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Effect on fluxes of the poloidally varying electrostatic potential

In these notes, we consider the following questions. Suppose neoclassical code #1 computes the particle and heat fluxes, totally ignoring the poloidal variation of the electrostatic potential. Then suppose neoclassical code #2 solves the drift-kinetic equation for the same driving radial gradients, but now including the parallel acceleration force from the poloidally varying potential in the kinetic equation. In code #2, several variants of the radial fluxes can be computed, as there is now both a radial magnetic and ExB drift, and the radial transport of the poloidally varying electrostatic energy becomes important. How exactly are the fluxes in code #1 and code #2 related, and what are the "correct" definitions of the fluxes to use in code #2? In the end we will find that code #2 ends up getting the same answer for both the particle and energy flux as code #1.

All of the discussion here is relevant to both tokamaks and stellarators. For simplicity here we will neglect the $\mathbf{v}_E \cdot \nabla f_1$ term which is important in stellarators but not (usually) in tokamaks. Some of the analysis may need to be re-examined when $\mathbf{v}_E \cdot \nabla f_1$ is retained.

Suppose we have a solution f_{s1} to the linear drift-kinetic equation in which there is no parallel variation of the electrostatic potential, as in DKES:

$$\upsilon_\parallel \nabla_\parallel f_{s1} - C_s\left[f_{s1}\right] = -\mathbf{v}_{ms} \cdot \nabla \psi \frac{\partial f_{s0}}{\partial \psi}. \tag{1}$$

Here,

$$f_{s0} = n_s(\psi)\left[\frac{m_s}{2\pi T_s(\psi)}\right]^{3/2} \exp\left(-\frac{m_s \upsilon^2}{2T_s(\psi)}\right) \tag{2}$$

is a stationary Maxwellian, s denotes species, and $\mathbf{v}_{ms}$ is the magnetic drift (not including any $\mathbf{E}\times\mathbf{B}$ drift). We neglect the radial electric field in (1) for simplicity. The solution f_{s1} gives rise to a certain particle flux

$$\Gamma_{ms} = \left\langle \int d^3\upsilon\, f_{s1} \mathbf{v}_{ms} \cdot \nabla \psi \right\rangle \tag{3}$$

and an energy flux

$$Q_{ms} = \left\langle \int d^3\upsilon\, f_{s1} \frac{m_s \upsilon^2}{2} \mathbf{v}_{ms} \cdot \nabla \psi \right\rangle. \tag{4}$$

(The subscript m on the fluxes (3)-(4) indicates these fluxes are associated with the radial *magnetic* drift.)

Now consider a more sophisticated drift-kinetic equation, in which parallel variation of the electrostatic potential Φ is retained. We denote the solution to this form of the kinetic equation by $\tilde{f}_{s1}$:

$$\upsilon_\parallel \nabla_\parallel \tilde{f}_{s1} - C_s\left[\tilde{f}_{s1}\right] = -\mathbf{v}_{ms} \cdot \nabla \psi \frac{\partial f_{s0}}{\partial \psi} - \upsilon_\parallel \frac{Z_s e}{T_s}\left(\nabla_\parallel \Phi\right) f_{s0}. \tag{5}$$

The solution to (5) can be written in terms of the solution to (1):

$$\tilde{f}_{s1} = f_{s1} - \frac{Z_s e \Phi}{T_s} f_{s0} . \tag{6}$$

If we evaluate the radial particle and energy fluxes associated with $\tilde{f}_{s1}$ caused by just the magnetic drifts (ignoring the radial $\mathbf{E}\times\mathbf{B}$ drift,) we do not get the same fluxes we got before:

$$\begin{aligned}\tilde{\Gamma}_{ms} &= \left\langle \int d^3\upsilon\, \tilde{f}_{s1} \mathbf{v}_{ms} \cdot \nabla\psi \right\rangle \\ &= \Gamma_{ms} - \frac{Z_s e}{T_s} \left\langle \Phi \int d^3\upsilon\, f_{s0} \mathbf{v}_{ms} \cdot \nabla\psi \right\rangle\end{aligned} \tag{7}$$

and

$$\begin{aligned}\tilde{Q}_{ms} &= \left\langle \int d^3\upsilon\, \tilde{f}_{s1} \frac{m_s \upsilon^2}{2} \mathbf{v}_{ms} \cdot \nabla\psi \right\rangle \\ &= Q_{ms} - \frac{Z_s e}{T_s} \left\langle \Phi \int d^3\upsilon\, f_{s0} \frac{m_s \upsilon^2}{2} \mathbf{v}_{ms} \cdot \nabla\psi \right\rangle.\end{aligned} \tag{8}$$

In other words, including the blue $\nabla_\parallel \Phi$ term in (5) causes the radial fluxes to change by the red terms in (7)-(8). Using

$$\mathbf{v}_{ms} \cdot \nabla\psi = \frac{m_s}{2 Z_s e B^3} \left(2\upsilon_\parallel^2 + \upsilon_\perp^2 \right) \mathbf{B} \times \nabla B \cdot \nabla\psi , \tag{9}$$

we can evaluate the $\int d^3\upsilon$ integrals in the red terms of (7)-(8), giving

$$\tilde{\Gamma}_{ms} = \Gamma_{ms} + n_{s0} \left\langle \Phi \mathbf{B} \times \nabla \left(\frac{1}{B^2} \right) \cdot \nabla\psi \right\rangle \tag{10}$$

and

$$\tilde{Q}_{ms} = Q_{ms} + \frac{5}{2} n_s T_s \left\langle \Phi \mathbf{B} \times \nabla \left(\frac{1}{B^2} \right) \cdot \nabla\psi \right\rangle. \tag{11}$$

Using the MHD equilibrium relation $\nabla\psi \cdot (\nabla \times \mathbf{B}) = 0$, along with

$$\langle \nabla \cdot \mathbf{Q} \rangle = \frac{1}{V'} \frac{d}{d\psi} \left(V' \langle \mathbf{Q} \cdot \nabla\psi \rangle \right), \tag{12}$$

then (10)-(11) can be integrated by parts to find

$$\tilde{\Gamma}_{ms} = \Gamma_{ms} - n_{s0} \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla\Phi \cdot \nabla\psi \right\rangle \tag{13}$$

and

$$\tilde{Q}_{ms} = Q_{ms} - \frac{5}{2} n_s T_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla\Phi \cdot \nabla\psi \right\rangle. \tag{14}$$

When Φ varies on a flux surface, there will also be a radial $\mathbf{E}\times\mathbf{B}$ drift. The particle and heat fluxes associated with this drift are, considering just f_{s0} as opposed to f_{s1},

$$\Gamma_{Es} = \left\langle \int d^3\upsilon\, f_{s0} \frac{1}{B^2} \mathbf{B} \times \nabla\Phi \cdot \nabla\psi \right\rangle = n_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla\Phi \cdot \nabla\psi \right\rangle \tag{15}$$

and

$$Q_{Es} = \left\langle \int d^3\upsilon\, f_{s0} \frac{m_s\upsilon^2}{2} \frac{1}{B^2} \mathbf{B}\times\nabla\Phi\cdot\nabla\psi \right\rangle = \frac{3}{2} n_s T_s \left\langle \frac{1}{B^2} \mathbf{B}\times\nabla\Phi\cdot\nabla\psi \right\rangle. \tag{16}$$

Combining (13)-(16), we find

$$\tilde{\Gamma}_{ms} + \Gamma_{Es} = \Gamma_{ms} \tag{17}$$

and

$$\tilde{Q}_{ms} + Q_{Es} = Q_{ms} - n_s T_s \left\langle \frac{1}{B^2} \mathbf{B}\times\nabla\Phi\cdot\nabla\psi \right\rangle. \tag{18}$$

Eq (17) shows that ignoring $\nabla_\| \Phi$ in the kinetic equation does not change the particle flux: we get the same particle flux if we do include $\nabla_\| \Phi$ and then account for the extra radial flux from the radial $\mathbf{E}\times\mathbf{B}$ drift. However, (18) shows that the flux of kinetic energy *does* change when $\nabla_\| \Phi$ is included in the kinetic equation, and this change cannot be compensated by accounting for the radial flux of kinetic energy from the radial $\mathbf{E}\times\mathbf{B}$ drift.

However, as Per points out in [1], the flux of *kinetic* energy (8) is less physically relevant than the flux of total (kinetic + potential) energy. This flux of total energy is

$$Q_{tot,s} = \left\langle \int d^3\upsilon\, f_s \left(\frac{m_s\upsilon^2}{2} + Z_s e\Phi \right) (\mathbf{v}_{ms} + \mathbf{v}_E)\cdot\nabla\psi \right\rangle \tag{19}$$

where $f_s = f_{s0} + \tilde{f}_{s1}$ is the total distribution function. Writing $\Phi = \Phi_0(\psi) + \Phi_1$ where $e\Phi_0 / T \sim 1$ and $e\Phi_1 / T \sim \rho_*$, and noting

$$\frac{\mathbf{v}_E\cdot\nabla\psi}{\mathbf{v}_{m,s}\cdot\nabla\psi} \sim \frac{e\Phi_1}{T} \sim \rho_* \tag{20}$$

(when $E_r = 0$), then formally the largest term in (19) in the ρ_* expansion is

$$Q_{tot0,s} = \left\langle \int d^3\upsilon\, f_{s0} \left(\frac{m_s\upsilon^2}{2} + Z_s e\Phi_0 \right) \mathbf{v}_{ms}\cdot\nabla\psi \right\rangle. \tag{21}$$

However (21) vanishes exactly. The next-order terms in (19) are

$$\begin{aligned} Q_{tot1,s} = & \left\langle \int d^3\upsilon\, f_{s0} Z_s e\Phi_1 \mathbf{v}_{ms}\cdot\nabla\psi \right\rangle + \left\langle \int d^3\upsilon\, \tilde{f}_{s1} \left(\frac{m_s\upsilon^2}{2} + Z_s e\Phi_0 \right) \mathbf{v}_{ms}\cdot\nabla\psi \right\rangle \\ & + \left\langle \int d^3\upsilon\, f_{s0} \left(\frac{m_s\upsilon^2}{2} + Z_s e\Phi_0 \right) \mathbf{v}_E\cdot\nabla\psi \right\rangle. \end{aligned} \tag{22}$$

Some of these terms are quantities we have already considered:

$$Q_{tot1,s} = Z_s e \left\langle \int d^3\upsilon\, f_{s0} \Phi_1 \mathbf{v}_{ms}\cdot\nabla\psi \right\rangle + \tilde{Q}_{ms} + Q_{Es} + Z_s e\Phi_0 \left(\tilde{\Gamma}_{ms} + \Gamma_{Es} \right). \tag{23}$$

Also, the first right-hand-side term in (23) is the same integral in red in (7), which we evaluated in going from (7) to (10) to (13). This term is one we neglected in computing the flux of kinetic energy (8), and it represents radial transport of the poloidally varying electrostatic energy. Thus, we find

$$Q_{tot1,s} = n_{s0} T_s \left\langle \frac{1}{B^2} \mathbf{B}\times\nabla\Phi\cdot\nabla\psi \right\rangle + \tilde{Q}_{ms} + Q_{Es} + Z_s e\Phi_0 \left(\tilde{\Gamma}_{ms} + \Gamma_{Es} \right). \tag{24}$$

As one can see from (17)-(18), (24) can be written in terms of the particle and heat fluxes that would be found in the absence of Φ_1:

$$Q_{tot1,s} = Q_{ms} + Z_s e \Phi_0 \Gamma_{ms} . \tag{25}$$

Thus, to leading order in ρ_*, the physically meaningful flux (19) should come out to be the same whether or not Φ_1 is included in the computation.

[1] Helander, PPCF **37**, 57 (1995).