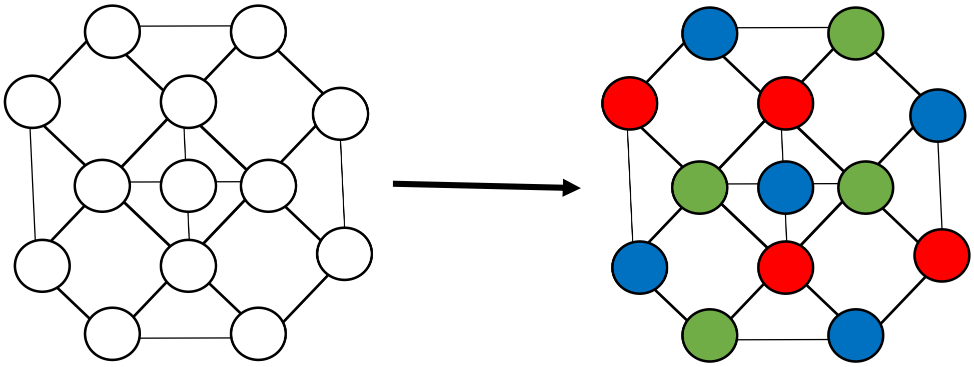
Problem 1)



To prove 3-coloring a planar graph is NP-complete, we can first show that the problem is in the class NP. A problem is in the class NP if we can verify it's solution in polynomial time. Above we must 3-color the given widget/graph. We go through all the vertices and check the neighbors are different colors and finally count the total number of colors. This algorithm scales linearly with the number of regions, so it is a polynomial check. Thus, PLANAR-3-COLOR ∈ NP.

Next we must show that the problem is NP-hard. To do this we will show the following reduction: 3-SAT $\leq_p$ PLANAR-3-COLOR. So, we have an input graph G from which we make a new graph G^ such that, G is 3-SAT if and only if G^ is PLANAR-3-COLOR. To make G^ replace all edge crossings in G with the above widget. The widget has the following properties: (1) opposite corners are the same color for every valid 3-COLORING and (2) any such coloring of the corners extend to a 3-COLORING of the entire widget. So, if an edge in G is crossed by multiple other edges, the gadgets that replace those crossings need to be linked together at the edges. This promotes the fact that the nodes at either end of the edge must be different colors. Thus, by replacing crossovers with the above widget we have made G^ PLANAR-3-COLORABLE! Then if we remove the widgets from G^ we get G! This reduction runs in polynomial time and we have G^ is PLANAR-3-COLOR so then G is 3-SAT thus, 3-SAT $\leq_p$ PLANAR-3-COLOR. This means that since we are able to solve the 3-SAT problem then we can solve the PLANAR-3-COLOR problem and the 3-SAT problem is an instance of the PLANAR-3-COLOR problem. Therefore, PLANAR-3-COLOR is also NP-hard thus making it NP-complete.

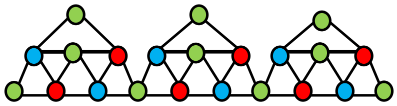
Citations: 1) https://courses.cs.washington.edu/courses/cse431/14sp/scribes/lec15.pdf, Author: James R. Lee, Title: Lecture 15, Date: May 20, 2014

2) http://math.stackexchange.com/questions/125136/how-is-the-graph-coloring-problem-np-complete, Author: Slaviks, Title: How is graph coloring problem NP-Complete?, Date: Mar. 27, 2012.

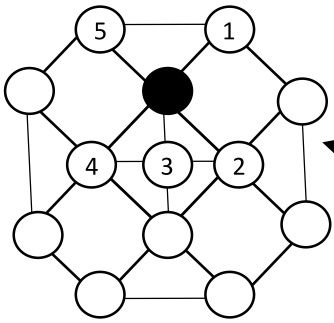
3) http://www.cs.bme.hu/~dmarx/papers/marx-focs2013-workshop-planar-talk.pdf, Author: Daniel Marx, Title: The square root phenomenon in planar graphs, Date: Oct. 26, 2013.

4) Chapter 34, Authors: CLRS, Title: Intro to Alg. 3E, Date: 2009

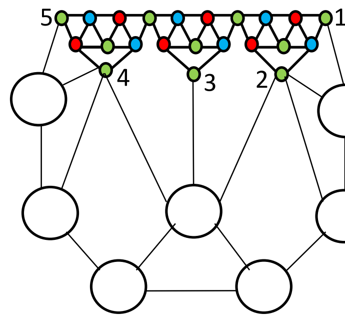
Problem 2)



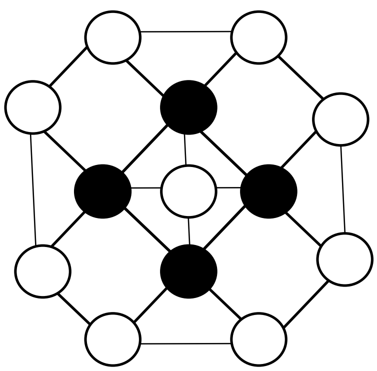
This figure shows that the widget given on the homework is 3-colorable.

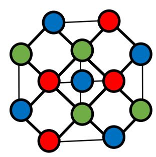


This figure shows how we replace a 5-degree vertex (colored black) with our widget.

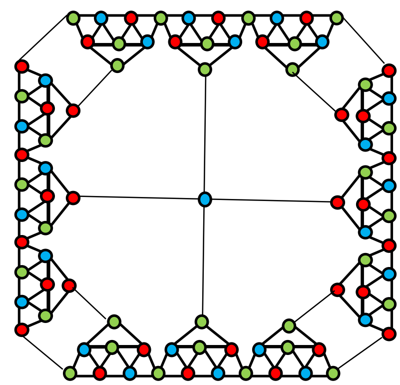


This transformation still has three 5-degree vertices that still need to be replaced.





This figure shows how we replace all four 5-egree vertices (colored black) with the widget. The result is still 3-colorable!



In the above figure vertices with degree of 5 are colored black to the left. Since there are four 5-degree vertices we will need to replace each of them with the provided widget so that no vertex in the graph has more than degree of 4. The resulting graph is shown to the right with 3-COLOR.

This reduction from a 5-degree vertex to a 4-degree vertex takes only polynomial time. Also taking the 4-degree widgets out will give us the same 5-degree widget again. Thus, since 3-coloring a 5-degree graph is an instance of 3-coloring a 4-degree graph then we can say that 3-coloring a graph with vertices having at most degree of 4 is NP-complete.

Citations: 1) https://courses.cs.washington.edu/courses/cse431/14sp/scribes/lec15.pdf, Author: James R. Lee, Title: Lecture 15, Date: May 20, 2014

2) Chapter 34, Authors: CLRS, Title: Intro to Alg. 3E, Date: 2009

3) page 4, Title: graph coloring slides, Author: Prof. Reingold, Date: Apr. 18, 2016