

## NTV

Consider the momentum equation summed over species,

$$\mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P} = \nabla \cdot (\rho \mathbf{V} \mathbf{V}) + \frac{\partial(\rho \mathbf{V})}{\partial t}. \quad (1)$$

We write

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \quad (2)$$

$$\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_1, \quad (3)$$

$$\mathbf{P} = p_0(\psi) \mathbf{I} + p_1(\psi) \mathbf{I} + \mathbf{\Pi}, \quad (4)$$

where

$$\mathbf{J}_0 \times \mathbf{B}_0 = p'_0(\psi) \nabla \psi. \quad (5)$$

Subtracting Eq. (5) from Eq. (1), we obtain

$$\frac{\partial \rho \mathbf{V}}{\partial t} = \mathbf{J}_1 \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}_1 - \nabla p_1 - \nabla \cdot (\rho \mathbf{V} \mathbf{V} + \mathbf{M} + \mathbf{\Pi}). \quad (6)$$

where we have used the vector identity  $\nabla B_1^2 = 2\mathbf{B}_1 \cdot \nabla \mathbf{B}_1 + 2\mathbf{B}_1 \times \nabla \times \mathbf{B}_1$  to write  $\mathbf{J}_1 \times \mathbf{B}_1 = -\nabla \cdot \mathbf{M}$ , with

$$\mathbf{M} = \frac{1}{\mu_0} \left( \frac{1}{2} B_1^2 \mathbf{I} - \mathbf{B}_1 \mathbf{B}_1 \right). \quad (7)$$

The two components of Eq. (6) in the directions along  $\mathbf{B}_0$  and  $\mathbf{J}_0$  are particularly interesting as we shall see, because several terms disappear upon taking the flux surface average. Observe that

$$\langle \mathbf{B}_0 \cdot \mathbf{J}_0 \times \mathbf{B}_1 \rangle = \langle \mathbf{B}_1 \cdot \nabla p_0 \rangle = -\langle (\nabla \times \mathbf{A}_1) \cdot \nabla p_0 \rangle = \langle \nabla \cdot (\nabla p_0 \times \mathbf{A}_1) \rangle = 0, \quad (8)$$

$$\langle \mathbf{J}_0 \cdot \mathbf{J}_1 \times \mathbf{B}_0 \rangle = -\langle \mathbf{J}_1 \cdot \nabla p_0 \rangle = \frac{1}{\mu_0} \langle (\nabla \times \mathbf{B}_1) \cdot \nabla p_0 \rangle = -\langle \nabla \cdot (\nabla p_0 \times \mathbf{B}_1) \rangle = 0, \quad (9)$$

$$\langle \mathbf{B}_0 \cdot \nabla p_1 \rangle = \langle \nabla \cdot (\mathbf{B}_0 p_1) \rangle = 0, \quad (10)$$

$$\langle \mathbf{J}_0 \cdot \nabla p_1 \rangle = \langle \nabla \cdot (\mathbf{J}_0 p_1) \rangle = 0 \quad (11)$$

if we neglect the displacement current. The flux surface average of the scalar product of Eq. (6) with  $\mathbf{B}_0$  and  $\mathbf{J}_0$  yields

$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{B}_0 \rangle}{\partial t} = -\langle \nabla \cdot (\rho \mathbf{V} \mathbf{V} + \mathbf{M} + \mathbf{\Pi}) \cdot \mathbf{B}_0 \rangle, \quad (12)$$

$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{J}_0 \rangle}{\partial t} = -\langle \nabla \cdot (\rho \mathbf{V} \mathbf{V} + \mathbf{M} + \mathbf{\Pi}) \cdot \mathbf{J}_0 \rangle. \quad (13)$$

Consequently, the same type of expression also holds for any linear combination  $\mathbf{L} = \alpha \mathbf{B}_0 + \beta \mathbf{J}_0$  of  $\mathbf{B}_0$  and  $\mathbf{J}_0$ ,

$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{L} \rangle}{\partial t} = -\langle \nabla \cdot (\rho \mathbf{V} \mathbf{V} + \mathbf{M} + \mathbf{\Pi}) \cdot \mathbf{L} \rangle \quad (14)$$

## Axisymmetry

In axisymmetry we can express  $\mathbf{B}$  and  $\mathbf{J}$  as

$$\mathbf{B}_0 = F(\psi)\nabla\phi + \iota\nabla\phi \times \nabla\psi, \quad (15)$$

$$\iota\mathbf{J}_0 = -p'_0(\psi) \left( \frac{\partial\mathbf{r}}{\partial\phi} + \Gamma(\psi)\mathbf{B}_0 \right), \quad (16)$$

where  $\phi$  is the geometrical toroidal angle and  $\psi$  the toroidal flux. The flux surface average of the parallel component of Eq. (16) gives us  $\Gamma(\psi) = (\langle \iota\mathbf{J}_0 \cdot \mathbf{B}_0 \rangle / p'_0 + F) / \langle B^2 \rangle$ . We are interested in the toroidal torque on the plasma, so in Eq. (14) we choose

$$\mathbf{L} = R\hat{\phi} = \frac{\partial\mathbf{r}}{\partial\phi} = -\frac{\iota}{p'_0}\mathbf{J}_0 - \Gamma\mathbf{B}_0 \quad (17)$$

## Non-axisymmetry

For an arbitrary 3D magnetic configuration, we would like to define a generalised vector  $\mathbf{L}$ , which in the limit of axisymmetry becomes  $R\hat{\phi}$ . One requirement for  $\mathbf{L}$  to be parallel to  $\hat{\phi}$  in this limit is that the streamlines of  $\mathbf{L}$  close on themselves toroidally. This also makes sense when we are not exactly at axisymmetry because we are not interested in any net poloidal torque component.

In Hamada coordinates  $(V, \vartheta, \varphi)$ , the streamlines of both  $\mathbf{B}_0$  and  $\mathbf{J}_0$  are straight, i.e.  $\mathbf{B}_0$  and  $\mathbf{J}_0$  are linear combinations of  $\partial\mathbf{r}/\partial\vartheta$  and  $\partial\mathbf{r}/\partial\varphi$  and vice versa. The sought linear combination of  $\mathbf{B}_0$  and  $\mathbf{J}_0$  whose streamlines close on themselves toroidally is thus  $\mathbf{e}_\varphi = \partial\mathbf{r}/\partial\varphi$ . Because of the above mentioned reasons,  $\mathbf{e}_\varphi$  becomes parallel to  $\hat{\phi}$  in the limit of axisymmetry (note that  $\nabla\varphi$  does not become parallel to  $\hat{\phi}$ ). Therefore, since  $\nabla \cdot \mathbf{e}_\varphi = 0$  and  $\nabla \cdot R\hat{\phi} = 0$ , we conclude that  $\mathbf{e}_\varphi \rightarrow cR\hat{\phi}$ , where the constant  $c = 1$  because  $\varphi$  and  $\phi$  are both  $2\pi$  periodic.

We now want to determine the constants  $\alpha$  and  $\beta$  in  $\mathbf{L} = \mathbf{e}_\varphi = \alpha\mathbf{B}_0 + \beta\mathbf{J}_0$ . In Hamada coordinates, we can express  $\mathbf{B}_0$  and  $\mathbf{J}_0$  as

$$\mathbf{B}_0 = \nabla\psi \times \nabla\vartheta + \iota(\psi)\nabla\varphi \times \nabla\psi = I(\psi)\nabla\vartheta + G(\psi)\nabla\varphi + \nabla H(\psi, \vartheta, \varphi), \quad (18)$$

$$\mu_0\mathbf{J}_0 = I'(\psi)\nabla\psi \times \nabla\vartheta - G'(\psi)\nabla\varphi \times \nabla\psi. \quad (19)$$

Moreover, the Jacobian is a flux function, so

$$V'(\psi) = \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \frac{1}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} = \frac{4\pi^2}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} \quad (20)$$

$$\mathbf{e}_\varphi = \frac{\nabla\psi \times \nabla\vartheta}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} = \frac{V'}{4\pi^2} \nabla\psi \times \nabla\vartheta, \quad (21)$$

and equilibrium implies

$$\mathbf{J}_0 \times \mathbf{B}_0 = -\frac{1}{\mu_0}(G' + \iota I')(\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi)\nabla\psi = p'_0\nabla\psi \quad (22)$$

$$G' + \iota I' = -\frac{V'}{4\pi^2}\mu_0 p'_0 \quad (23)$$

The equation  $\mathbf{e}_\varphi = \alpha\mathbf{B}_0 + \beta\mathbf{J}_0$  becomes

$$\frac{V'}{4\pi^2}\nabla\psi \times \nabla\vartheta = \alpha(\nabla\psi \times \nabla\vartheta + \iota\nabla\varphi \times \nabla\psi) + \frac{\beta}{\mu_0}(I'\nabla\psi \times \nabla\vartheta - G'\nabla\varphi \times \nabla\psi) \quad (24)$$

yielding

$$\alpha = \frac{V'}{4\pi^2} \frac{G'}{G' + \iota I'} = -\frac{G'}{\mu_0 p'_0}, \quad (25)$$

$$\beta = \frac{V'}{4\pi^2} \frac{\iota}{G' + \iota I'} = -\frac{\iota}{p'_0}, \quad (26)$$

i.e.,

$$\mathbf{e}_\varphi = -\frac{\iota}{p'_0} \mathbf{J}_0 - \frac{G'}{\mu_0 p'} \mathbf{B}_0. \quad (27)$$

Note that in the expressions for  $\alpha$  and  $\beta$ , the flux functions  $G(\psi)$  and  $I(\psi)$  are the same as in Boozer coordinates  $(\psi, \theta, \zeta)$ , where

$$\mathbf{B}_0 = \nabla\psi \times \nabla\theta + \iota \nabla\zeta \times \nabla\psi = I(\psi) \nabla\theta + G(\psi) \nabla\zeta + \kappa(\psi, \theta, \zeta) \nabla\psi. \quad (28)$$

### The toroidal viscosity

We now turn our attention to the toroidal viscosity term in Eq. (14),

$$\langle \mathbf{e}_\varphi \cdot \nabla \cdot \mathbf{\Pi} \rangle = \alpha \langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle + \beta \langle \mathbf{J} \cdot \nabla \cdot \mathbf{\Pi} \rangle. \quad (29)$$

where we have dropped the index 0 on  $\mathbf{B}_0$ ,  $\mathbf{J}_0$  and  $p_0$ . First, we note that  $\mathbf{\Pi} = \tilde{p}(\mathbf{I}/3 - B^{-2} \mathbf{B} \mathbf{B})$ , where  $\tilde{p} \equiv p_\perp - p_\parallel$ , which implies that

$$\nabla \cdot \mathbf{\Pi} = \frac{1}{3} \nabla \tilde{p} - \nabla \frac{\tilde{p}}{B^2} \cdot \mathbf{B} \mathbf{B} - \tilde{p} \left( B^{-1} \nabla B + \frac{\mu_0}{B^2} \nabla p \right) \quad (30)$$

$$\mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} = -\frac{2}{3} \mathbf{B} \cdot \nabla \tilde{p} + \frac{\tilde{p}}{B} \nabla B \quad (31)$$

$$\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi} \rangle = \langle \tilde{p} \nabla_\parallel B \rangle \quad (32)$$

$$\langle \mathbf{J} \cdot \nabla \cdot \mathbf{\Pi} \rangle = -\left\langle J_\parallel \mathbf{B} \mathbf{B} \cdot \nabla \frac{\tilde{p}}{B^2} \right\rangle - \left\langle \frac{\tilde{p}}{B} \mathbf{J} \cdot \nabla B \right\rangle \quad (33)$$

In the last expression, we can replace  $\mathbf{J}$  with  $\mathbf{J} = J_\parallel B^{-1} \mathbf{B} + p' B^{-2} (\mathbf{B} \times \nabla\psi)$  where  $\nabla \cdot \mathbf{J} = 0$  gives  $\mathbf{B} \cdot \nabla (J_\parallel / B) = 2p' B^{-3} \mathbf{B} \times \nabla\psi \cdot \nabla B$ . If we also use that  $\langle a \mathbf{B} \cdot \nabla b \rangle = -\langle b \mathbf{B} \cdot \nabla a \rangle$ , we can write

$$\begin{aligned} \langle \mathbf{J} \cdot \nabla \cdot \mathbf{\Pi} \rangle &= \left\langle \frac{\tilde{p}}{B^2} \mathbf{B} \cdot \nabla (J_\parallel B) \right\rangle - \left\langle \frac{\tilde{p}}{B^2} J_\parallel \mathbf{B} \cdot \nabla B \right\rangle - p' \left\langle \frac{\tilde{p}}{B^3} \mathbf{B} \times \nabla\psi \cdot \nabla B \right\rangle = \\ &= \left\langle \frac{\tilde{p}}{B^2} J_\parallel \mathbf{B} \cdot \nabla B \right\rangle + p' \left\langle \frac{\tilde{p}}{B^3} \mathbf{B} \times \nabla\psi \cdot \nabla B \right\rangle = \\ &= \frac{1}{2} \left\langle \frac{\tilde{p}}{B^2} \mathbf{B} \cdot \nabla (J_\parallel B) \right\rangle. \end{aligned} \quad (34)$$

We obtain the torque

$$\tau \equiv \langle \mathbf{e}_\varphi \cdot \nabla \cdot \mathbf{\Pi} \rangle = -\frac{G'}{\mu_0 p'} \langle \tilde{p} \nabla_\parallel B \rangle - \frac{\iota}{2p'} \left\langle \frac{\tilde{p}}{B} \nabla_\parallel (J_\parallel B) \right\rangle \quad (35)$$

Note that the two terms cancel in axisymmetry, because equating Eqs. (17) and (27), we get  $\Gamma = G' / (\mu_0 p')$ , and the scalar product of Eq. (16) with  $\mathbf{B}$  gives  $-\iota J_\parallel B / p' = F + \Gamma B^2$ .

## Implementation

Define  $\gamma \equiv G' / (\mu_0 p')$  and  $u \equiv \iota J_{\parallel} / (B p')$ . The torque becomes

$$\tau = \langle \mathbf{e}_{\varphi} \cdot \nabla \cdot \mathbf{\Pi} \rangle = -\gamma \left\langle \tilde{p} \nabla_{\parallel} B \right\rangle - \frac{1}{2} \left\langle \frac{\tilde{p}}{B} \nabla_{\parallel} (u B^2) \right\rangle \quad (36)$$

To calculate this, we first need to determine the quantity  $u$  from the equation

$$\mathbf{B} \cdot \nabla u = 2 \iota B^{-3} \mathbf{B} \times \nabla \psi \cdot \nabla B. \quad (37)$$

Henceforth, we employ Boozer coordinates, in which the above equation corresponds to

$$\left( \iota \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \zeta} \right) = 2 \frac{\iota}{B^3} \left( G \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \zeta} \right). \quad (38)$$

In SFINCS normalisation, we have  $G = \hat{G} \bar{R} \bar{B}$ ,  $B = \hat{B} \bar{B}$ ,  $u = \hat{u} \bar{R} / \bar{B}$  and

$$\left( \iota \frac{\partial \hat{u}}{\partial \theta} + \frac{\partial \hat{u}}{\partial \zeta} \right) = 2 \frac{\iota}{\hat{B}^3} \left( \hat{G} \frac{\partial \hat{B}}{\partial \theta} - \hat{I} \frac{\partial \hat{B}}{\partial \zeta} \right). \quad (39)$$

The torque becomes

$$\begin{aligned} \tau &= -\gamma \left\langle \frac{\tilde{p}}{B \sqrt{g}} \left( \iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) \right\rangle - \frac{1}{2} \left\langle \frac{\tilde{p}}{B^2 \sqrt{g}} \left( \iota \frac{\partial (u B^2)}{\partial \theta} + \frac{\partial (u B^2)}{\partial \zeta} \right) \right\rangle = \\ &= -\frac{1}{V'} \int d\theta d\zeta \frac{\tilde{p}}{B^2} \left[ \gamma B \left( \iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) + \frac{1}{2} \left( \iota \frac{\partial (u B^2)}{\partial \theta} + \frac{\partial (u B^2)}{\partial \zeta} \right) \right] \end{aligned} \quad (40)$$

In the normalisations used in the SFINCS single species documentation,

$$\begin{aligned} \tilde{p} &= p_{\perp} - p_{\parallel} = m \int d^3 v f \left( \frac{v_{\perp}^2}{2} - v_{\parallel}^2 \right) = \frac{2 \Delta \hat{T}^{3/2} n}{\sqrt{\pi} \hat{\psi}_a} \int_{-1}^1 d\xi \int_0^{\infty} dx x^2 \hat{f} m \left( \frac{v_{\perp}^2}{2} - v_{\parallel}^2 \right) = \\ &= \left\{ m \left( \frac{v_{\perp}^2}{2} - v_{\parallel}^2 \right) = \hat{T} \bar{T} \frac{x^2}{2} (1 - 3\xi^2) = -\hat{T} \bar{T} x^2 P_2(\xi) \right\} = \\ &= -\bar{T} n \frac{2 \Delta \hat{T}^{5/2}}{\sqrt{\pi} \hat{\psi}_a} \int_{-1}^1 d\xi \int_0^{\infty} dx x^4 P_2(\xi) \hat{f}. \end{aligned} \quad (41)$$

Note the following about the Legendre polynomial  $P_2$ ,

$$\int_{-1}^1 d\xi P_2^2(\xi) = \frac{2}{5}, \quad (42)$$

so that with

$$\hat{f} = \sum_{l=0}^{\infty} f_l P_l(\xi) \quad (43)$$

we obtain

$$\int_{-1}^1 d\xi P_2(\xi) \hat{f} = \frac{2}{5} f_2. \quad (44)$$

If we define  $\gamma = \hat{\gamma} \bar{R} / \bar{B}$  we can write

$$\tau = \bar{T} n \frac{2 \Delta \hat{T}^{5/2} \bar{R}}{V' \sqrt{\pi} \hat{\psi}_a \bar{B}} \int d\theta d\zeta \frac{1}{\hat{B}^2} \left[ \hat{\gamma} \hat{B} \left( \iota \frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta} \right) + \frac{1}{2} \left( \iota \frac{\partial (\hat{u} \hat{B}^2)}{\partial \theta} + \frac{\partial (\hat{u} \hat{B}^2)}{\partial \zeta} \right) \right] \frac{2}{5} \int_0^{\infty} dx x^4 f_2 \quad (45)$$

Define  $\hat{V}' = \int d\theta d\zeta \hat{B}^{-2}$  and note that

$$V' = \int d\theta d\zeta \frac{G + \iota I}{B^2} = \frac{\bar{R}}{\bar{B}} (\hat{G} + \iota \hat{I}) \hat{V}' \quad (46)$$

We define the normalised torque in the single species code

$$\begin{aligned} \hat{\tau} &= \frac{1}{nT} \langle \mathbf{e}_\varphi \cdot \nabla \cdot \mathbf{\Pi} \rangle \frac{\hat{\psi}_a (\hat{G} + \iota \hat{I}) \hat{V}'}{2\Delta} = \\ &= \frac{\hat{T}^{5/2}}{\sqrt{\pi}} \int d\theta d\zeta \frac{1}{\hat{B}^2} \left[ \hat{\gamma} \hat{B} \left( \iota \frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta} \right) + \frac{1}{2} \left( \iota \frac{\partial(\hat{u} \hat{B}^2)}{\partial \theta} + \frac{\partial(\hat{u} \hat{B}^2)}{\partial \zeta} \right) \right] \frac{2}{5} \int_0^\infty dx x^4 f_2 \end{aligned} \quad (47)$$

In the multi-species code some normalisations are different, in particular, the definition of  $\hat{f}$  differs in the following way,

$$\hat{f}^{\text{multi}} = \frac{\bar{v}^3}{\bar{n}} f = \frac{\hat{m}^{3/2} \hat{n}}{\pi^{3/2} \hat{\psi}_a} \hat{f}. \quad (48)$$

A suitable normalised torque in the multiple species code is

$$\begin{aligned} \hat{\tau}^{\text{multi}} &= \frac{1}{\bar{n}T} \langle \mathbf{e}_\varphi \cdot \nabla \cdot \mathbf{\Pi} \rangle = \\ &= \frac{2\pi \hat{T}^{5/2}}{\hat{m}^{3/2} (\hat{G} + \iota \hat{I}) \hat{V}'} \int d\theta d\zeta \frac{1}{\hat{B}^2} \left[ \hat{\gamma} \hat{B} \left( \iota \frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta} \right) + \frac{1}{2} \left( \iota \frac{\partial(\hat{u} \hat{B}^2)}{\partial \theta} + \frac{\partial(\hat{u} \hat{B}^2)}{\partial \zeta} \right) \right] \frac{2}{5} \int_0^\infty dx x^4 f_2^{\text{multi}} \end{aligned} \quad (49)$$