Illinois Institute of Technology
Department of Computer Science

Homework 9 Solutions

CS 430 Introduction to Algorithms
Spring Semester, 2016

1. **Problem 1**

   Solution: Consider a a bipartite graph of vertices a_1, a_2, ... , a_n and b_1, b_2, ... , b_n with edges (a_i, a_j) for all $i \neq j$. In optimal coloring, we just need to use two colors: color a_1, a_2, ... , a_n using one color; color b_1, b_2, ... , b_n with the other.

   The greedy algorithm, however may, color the vertices with $\frac{n}{2}$ colors for certain ordering of vertices. Let the vertices be in the following order: a_1, b_1, a_2, b_2, ..., a_n, b_n. According to the greedy algorithm, a_1 and b_1 will be given the same color. a_2 and b_2 will be given a new color as there is edge (a_2, b_1). a_3 and b_3 will be given a new color as where are edge (a_3, b_1) and (a_3, b_2) and so on. To color all the vertices, we need $\frac{n}{2}$ colors. The approximation ratio is $\frac{n}{4}$. Thus the greedy algorithm cannot achieve a constant approximation ratio.

2. **Problem 2**

   Solution: For a graph $G = (V, E)$, it is straight forward that there always exists a optimal coloring (of all coloring, the one that uses minimum number of colors). Suppose the optimal coloring use k colors to color G. Let the vertices of G are grouped by V_1, V_2, ... V_k. We require that the vertices in the same group associated with the same color by the optimal coloring. The vertices in different group are colored by different colors. Then we order the vertices by V_1, V_2, ... V_k. The greedy algorithm will always color the vertices in the same group with the same color and vertices in different groups with different colors. Thus the greedy strategy will use k colors which is the minimum number of colors to use.

3. **Problem 3 Solution:** We can reduce the graph coloring problem to the optimal ordering. Thus the optimal ordering is NP-Hard. The decision version of the optimal ordering, where we just need to verify if an ordering can result in a k-color coloring, is in NP.