MM3110 Assignment 2

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1 Problem 1

The 2×2 aluminium slab can be represented as a 4×4 grid where each square of the grid has a dimension of 0.5 cm × 0.5 cm. Each grid point represents the temperature at the respective coordinates.

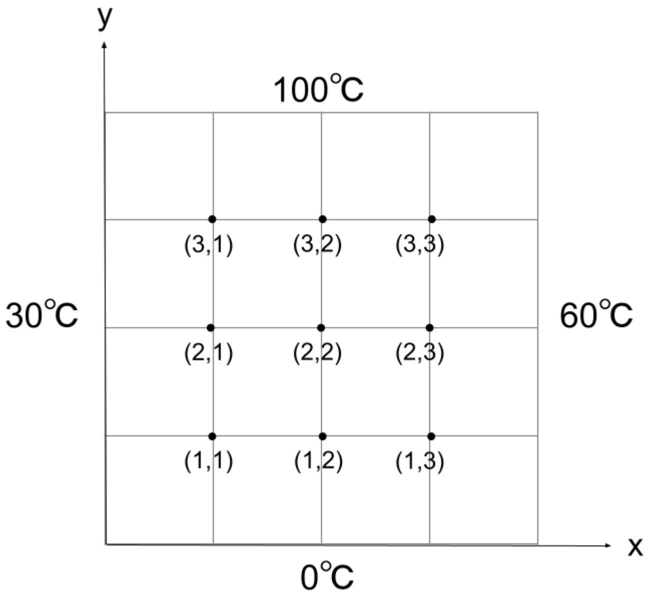


Figure 1: A 2×2 aluminium slab represented as a 4×4 grid. Each square of the grid has a dimension of 0.5 cm × 0.5 cm

Since the slab is at steady state and has a square geometry, we can use the **Laplace difference equation** to determine the temperature $T_{(i,j)}$ at the (i,j) in the grid.

$$T_{(i+1,j)} + T_{(i-1,j)} + T_{(i,j+1)} + T_{(i,j-1)} - 4T_{(i,j)} = 0 \tag{1}$$

The equations obtained using the Laplace difference equation to determine the unknown temperatures within the slab are :

$$T_{(2,1)} + T_{(1,2)} - 4T_{(1,1)} + 30 = 0 \tag{2}$$

$$T_{(2,2)} + T_{(1,1)} + T_{(3,1)} - 4T_{(2,1)} = 0 \tag{3}$$

$$T_{(3,2)} + T_{(2,1)} - 4T_{(3,1)} + 60 = 0 \tag{4}$$

$$T_{(1,3)} + T_{(1,1)} + T_{(2,2)} - 4T_{(1,2)} + 30 = 0 \tag{5}$$

$$T_{(3,2)} + T_{(1,2)} + T_{(2,3)} + T_{(2,1)} - 4T_{(2,2)} = 0 \tag{6}$$

$$T_{(2,2)} + T_{(3,1)} + T_{(3,3)} - 4T_{(3,2)} + 60 = 0 \tag{7}$$

$$T_{(2,3)} + T_{(1,2)} - 4T_{(1,3)} + 130 = 0 \tag{8}$$
$$T_{(3,3)} + T_{(1,3)} + T_{(2,2)} - 4T_{(2,3)} + 100 = 0 \tag{9}$$
$$T_{(2,3)} + T_{(3,2)} - 4T_{(3,3)} + 160 = 0 \tag{10}$$

These equations can be simultaneously solved in MATLAB by using the function **solve** in the **Symbolic Math Toolbox**. The solution obtained is :

$$T_{(1,1)} = 24.29°C$$
$$T_{(2,1)} = 26.70°C$$
$$T_{(3,1)} = 35.00°C$$
$$T_{(1,2)} = 40.45°C$$
$$T_{(2,2)} = 47.50°C$$
$$T_{(3,2)} = 53.30°C$$
$$T_{(1,3)} = 60.00°C$$
$$T_{(2,3)} = 69.55°C$$
$$T_{(3,3)} = 70.71°C$$

2 Problem 2

To find the cube root of any positive integer, say n, look for the two perfect cube numbers a^3 & b^3 such that $a^3 \leq n < b^3$. This implies that $\sqrt[3]{n}$ is greater than or equal to a.
Assuming that n is a non-perfect number, we can find $\sqrt[3]{n}$ by

$$\sqrt[3]{n} = a + \frac{n - a^3}{a^2 \times 3} \tag{11}$$

This approach can be implemented in MATLAB by using a for loop to iterate over perfect cubes (I used cubes of the first 100 natural numbers), and an if loop to find a and compute an estimate of $\sqrt[3]{n}$.
The table below verifies my MATLAB code.

Non-perfect cube n	**$\sqrt[3]{n}$ estimated in MATLAB**	**Actual $\sqrt[3]{n}$ value**	**Absolute Error**
765	9.1481	9.1458	0.0023
98924	46.2502	46.2488	0.0014
418657	74.8177	74.8088	0.0089

3 Problem 3

The values of melting point, mass density and specific heat of gold I considered are $1064°C$, $19.32gcm^{-3}$ and $0.129J/g°C$ respectively. I assumed that the initial temperature of the sphere T_0 is 30°.

3.1 Problem 3(a)

Using the inputs given the problem, we can write the following equations:

$$Rate\, of\; Heat\; conduction\; (In) = Q_0 \tag{12}$$
$$Rate\, of\; Heat\; conduction\; (Out) = 0 \tag{13}$$
$$Rate\, of\; heat\; consumption = \rho V C_P \frac{dT}{dt} \tag{14}$$

Writing the energy balance equation

$$Rate\, of\; Heat\; conduction\; (In) - Rate\, of\; Heat\; conduction\; (Out) = Rate\, of\; heat\; consumption \tag{15}$$
$$Q_0 = \rho V C_P \frac{dT}{dt} \tag{16}$$
$$Q_0 dt = \rho V C_P dT \tag{17}$$

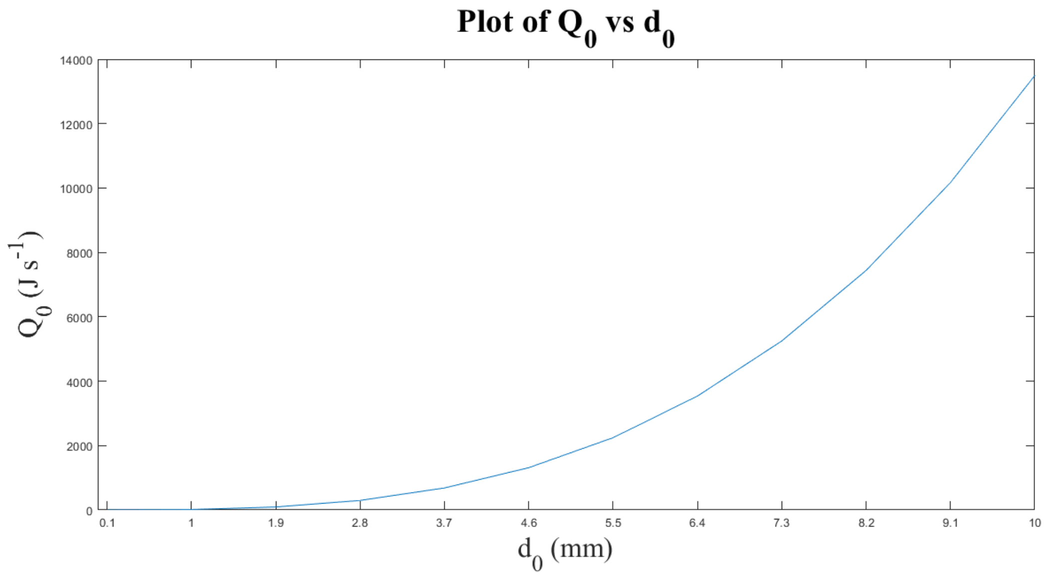
Integrating the RHS from t = 0 to t = t_m = 100ms = 0.1s and the LHS from T = T_0 = 30° to the melting point $T_m = 1064°$

$$\int_0^{t_m} Q_0 \, dt = \int_{T_0}^{T_m} \rho V C_P \, dT \tag{18}$$

On solving the above equation, The final expression of Q_0 in terms of the diameter of the sphere d_0 is

$$Q_0 = \frac{4}{3}\pi(\frac{d_0}{2})^3\rho C_P \frac{(T_0 - T_m)}{t_m} \tag{19}$$

A plot of Q_0 vs d_0, for d_0 in the range of 0.1 - 10 mm, is given below.



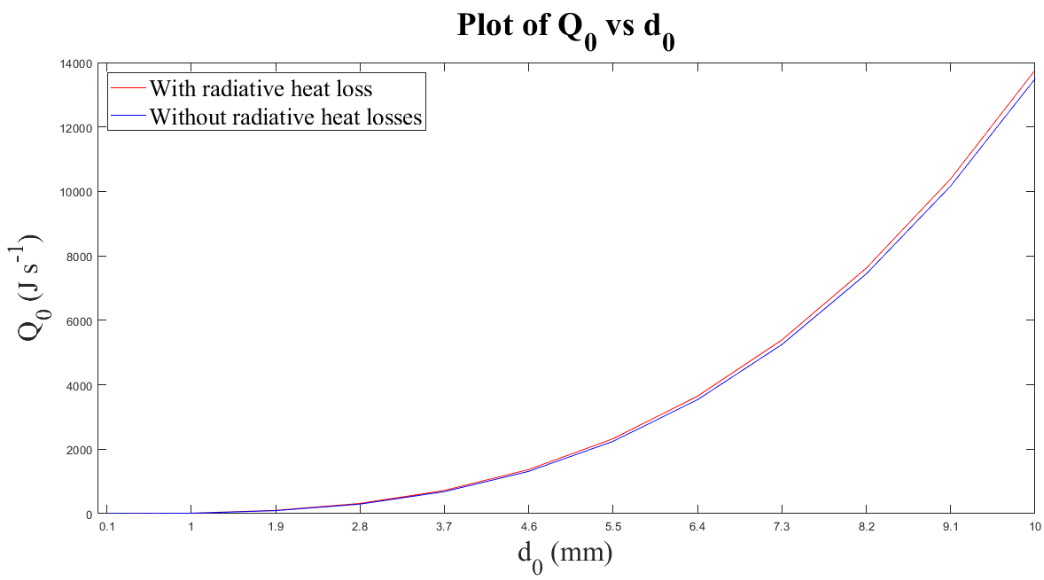
The plot is consistent with the cubic dependence on d_0.

3.2 Problem 3(b)

To account for radiative heat losses, we use the **Stefan-Boltzmann Law**. Using approximations to make calculations simpler, equation (19) now becomes

$$Q_0 t_m - ((T_m + 273)^4 - (T_0 + 273)^4)e\sigma A = \rho C_P V(T_m - T_0) \tag{20}$$

In the above equation, e is the emissivity of the gold sphere, which I take as 0.47, σ is the Stefan-Boltzmann constant and A is the surface area of the sphere.
The new plot of Q_0 vs d_0 is given below along with the plot obtained in part (a).



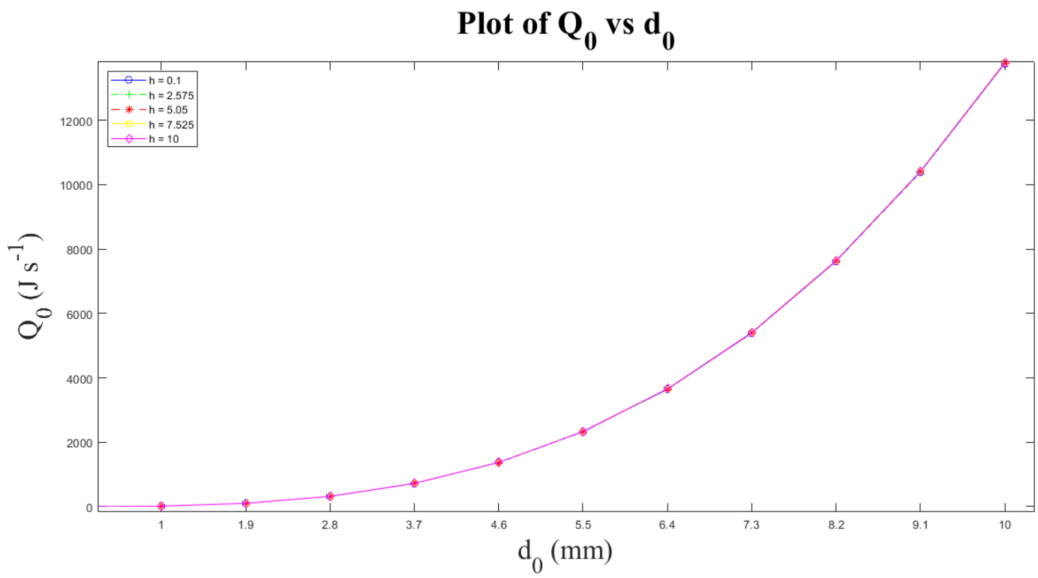
It can be observed that the new plot has higher values of Q_0, which is consistent with the fact that for the sphere to melt in the same time, the input heat now required would be greater given that radiative heat losses are occurring.

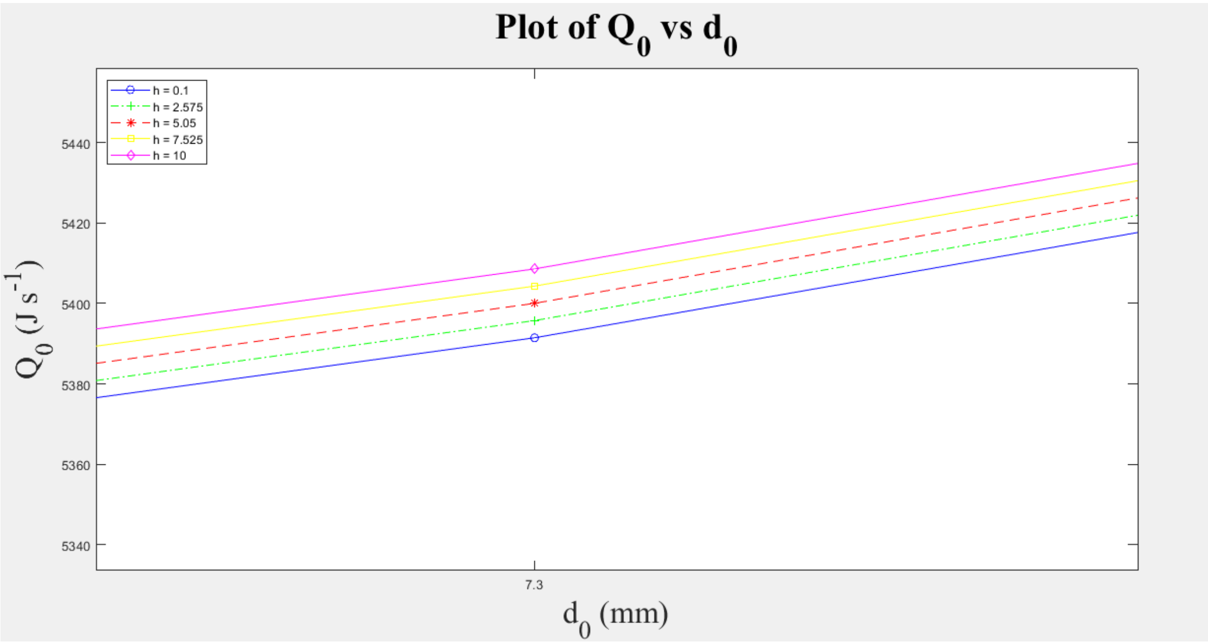
3.3 Problem 3(c)

To account for convective heat losses in addition to our previous conditions, the above equation is modified to

$$Q_0 t_m - ((T_m + 273)^4 - (T_0 + 273)^4)e\sigma A - h(T_m - T_0) = \rho C_P V (T_m - T_0) \tag{21}$$

where h is the heat transfer coefficient.
The plot of Q_0 vs d_0 obtained for h = 0.1, 2.575, 5.05, 7.525 and 10 $Wm^{-1}K^{-1}$ is given below. The magnification of this plot around d_0 = 7.3 mm can be found below it.

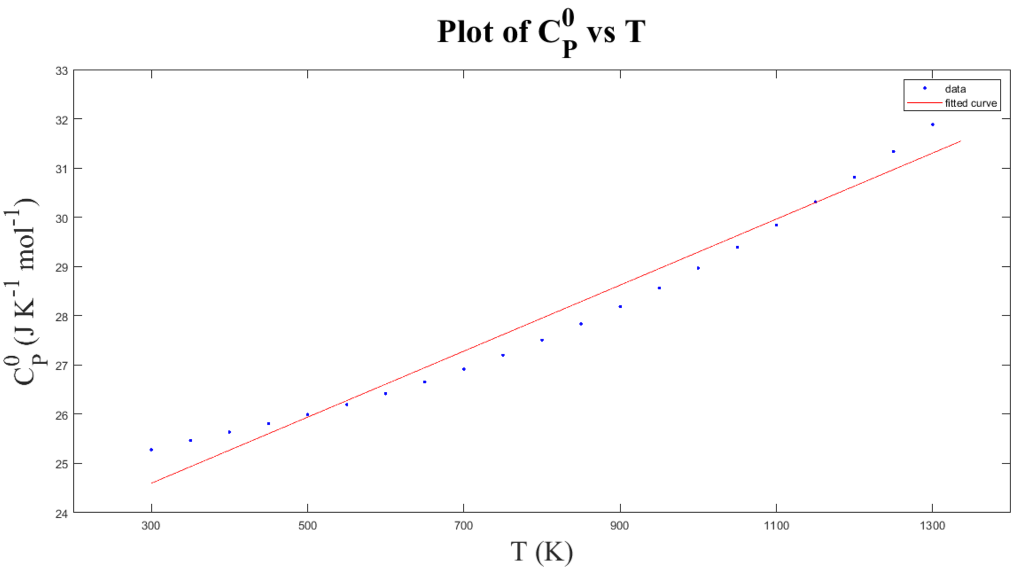




It can be observed that as the heat transfer coefficient increases Q_0 increases, which is consistent with the fact that for the sphere to melt in the same time, the input heat required would be greater given that convective heat losses increase with increasing heat transfer coefficient.

3.4 Problem 3(d)

Using MATLAB's **fit** function to fit the data given in Table 18 in J.W. Arblaster, Thermodynamic Properties of Gold, J. Phase Equi. Diff. 37, 229 (2016), I got the following linear fit of C_P^0 vs T



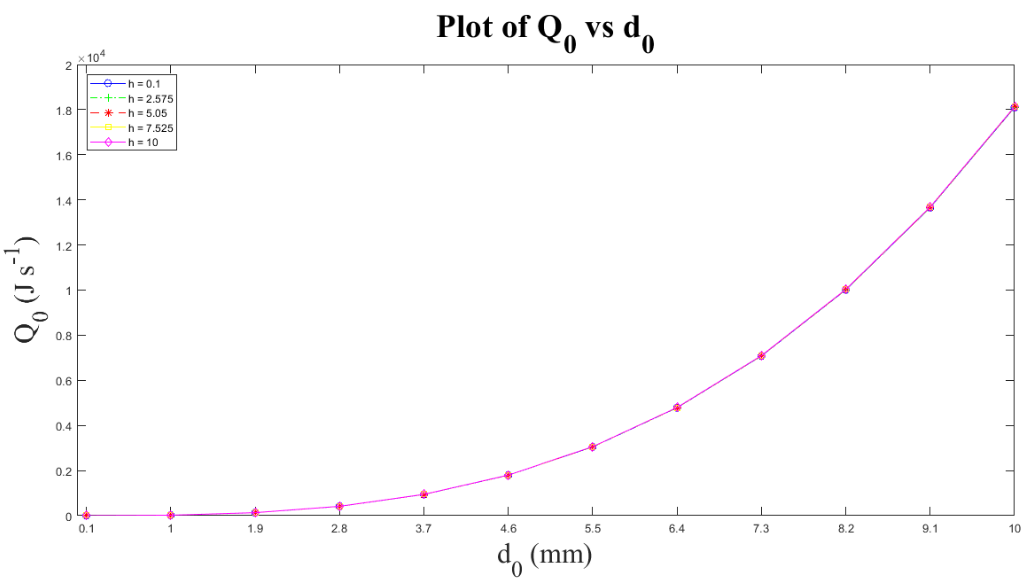
The values of a and b obtained are 22.59 and 6.705×10^{-3} respectively. Therefore, the following linear relationship for C_P^0 obtained is

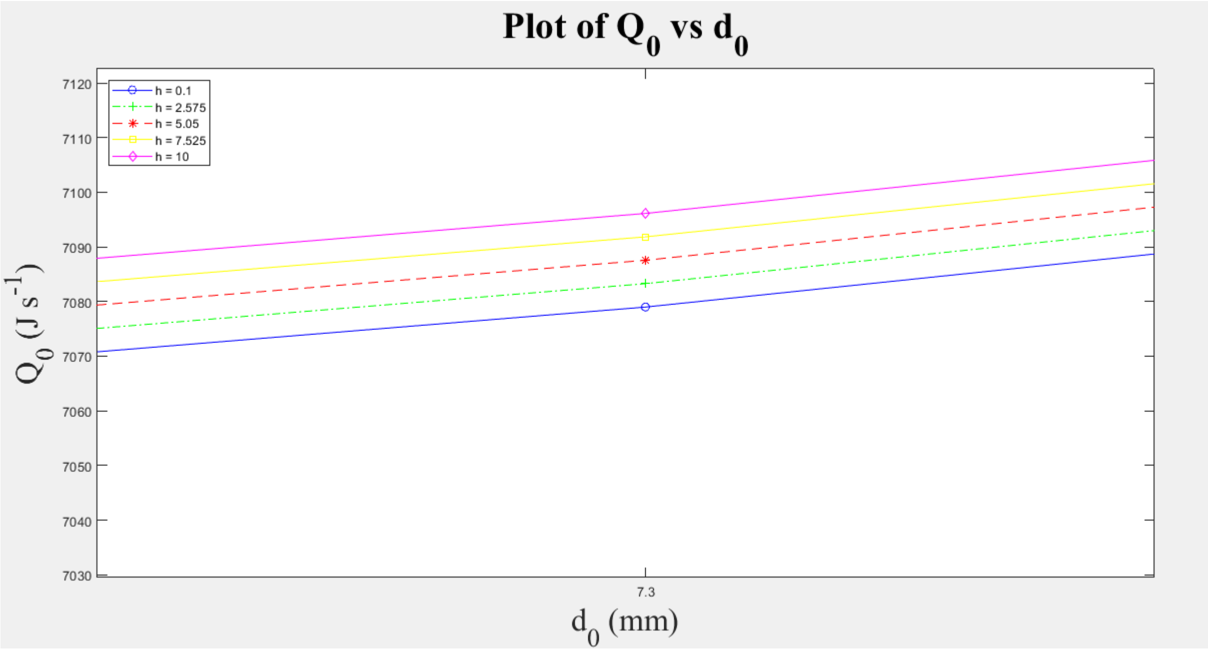
$$C_P^0 = 22.59 + 6.705 \times 10^{-3} T \, JK^{-1} mol^{-1} \tag{22}$$

On repeating part (c) using the above relation, equation (21) changes to

$$Q_0 t_m - ((T_m + 273)^4 - (T_0 + 273)^4) e\sigma A - h(T_m - T_0) = \rho V \frac{((C_P^0|_{1337} 1337) - (C_P^0|_{303} 303))}{197} \tag{23}$$

and the following plot is obtained. The magnification of this plot around $d_0 = 7.3$ mm can be found below it.





It can be observed that the values of Q_0 obtained by using the above linear relationship of C_P^0 and T are higher than those obtained when C_P^0 is assumed to be a constant $(0.129J/g°C)$.