Notes on NTV

Here, we derive an expression for the toroidal torque $\langle \mathbf{e}_\zeta \cdot (\nabla \cdot \mathbf{P}) \rangle$ where we use the the contravariant toroidal basis vector $\mathbf{e}_\zeta = \partial \mathbf{r}/\partial \zeta = J \nabla \Psi \times \nabla \theta$ in Boozer coordinates, which are the coordinates used in SFINCS. Note that $\mathbf{e}_\zeta$ depends on the choice of coordinates, i.e., it is for instance different in Hamada coordinates, which are often used in NTV calculations (Shaing & Callen 1983).

Following Lewandowski et al. (2001), we start by expressing the pressure tensor $\mathbf{P}$ as

$$\mathbf{P} = p_\parallel \mathbf{bb} + p_\perp (\mathbf{I} - \mathbf{bb}), \tag{1}$$

where $\mathbf{I}$ is the unit tensor and $\mathbf{b} = \mathbf{B}/B$. We define $\tilde{p} \equiv (p_\parallel - p_\perp)/B^2$, so that

$$\mathbf{P} = \tilde{p}\mathbf{BB} + p_\perp \mathbf{I}, \tag{2}$$

and the divergence of the pressure tensor becomes

$$\begin{aligned} \nabla \cdot \mathbf{P} &= \mathbf{B}(\mathbf{B} \cdot \nabla \tilde{p}) + \tilde{p} \nabla \cdot (\mathbf{BB}) + \nabla p_\perp = \\ &= \mathbf{B}(\mathbf{B} \cdot \nabla \tilde{p}) + \tilde{p} \left(\frac{1}{2} \nabla B^2 - \mathbf{B} \times \nabla \times \mathbf{B} \right) + \nabla p_\perp = \\ &= \mathbf{B}(\mathbf{B} \cdot \nabla \tilde{p}) + \tilde{p} \left(\frac{1}{2} \nabla B^2 + \mu_0 \nabla p_0 \right) + \nabla p_\perp \end{aligned} \tag{3}$$

In Boozer coordinates $\mathbf{B} = G(\psi)\nabla\zeta + I(\psi)\nabla\theta + \beta(\psi,\theta,\zeta)\nabla\psi$, and the Jacobian is $J = (G + \iota I)/B^2$. The scalar product of the divergence of the pressure tensor with $\mathbf{e}_\zeta$ becomes

$$\mathbf{e}_\zeta \cdot \nabla \cdot \mathbf{P} = G\mathbf{B} \cdot \nabla \tilde{p} + \frac{\tilde{p}}{2} \frac{\partial B^2}{\partial \zeta} + \frac{\partial p_\perp}{\partial \zeta}. \tag{4}$$

Note that $\langle \mathbf{B} \cdot \nabla F \rangle = 0$ for any function F, and that $B^{-2} \partial B^2 / \partial \zeta = -J^{-1} \partial J / \partial \zeta$ so that

$$\left\langle \tilde{p} \frac{\partial B^2}{\partial \zeta} \right\rangle = \int \tilde{p} \frac{\partial B^2}{\partial \zeta} J d\theta d\zeta = - \int (p_\parallel - p_\perp) \frac{\partial J}{\partial \zeta} d\theta d\zeta = \left\langle \frac{\partial}{\partial \zeta} (p_\parallel - p_\perp) \right\rangle. \tag{5}$$

We can conclude that

$$\begin{aligned} &\langle \mathbf{e}_\zeta \cdot (\nabla \cdot \mathbf{P}) \rangle = \frac{1}{2} \left\langle \frac{\partial}{\partial \zeta} (p_\parallel + p_\perp) \right\rangle = \frac{1}{2} \int \frac{\partial}{\partial \zeta} (p_\parallel + p_\perp) J d\zeta d\theta = \\ &= -\frac{1}{2} \int (p_\parallel + p_\perp) \frac{\partial \ln J}{\partial \zeta} J d\zeta d\theta = \int (p_\parallel + p_\perp) \frac{\partial \ln B}{\partial \zeta} J d\zeta d\theta = \left\langle \frac{p_\parallel + p_\perp}{B} \frac{\partial B}{\partial \zeta} \right\rangle. \end{aligned} \tag{6}$$

In the normalisations used in the SFINCS single species documentation,

$$p_\parallel + p_\perp = m\int d^3v\ f\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right) = \frac{2\Delta\hat{T}^{3/2}n}{\sqrt{\pi}\hat{\psi}_a}\int_{-1}^{1}d\xi\int_0^\infty dx\ x^2\hat{f}m\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right) =$$
$$= \left\{m\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right) = \hat{T}\bar{T}x^2(1+\xi^2)\right\} = \bar{T}n\frac{2\Delta\hat{T}^{5/2}}{\sqrt{\pi}\hat{\psi}_a}\int_{-1}^{1}d\xi\int_0^\infty dx\ x^4(1+\xi^2)\hat{f}. \tag{7}$$

Note the following about the Legendre polynomials P_l,

$$1+\xi^2 = \frac{4}{3}P_0(\xi) + \frac{2}{3}P_2(\xi) \tag{8}$$
$$\int_{-1}^{1}d\xi P_0^2(\xi) = 2 \tag{9}$$
$$\int_{-1}^{1}d\xi P_2^2(\xi) = \frac{2}{5}, \tag{10}$$

so that with

$$\hat{f} = \sum_{l=0}^{\infty} f_l P_l(\xi) \tag{11}$$

we obtain

$$\int_{-1}^{1}d\xi(1+\xi^2)\hat{f} = \frac{8}{3}f_0 + \frac{4}{15}f_2. \tag{12}$$

We can write

$$\langle \mathbf{e}_\zeta \cdot (\nabla\cdot\mathbf{P})\rangle \equiv \frac{2\Delta\bar{T}}{\hat{\psi}_a}\frac{n}{\hat{V}'}\cdot \mathrm{NTV}, \tag{13}$$

where the normalised torque NTV is

$$\begin{aligned}\mathrm{NTV} &= \frac{\hat{T}^{5/2}}{\sqrt{\pi}}\int_0^{2\pi}d\zeta\int_0^{2\pi}d\theta\int_{-1}^{1}d\xi\int_0^\infty dx\ x^4(1+\xi^2)\hat{f}\frac{1}{\hat{B}^3}\frac{\partial\hat{B}}{\partial\zeta} = \\ &= \frac{\hat{T}^{5/2}}{\sqrt{\pi}}\int_0^{2\pi}d\zeta\int_0^{2\pi}d\theta\int_0^\infty dx\ x^4\left(\frac{8}{3}f_0 + \frac{4}{15}f_2\right)\frac{1}{\hat{B}^3}\frac{\partial\hat{B}}{\partial\zeta}.\end{aligned} \tag{14}$$

Compare with normalised particle flux, which is a similar quantity, defined in the single species code as

$$\mathrm{particleFlux} = \frac{\hat{\Psi}_a V'}{\Delta^2 n\sqrt{2\bar{T}/m}\bar{R}^2}\left\langle\int d^3v f\mathbf{v}_d\cdot\nabla\Psi\right\rangle =$$
$$-\frac{\hat{T}^{5/2}}{\sqrt{\pi}}\int_0^{2\pi}d\zeta\int_0^{2\pi}d\theta\int_{-1}^{1}d\xi\int_0^\infty dx\ x^4(1+\xi^2)\hat{f}\frac{1}{\hat{B}^3}\left[\hat{G}\frac{\partial\hat{B}}{\partial\theta} - \hat{I}\frac{\partial B}{\partial\zeta}\right] \tag{15}$$

and in the multi species code as

$$\begin{aligned}
\text{particleFlux}_s &= \frac{1}{\bar{n}\bar{v}\bar{R}\bar{B}} \left\langle \int d^3v f_s \mathbf{v}_d \cdot \nabla\Psi \right\rangle \\
&= -\frac{\pi\tilde{\Delta}\hat{T}_s^{\ 5/2}}{Z_s \hat{m}_s \hat{V}'} \frac{1}{\hat{G}+\iota\hat{I}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4(1+\xi^2)\hat{f}_s \frac{1}{\hat{B}^3} \left[\hat{G}\frac{\partial\hat{B}}{\partial\theta} - \hat{I}\frac{\partial B}{\partial\zeta}\right] \qquad (16)
\end{aligned}$$

where the definitions of $\hat{f}$ and Δ in the two codes differ in the following way,

$$\hat{f}_s = \frac{\tilde{\Delta}\hat{m}^2\hat{n}}{Z\pi^{3/2}\hat{\psi}_a}\hat{f} = \frac{\bar{v}^3}{\bar{n}} f, \qquad (17)$$

$$\tilde{\Delta} = \frac{Z_s}{\sqrt{\hat{m}}}\Delta = \bar{m}\bar{v}/(e\bar{B}\bar{R}). \qquad (18)$$

For the NTV torque, we may define

$$\langle \mathbf{e}_\zeta \cdot (\nabla \cdot \mathbf{P}_s)\rangle \equiv \bar{T}\bar{n} \cdot \text{NTV}_s. \qquad (19)$$

Comparing with the previous definition (13) we have

$$\begin{aligned}
\text{NTV}_s &= \frac{2\tilde{\Delta}\sqrt{\hat{m}_s}\hat{n}}{Z_s\hat{\psi}_a\hat{V}'} \cdot \text{NTV} = \\
&= \frac{2\tilde{\Delta}\sqrt{\hat{m}_s}\hat{n}}{Z_s\hat{\psi}_a\hat{V}'} \frac{\hat{T}_s^{5/2}}{\sqrt{\pi}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4(1+\xi^2)\hat{f}\frac{1}{\hat{B}^3}\frac{\partial\hat{B}}{\partial\zeta} = \\
&= \frac{2\pi\hat{T}_s^{5/2}}{\hat{m}_s^{3/2}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4(1+\xi^2)\hat{f}_s\frac{1}{\hat{B}^3}\frac{\partial\hat{B}}{\partial\zeta} \qquad (20)
\end{aligned}$$

REFERENCES

Lewandowski, J. L. V., Williams, J., Boozer, A. H., & Lin, Z. 2001, Phys. Plasmas, 8, 2849

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