Module 16: Support Vector Machines (SVMs)

Video 2: The Kernel Trick

Equations

In this video, Dr. Gomes explains the mathematics behind two approaches for the introduction of nonlinearity into linear regression. The equations introduced are summarized below.

Equation 1: Derivative of the slope of the loss function

$$\frac{\partial d}{\partial p} = 0 \qquad \boxed{\sum_i (\beta^T \phi(x_i) - y_i)\phi(x_i) + \lambda\beta = 0}$$

Remove the remaining summation from the notation by defining a vector form for y_i and a matrix form for $\phi(x_i)$.

Define Φ as an N by M matrix that holds the features for each of the x_i data points as rows:

$$\Phi = \begin{bmatrix} \phi^T_{(x_i)} \\ \vdots \\ \phi^T_{(x_N)} \end{bmatrix} \in \mathbb{R}^{N \times M}$$

And define a column vector Y to hold the output data:

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

Using these definitions, equation 1 becomes

$$\Phi^T \Phi \beta - \Phi^T y + \lambda \beta = 0$$

And then solve for β:

$$\beta = (\Phi^T \Phi + \lambda I_M)^{-1} \Phi^T Y$$

Once the optimal β is computed, the training data can be discarded, and the M coefficients β_0 through β_{M-1} and use them to make a prediction for a new data point x_n using:

$$y(x_n) = \beta^T \phi(x_n)$$

Equation 2: Definition of the Alphas (α_{i})

Define an alternative set of parameters α_i:

$$\alpha_i := -\frac{1}{\lambda}(y(x_i) - y_i) = \boxed{-\frac{1}{\lambda}(\beta^T \phi(x_i) - y_i)} \quad i = 1 \ldots N$$

Plug this definition into equation 1 and divide by λ to get:

$$\sum_i (-\lambda \alpha_i) \phi(x_i) + \lambda \beta = 0$$

And cancel λ to obtain equation 3.

Equation 3: The expression between optimal Betas and optimal Alphas

$$\beta = \sum_i \alpha_i \phi(x_i) = \boxed{\Phi^T \alpha} \qquad \alpha = \begin{bmatrix} \alpha_i \\ \vdots \\ \alpha_N \end{bmatrix}$$

Equation 4: Definition of the alphas [equation 2] in Matrix form

$$\boxed{-\lambda\alpha = \Phi\beta - Y}$$

Use equation 3 to eliminate β in equation 4:

$$-\lambda\alpha = \Phi\Phi^T\alpha - Y$$

And solve for α:

$$\alpha = (\Phi\Phi^T - \lambda I)^{-1}Y$$