrac{\ln x^{157}}{x}$

A) 0 B) e C) -1 D) 1

7) $\lim\limits_{x \to \infty} \dfrac{x^{66}}{e^x}$

A) 0 B) ∞ C) 66e D) -1

8) $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^5}$

A) 0 B) e C) 1 D) 5e

9) $\lim\limits_{x \to \infty} \dfrac{\displaystyle\int_1^x \sqrt{1 + e^{-t}}\, dt}{25x}$

A) $\dfrac{1}{25}$ B) 25 C) $\dfrac{1}{27}$ D) $\dfrac{2}{25}$

10) $\lim\limits_{x \to 0^+} \left(\dfrac{1}{2}8^t + \dfrac{1}{2}9^t\right)^{1/t}$

A) 8.485 B) 2.828 C) 3.000 D) 17.605

8.3 Improper Integrals: Infinite Limits of Integration

1 Evaluate Improper Integral I

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Evaluate the improper integral or state that it diverges.

1) $\displaystyle\int_1^{\infty} \frac{24}{8x(x + 1)^2}\, dx$

A) 0.578 B) 1.569 C) -1.569 D) diverges

2) $\displaystyle\int_{6}^{\infty} \frac{dx}{x^2 - 25}$

A) $\frac{1}{10} \ln 11$ B) $\frac{1}{10} \ln 6$ C) $-\frac{1}{5} \ln 11$ D) $\frac{1}{10} \ln \frac{1}{6}$

3) $\displaystyle\int_{3}^{\infty} \frac{dt}{t^2 - 2t}$

A) $\frac{1}{2} \ln 3$ B) $-\frac{1}{2} \ln 3$ C) $\frac{1}{3} \ln 2$ D) $2 \ln 3$

4) $\displaystyle\int_{0}^{\infty} \frac{2dx}{25 + x^2}$

A) $\frac{\pi}{5}$ B) 0 C) $\frac{\pi}{25}$ D) $\pi + 5$

5) $\displaystyle\int_{-\infty}^{0} \frac{23}{(x - 1)^2}\, dx$

A) 23 B) 46 C) -23 D) diverges

6) $\displaystyle\int_{-\infty}^{\infty} \frac{12x}{(x^2 - 1)^2}\, dx$

A) 0 B) 12 C) 24 D) diverges

7) $\displaystyle\int_{1}^{\infty} \frac{1}{x(x^2 + 6)}\, dx$

A) $\frac{\ln 7}{12}$ B) $\ln 7$ C) $\ln 5$ D) diverges

8) $\displaystyle\int_{-\infty}^{-4} \frac{8}{x^5}\, dx$

A) $-\frac{1}{128}$ B) $\frac{1}{8192}$ C) $\frac{1}{32}$ D) diverges

9) $\displaystyle\int_{1}^{\infty} \frac{dx}{x^{3.929}}$

A) $\frac{1}{2.929}$ B) $\frac{1}{4.929}$ C) $\frac{1}{3.929}$ D) diverges

10) $\displaystyle\int_{7}^{\infty} \frac{dx}{(x - 6)(x - 5)}$

A) $\ln 2$ B) $\ln 6$ C) $\frac{1}{2} \ln 7$ D) $-\frac{1}{6} \ln 2$

2 Evaluate Improper Integral II

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Evaluate the improper integral or state that it diverges.

1) $\displaystyle\int_{-\infty}^{0} \frac{dx}{(49 + x)\sqrt{x}}$

A) $-\frac{\pi}{7}$ B) $\frac{\pi}{7}$ C) 0 D) -7π

2) $\displaystyle\int_{0}^{\infty} 11e^{-11x}\,dx$

A) 1 B) 0 C) -1 D) diverges

3) $\displaystyle\int_{-\infty}^{\infty} x^7e^{-x^8}\,dx$

A) 0 B) $-\frac{1}{4}$ C) $\frac{1}{8}$ D) diverges

4) $\displaystyle\int_{-\infty}^{0} \frac{24}{(x - 1)}\,dx$

A) -48 B) -24 C) 24 D) diverges

5) $\displaystyle\int_{-\infty}^{\infty} \frac{3x^2}{(x^2 - 1)^2}\,dx$

A) 6 B) 3 C) 0 D) diverges

6) $\displaystyle\int_{-\infty}^{0} 2xe^{3x}\,dx$

A) -0.2222 B) 0.3333 C) -4.667 D) diverges

7) $\displaystyle\int_{0}^{\infty} \frac{36(1 + \tan^{-1}x)}{1 + x^2}\,dx$

A) $18\pi\left(1 + \frac{\pi}{4}\right)$ B) $18\left(1 + \frac{\pi}{2}\right)^2$ C) 18π D) $36\ln\left(1 + \frac{\pi}{2}\right)$

8) $\displaystyle\int_{-\infty}^{0} 9e^x \sin x\,dx$

A) $-\frac{9}{2}$ B) 0 C) $\frac{9}{2}$ D) -9

9) $\displaystyle\int_1^{\infty} \frac{16}{(1 + x^2)\tan^{-1}x}\,dx$

A) $16 \ln 2$ B) $8\left(1 + \frac{\pi}{2}\right)^2$ C) $16 \ln\left(1 + \frac{\pi}{2}\right)$ D) $16 \ln \frac{\pi}{2}$

10) $\displaystyle\int_0^{\infty} 10xe^{2x}\,dx$

A) 1.6667 B) 1.3333 C) 2.6667 D) diverges

11) $\displaystyle\int_{-\infty}^{e} 15e^{-x}\,dx$

A) 30 B) -15 C) 15 D) diverges

12) $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x}\,dx$

A) 0 B) -2 C) 2 D) diverges

3 Solve Apps: Improper Integrals

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Solve the problem.

1) Find the area of the region in the first quadrant between the curve $y = e^{-3x}$ and the x-axis.

A) $\frac{1}{3}$ B) 3 C) 1 D) $\frac{1}{3}e$

2) Find the area of the region bounded by the curve $y = 10x^{-2}$, the x-axis, and on the left by $x = 1$.

A) 10 B) 20 C) 5 D) 100

3) Find the area under the curve $y = \dfrac{1}{(x + 1)^{3/2}}$ bounded on the left by $x = 24$.

A) $\frac{2}{5}$ B) $\frac{1}{12}$ C) 10 D) $\frac{1}{5}$

4) Find the area under $y = \dfrac{8}{1 + x^2}$ in the first quadrant.

A) 4π B) 8π C) 16π D) 2π

5) Find the area between the graph of $y = \dfrac{18}{(x - 1)^2}$ and the x-axis, for $-\infty < x \le 0$.

A) 18 B) 36 C) 1 D) 9

6) According to Newton's Inverse Square Law, the force exerted by the earth on a space capsule is $-\frac{k}{x^2}$, where x is the distance (in miles, for instance) from the capsule to the center of the earth. The force F(x) required to lift the capsule is given by $F(x) = \frac{k}{x^2}$. How much work is done in propelling a 1500-pound capsule out of the earth's gravitational field? Note: The radius of the earth is 3960 miles.

A) 5.94×10^6 B) 7.92×10^6 C) 9.90×10^6 D) 1.98×10^6

7) An electrical component has a lifetime given by the probability density function function $f(x) = \begin{cases} 0.01e^{-0.01x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$, where x is measured in hours. What is the probability that the component will work for at least 21 hours?

A) 0.811 B) 0.795 C) 0.787 D) 0.827

4 *Know Concepts: Improper Integrals

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

Provide an appropriate response.

1) A student wishes to find the integral $\int_0^{+\infty} f(x)\,dx$ of a function that has the property limit $\lim\limits_{x \to +\infty} f(x) = 1$. Why can this not be done?

2) A student wishes to take the integral over all real numbers of $f(x) = \begin{cases} \frac{1}{x}, & \text{if } x < 0 \; \frac{1}{x^2}, \text{if } x > 0 \end{cases}$ and claims this is zero because $-\infty + \infty$ equals zero. What is wrong with this thinking?

3) A student needs $\int_{-\infty}^{+\infty} e^{-|x|}\,dx$. Is this integral the same as $2\int_0^{+\infty} e^{-|x|}\,dx$, and if so, why?

4) The Cauchy density function, $f(x) = \frac{1}{\pi(1 + x^2)}$, occurs in probability theory. Show that $\int_{-\infty}^{\infty} \frac{1}{\pi(1 + x^2)}\,dx = 1$.

5) The standard normal probability density function is defined by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

(a) Show that $\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}$.

(b) Use the result in part (a) to show that the standard normal probability density function has mean 0.

6) The standard normal probability density function is defined by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

(a) Use the fact that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$ to show that $\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x^2\, e^{-x^2/2}\,dx = \frac{1}{2}$.

(b) Use the result in part (a) to show that the standard normal probability density function has variance 1.

8.4 Improper Integrals: Infinite Integrands

1 Evaluate Improper Integral

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Evaluate the improper integral or state that it diverges.

1) $\displaystyle\int_0^{64} \frac{dx}{\sqrt{64 - x}}$

A) 16 B) 8 C) −16 D) diverges

2) $\displaystyle\int_{-1}^{27} \frac{dx}{x^{2/3}}$

A) 12 B) 6 C) 4 D) diverges

3) $\displaystyle\int_0^{3} \frac{dx}{\sqrt{9 - x^2}}$

A) $\frac{\pi}{2}$ B) 6 C) $\frac{\pi}{6}$ D) diverges

4) $\displaystyle\int_6^{12} \frac{dx}{x\sqrt{x^2 - 36}}$

A) $\frac{\pi}{18}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{12}$ D) diverges

5) $\displaystyle\int_0^{4} x \ln 6x \, dx$

A) $8 \ln 24 - 4$ B) $8 \ln 24 - 8$ C) $-\frac{1}{2} \ln 24 + 4$ D) diverges

6) $\displaystyle\int_0^{9} \frac{x}{\sqrt{81 - x^2}} dx$

A) 9 B) −9 C) 81 D) diverges

7) $\displaystyle\int_0^{1} \frac{1}{x^{20}} dx$

A) 0 B) $\frac{1}{20}$ C) $-\frac{1}{20}$ D) diverges

8) $\displaystyle\int_{-7}^{1} \frac{1}{x^4} dx$

A) 0 B) $\frac{1}{4}$ C) 1 D) diverges

9) $\int_{0}^{\ln 11} \frac{e^x}{\sqrt{e^x - 1}}\, dx$

A) $2\sqrt{11 - 1}$ B) $\sqrt{11 - 1}$ C) $\frac{1}{2\sqrt{11 - 1}}$ D) diverges

10) $\int_{0}^{0.3} \frac{\cos x}{x^2}\, dx$

A) 1 B) 0.3 C) 0 D) diverges

2 Solve Apps: Improper Integrals

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Find the area.

1) Find the area of the region in the first quadrant between the curve $y = e^{-2x}$ and the x-axis.

A) $\frac{1}{2}$ B) 2 C) 1 D) $\frac{1}{2}e$

2) Find the area of the region bounded by the curve $y = 6x^{-2}$, the x-axis, and on the left by $x = 1$.

A) 6 B) 12 C) 3 D) 36

3) Find the area under the curve $y = \frac{1}{(x + 1)^{3/2}}$ bounded on the left by $x = 3$.

A) 1 B) $\frac{2}{3}$ C) 4 D) $\frac{1}{2}$

4) Find the area under $y = \frac{10}{1 + x^2}$ in the first quadrant.

A) 5π B) 10π C) 20π D) $\frac{5}{2}\pi$

5) Find the area between the graph of $y = \frac{5}{(x - 1)^2}$ and the x-axis, for $-\infty < x \le 0$.

A) 5 B) 10 C) 1 D) $\frac{5}{2}$

3 Determine Convergence

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Use the Comparison Test to determine whether the improper integral converges or diverges.

1) $\int_{-3}^{10} \frac{dx}{(x + 1)^{1/3}}$

A) converges B) diverges

2) $\int_0^9 \frac{x^2\,dx}{\sqrt{81 - x^2}}$

A) converges B) diverges

3) $\int_0^{\ln 10} x^{-3} e^{1/x^2}\,dx$

A) converges B) diverges

4) $\int_0^7 \frac{dx}{49 - x^2}$

A) converges B) diverges

5) $\int_0^{\pi/2} \sec\theta\,d\theta$

A) converges B) diverges

6) $\int_0^{\pi/2} \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$

A) converges B) diverges

7) $\int_{-1}^1 \frac{1}{x \ln|x|}dx$

A) converges B) diverges

8) $\int_0^6 (x - 5)^{-4/3}\,dx$

A) converges B) diverges

9) $\int_0^3 \frac{dx}{|x - 2|}\,dx$

A) converges B) diverges

10) $\int_0^6 \frac{e^{-\sqrt{x - 5}}}{\sqrt{x - 5}}\,dx$

A) converges B) diverges

11) $\int_0^\pi \frac{\sin\theta\,d\theta}{(\pi - \theta)^{1/9}}$

A) converges B) diverges

4 *Know Concepts: Improper Integrals

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

Provide an appropriate response.

1) A student claims that $\int_a^b f(x)\,dx$ always exists, as long as a and b are both positive. Refute this by giving an example of a function for which this is not true.

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

2) A student knows that $\int_a^{+\infty} f(x)\,dx$ diverges, but needs to investigate $\int_a^{+\infty} g(x)\,dx$, where $g(x) = \frac{f(x)}{69}$. Does this integral necessarily also diverge?

A) Yes B) No

3) A student knows that $\int_a^{+\infty} f(x)\,dx$ converges. Does $\int_{-\infty}^{a} f(x)\,dx$ also necessarily converge?

A) Yes B) No

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

4) i) Show that $\int_0^{\infty} \frac{4x^3}{x^4 + 1}dx$ diverges, and hence $\int_{-\infty}^{\infty} \frac{4x^3}{x^4 + 1}dx$ diverges.

ii) Show that $\lim\limits_{b\to\infty} \int_{-b}^{b} \frac{4x^3}{x^4 + 1}dx = 0.$

5) (a) Find the values of p for which $\int_0^1 \frac{1}{x^p}\,dx$ converges.

(b) Find the values of p for which $\int_0^1 \frac{1}{x^p}\,dx$ diverges.

6) Explain why the integral $\int_0^5 \frac{dx}{x^2 - 3x - 4}$ is improper, but the integral $\int_0^3 \frac{dx}{x^2 - 3x - 4}$ is proper.

Ch. 8 Indeterminate Forms and Improper Integrals
Answer Key

8.1 Indeterminate Forms of Type 0/0

1 Use L'Hopital's Rule To Find Limit I

1) A
2) A
3) A
4) A
5) A
6) A
7) A
8) A
9) A
10) A

2 Use L'Hopital's Rule To Find Limit II

1) A
2) A
3) A
4) A
5) A
6) A
7) A
8) A
9) A
10) A

3 Find Continuous Extension

1) A
2) A
3) A

4 *Know Concepts: Indeterminate Forms

1) Choice (a) is incorrect. L'Hopital's rule cannot be applied to $\lim\limits_{x\to 4} \frac{x - 4}{x^2 - 4}$ because it corresponds to $\frac{0}{12}$ which is not an indeterminate form. Choice (b) is correct.

2) Choice (a) is correct. L'Hopital's rule can be applied to $\lim\limits_{x\to -2} \frac{x + 2}{x^2 - 4}$ since it corresponds to the indeterminate form $\frac{0}{0}$. Choice (b) is incorrect because $(-2)^2 = 4$ not -4.

3) $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to 0} \frac{2}{1} = 2$ and $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} \frac{2x + 4}{x + 1} = 4$. L'Hopital's Rule does not apply to either limit. Neither limit is an indeterminate form.

4) L'Hopital's Rule cannot be applied to $\lim\limits_{x\to -2} \frac{3x^2 - 4x}{6x - 6}$ because it corresponds to $\frac{20}{-18}$ which is not an indeterminate form.

8.2 Other Indeterminate Forms

1 Use L'Hopital's Rule To Find Limit I

1) A
2) A
3) A
4) A
5) A
6) A

7) A
8) A
9) A
10) A

2 Use L'Hopital's Rule To Find Limit II

1) A
2) A
3) A
4) A
5) A
6) A
7) A
8) A
9) A
10) A

8.3 Improper Integrals: Infinite Limits of Integration

1 Evaluate Improper Integral I

1) A
2) A
3) A
4) A
5) A
6) A
7) A
8) A
9) A
10) A

2 Evaluate Improper Integral II

1) A
2) A
3) A
4) D
5) D
6) A
7) A
8) A
9) A
10) D
11) D
12) D

3 Solve Apps: Improper Integrals

1) A
2) A
3) A
4) A
5) A
6) A
7) A

4 *Know Concepts: Improper Integrals

1) The only way the limit of the integral can exist is if the limit of the function is zero.
2) Infinity cannot be added like this.
3) Yes, the function is symmetric about the y-axis.

4) $\int_0^{\infty} \frac{1}{\pi(1+x^2)}\,dx = \lim_{b\to\infty} \frac{1}{\pi}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \frac{1}{\pi}\left[\tan^{-1}b - \tan^{-1}0\right] = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$. Since f(x) is an even function, $\int_{-\infty}^{0} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2}$ and $\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2} + \frac{1}{2} = 1$.

5) (a) $\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{\sqrt{2\pi}} x\, e^{-x^2/2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\right]_0^b = \frac{1}{\sqrt{2\pi}}$.

(b) Since $y = \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}$ is an odd function, $\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx = -\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx = -\frac{1}{\sqrt{2\pi}}$. The mean of the distribution is equal to $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx + \int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x\, e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} + -\frac{1}{\sqrt{2\pi}} = 0$.

6) (a) Since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,dx = 1$ and f(x) is an even function, $\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,dx = \frac{1}{2}$. Then

$\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x^2\, e^{-x^2/2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{\sqrt{2\pi}}\, x^2\, e^{-x^2/2}\,dx = \lim_{b\to\infty} \frac{1}{\sqrt{2\pi}}\left[\left[-x\, e^{-x^2/2}\right]_0^b + \int_0^b e^{-x^2/2}\,dx\right] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2/2}\,dx = \frac{1}{2}$.

(b) The variance of the distribution is equal to $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x^2\, e^{-x^2/2}\,dx = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}}\, x^2\, e^{-x^2/2}\,dx + \int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, x^2\, e^{-x^2/2}\,dx = \frac{1}{2} + \frac{1}{2} = 1$.

8.4 Improper Integrals: Infinite Integrands

1 Evaluate Improper Integral

1) A
2) A
3) A
4) A
5) A
6) A
7) D
8) D
9) A
10) D

2 Solve Apps: Improper Integrals

1) A
2) A
3) A
4) A
5) A

3 Determine Convergence

1) A
2) A
3) B

4) B
5) B
6) A
7) B
8) B
9) B
10) A
11) A

4 *Know Concepts: Improper Integrals

1) Answers will vary, but $f(x) = \dfrac{1}{(x - c)^d}$ where $a < c < b$ and d a positive integer is a family of examples.

2) A
3) B
4) i) $\displaystyle\int_0^\infty \frac{4x^3}{x^4 + 1}dx = \lim_{b\to\infty} \int_0^b \frac{4x^3}{x^4 + 1}dx = \lim_{b\to\infty} \ln(x^4 + 1) \Big|_0^b = \lim_{b\to\infty} (\ln(b^4 + 1) - \ln 1) = \infty$

ii) $\displaystyle\lim_{b\to\infty} \int_{-b}^b \frac{4x^3}{x^4 + 1}dx = \lim_{b\to\infty} \ln(x^4 + 1) \Big|_{-b}^b = \lim_{b\to\infty} (\ln(b^4 + 1) - \ln(b^4 + 1)) = \lim_{b\to\infty} 0 = 0$

5) (a) The integral converges if $p < 1$.
(b) The integral diverges if $p \ge 1$.

6) The graph of the function $f(x) = \dfrac{1}{x^2 - 3x - 4}$ has a vertical asymptote at $x = 4$. The first integral is improper because the integrand is unbounded on the interval [0, 5]. The second integral is proper because the integrand is continuous on the interval [0, 3].