March 26, 2014

NTV

Consider the momentum equation summed over species,

$$\boldsymbol{J}\times\boldsymbol{B}-\nabla\cdot\mathbf{P}=\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V})+\frac{\partial(\rho\boldsymbol{V})}{\partial t}. \tag{1}$$

We write

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{B}_0+\boldsymbol{B}_1, && (2)\\ \boldsymbol{J} &= \boldsymbol{J}_0+\boldsymbol{J}_1, && (3)\\ \mathbf{P} &= p_0(\psi)\mathbf{I}+p_1(\psi)\mathbf{I}+\mathbf{\Pi}, && (4)\end{aligned}$$

where

$$\boldsymbol{J}_0\times\boldsymbol{B}_0=p_0'(\psi)\nabla\psi. \tag{5}$$

Subtracting Eq. (5) from Eq. (1), we obtain

$$\frac{\partial\rho\boldsymbol{V}}{\partial t}=\boldsymbol{J}_1\times\boldsymbol{B}_0+\boldsymbol{J}_0\times\boldsymbol{B}_1-\nabla p_1-\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi}). \tag{6}$$

where we have used the vector identity $\nabla B_1^2=2\boldsymbol{B}_1\cdot\nabla\boldsymbol{B}_1+2\boldsymbol{B}_1\times\nabla\times\boldsymbol{B}_1$ to write $\boldsymbol{J}_1\times\boldsymbol{B}_1=-\nabla\cdot\mathbf{M}$, with

$$\mathbf{M}=\frac{1}{\mu_0}\left(\frac{1}{2}B_1^2\mathbf{I}-\boldsymbol{B}_1\boldsymbol{B}_1\right). \tag{7}$$

The two components of Eq. (6) in the directions along $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are particularly interesting as we shall see, because several terms disappear upon taking the flux surface average. Observe that

$$\begin{aligned}\langle\boldsymbol{B}_0\cdot\boldsymbol{J}_0\times\boldsymbol{B}_1\rangle &= \langle\boldsymbol{B}_1\cdot\nabla p_0\rangle=-\langle(\nabla\times\boldsymbol{A}_1)\cdot\nabla p_0\rangle=\langle\nabla\cdot(\nabla p_0\times\boldsymbol{A}_1)\rangle=0, && (8)\\ \langle\boldsymbol{J}_0\cdot\boldsymbol{J}_1\times\boldsymbol{B}_0\rangle &= -\langle\boldsymbol{J}_1\cdot\nabla p_0\rangle=\frac{1}{\mu_0}\langle(\nabla\times\boldsymbol{B}_1)\cdot\nabla p_0\rangle=-\langle\nabla\cdot(\nabla p_0\times\boldsymbol{B}_1)\rangle=0, && (9)\\ \langle\boldsymbol{B}_0\cdot\nabla p_1\rangle &= \langle\nabla\cdot(\boldsymbol{B}_0p_1)\rangle=0, && (10)\\ \langle\boldsymbol{J}_0\cdot\nabla p_1\rangle &= \langle\nabla\cdot(\boldsymbol{J}_0p_1)\rangle=0 && (11)\end{aligned}$$

if we neglect the displacement current. The flux surface average of the scalar product of Eq. (6) with $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ yields

$$\begin{aligned}\frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{B}_0\rangle}{\partial t} &= -\langle\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\cdot\boldsymbol{B}_0\rangle, && (12)\\ \frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{J}_0\rangle}{\partial t} &= -\langle\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\cdot\boldsymbol{J}_0\rangle. && (13)\end{aligned}$$

Consequently, the same type of expression also holds for any linear combination $\boldsymbol{L}=\alpha\boldsymbol{B}_0+\beta\boldsymbol{J}_0$ of $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$,

$$\frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{L}\rangle}{\partial t}=-\langle\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\cdot\boldsymbol{L}\rangle \tag{14}$$

Axisymmetry

In axisymmetry we can express $\boldsymbol{B}$ and $\boldsymbol{J}$ as

$$\boldsymbol{B}_0 = F(\psi)\nabla\phi + \nabla\phi \times \nabla\psi, \tag{15}$$

$$\boldsymbol{J}_0 = -p_0'(\psi)\frac{\partial \boldsymbol{r}}{\partial \phi} + K(\psi)\boldsymbol{B}_0, \tag{16}$$

where ϕ is the geometrical toroidal angle. The flux surface average of the parallel component of Eq. (16) gives us $K(\psi) = (\langle \boldsymbol{J}_0 \cdot \boldsymbol{B}_0\rangle + p_0' F)/\langle B^2\rangle$. We are interessed in the toroidal torque on the plasma, so in Eq. (14) we chose

$$\boldsymbol{L} = R\hat{\phi} = \frac{\partial \boldsymbol{r}}{\partial \phi} = -\frac{1}{p_0'}\boldsymbol{J}_0 + \frac{K}{p_0'}\boldsymbol{B}_0 \tag{17}$$

Non-axisymmetry

For an arbitrary 3D magnetic configuration, we would like to define a generalised vector $\boldsymbol{L}$, which in the limit of axisymmetry becomes $R\hat{\phi}$. One requirement for $\boldsymbol{L}$ to be parallel to $\hat{\phi}$ in this limit is that the streamlines of $\boldsymbol{L}$ close on themselves toroidally. This also makes sense when we are not exactly at axisymmentry because we are not interested in any net poloidal torque component.

In Hamada coordinates (V, ϑ, φ), the streamlines of both $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are straight, i.e. $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are linear combinations of $\partial\boldsymbol{r}/\partial\vartheta$ and $\partial\boldsymbol{r}/\partial\varphi$ and vice versa. The sought linear combination of $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ whose streamlines close on themselves toroidally is thus $\boldsymbol{e}_\varphi = \partial\boldsymbol{r}/\partial\varphi$. Because of the above mentioned reasons, $\boldsymbol{e}_\varphi$ becomes parallel to $\hat{\phi}$ in the limit of axisymmetry (note that $\nabla\varphi$ does not become parallel to $\hat{\phi}$). Therefore, since $\nabla \cdot \boldsymbol{e}_\varphi = 0$ and $\nabla \cdot R\hat{\phi} = 0$, we conclude that $\boldsymbol{e}_\varphi \to cR\hat{\phi}$, where the constant $c = 1$ because φ and ϕ are both 2π periodic.

We now want to determine the constants α and β in $\boldsymbol{L} = \boldsymbol{e}_\varphi = \alpha\boldsymbol{B}_0 + \beta\boldsymbol{J}_0$. In Hamada coordinates, we can express $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ as

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\vartheta + \iota(\psi)\nabla\varphi \times \nabla\psi = I(\psi)\nabla\vartheta + G(\psi)\nabla\varphi + \nabla H(\psi, \vartheta, \varphi), \tag{18}$$

$$\mu_0\boldsymbol{J}_0 = I'(\psi)\nabla\psi \times \nabla\vartheta - G'(\psi)\nabla\varphi \times \nabla\psi. \tag{19}$$

Moreover, the Jacobian is a flux function, so

$$V'(\psi) = \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \frac{1}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} = \frac{4\pi^2}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} \tag{20}$$

$$\boldsymbol{e}_\varphi = \frac{\nabla\psi \times \nabla\vartheta}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} = \frac{V'}{4\pi^2}\nabla\psi \times \nabla\vartheta, \tag{21}$$

and equilibrium implies

$$\boldsymbol{J}_0 \times \boldsymbol{B}_0 = -\frac{1}{\mu_0}(\iota I' + G')(\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi)\nabla\psi = p_0'\nabla\psi \tag{22}$$

$$\iota I' + G' = -\frac{V'}{4\pi^2}\mu_0 p_0' \tag{23}$$

The equation $\boldsymbol{e}_\varphi = \alpha\boldsymbol{B}_0 + \beta\boldsymbol{J}_0$ becomes

$$\frac{V'}{4\pi^2}\nabla\psi \times \nabla\vartheta = \alpha\left(\nabla\psi \times \nabla\vartheta + \iota\nabla\varphi \times \nabla\psi\right) + \frac{\beta}{\mu_0}\left(I'\nabla\psi \times \nabla\vartheta - G'\nabla\varphi \times \nabla\psi\right) \tag{24}$$

yielding

$$
\begin{aligned}
\alpha &= \frac{V'}{4\pi^2}\frac{G'}{G'+\iota I'} = -\frac{G'}{\mu_0 p_0'}, && (25)\\
\beta &= \frac{V'}{4\pi^2}\frac{\iota}{G'+\iota I'} = -\frac{\iota}{p_0'}, && (26)
\end{aligned}
$$

i.e.,

$$
\boldsymbol{e}_\varphi = -\frac{\iota}{p_0'}\boldsymbol{J}_0 - \frac{G'}{\mu_0 p'}\boldsymbol{B}_0. \tag{27}
$$

Note that in the expressions for α and β, the flux functions $G(\psi)$ and $I(\psi)$ are the same as in Boozer coordinates (ψ, θ, ζ), where

$$
\boldsymbol{B}_0 = \nabla\psi\times\nabla\theta + \iota\nabla\zeta\times\nabla\psi = I(\psi)\nabla\theta + G(\psi)\nabla\zeta + \kappa(\psi,\theta,\zeta)\nabla\psi. \tag{28}
$$

The toroidal viscosity

We now turn our attention to the toroidal viscosity term in Eq. (14),

$$
\langle \boldsymbol{e}_\varphi\cdot\nabla\cdot\boldsymbol{\Pi}\rangle = \alpha\langle\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle + \beta\langle\boldsymbol{J}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle. \tag{29}
$$

where we have dropped the index 0 on $\boldsymbol{B}_0$, $\boldsymbol{J}_0$ and p_0. First, we note that $\boldsymbol{\Pi} = \tilde{p}(\mathbf{I}/3 - B^{-2}\boldsymbol{BB})$, where $\tilde{p} \equiv p_\perp - p_\parallel$, which implies that

$$
\begin{aligned}
\nabla\cdot\boldsymbol{\Pi} &= \frac{1}{3}\nabla\tilde{p} - \nabla\frac{\tilde{p}}{B^2}\cdot\boldsymbol{BB} - \tilde{p}\left(B^{-1}\nabla B + \frac{\mu_0}{B^2}\nabla p\right) && (30)\\
\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi} &= -\frac{2}{3}\boldsymbol{B}\cdot\nabla\tilde{p} + \frac{\tilde{p}}{B}\nabla B && (31)\\
\langle\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle &= \langle\tilde{p}\nabla_\parallel B\rangle && (32)\\
\langle\boldsymbol{J}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle &= -\left\langle J_\parallel B\boldsymbol{B}\cdot\nabla\frac{\tilde{p}}{B^2}\right\rangle - \left\langle\frac{\tilde{p}}{B}\boldsymbol{J}\cdot\nabla B\right\rangle && (33)
\end{aligned}
$$

In the last expression, we can replace $\boldsymbol{J}$ with $\boldsymbol{J} = J_\parallel B^{-1}\boldsymbol{B} + p'B^{-2}(\boldsymbol{B}\times\nabla\psi)$ where $\nabla\cdot\boldsymbol{J} = 0$ gives $\boldsymbol{B}\cdot\nabla(J_\parallel/B) = 2p'B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B$. If we also use that $\langle a\boldsymbol{B}\cdot\nabla b\rangle = -\langle b\boldsymbol{B}\cdot\nabla a\rangle$, we can write

$$
\begin{aligned}
\langle\boldsymbol{J}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle &= \left\langle\frac{\tilde{p}}{B^2}\boldsymbol{B}\cdot\nabla(J_\parallel B)\right\rangle - \left\langle\frac{\tilde{p}}{B^2}J_\parallel\boldsymbol{B}\cdot\nabla B\right\rangle - p'\left\langle\frac{\tilde{p}}{B^3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle = \\
&= \left\langle\frac{\tilde{p}}{B^2}J_\parallel\boldsymbol{B}\cdot\nabla B\right\rangle + p'\left\langle\frac{\tilde{p}}{B^3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle = \\
&= \frac{1}{2}\left\langle\frac{\tilde{p}}{B^2}\boldsymbol{B}\cdot\nabla(J_\parallel B)\right\rangle. && (34)
\end{aligned}
$$

We obtain the torque

$$
\tau \equiv \langle\boldsymbol{e}_\varphi\cdot\nabla\cdot\boldsymbol{\Pi}\rangle = -\frac{G'}{\mu_0 p'}\left\langle\tilde{p}\nabla_\parallel B\right\rangle - \frac{\iota}{2p'}\left\langle\frac{\tilde{p}}{B}\nabla_\parallel(J_\parallel B)\right\rangle \tag{35}
$$

Implementation

Define $\gamma \equiv G'/(\mu_0 p')$ and $u \equiv \iota J_\parallel/(Bp')$. The torque becomes

$$\tau = \langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi} \rangle = -\gamma \left\langle \tilde{p} \nabla_\parallel B \right\rangle - \frac{1}{2} \left\langle \frac{\tilde{p}}{B} \nabla_\parallel (uB^2) \right\rangle \tag{36}$$

To calculate this, we first need to determine the quantity u from the equation

$$\boldsymbol{B} \cdot \nabla u \;=\; 2\iota B^{-3} \boldsymbol{B} \times \nabla\psi \cdot \nabla B. \tag{37}$$

Henceforth, we employ Boozer coordinates, in which the above equation corresponds to

$$\left(\iota \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \zeta} \right) = 2 \frac{\iota}{B^3} \left(G \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \zeta} \right). \tag{38}$$

In SFINCS normalisation, we have $G = \hat{G}\bar{R}\bar{B}$, $B = \hat{B}\bar{B}$, $u = \hat{u}\bar{R}/\bar{B}$ and

$$\left(\iota \frac{\partial \hat{u}}{\partial \theta} + \frac{\partial \hat{u}}{\partial \zeta} \right) = 2 \frac{\iota}{\hat{B}^3} \left(\hat{G} \frac{\partial \hat{B}}{\partial \theta} - \hat{I} \frac{\partial \hat{B}}{\partial \zeta} \right). \tag{39}$$

The torque becomes

$$\begin{aligned} \tau &= -\gamma \left\langle \frac{\tilde{p}}{B\sqrt{g}} \left(\iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) \right\rangle - \frac{1}{2} \left\langle \frac{\tilde{p}}{B^2\sqrt{g}} \left(\iota \frac{\partial (uB^2)}{\partial \theta} + \frac{\partial (uB^2)}{\partial \zeta} \right) \right\rangle = \\ &= -\frac{1}{V'} \int d\theta d\zeta \frac{\tilde{p}}{B^2} \left[\gamma B \left(\iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) + \frac{1}{2} \left(\iota \frac{\partial (uB^2)}{\partial \theta} + \frac{\partial (uB^2)}{\partial \zeta} \right) \right] \end{aligned} \tag{40}$$

In the normalisations used in the SFINCS single species documentation,

$$\begin{aligned} &\tilde{p} = p_\perp - p_\parallel = m \int d^3v \; f \left(\frac{v_\perp^2}{2} - v_\parallel^2 \right) = \frac{2\Delta \hat{T}^{3/2} n}{\sqrt{\pi}\hat{\psi}_a} \int_{-1}^{1} d\xi \int_0^\infty dx \; x^2 \hat{f} m \left(\frac{v_\perp^2}{2} - v_\parallel^2 \right) = \\ &= \left\{ m \left(\frac{v_\perp^2}{2} - v_\parallel^2 \right) = \hat{T}\bar{T} \frac{x^2}{2} (1 - 3\xi^2) = -\hat{T}\bar{T} x^2 P_2(\xi) \right\} = \\ &= -\bar{T} n \frac{2\Delta \hat{T}^{5/2}}{\sqrt{\pi}\hat{\psi}_a} \int_{-1}^{1} d\xi \int_0^\infty dx \; x^4 P_2(\xi) \hat{f}. \end{aligned} \tag{41}$$

Note the following about the Legendre polynomial P_2,

$$\int_{-1}^{1} d\xi P_2^2(\xi) = \frac{2}{5}, \tag{42}$$

so that with

$$\hat{f} = \sum_{l=0}^{\infty} f_l P_l(\xi) \tag{43}$$

we obtain

$$\int_{-1}^{1} d\xi P_2(\xi) \hat{f} = \frac{2}{5} f_2. \tag{44}$$

If we define $\gamma = \hat{\gamma}\bar{R}/\bar{B}$ we can write

$$\tau \;=\; \bar{T} n \frac{2\Delta \hat{T}^{5/2} \bar{R}}{V'\sqrt{\pi}\hat{\psi}_a \bar{B}} \int d\theta d\zeta \frac{1}{\hat{B}^2} \left[\hat{\gamma} \hat{B} \left(\iota \frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta} \right) + \frac{1}{2} \left(\iota \frac{\partial (\hat{u}\hat{B}^2)}{\partial \theta} + \frac{\partial (\hat{u}\hat{B}^2)}{\partial \zeta} \right) \right] \frac{2}{5} \int_0^\infty dx \; x^4 f_2 \tag{45}$$

Define $\hat{V}' = \int d\theta d\zeta\ \hat{B}^{-2}$ and note that

$$V' = \int d\theta d\zeta\ \frac{G + \iota I}{B^2} = \frac{\bar{R}}{\bar{B}} \left(\hat{G} + \iota \hat{I}\right) \hat{V}' \tag{46}$$

We define the normalised torque in the single species code

$$\begin{aligned}\hat{\tau} &= \frac{1}{n\bar{T}} \langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi} \rangle \frac{\hat{\psi}_a(\hat{G} + \iota \hat{I})\hat{V}'}{2\Delta} = \\ &= \frac{\hat{T}^{5/2}}{\sqrt{\pi}} \int d\theta d\zeta \frac{1}{\hat{B}^2} \left[\hat{\gamma}\hat{B} \left(\iota \frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta} \right) + \frac{1}{2} \left(\iota \frac{\partial (\hat{u}\hat{B}^2)}{\partial \theta} + \frac{\partial (\hat{u}\hat{B}^2)}{\partial \zeta} \right) \right] \frac{2}{5} \int_0^\infty dx\ x^4 f_2\end{aligned} \tag{47}$$

In the multi-species code some normalisations are different, in particular, the definition of $\hat{f}$ differs in the following way,

$$\hat{f}^{\,\mathrm{multi}} = \frac{\bar{v}^3}{\bar{n}} f = \frac{\hat{m}^{3/2}\hat{n}}{\pi^{3/2}\hat{\psi}_a} \hat{f}. \tag{48}$$

A suitable normalised torque in the multiple species code is

$$\begin{aligned}\hat{\tau}^{\,\mathrm{multi}} &= \frac{1}{\bar{n}\bar{T}} \langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi} \rangle = \\ &= \frac{2\pi \hat{T}^{5/2}}{\hat{m}^{3/2}(\hat{G} + \iota \hat{I})\hat{V}'} \int d\theta d\zeta \frac{1}{\hat{B}^2} \left[\hat{\gamma}\hat{B} \left(\iota \frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta} \right) + \frac{1}{2} \left(\iota \frac{\partial (\hat{u}\hat{B}^2)}{\partial \theta} + \frac{\partial (\hat{u}\hat{B}^2)}{\partial \zeta} \right) \right] \frac{2}{5} \int_0^\infty dx\ x^4 f_2^{\mathrm{multi}}\end{aligned} \tag{49}$$