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EQUILIBRIUM AND STABILITY OF LOW-SHEAR STELLARATORS

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To Professor Arnulf Schlüter on his 65th Birthday (≙ 1987)

1. Introduction

Four main types of stellarator configuration are currently being actively studied [1]: i) high-transform, high-shear-stabilized configurations, typified by, for example, Heliotron-E [2], ii) moderate-transform, shear/magnetic-well-stabilized configurations, typified by, for example, L-2 [3] and ATF [4], iii) moderate-transform, low-shear, reduced-parallel-current-density, magnetic-well-stabilized configurations, typified by, for example, W VII-AS [5] and Helias [6], iv) high-transform, low-shear, magnetic-well-stabilized configurations, typified by, for example, Sheila [7] and TJ-II [8]. From the point of view of MHD equilibrium and stability theory i) and ii) as well as iii) and iv) have much in common. The first group is amenable to the stellarator expansion [9] so that a large amount of significant results pertaining to equilibrium and stability properties was obtained within the framework of this expansion and has been extensively reviewed [10, 11]. One of the main problems associated with these types of configuration is nonlinear resistive MHD theory [12, 13], which may hold the key to understanding them. Wakatani's paper in these proceedings addresses such configurations. The second group [iii) and iv)] of configurations has to be treated fully three-dimensionally, although the results for strictly helically symmetric equilibria with spatial magnetic axis (helically symmetric Heliacs) are instructive with respect to the stability properties of these systems. This paper describes results for this second group of configurations.

2. Equilibrium Considerations

Three aspects of the equilibrium properties of stellarators are well described by today's 3D flux-variable based codes [14, 15, 16] supplemented by evaluation of the parallel current density [17]: the Shafranov shift, the change of rotational transform with β, and the pressure-driven parallel current density near rational values of the rotational transform.

The Shafranov shift yields the simple equilibrium-β estimate ι^2/A (ι rotational transform, A aspect ratio). In advanced stellarators this limit is overcome by reduction of the parallel current density; W VII-AS and Helias rely on this principle. In Heliacs the Shafranov shift is made harmless by the large value of ι, typically $1 < \iota < 2$. In Figs. 1 to 3 results pertaining to this aspect of equilibrium are shown for W VII-AS ($\langle\beta\rangle = 0.015$), Helias ($\langle\beta\rangle = 0.15$), and TJ-II ($\langle\beta\rangle = 0.05$). The β-value in W VII-AS is close to the simple estimate ι^2/A; local pressure gradients are different by one order of magnitude. The Helias example shows that configurations can be found in which the Shafranov shift does not impose a practical limit on the equilibrium β-value. While this is also true of the Heliaç example, the symmetry of the magnetic surfaces is significantly violated at high β.

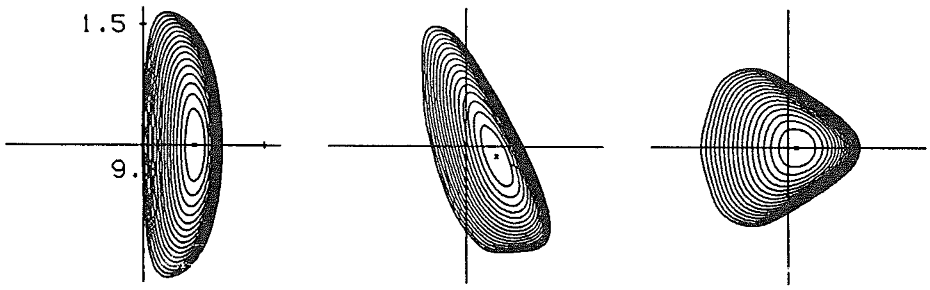

Fig. 1: Flux surface cross-sections at the beginning of, at quarter of, and half a period of a W VII-AS equilibrium at $\langle\beta\rangle = 0.015$ and with an approximately parabolic pressure profile.

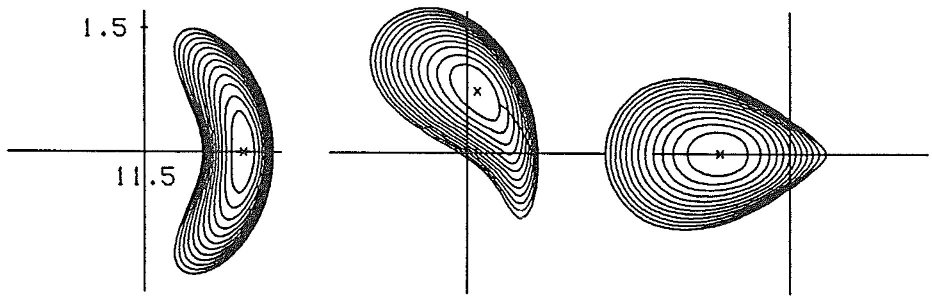

Fig. 2: Flux surface cross-sections of a Helias equilibrium [18] with $N = 5$, $A = 11.5$, $R_{0,1} = 0.8$, $Z_{0,1} = 0.4$, $\Delta_{1,0} = 0.1$, $\Delta_0 = 0.07$, $\Delta_{2,0} = 0.05$, $\Delta_{1,-1} = 0.39$, $\Delta_{2,-1} = 0.24$, $\Delta_{2,-2} = 0.07$; the formula for the Helias boundary shape is given in [6]; $\langle\beta\rangle = 0.15$, parabolic profile.

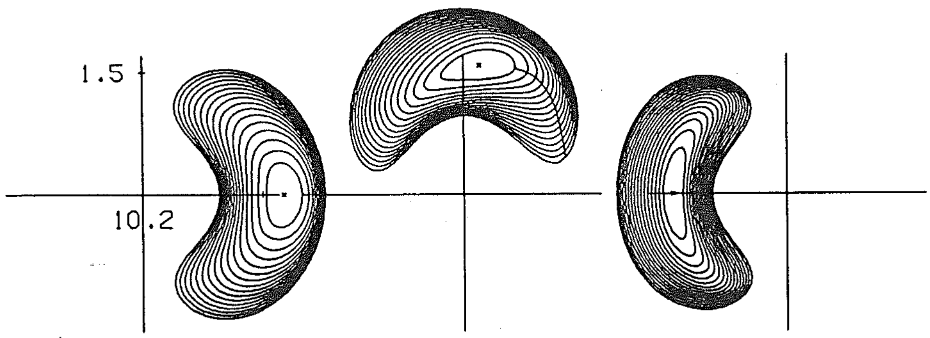

Fig. 3: Flux surface cross-sections of a TJ-II equilibrium at $\langle\beta\rangle = 0.05$; bell-shaped pressure profile [19].

The rotational transform depends on β for genuine stellarator equilibria with identically vanishing net toroidal current. This effect is quite strong in W VII-AS, where a configuration with virtually vanishing shear at $\beta = 0$ changes into one with $[\iota(1) - \iota(0)]/\iota(0) \approx -0.2$; see Fig. 4.

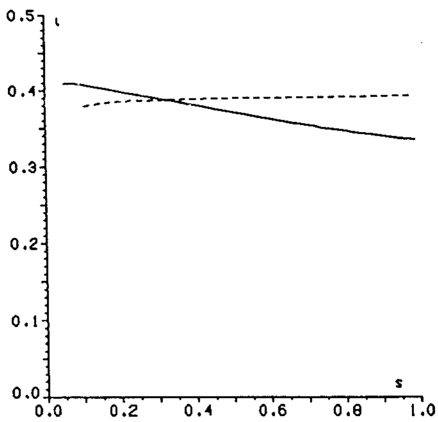

Fig. 4: ι profiles of W VII-AS at $\beta = 0$ (- - -), 0.015 (—); parabolic pressure profile.

In Helias the parallel current density is much more reduced than in W VII-AS so that finite-β effects on the shape of the magnetic surfaces are less pronounced; accordingly, the effect of β on ι is small. Figure 5 shows that the shear is nearly unchanged and that the change in rotational transform is $\lesssim \langle\beta\rangle$.

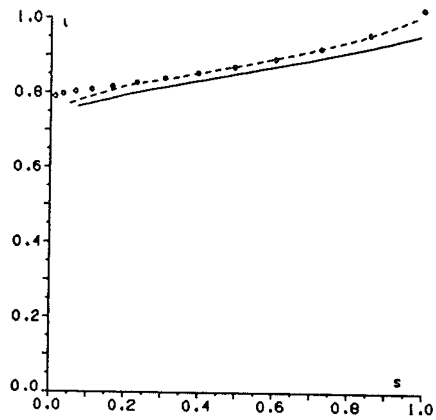

Fig. 5: ι-profiles of the Helias configuration shown in Fig. 2 with $\langle\beta\rangle = 0$ (- - -), 0.09 (—). The circles are results from a NESCOIL run [20].

In the TJ-II example, the situation is comparable. Here, it is of particular importance that the change of ι be small, otherwise low-order resonances are encountered; $\iota = \frac{3}{2}$ in the particular example considered here.

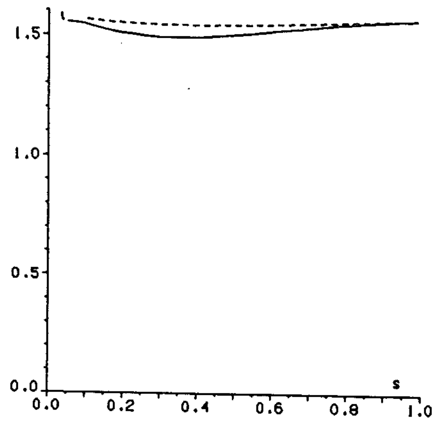

Fig. 6: ι-profiles of the TJ-II configuration shown in Fig. 3 with $\langle\beta\rangle = 0$ (- - -), 0.05 (—).

Near a resonant surface, i.e. a rational value of the rotational transform, a finite pressure gradient drives a diverging parallel current density, which is of particular importance in devices with a sizeable rotational transform per period. By way of example, Figs. 7 and 8 show a small-shear Helias equilibrium [21] in which at $\langle\beta\rangle = 0.03$ the $\iota_p = \frac{1}{6}$ resonance occurs.

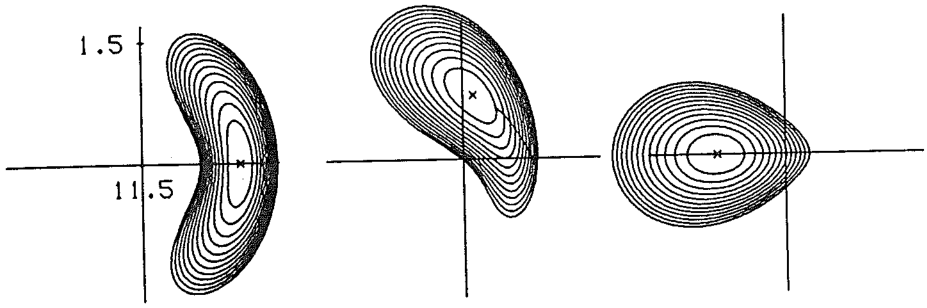

Fig. 7: Flux surfaces of a Helias equilibrium with very small shear; the parameters are $N = 5$, $A = 11.5$, $R_{0,1} = 0.805$, $Z_{0,1} = 0.48$, $\Delta_{1,0} = 0.15$, $\Delta_0 = 0.014$, $\Delta_{1,-1} = 0.355$, $\Delta_{2,0} = 0.1$, $\Delta_{2,-1} = 0.26$, $\Delta_{2,-2} = 0.04$, $\langle\beta\rangle = 0.05$.

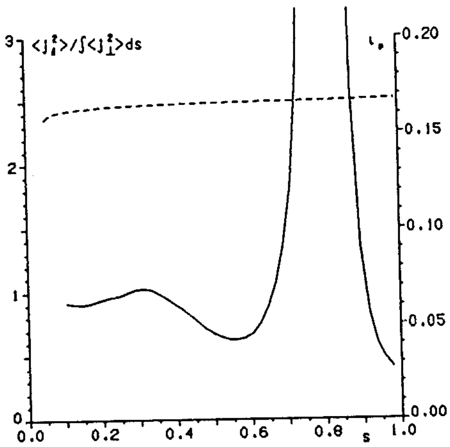

Fig. 8: Normalized measure of parallel current density $< j_\parallel^2 > / \int < j_\perp^2 > ds$ and ι_p (- - -) as functions of s for the equilibrium shown in Fig. 7; $< \ldots >= \int d\theta d\phi \frac{\sqrt{g}}{|\nabla s|^2}(\ldots)$; $\langle\beta\rangle = 0.03$.

Clearly, there are two ways to avoid this intolerable situation. The one is to avoid the occurrence of resonances of this order, as shown in Fig. 9. The TJ-II example at $\langle\beta\rangle = 0.02$ can also be characterized in this way, since $\iota_p = \frac{3}{8}$ and $\iota_p = \frac{2}{5}$ are carefully avoided.

The other possibility is to consider an equilibrium with somewhat stronger shear, e.g. that in Fig. 2, and eliminate the divergence of the parallel current density by pressure profile flattening near the resonance [6, 22]. Figure 10 shows a pressure profile obtained in this way for the configuration of Fig. 2.

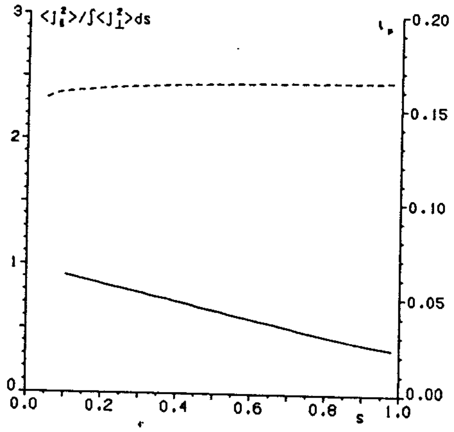

Fig. 9: Same as Fig. 8, but $\langle\beta\rangle = 0.05$.

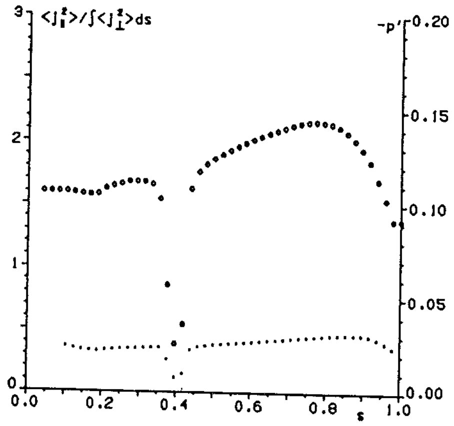

Fig. 10: Profiles of $-\mathrm{d}p/\mathrm{d}s$ ($\circ\ \circ\ \circ$) flattened around $\iota_p = \frac{1}{6}$ for the configuration shown in Fig. 2, $\langle\beta\rangle = 0.09$, and $< j_{\parallel}^2 > / \int < j_{\perp}^2 > \mathrm{d}s$ (...).

A fourth aspect of equilibrium properties of stellarators is the size of the islands occurring at rational ι in finite-β equilibria. It is this that constitutes the main risk in the assessment of equilibrium properties; its analytical and computational treatment is still being developed [23, 24, 25]. However, one has to be aware that in equilibria in which medium-order resonances (e.g. $\iota_p = \frac{1}{6}$) are avoided (see the above examples) a computational treatment of island sizes will be very difficult. Also, the simple MHD picture may be corrected by neoclassical theory [26].

3. Stability Considerations

The MHD stability behaviour of stellarators without net toroidal current through each magnetic surface is completely different from that found in tokamaks.

This can be seen from the following form of the energy principle (for a derivation see Appendix 2) given in Boozer's coordinates [27] with $s \propto F_T$ and two scalar perturbation functions ξ^s and η:

$$\delta^2 W = \frac{1}{2}\int\int\int \sqrt{g}\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi(\vec{C}^2 - A\xi^{s2})$$

$$\vec{C} = C^s\vec{r},_s + C_\perp \frac{\nabla s \times \vec{B}}{|\nabla s| B} + C_\parallel \frac{\vec{B}}{B}$$

$$C^s = \vec{B}\cdot\nabla\xi^s$$

$$C_\perp = -\frac{|\nabla s|}{B\sqrt{g}}[\sqrt{g}\vec{B}\cdot\nabla\eta - F_T' F_P'' \xi^s + \frac{\vec{j}\cdot\vec{B}}{|\nabla s|^2}\sqrt{g}\xi^s]$$

$$C_\parallel = \frac{1}{B\sqrt{g}}(-I\eta,_\theta + IF_T'\xi^s,_s - p'\sqrt{g}\xi^s)$$

$$\sqrt{g}A = \frac{\vec{j}^2\sqrt{g}}{|\nabla s|^2} - F_P''\tilde{\beta},_\theta + \sqrt{g}\vec{B}\cdot\nabla\frac{\tilde{\beta},_\theta g^{\theta s} - (I' - \tilde{\beta},_\phi)g^{\phi s}}{|\nabla s|^2} - p'\sqrt{g},_s$$

Since I', $\tilde{\beta}$, $\vec{j}\cdot\vec{B} \propto p'$, the modes are purely pressure-driven. For small shear and localization of the modes around mode rational surfaces the contribution of $\vec{C}^2$ becomes small [28] so that

$$p'V'' - \int \mathrm{d}\theta\,\mathrm{d}\phi\frac{\vec{j}^2\sqrt{g}}{|\nabla s|^2} > 0$$

becomes a good approximation of a necessary stability criterion. This behaviour leads one to expect that only the resonant modes are dangerous and that the local modes will formally be the most unstable ones. Indeed, detailed mode analyses in helically symmetric equilibria [29] and toroidal $\ell = 2$ stellarators treated by means of the stellarator expansion [30, 31] verify this conjecture. Moreover, these investigations showed that fixed- and free-boundary modes are of the same nature, which, again, is in keeping with the fact that the stability condition for local surface modes ("peeling" modes) [32] is identical with the above stability criterion. It is also identical with the applicability condition of a sufficient stability criterion [33]. Furthermore, it is identical with the resistive interchange criterion [34]. This may be of particular importance because it has been conjectured [24] that saturation of island growth (as a function of β) is connected with resistive interchange stability. Indeed, the above criterion provides a natural means of pressure profile flattening [6, 22] because marginal stability leads to the profile $[< \ldots >= \int \mathrm{d}\theta \mathrm{d}\phi \frac{\sqrt{g}}{|\nabla s|^2}(\ldots)]$

$$p = \int_1^s V'' \frac{p'^2}{< \vec{j}^2 >} \mathrm{d}s'$$

which is characterized by vanishing current density near rational surfaces

$$p' \propto (s - s_r)^2 V'', \quad \vec{j}^2 \propto (s - s_r)^2$$

This regularization of the current density is analogous to that obtained with the classical diffusion argument [35] that the transport be finite across rational surfaces. With the help of the above procedure a significant deterioration of β, as rational values of ι are crossed by variation of ι in 3D small-shear equilibria, was demonstrated [22].

The above form of $\delta^2 W$ explicitly shows that ballooning modes may also be dangerous because $\sqrt{g}_{,s}$ also represents locally outwardly increasing or decreasing B^2, i.e. locally favourable and unfavourable magnetic curvature. The ballooning mode criterion in 3D equilibria [36] can indeed be written in an analogous form [37]; see Appendix 3. Previous results on Mercier modes (approximately equivalent to resistive interchanges for the low-shear stellarators considered here) indicate that although ballooning modes formally include Mercier modes, ballooning instability proper does not occur at β-values lower than those imposed by resistive interchange stability.

The above picture of the MHD stability of low-shear stellarators served as an essential part in developing the Helias concept. Here, this picture is illustrated with W VII-AS, Helias and TJ-II equilibria. Marginality with respect to the resistive interchange modes is characterized by a value of the stability criterion which is approximately one order of magnitude smaller than $min[< j_\perp^2 >, < j_\parallel^2 >]$. The actual value plotted is the exponent (shifted by $\frac{1}{2}$) occurring in the asymptotic theory of ballooning modes (for details see Appendix 3). The ballooning modes considered are selected with the following argument. Usually, the total transform ι is of order one and exhibits some shear. A rational value of ι is picked from the available range in such a way that, for $\iota_r = \frac{n_\iota}{m_\iota}$, n_ι and m_ι are as small as possible, e.g. $n_\iota = 3$, $m_\iota = 8$ in the example of W VII-AS. Since the rotational transform per period $\iota_{pr} = \frac{n_\iota}{m_\iota N}$, where N is the number of field periods, the pressure-driven singular current density near this resonance is usually numerically not resolved so that the stability of a mode is considered which is virtually unaffected by the theoretically required profile flattening around this high-order resonance. This separation of scales becomes impossible for small m_ι and N. So, in the TJ-II example, $\iota_r = \frac{14}{9}$ is selected instead of $\iota_r = \frac{3}{2}$. The ballooning stability behaviour is characterized by the solution F of the one-dimensional ballooning equation; a zero of F indicates ballooning instability; the ballooning character prevails if $F = 0$ at $\bar{\phi} < n_\iota^{-1}$.

In Figs. 11 to 13 results from W VII-AS are shown for the resistive interchange criterion and the $\frac{3}{8}$ ballooning mode. $\langle\beta\rangle = 0.015$ is approximately marginal with respect to resistive interchanges as well as the ballooning mode. Figure 13 shows the structure of periodic functions occurring in the ballooning criterion: the equilibrium period, the poloidal period, and the period given by the field line closure are clearly distinguishable.

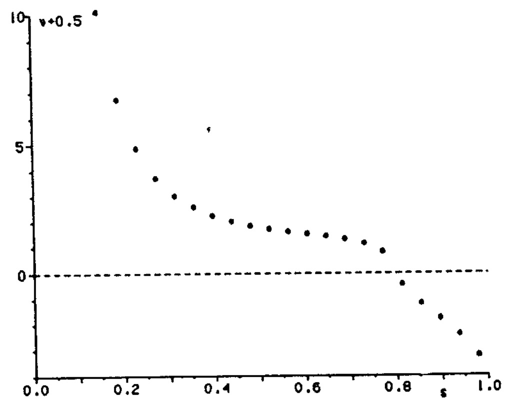

Fig. 11: The resistive interchange stability criterion as a function of s in W VII-AS at $\langle\beta\rangle = 0.015$. The actual value plotted is the exponent (shifted by $\frac{1}{2}$) occurring in the asymptotic theory of ballooning modes (for details see Appendix 3); if the stability criterion is violated, its absolute value (giving the imaginary part of ν) is used.

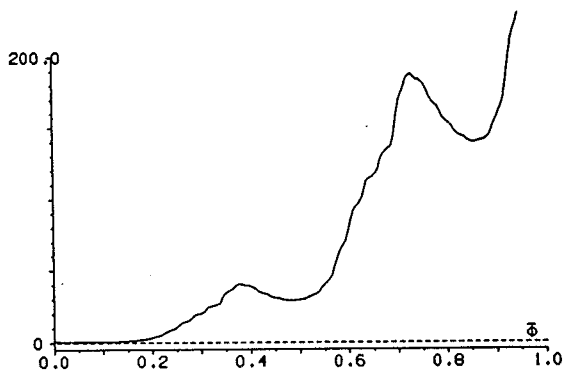

Fig. 12: The ballooning solution F as a function of the contracted variable $\bar{\phi} = \frac{\phi}{m_\iota N}$; $N = 5$, $m_\iota = 8$, $n_\iota = 3$; $\phi = 0$ corresponds to the toroidally outermost point in the equatorial plane at the beginning of a field period (see e.g. Fig. 1). Shown is the solution for W VII-AS with $\langle\beta\rangle = 0.015$.

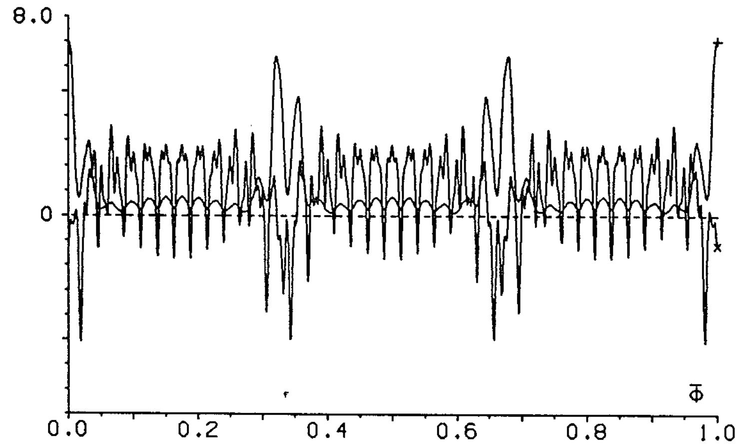

Fig. 13: The functions a (+) and $\tilde{D}$ × for W VII-AS at $\langle\beta\rangle = 0.015$.

As a Helias example, the configuration of Fig. 2 with the pressure profile of Fig. 10 is shown in Fig. 14 (resistive interchange stability) and Fig. 15 (ballooning stability at $\iota = \frac{6}{7}$).

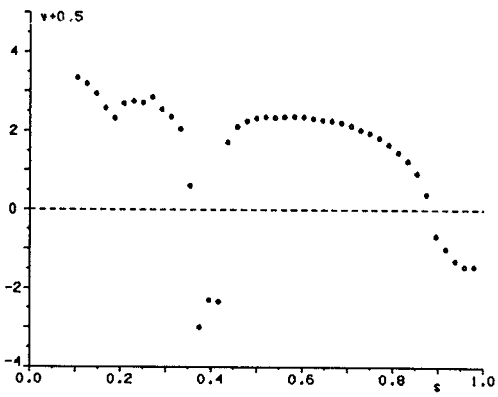

Fig. 14: Same as Fig. 11 but for the configuration of Fig. 2 with the pressure profile of Fig. 10.

As a TJ-II example the configuration of Fig. 3 is considered which is marginal at $\langle\beta\rangle \approx 0.02$ with respect to resistive interchanges. The $\frac{14}{9}$ ballooning mode is shown in Fig. 17.

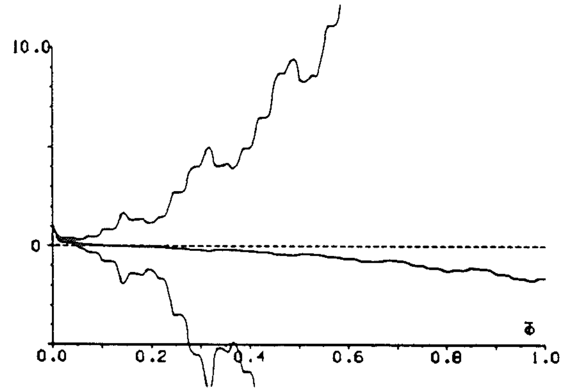

Fig. 15: Same as Fig. 12 but for the configuration of Fig. 2 with $m_\iota = 7$, $n_\iota = 6$, and $\langle\beta\rangle \approx 0.05$ (uppermost curve), 0.06 (middle curve), 0.07 (lower curve).

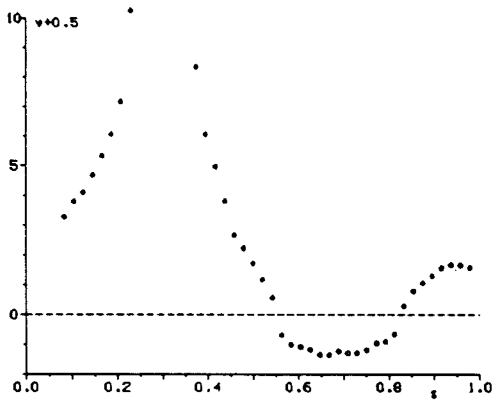

Fig. 16: Same as Fig. 11 but for the configuration of Fig. 3 with $\langle\beta\rangle = 0.02$. The singularity comes from a zero in the shear.

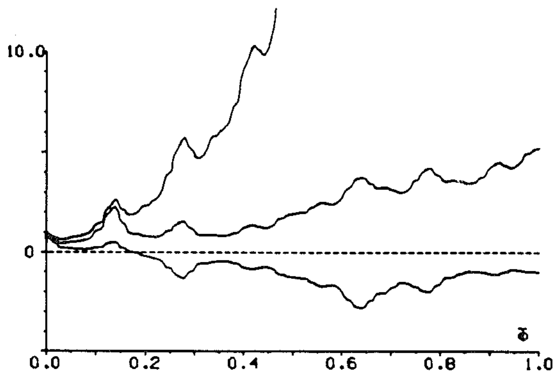

Fig. 17: Same as Fig. 12 but for the configuration of Fig. 3 with $m_\iota = 9$, $n_\iota = 14$, and $\langle\beta\rangle = 0.01$ (uppermost curve), 0.015 (middle curve), 0.02 (lower curve).

4. Some New Helias Configurations

Two of the questions arising in the context of the available parameter space for Helias equilibria are the range of the rotational transform and the lower limit of the aspect ratio in these configurations. Three examples are given here which show that there is considerable freedom in the choice of a Helias configuration which is stable with respect to resistive interchanges up to $\langle\beta\rangle \approx 0.05$ and that a selection will have to be made with additional physics and engineering arguments taken into account.

Figure 18 shows an example with rather large rotational transform $1.1 \lesssim \iota \lesssim 1.2$. The lowest-order rational value of transform per period occurring is $\frac{2}{9}$. The shear could also be increased to $1 \lesssim \iota \lesssim \frac{5}{4}$ so that $\iota = 1$ could be used as control parameter in the centre and the $\iota_p = \frac{1}{4}$ island structure at the outer boundary.

Figure 19 shows an equilibrium with 4 periods, smaller aspect ratio ($A \approx 8$), and rotational transform $0.8 \lesssim \iota \lesssim 0.9$ at $\langle\beta\rangle = 0.045$. Figure 21a shows that, despite the stronger toroidal effect, the transform profiles at $\langle\beta\rangle = 0$ and 0.045 are still similar.

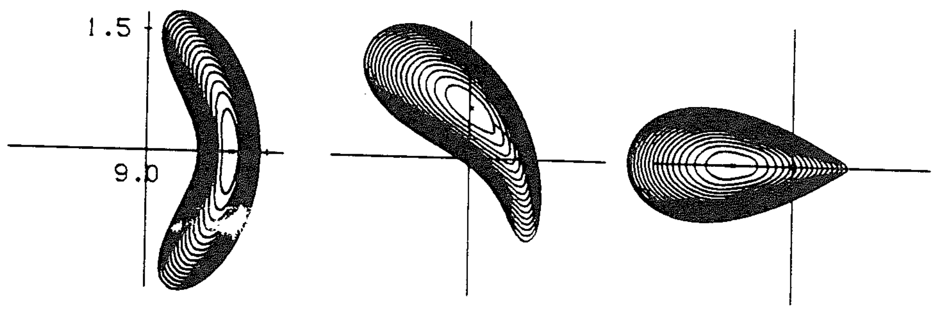

Fig. 18: A Helias equilibrium with $N = 5$, $1.1 \lesssim \iota \lesssim 1.2$. If the averages of the average radius at $v = 0$ and $v = \frac{1}{2}$ are used to define an aspect ratio, then $A_{a_0, a_{\frac{1}{2}}} \approx 9.7$; $\langle\beta\rangle = 0.05$; approximately bell-shaped pressure profile.

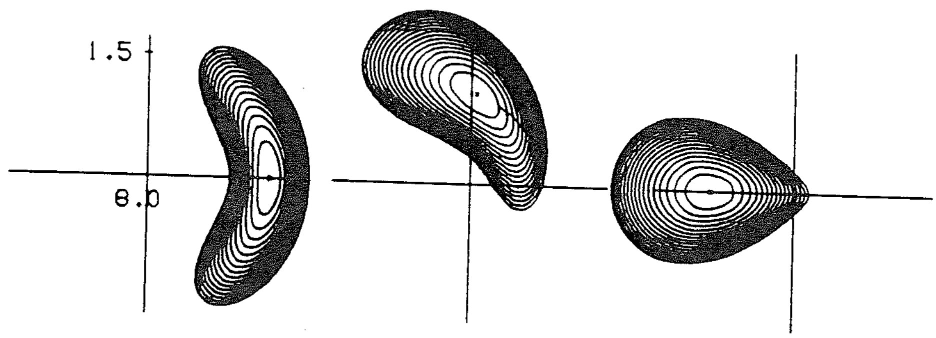

Fig. 19: A Helias equilibrium with $N = 4$, $0.8 \lesssim \iota \lesssim 0.9$, $\langle\beta\rangle = 0.045$, $A_{a_0, a_{\frac{1}{2}}} \approx 8.1$; parabolic pressure profile.

Finally, Fig. 20 shows an equilibrium with 4 periods and $0.7 \lesssim \iota \lesssim 0.8$ at $\langle\beta\rangle = 0.05$; Fig. 21b the corresponding ι-profiles. Their ballooning stability properties remain to be studied.

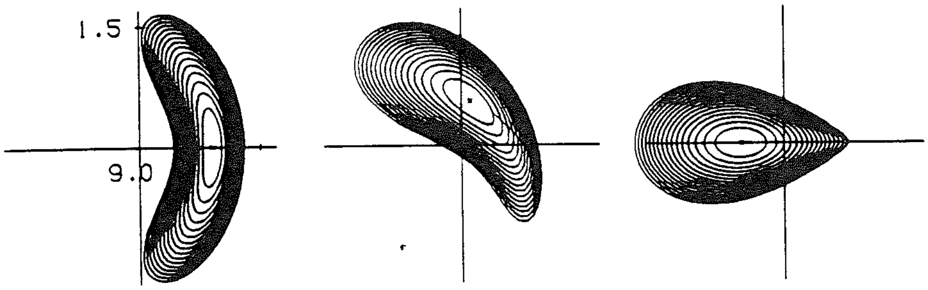

Fig. 20: A Helias equilibrium with $N = 4$, $0.7 \lesssim \iota \lesssim 0.8$, $\langle\beta\rangle = 0.05$, $A_{a_0,a_{\frac{1}{2}}} \approx 9.5$; parabolic pressure profile.

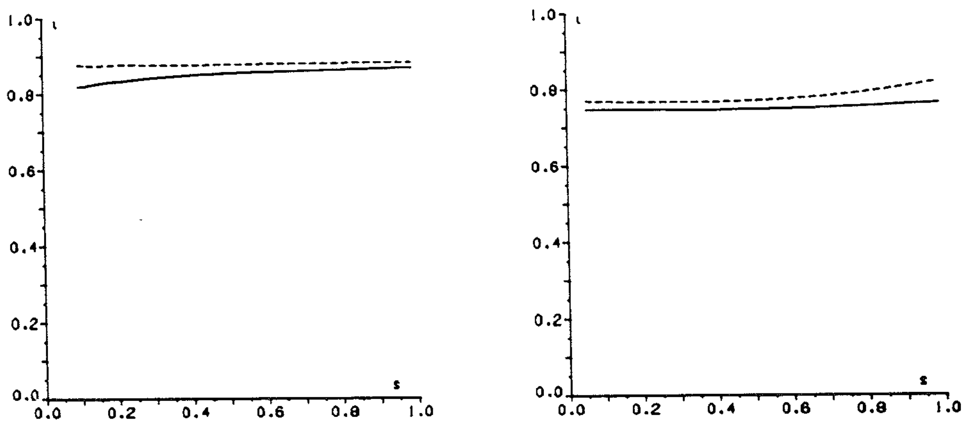

Fig. 21: Rotational transform profiles for the cases of Fig. 19 (a) and Fig. 20 (b) at $\beta = 0$ (- - -) and $\langle\beta\rangle = 0.045, 0.05$ (—), respectively.

5. Conclusions

It appears that low-shear stellarators are a sufficiently wide class to offer desirable equilibrium properties such as small Shafranov shift, small change of ι with β, and avoidance of low-order rational values of the rotational transform. This class of stellarators is characterized by magnetic-well stabilization, which appears to be sufficient for stability at sufficiently high β-values only if the parallel current density is reduced in relation to the moderate-transform $\ell = 2$ stellarator, e.g. either by appropriate choice of the magnetic geometry (Helias) or by sufficiently large value of ι (Heliac).

The assessment of the MHD properties of these low-shear stellarators is incomplete mainly because of three aspects which have not been considered here: i) Formation of islands due to the equilibrium currents may deteriorate the quality of the magnetic surfaces [23, 24]. Application of the estimates given in [24] yields small island size for the $\iota_p = \frac{1}{6}$ island of the equilibrium shown in Fig. 2. ii) The stability behaviour of global MHD modes remains to be studied. With the present evidence the approach used in this work seems to yield equilibria which should be stable to them. However, as is known from high-β stellarator research [38], their occurrence would be fatal for the β-values envisaged here. With the experience gained by evaluating the ballooning modes and with the rather simple structure of the simplified energy functional (see Sec. 3) their treatment should soon become feasible. A typical mode to be considered will be the $n = 3$, $m = 4$ mode in the Helias equilibrium of Figs. 20 and 21. iii) Calculations of the bootstrap current [39, 40] show that it may endanger the concept of the stellarator with vanishing net longitudinal current through each magnetic surface. Future (experimental and theoretical) work will have to determine this current particularly in the long-mean-free-path regime and whether there are stellarator geometries, possibly within the Helias class of stellarators [41], which avoid it.

Aspects of the stellarators considered here which are treated in P. Merkel's paper (these proceedings) are their behaviour as free-boundary equilibria, the quality of their vacuum magnetic surfaces and their realization with (modular) coils.

6. Acknowledgements

The computations presented here only became possible with the advent of codes which exploit the assumption of nested surfaces by using a flux variable for the computation of 3D equilibria. We are thus much indebted to F. Bauer, O. Betancourt and P. Garabedian for providing us with the BETA code and to S.P. Hirshman for providing us with the MOMCON and VMEC Fourier codes.

7. Appendix 1

Here, the 3D equilibrium relations are described in Boozer's coordinates [25]. These poloidal (θ) and toroidal (ϕ) coordinates are constructed from the usual poloidal (u) and toroidal ($v \propto$ toroidal angle of the cylinder coordinate system) coordinates of 3D flux-variable codes [14, 15, 16] in the following way. The condition that $\vec{B}$ lie in magnetic surfaces $s =$ const is realized by the Clebsch representation

$$\vec{B} = \nabla s \times \nabla \psi$$

with the flux function ψ given by

$$\psi = -F_T' u + F_P' v + \lambda(s, u, v)$$
$$F_T = F_T(s), \; F_P = F_P(s), \; ' = \frac{\mathrm{d}}{\mathrm{d}s}$$

with the single-valued function λ determined from the linear elliptic equation $B_{u,v} - B_{v,u} = 0$ $(,\cdots = \partial/\partial\cdots)$ which guarantees that $\vec{\nabla} \times \vec{B}$ lies in the s-surfaces. With this condition satisfied, the currents

$$J = \int B_u \mathrm{d}u, \; I = \int B_v \mathrm{d}v, \; J = J(s), \; I = I(s)$$

tor. current   pol. current.

are well-defined so that the complementary representation of $\vec{B}$

$$\vec{B} = \nabla\chi + \beta\nabla s$$

leads to

$$\chi = Ju + Iv + \tilde{\chi}(s, u, v)$$

with $\tilde{\chi}$ given by

$$\tilde{\chi}_{,u} = B_u - J, \; \tilde{\chi}_{,v} = B_v - I$$

The functions λ and $\tilde{\chi}$ are used to construct

$$\theta = u + \tilde{\theta}$$
$$\phi = v + \tilde{\phi}$$
$$\tilde{\theta} = (F_P'\tilde{\chi} - I\lambda)/(F_T' I + F_P' J)$$
$$\tilde{\phi} = (F_T'\tilde{\chi} + J\lambda)/(F_T' I + F_P' J)$$

Thus, $\vec{r} = \vec{r}(s, \theta, \phi)$ can be computed in Boozer's coordinates. Note that these coordinates are well behaved in the magnetic surfaces in contrast to Hamada's coordinates [42], which become singular for rational values of the rotational transform if p' is finite. In these coordinates the relations

$$-(\theta - \iota\phi)F_T' = \psi = -F_T'\theta + F_P'\phi \qquad \vec{B} = -F_T'\nabla s \times \nabla\theta + F_P'\nabla s \times \nabla\phi$$
$$\iota = \frac{F_P'}{F_T'} \qquad \chi = J\theta + I\phi$$

hold so that the field lines and their orthogonals are straight lines. In s, θ, ϕ coordinates one has

$$\vec{B} = -\frac{F_T'}{\sqrt{g}}\vec{r}_{,\phi} - \frac{F_P'}{\sqrt{g}}\vec{r}_{,\theta}$$
$$= I\nabla\phi + J\nabla\theta + \tilde{\beta}\nabla s$$

$$\sqrt{g}\vec{B} \cdot \nabla = -F_T'\partial_\phi - F_P'\partial_\theta \hat{=} -F_T'(n + m\iota)$$

$$B^2 = \nabla s \times \nabla(-F_T'\theta + F_P'\phi) \cdot \nabla(J\theta + I\phi)$$
$$= \nabla s \times \nabla\theta \cdot \nabla\phi \, (-F_T' \cdot I - F_P' J) = -(\nabla s \times \nabla\theta \cdot \nabla\phi)(F_T' I + F_P' J)$$
$$\Rightarrow \frac{1}{\sqrt{g}} \equiv \nabla s \times \nabla\theta \cdot \nabla\phi$$

$$B^2 = (\nabla s \times \nabla\psi) \cdot \nabla\chi = -\frac{1}{\sqrt{g}}(F_T' I + F_P' J)$$

where $\tilde{\beta}(s,\ \theta,\ \phi)$ is single-valued and obeys

$$\tilde{\beta} = -\frac{1}{\sqrt{g}}(F_T' g_{\phi s} + F_P' g_{\theta s})$$ $\vec{B} \cdot \vec{r}_{,s} = \left(-\frac{F_T'}{\sqrt{g}} \vec{r}_{,\phi} - \frac{F_P'}{\sqrt{g}} \vec{r}_{,\theta}\right) \cdot \vec{r}_{,s} = \tilde{\beta} \, \nabla s \cdot \vec{r}_{,s}$

or, equivalently,

$$\tilde{\beta} = -\frac{1}{|\nabla s|^2}(J g^{\theta s} + I g^{\phi s})$$ $\vec{B} \cdot \nabla s = 0 = (I\nabla\phi + J\nabla\theta + \tilde{\beta}\nabla s) \cdot \nabla s$

Finally, the equilibrium equation $(\vec{\nabla} \times \vec{B}) \times \vec{B} = \nabla p$ leads to

$$(J' - \tilde{\beta}_{,\theta})F_P' + (I' - \tilde{\beta}_{,\phi})F_T' = p'\sqrt{g}$$

so that the one-dimensional force balance (determining p')

$$J' F_P' + I' F_T' = p' V'$$

and the magnetic differential equation (determining $\tilde{\beta}$)

$$\sqrt{g}\vec{B} \cdot \nabla\tilde{\beta} = p'(\sqrt{g} - V'),\ \tilde{\beta}_{mn} = -\frac{p'\sqrt{g}_{mn}}{nF_T' + mF_P'}$$

are obtained. The parallel current density and $\tilde{\beta}$ are related by

$\iint X \, d\theta \, d\varphi = \frac{JI' - IJ'}{F_T' I + F_P' J}$

$$X = \vec{j} \cdot \vec{B}/B^2 = (-J\tilde{\beta}_{,\phi} + I\tilde{\beta}_{,\theta})/(F_T' I + F_P' J) + (JI' - IJ')/(F_T' I + F_P' J)$$

Consistency of the equilibrium description requires that the two equations obtained for $\tilde{\beta}$ be equivalent. This means in particular that the singular behaviour of $\tilde{\beta}$ near rational surfaces inferred from its magnetic differential equation requires $\vec{r}_{,s}(s, \theta, \phi)$ to be singular in these coordinates. Three remarks are useful here: i) $\tilde{\beta}_{mn}$ as given here only contains the inhomogeneous solution, while the homogeneous δ-solution [24], which describes the force-free currents necessary for rational magnetic surfaces to exist, is not considered. ii) Equilibria which are stable with respect to resistive interchange modes have vanishing $\tilde{\beta}$ (as obtained from the magnetic differential equation) near resonant surfaces instead of diverging $\tilde{\beta}$. iii) 3D equilibrium codes relying on energy minimization and radial (s) differencing will be insensitive to the singular behaviour of the current density since $\vec{B}^2$ is integrable; therefore, $\tilde{\beta}$ (and X) should be evaluated from its magnetic differential equation and not from the equilibrium metric.

8. Appendix 2

Here, the MHD energy principle

$$\delta^2 W = \frac{1}{2}\int\int\int d^3\tau$$
$$\left\{ \left|\nabla \times (\vec{\xi}\times\vec{B}) + \frac{\vec{j}\times\nabla s}{|\nabla s|^2}\vec{\xi}\cdot\nabla s\right|^2 + \gamma p(\vec{\nabla}\cdot\vec{\xi})^2 - A(\vec{\xi}\cdot\nabla s)^2\right\}$$

$$A = 2|\nabla s|^{-4}(\vec{j}\times\nabla s)\cdot(\vec{B}\cdot\nabla)\nabla s$$

for the plasma energy is given in a form suitable for discussing the stability of low-shear equilibria. Boozer's coordinates s, θ, ϕ (for details see Appendix 1) satisfying

$$\begin{aligned}\vec{B} &= -\frac{F_T'}{\sqrt{g}}\vec{r},_\phi - \frac{F_P'}{\sqrt{g}}\vec{r},_\theta \\ &= I\nabla\phi + J\nabla\theta + \tilde{\beta}\nabla s\end{aligned}$$

are used and the displacement vector $\vec{\xi}$ is decomposed into scalar quantities ξ, η, μ given by

$$\sqrt{g}\vec{\xi} = \xi\vec{r},_s + \frac{-F_T'\eta\sqrt{g} + F_P'\mu}{F_T'^2 + F_P'^2}\vec{r},_\theta + \frac{F_P'\eta\sqrt{g} + F_T'\mu}{F_T'^2 + F_P'^2}\vec{r},_\phi$$

This decomposition results in

$$(F_T'^2 + F_P'^2)\sqrt{g}\vec{\nabla}\cdot\vec{\xi} = (F_T'^2 + F_P'^2)\xi,_s - \sqrt{g}\vec{B}\cdot\nabla\mu + F_P'(\eta\sqrt{g}),_\phi - F_T'(\eta\sqrt{g}),_\theta$$

With the integrability condition

$$\int \xi \mathrm{d}\theta\mathrm{d}\phi = 0$$

a magnetic differential equation is left for μ which guarantees $\vec{\nabla}\cdot\vec{\xi}$ to be arbitrarily small if the equilibrium has some shear (see, for example, [28, 32]). With $(\xi^s = \xi/\sqrt{g})$

$$\vec{\xi}\times\vec{B} = \eta\nabla s + F_T'\xi^s\nabla\theta - F_P'\xi^s\nabla\phi$$

the perturbed magnetic energy is given by

$$\sqrt{g}\vec{\nabla}\times(\vec{\xi}\times\vec{B}) = \sqrt{g}(\vec{B}\cdot\nabla\xi^s)\vec{r},_s + [\eta,_\phi + (F_P'\xi^s),_s]\vec{r},_\theta + [-\eta,_\theta + (F_T'\xi^s),_s]\vec{r},_\phi$$

The decomposition

$$\vec{C} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}) + \frac{\vec{j}\times\nabla s}{|\nabla s|^2}\xi^s = C^s\vec{r},_s + C_\perp\frac{\nabla s\times\vec{B}}{|\nabla s|B} + C_\parallel\frac{\vec{B}}{B}$$

then leads to the expressions

$$C^s = \vec{B}\cdot\nabla\xi^s$$
$$C_\perp = -\frac{|\nabla s|}{B\sqrt{g}}[\sqrt{g}\vec{B}\cdot\nabla\eta - (F_T'F_P'' - F_T''F_P')\xi^s + \frac{\vec{j}\cdot\vec{B}}{|\nabla s|^2}\sqrt{g}\xi^s]$$
$$C_\parallel = \frac{1}{B\sqrt{g}}[J\eta,_\phi - I\eta,_\theta + (F_T'I + F_P'J)\xi^s,_s + (JF_P'' + IF_T'')\xi^s - p'\sqrt{g}\xi^s]$$

Finally, one has

$$\sqrt{g}A = \frac{\vec{j}^2\sqrt{g}}{|\nabla s|^2} + F_T''I' + F_P''J' - F_T''\tilde{\beta},_\phi - F_P''\tilde{\beta},_\theta$$
$$+\sqrt{g}\vec{j}\cdot\nabla\frac{F_P'g^{\phi s} - F_T'g^{\theta s}}{|\nabla s|^2} - p'\sqrt{g}\vec{\nabla}\cdot\frac{\nabla s}{|\nabla s|^2}$$

or, equivalently,

$$\sqrt{g}A = \frac{\vec{j}^2\sqrt{g}}{|\nabla s|^2} + F_T''I' + F_P''J' - F_T''\tilde{\beta},_\phi - F_P''\tilde{\beta},_\theta$$
$$+\sqrt{g}\vec{B}\cdot\nabla\frac{\sqrt{g}j^\phi g^{\theta s} - \sqrt{g}j^\theta g^{\phi s}}{|\nabla s|^2} - p'\sqrt{g},_s$$

so that

$$\int\sqrt{g}A\mathrm{d}\theta\mathrm{d}\phi = \int\frac{\vec{j}^2\sqrt{g}}{|\nabla s|^2}\mathrm{d}\theta\mathrm{d}\phi + F_T''I' + F_P''J' - p'V''$$

For vanishing longitudinal current ($J \equiv J' \equiv 0$), the above formulae suggest a heuristic argument for a necessary stability criterion for low-shear stellarators: taking the perturbation as nearly resonant ($\vec{B}\cdot\nabla\xi^s \approx 0$), considering $C_\perp$ as equation for η,

$$\sqrt{g}\vec{C}_\perp^2 \sim \frac{|\nabla s|^2}{B^2\sqrt{g}}(\int\frac{\vec{j}\cdot\vec{B}}{|\nabla s|^2}\sqrt{g}\mathrm{d}\theta\mathrm{d}\phi)^2 \ll \int\frac{\vec{j}^2\sqrt{g}}{|\nabla s|^2}\mathrm{d}\theta\mathrm{d}\phi$$

and $C_\parallel \approx 0$ as equation for the radial behaviour of ξ^s, one will obtain $\vec{C}^2$ so small that

$$\int\sqrt{g}A\mathrm{d}\theta\mathrm{d}\phi < 0$$

will be an approximation of a necessary stability criterion (see also Appendix 3).

A complete list of the three-dimensional equilibrium functions occurring in the stability functional is given by

$$\sqrt{g},\ \sqrt{g},_s,\ g_{ss},\ g_{s\theta},\ g^{ss},\ g^{s\theta}$$

if use is made of the equilibrium relations of Appendix 1.

$$\sqrt{g}\ \vec{B}\cdot\nabla\tilde{\beta} = p'(\sqrt{g} - V')$$

$$\tilde{\beta} = -(F_T' g_{\phi s} + F_p' g_{\theta s})\frac{1}{\sqrt{g}}$$

$$-p'\frac{1}{B^2}(IF_T'' + JF_p'') = \sqrt{g}\frac{1}{V'}\left[I'F_T'' + J'F_p'' + \iota' F_T'^2 (I'J - J'I)/(IF_T' + JF_p')\right]$$

9. Appendix 3

Here, the formulae for the necessary stability criteria used are given; the ballooning formalism [36] is used for conciseness since Mercier modes and resistive interchanges are formally special cases. In Boozer's coordinates the local shear σ can be written in the form ([37], a mistake in $\tilde{D}$, see below, is corrected) $(\tilde{\beta}B^2 \to \tilde{\beta}B^{-2})$

$$\begin{aligned}\sigma &= |\nabla s|^{-4}(\nabla s\times\vec{B})\cdot\vec{\nabla}\times(\nabla s\times\vec{B})\\ &= F_T'^2\iota'/\sqrt{g} + \vec{B}\cdot\nabla(Ig_{\theta s}/\sqrt{g}|\nabla s|^2 - Jg_{\phi s}/\sqrt{g}|\nabla s|^2)\end{aligned}$$

The ballooning equation can then be reduced to the following equation for F (which indicates instability if F vanishes twice):

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left\{a^{-1}[1+(\bar{\sigma}\phi+\tilde{\sigma})^2]\frac{\mathrm{d}F}{\mathrm{d}\phi}\right\} + (\bar{D}\phi + \tilde{D})F = 0$$

Here, ϕ is used as the variable along the field line and the coefficients a^{-1}, $\bar{\sigma}$, $\tilde{\sigma}$, $\bar{D}$, $\tilde{D}$ have to be used as functions of this variable:

$$\begin{aligned}a &= \sqrt{g}|\nabla s|^2\\ \bar{\sigma} &= -F_T'\iota'|\nabla s|^2/B\\ \tilde{\sigma} &= (Ig_{\theta s} - Jg_{\phi s})/\sqrt{g}B\\ \bar{D} &= p'\iota' F_T'^{-1}(IB^{-2}{}_{,\theta} - JB^{-2}{}_{,\phi}) \qquad \text{PK12+PK11} \qquad 2\\ \tilde{D} &= [\underbrace{\sqrt{g}p'^2/B^2}_{\text{PK14}} - \underbrace{p'\sqrt{g}_{,s}}_{\text{PK13}} - p'B^{-2}(IF_T'' + JF_P'') + p'\sqrt{g}\vec{B}\cdot\nabla(\tilde{\beta}B^{-2})]/F_T'^2\end{aligned}$$

Note that $\bar{D} = -F_T'^{-1}\iota'\sqrt{g}\vec{B}\cdot\nabla X$. So, $\bar{\sigma}$ represents the stabilizing influence of the shear, $\tilde{D}$ (for stellarators with vanishing net toroidal current) mainly the stabilizing influence of an outwardly increasing B^2 diminished by the diamagnetic effect, and $\bar{D}$ the terms connected with $j_\parallel$ and with the shear known from Mercier's criterion [28]. In accordance with the explicit resemblance of this form of the ballooning equation with Mercier's criterion, the well-known asymptotic analysis, $F \propto \phi^\nu + F_1\phi^{\nu-1} + \ldots$, readily recovers that criterion in the form of the indicial equation

Achtung:

pomers $= \nu + \frac{1}{2}$

$$\nu = -.5 \pm [(.5 - \langle aD/\bar{\sigma}^2\rangle)^2 - \langle a/\bar{\sigma}^2\rangle(\langle aD^2/\bar{\sigma}^2\rangle + \langle\tilde{D}\rangle - \langle D\rangle)]^{\frac{1}{2}} \qquad \langle f\rangle = \iint f\, d\theta\, d\varphi$$

PK15 ?

where $D = \int \bar{D}d\phi = \iota' X$. Note that

$$\langle D\rangle = \iota' X_0 = \iota'(I'J - J'I)/(IF_T' + JF_P')$$

and that X_0 can be omitted from D since it cancels out. When the stabilizing influence of the squared term is omitted, the resistive interchange criterion [34] is obtained in the form ($J \equiv 0$, $F_T' \propto s$)

$$\langle a/\bar{\sigma}^2\rangle(-\langle aD^2/\bar{\sigma}^2\rangle - \frac{p'^2}{F_T'^2}\langle\frac{\sqrt{g}}{B^2}\rangle + \frac{p'V''}{F_T'^2}) > 0$$

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