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Matt Landreman

Transport matrix in SFINCS and Beidler et al (2011)

In these notes we first derive the relationships between the transport matrix elements in the SFINCS single-species documentation and the matrix elements defined in eq (4) of Beidler et al NF (2011). Next, we show that in axisymmetry and the limit of high collisionality, the transport matrices given in the two references agree. Since Beidler uses SI units whereas the SFINCS documentation uses Gaussian units, we replace $B \to Bc$ everywhere in the SFINCS formulae to convert them to SI, which means we also replace $G \to Gc$, $I \to Ic$, and $\psi \to \psi c$.

Relating sfincs transport matrix elements to Beidler's notation

Examining (4) in Beidler and the equations that follow it, we can schematically write each element of the L transport matrix by setting 2 of the 3 thermodynamic forces to 0, giving

$$L_{11}^{B} = -\frac{1}{n}\frac{\left\langle \Gamma \cdot \nabla \psi \right\rangle \frac{dr}{d\psi}}{\left(\frac{1}{n}\frac{dn}{d\psi} + \frac{Ze}{T}\frac{d\Phi}{d\psi} - \frac{3}{2T}\frac{dT}{d\psi} \right)\frac{d\psi}{dr}}, \tag{1}$$

$$L_{12}^{B} = -\frac{1}{n}\frac{\left\langle \Gamma \cdot \nabla \psi \right\rangle \frac{dr}{d\psi}}{\frac{1}{T}\frac{dT}{d\psi}\frac{d\psi}{dr}}, \tag{2}$$

$$L_{13}^{B} = \frac{1}{n}\frac{\left\langle \Gamma \cdot \nabla \psi \right\rangle \frac{dr}{d\psi}}{\frac{ZeB_0}{T}\frac{\left\langle E_{\parallel} B \right\rangle}{\left\langle B^2 \right\rangle}}, \tag{3}$$

$$L_{21}^{B} = -\frac{1}{n}\frac{\left\langle \frac{1}{T}\mathbf{q} \cdot \nabla \psi \right\rangle \frac{dr}{d\psi}}{\left(\frac{1}{n}\frac{dn}{d\psi} + \frac{Ze}{T}\frac{d\Phi}{d\psi} - \frac{3}{2T}\frac{dT}{d\psi} \right)\frac{d\psi}{dr}}, \tag{4}$$

$$L_{22}^{B} = -\frac{1}{n}\frac{\left\langle \frac{1}{T}\mathbf{q} \cdot \nabla \psi \right\rangle \frac{dr}{d\psi}}{\frac{1}{T}\frac{dT}{d\psi}\frac{d\psi}{dr}}, \tag{5}$$

$$L_{23}^{B} = \frac{1}{n} \frac{\left\langle \frac{1}{T} \mathbf{q} \cdot \nabla \psi \right\rangle \frac{dr}{d\psi}}{\frac{ZeB_0}{T} \frac{\left\langle E_{\|} B \right\rangle}{\left\langle B^2 \right\rangle}}, \tag{6}$$

$$L_{31}^{B} = -\frac{1}{n} \frac{\frac{n\left\langle V_{\|} B \right\rangle}{B_0}}{\left(\frac{1}{n} \frac{dn}{d\psi} + \frac{Ze}{T} \frac{d\Phi}{d\psi} - \frac{3}{2T} \frac{dT}{d\psi} \right) \frac{d\psi}{dr}}, \tag{7}$$

$$L_{32}^{B} = -\frac{1}{n} \frac{\frac{n\left\langle V_{\|} B \right\rangle}{B_0}}{\frac{1}{T} \frac{dT}{d\psi} \frac{d\psi}{dr}}, \tag{8}$$

and

$$L_{33}^{B} = \frac{\left\langle V_{\|} B \right\rangle T \left\langle B^2 \right\rangle}{eB_0^2 \left\langle E_{\|} B \right\rangle}. \tag{9}$$

The superscript B has been added to L to distinguish Beidler's transport matrix L_{ij}^{B} from the SFINCS transport matrix L_{ij}^{S}. Beidler says "R_0 and B_0 are reference values of the torus major radius and magnetic field strength, respectively." In SFINCS, B_0 is defined specifically as the $(0,0)$ Fourier harmonic of the Boozer spectrum. It seems safe to take Beidler's B_0 to have this same definition.

For comparison, the sfincs definitions are

$$L_{11}^{S} = \frac{\frac{Ze(G+\iota I)}{nTG} \left\langle \Gamma \cdot \nabla \psi \right\rangle}{\frac{GT}{ZeB_0 \upsilon_i} \left(\frac{1}{n} \frac{dn}{d\psi} + \frac{e}{T} \frac{d\Phi}{d\psi} - \frac{3}{2T} \frac{dT}{d\psi} \right)}, \tag{10}$$

$$L_{12}^{S} = \frac{\frac{Ze(G+\iota I)}{nTG} \left\langle \Gamma \cdot \nabla \psi \right\rangle}{\frac{GT}{ZeB_0 \upsilon_i} \frac{1}{T} \frac{dT}{d\psi}}, \tag{11}$$

$$L_{13}^{S} = \frac{\frac{Ze(G+\iota I)}{nTG} \left\langle \Gamma \cdot \nabla \psi \right\rangle}{\frac{Ze}{T} (G+\iota I) \frac{\left\langle E_{\|} B \right\rangle}{\left\langle B^2 \right\rangle}} = \frac{\left\langle B^2 \right\rangle \left\langle \Gamma \cdot \nabla \psi \right\rangle}{nG \left\langle E_{\|} B \right\rangle}, \tag{12}$$

$$L_{21}^{S} = \frac{\dfrac{Ze\left(G+\iota I\right)}{nTG}\left\langle \dfrac{1}{T}\mathbf{q}\cdot\nabla\psi \right\rangle}{\dfrac{GT}{ZeB_0\upsilon_i}\left(\dfrac{1}{n}\dfrac{dn}{d\psi}+\dfrac{e}{T}\dfrac{d\Phi}{d\psi}-\dfrac{3}{2T}\dfrac{dT}{d\psi}\right)}, \tag{13}$$

$$L_{22}^{S} = \frac{\dfrac{Ze\left(G+\iota I\right)}{nTG}\left\langle \dfrac{1}{T}\mathbf{q}\cdot\nabla\psi \right\rangle}{\dfrac{GT}{ZeB_0\upsilon_i}\dfrac{1}{T}\dfrac{dT}{d\psi}}, \tag{14}$$

$$L_{23}^{S} = \frac{\dfrac{Ze\left(G+\iota I\right)}{nTG}\left\langle \dfrac{1}{T}\mathbf{q}\cdot\nabla\psi \right\rangle}{\dfrac{Ze}{T}\left(G+\iota I\right)\dfrac{\left\langle E_\parallel B\right\rangle}{\left\langle B^2\right\rangle}} = \frac{\left\langle B^2\right\rangle\left\langle \dfrac{1}{T}\mathbf{q}\cdot\nabla\psi \right\rangle}{nG\left\langle E_\parallel B\right\rangle}, \tag{15}$$

$$L_{31}^{S} = \frac{\dfrac{1}{\upsilon_i B_0}\left\langle V_\parallel B\right\rangle}{\dfrac{GT}{ZeB_0\upsilon_i}\left(\dfrac{1}{n}\dfrac{dn}{d\psi}+\dfrac{e}{T}\dfrac{d\Phi}{d\psi}-\dfrac{3}{2T}\dfrac{dT}{d\psi}\right)}, \tag{16}$$

$$L_{32}^{S} = \frac{\dfrac{1}{\upsilon_i B_0}\left\langle V_\parallel B\right\rangle}{\dfrac{GT}{ZeB_0\upsilon_i}\dfrac{1}{T}\dfrac{dT}{d\psi}}, \tag{17}$$

and

$$L_{33}^{S} = \frac{\left\langle V_\parallel B\right\rangle}{\upsilon_i B_0}\frac{T\left\langle B^2\right\rangle}{e\left(G+\iota I\right)\left\langle E_\parallel B\right\rangle}. \tag{18}$$

Comparing the 2 sets of definitions, then, we find

$$L_{11}^{B} = -\left(\frac{dr}{d\psi}\right)^2 \frac{T^2G^2}{Z^2e^2B_0\upsilon_i\left(G+\iota I\right)}L_{11}^{S}, \tag{19}$$

$$L_{12}^{B} = -\left(\frac{dr}{d\psi}\right)^2 \frac{T^2G^2}{Z^2e^2B_0\upsilon_i\left(G+\iota I\right)}L_{12}^{S}, \tag{20}$$

$$L_{13}^{B} = \frac{TG}{ZeB_0} \frac{dr}{d\psi} L_{13}^{S}, \tag{21}$$

$$L_{21}^{B} = -\left(\frac{dr}{d\psi}\right)^2 \frac{T^2 G^2}{Z^2 e^2 B_0 \upsilon_i \left(G + \iota I\right)} L_{21}^{S}, \tag{22}$$

$$L_{22}^{B} = -\left(\frac{dr}{d\psi}\right)^2 \frac{T^2 G^2}{Z^2 e^2 B_0 \upsilon_i \left(G + \iota I\right)} L_{22}^{S}, \tag{23}$$

$$L_{23}^{B} = \frac{TG}{ZeB_0} \frac{dr}{d\psi} L_{23}^{S}, \tag{24}$$

$$L_{31}^{B} = -\frac{TG}{ZeB_0} \frac{dr}{d\psi} L_{31}^{S}, \tag{25}$$

$$L_{32}^{B} = -\frac{TG}{ZeB_0} \frac{dr}{d\psi} L_{32}^{S}, \tag{26}$$

$$L_{33}^{B} = \frac{\upsilon_i \left(G + \iota I\right)}{B_0} L_{33}^{S}. \tag{27}$$

The relations (19)-(27) may be useful for relating the transport matrix output by SFINCS to DKES or other codes. Notice (20) and (22) have the same sign, whereas (24) and (26) have opposite sign, as do (21) and (25).

Beidler's energy-integrated matrix elements for axisymmetry

From p13 of Beidler, at large collisionality the monoenergetic coefficients are

$$D_{11} = \frac{\pi}{4} \frac{\upsilon_d^2 R_0}{\upsilon \iota} \frac{32}{3\pi} \nu_*, \tag{28}$$

$$D_{31} = 0, \tag{29}$$

and

$$D_{33} = \frac{\upsilon^2}{3\nu} \frac{\left\langle B^2 \right\rangle}{B_0^2}, \tag{30}$$

where

$$\nu_* = \frac{R_0 \nu}{\iota \upsilon}, \tag{31}$$

$$\upsilon_D = \frac{m\upsilon^2}{2ZeR_0 B_0}, \tag{32}$$

$$\nu = \nu_D \hat{\nu}(x) = \frac{3\sqrt{\pi}}{4} \nu_{ii} \hat{\nu}(x), \tag{33}$$

$$\hat{\nu}(x) = \frac{1}{x^3}\left[erf\,(x)\left(1 - \frac{1}{2x^2}\right) + \frac{1}{x\sqrt{\pi}} e^{-x^2} \right], \tag{34}$$

$$\nu_D = \frac{\sqrt{2}\pi n Z^4 e^4 \ln\Lambda}{(4\pi\varepsilon_0)^2 m^{1/2} T^{3/2}} = \frac{3\sqrt{\pi}}{4}\nu_{ii}, \tag{35}$$

$$\nu_{ii} = \frac{4\sqrt{2\pi} n Z^4 e^4 \ln\Lambda}{(4\pi\varepsilon_0)^2 3 m^{1/2} T^{3/2}}. \tag{36}$$

Here, $x = \upsilon / \upsilon_i$ where $\upsilon_i = \sqrt{2T/m}$. This definition of the pitch-angle scattering frequency agrees with (3.45) in Per's textbook.

On p3, Beidler says the energy-integrated transport matrix is related to the monoenergetic coefficients by

$$L_{ij}^B = \frac{2}{\sqrt{\pi}} \int_0^\infty dK \sqrt{K} e^{-K} D_{ij} h_i h_j \tag{37}$$

where $h_1 = h_3 = 1$ and $h_2 = K$. Switching the integration variable to the speed $x = \sqrt{K}$,

$$L_{ij}^B = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\; x^2 e^{-x^2} D_{ij} h_i h_j. \tag{38}$$

Computing the energy integrals of (28)-(30) gives

$$L_{11}^B = 2\frac{\nu_{ii} m^2 \upsilon_i^2}{Z^2 e^2 B_0^2 \iota^2} \underbrace{\int_0^\infty dx\; x^4 e^{-x^2} \hat{\nu}}_{0.26642} = 2.131 \frac{\nu_{ii} T^2}{Z^2 e^2 B_0^2 \iota^2 \upsilon_i^2}, \tag{39}$$

$$L_{12}^B = L_{21}^B = 2\frac{\nu_{ii} m^2 \upsilon_i^2}{Z^2 e^2 B_0^2 \iota^2} \underbrace{\int_0^\infty dx\; x^6 e^{-x^2} \hat{\nu}}_{0.353553} = 2.828 \frac{\nu_{ii} T^2}{Z^2 e^2 B_0^2 \iota^2 \upsilon_i^2}, \tag{40}$$

$$L_{22}^B = 2\frac{\nu_{ii} m^2 \upsilon_i^2}{Z^2 e^2 B_0^2 \iota^2} \underbrace{\int_0^\infty dx\; x^8 e^{-x^2} \hat{\nu}}_{0.795495} = 6.364 \frac{\nu_{ii} T^2}{Z^2 e^2 B_0^2 \iota^2 \upsilon_i^2}, \tag{41}$$

and

$$L_{33}^B = \upsilon_i^2 \frac{\langle B^2 \rangle}{B_0^2} \frac{4}{3\sqrt{\pi}\nu_D} \underbrace{\int_0^\infty dx\; x^4 e^{-x^2} \frac{1}{\hat{\nu}}}_{3.57747} = 3.57747 \left(\frac{4}{3\sqrt{\pi}}\right)^2 \frac{\upsilon_i^2}{\nu_{ii}} \frac{\langle B^2 \rangle}{B_0^2} = 2.02443 \frac{\upsilon_i^2}{\nu_{ii}} \frac{\langle B^2 \rangle}{B_0^2}, \tag{42}$$

To compare these results to the SFINCS documentation, we must recall in SFINCS that

$$\nu' = \frac{\nu_{ii}(G + \iota I)}{\upsilon_i B_0}. \tag{43}$$

Then using (19)-(27) in these notes, and using (155) in the SFINCS single-species documentation, we find the SFINCS predictions for Beidler's matrix elements are

$$L_{11}^{B} = 2.13135\frac{\varepsilon^2}{\iota^2}\left(\frac{dr}{d\psi}\right)^2 \frac{\nu_{ii} T^2 G^2}{Z^2 e^2 B_0^2 \upsilon_i^2}, \tag{44}$$

$$L_{22}^{B} = 6.363961\frac{\varepsilon^2}{\iota^2}\left(\frac{dr}{d\psi}\right)^2 \frac{\nu_{ii} T^2 G^2}{Z^2 e^2 B_0^2 \upsilon_i^2}, \tag{45}$$

$$L_{33}^{B} = 2.0244\frac{\upsilon_i^2}{\nu_{ii}}. \tag{46}$$

It is now apparent that (44)-(46) equal (39)-(42) if make the following replacements, both of which seem reasonable:

$$G\ dr / d\psi \rightarrow \varepsilon \tag{47}$$

$$\left\langle B^2 \right\rangle / B_0^2 \rightarrow 1. \tag{48}$$

The replacement (47) follows from equating 2 relations for the toroidal flux: $2\pi\psi \approx \pi r^2 B$, noting $G \approx RB$. Both (155) in the SFINCS documentation and Beidler agree that L_{13}, L_{23}, L_{31}, and L_{32} are nearly 0. Thus, the SFINCS documentation and Beidler agree on all the ion transport matrix elements in axisymmetry at high collisionality.