Illinois Institute of Technology
Department of Computer Science

Solutions to Homework Assignment 8

CS 430 Introduction to Algorithms
Spring Semester, 2016

1. It is easy to show that the gadget itself is 3-colorable as Fig. 1. Furthermore, this gadget has a symmetric

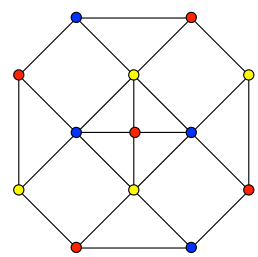

Figure 1: 3-colorable gadget

property, which is the color of the node at the 11:00 clock postion could be always the same as the node at the 5:00 clock postion, and the color of the node at the 1:00 clock position could be always the same as the node at the 7:00 clock position, and so on. Thus we can always replace a crossed edge (u, v) with u embedded at a corner of the gadget and v connected to the opposite corner with an edge, as Fig. 2, which takes time $O(E^2)$. The resulting graph is planar and the symmetric property of the gadget implies that u and v cannot be assigned the same color. We have shown a reduction of the 3-colorability

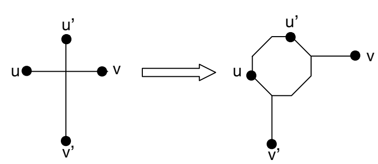

Figure 2: Replacing a cross with the gadget

for an arbitrary graph to the same problem for a planar by using this gadget. Since we can verify a color assignment by checking if the end points of every edge have different colors in $O(E)$ time, 3-coloring a planar graph belongs to the class NP. Because 3-colorable is NP-complete for general graphs, we can prove that 3-colorable is NP-complete for planar graphs.

2. In each trestle-like graph, the degree of "outer" vertices is at most 2, and the degree of "inner" vertices is at most 4. Since the two outer vertices have degree 2, two copies of trestle-like graphs can be fused at those vertices without increasing the degree of any vertex beyond 4. This graph will be 3-colorable if and only if the outer vertices all have the same color, as Figure 3.

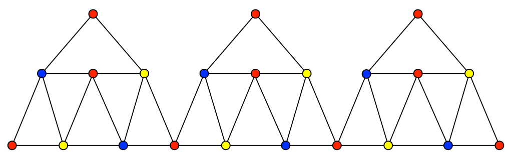

Figure 3: 3-colorable gadget

For a vertex v of degree d, the d edges incident to v can be distributed among the d outer vertices of the graph representing v without changing the colorability properties, as Figure 4.

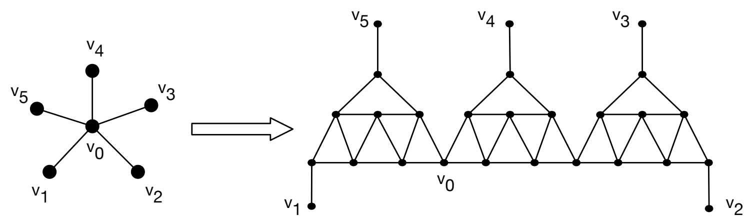

Figure 4: Replacing a vertex of degree 5 with the gadget

In the solution to Question 1, we have shown how to transform any graph to a planner graph, and we can use trestle-like graph to replace any vertex with degree $d > 4$. Therefore, an arbitrary graph can transform to a planner graph with degree at most 4 in $O(E^2 + V)$ time. Since we can verify a color assignment by checking if the end points of every edge have different colors in $O(E)$ time, 3-coloring a planar graph belongs to the class NP. Because 3-colorable is NP-complete for general graphs, we can prove that 3-colorable is NP-complete for planar graphs with degree at most 4.