Equation

We wish to find the function u in Eq. (8) of [1],

$$\nabla_{\|} u = \frac{2}{B^2} \mathbf{b} \times \nabla\psi \cdot \nabla \ln B \quad \Leftrightarrow \quad \mathbf{b} \cdot \nabla u = \frac{2}{B^4} \nabla B \times \mathbf{B} \cdot \nabla\psi. \tag{1}$$

The magnetic field can be written

$$\mathbf{B} = \nabla\psi \times \nabla\theta + \iota \nabla\varphi \times \nabla\psi, \tag{2}$$

where ψ is the toroidal flux divided by 2π, and in Boozer coordinates

$$\mathbf{B} = I(\psi)\nabla\theta + J(\psi)\nabla\varphi + K(\psi,\theta,\varphi)\nabla\psi. \tag{3}$$

In Boozer coordinates the Jacobian is given by

$$g^{1/2} = \frac{1}{\nabla\psi \times \nabla\theta \cdot \nabla\varphi} = \frac{\iota I + J}{B^2}. \tag{4}$$

Using

$$\nabla B = \frac{\partial B}{\partial \psi}\nabla\psi + \frac{\partial B}{\partial \theta}\nabla\theta + \frac{\partial B}{\partial \varphi}\nabla\varphi \tag{5}$$

and Eq. 3 for the magnetic field we can write the RHS of Eq. 1 as

$$\begin{aligned} \frac{2}{B^4}\nabla B \times \mathbf{B} \cdot \nabla\psi &= \frac{2}{B^4}\left(I \frac{\partial B}{\partial \varphi} \nabla\varphi \times \nabla\theta \cdot \nabla\psi + J \frac{\partial B}{\partial \theta} \nabla\theta \times \nabla\varphi \cdot \nabla\psi \right) = \\ &= \frac{2}{B^4} g^{-1/2} \left(J \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \varphi} \right) = \frac{2}{B^2} \frac{1}{\iota I + J} \left(J \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \varphi} \right). \end{aligned} \tag{6}$$

Using Eq. 2 instead for the magnetic field the LHS can be written

$$\begin{aligned} \mathbf{b} \cdot \nabla u &= \frac{1}{B}\mathbf{B} \cdot \nabla u = \frac{1}{B}\left(\nabla\psi \times \nabla\theta + \iota\nabla\varphi \times \nabla\psi\right) \cdot \left(\frac{\partial u}{\partial \psi}\nabla\psi + \frac{\partial u}{\partial \theta}\nabla\theta + \frac{\partial u}{\partial \varphi}\nabla\varphi \right) = \\ &= \frac{1}{B}\left(\iota \frac{\partial u}{\partial \theta} \nabla\varphi \times \nabla\psi \cdot \nabla\theta + \frac{\partial u}{\partial \varphi} \nabla\psi \times \nabla\theta \cdot \nabla\varphi \right) = \frac{1}{B} g^{-1/2} \left(\iota \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \varphi} \right) = \frac{B}{\iota I + J}\left(\iota \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \varphi} \right) \end{aligned} \tag{7}$$

Setting Eqs. 6 and 7 equal we obtain

$$\frac{2}{B^3}\left(J \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \varphi} \right) = \left(\iota \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \varphi} \right). \tag{8}$$

A homogeneous solution to Eq. 8 would be

$$u_h(\psi,\theta,\varphi) = u_h(\psi,\theta - \iota\varphi) \equiv u_h(\psi,\alpha), \tag{9}$$

where $\alpha = \theta - \iota\varphi$ is constant along a magnetic field line. Consequently on an irrational surface, where a field line trace out the whole surface, the homogeneous solution must be a flux function

$$u_h(\psi,\theta,\varphi) = u_h(\psi). \tag{10}$$

Solution

To solve Eq. 8 we assume that the solution can be written as a cosine Fourier series in a similar way to how the magnetic field is represented

$$u\left(\psi,\theta,\varphi\right)=\sum_{n,m}u_{n,m}\left(\psi\right)\cos\left(m\theta-n\varphi\right), \tag{11}$$

$$B\left(\psi,\theta,\varphi\right)=\sum_{n,m}B_{n,m}\left(\psi\right)\cos\left(m\theta-n\varphi\right). \tag{12}$$

Then

$$\frac{\partial u}{\partial\theta}=\sum_{n,m}-m\,u_{n,m}\left(\psi\right)\sin\left(m\theta-n\varphi\right), \tag{13}$$

$$\frac{\partial u}{\partial\varphi}=\sum_{n,m}n\,u_{n,m}\left(\psi\right)\sin\left(m\theta-n\varphi\right). \tag{14}$$

The LHS of Eq. 8 is

$$\frac{2}{B^3}\left(J\frac{\partial B}{\partial\theta}-I\frac{\partial B}{\partial\varphi}\right)=\left(-J\frac{\partial B^{-2}}{\partial\theta}+I\frac{\partial B^{-2}}{\partial\varphi}\right), \tag{15}$$

and we can define a new cosine Fourier series

$$B^{-2}\left(\psi,\theta,\varphi\right)\equiv h\left(\psi,\theta,\varphi\right)=\sum_{n,m}h_{n,m}\left(\psi\right)\cos\left(m\theta-n\varphi\right), \tag{16}$$

and solve

$$\left(-J\frac{\partial h}{\partial\theta}+I\frac{\partial h}{\partial\varphi}\right)=\left(\iota\frac{\partial u}{\partial\theta}+\frac{\partial u}{\partial\varphi}\right). \tag{17}$$

Substituting Eqs. 13 and 14 and the equivalent for h into Eq. 17 yields

$$\sum_{n,m}\left(Jm+nI\right)h_{n,m}\left(\psi\right)\sin\left(m\theta-n\varphi\right)=\sum_{n,m}\left(-\iota m+n\right)u_{n,m}\left(\psi\right)\sin\left(m\theta-n\varphi\right) \tag{18}$$

hence the cosine Fourier coefficients in Eq. 11 is found from the Fourier coefficients of $h=B^{-2}$,

$$u_{n,m}\left(\psi\right)=\frac{Jm+nI}{n-\iota m}h_{n,m}\left(\psi\right). \tag{19}$$

Now

$$\int_0^{2\pi}\int_0^{2\pi}\cos\left(M\theta-N\varphi\right)\sum_{n,m}h_{n,m}\left(\psi\right)\cos\left(m\theta-n\varphi\right)d\theta d\varphi=$$

$$=h_{N,M}\left(\psi\right)\int_0^{2\pi}\int_0^{2\pi}\cos^2\left(M\theta-N\varphi\right)d\theta d\varphi=\begin{cases}4\pi^2h_{N,M}\left(\psi\right) & M=N=0\\ 2\pi^2h_{N,M}\left(\psi\right) & \text{otherwise}\end{cases}. \tag{20}$$

Therefore

$$h_{N,M}\left(\psi\right)=\begin{cases}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}B^{-2}d\theta d\varphi & M=N=0\\ \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos\left(M\theta-N\varphi\right)B^{-2}d\theta d\varphi & \text{otherwise}\end{cases}. \tag{21}$$

However from Eq. 19 it is clear that $u_{0,0}\left(\psi\right)$ can not be defined, but $u_{0,0}\left(\psi\right)$ is simply a flux-function constant and can thus be put $u_{0,0}\left(\psi\right)=0$.

Rational surface

On a rational surface, where $\iota = N/M$ we observe from Eq. 19 that $u_{N,M}(\psi)$ is infinite. However using $\alpha = \theta - \iota\varphi$ in the Fourier coefficient we obtain

$$h_{N,M}(\psi) = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos(M\theta - N\varphi)\,B^{-2}d\theta d\varphi = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos(M\alpha + M\iota\varphi - N\varphi)\,B^{-2}d\alpha d\varphi =$$
$$= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos(M\alpha)\,B^{-2}d\alpha d\varphi = \frac{1}{2\pi^2}\int_0^{2\pi}\cos(M\alpha)\,d\alpha\int_0^{2\pi}B^{-2}d\varphi. \quad (22)$$

We now note that from Eqs. 2 and 4

$$\mathbf{b}\cdot\nabla\varphi = \frac{1}{B}\mathbf{B}\cdot\nabla\varphi = \frac{1}{B}\nabla\psi\times\nabla\theta\cdot\nabla\varphi = \frac{B}{\iota I + J}, \quad (23)$$

and from [2] p.14

$$d\varphi = \mathbf{b}\cdot\nabla\varphi dl = \frac{B}{\iota I + J}dl \quad (24)$$

Eq. 22 can be rewritten as

$$h_{N,M}(\psi) = \frac{1}{2\pi^2}\int_0^{2\pi}\cos(M\alpha)\,\frac{1}{\iota I + J}d\alpha\oint\frac{dl}{B}. \quad (25)$$

From [2] Eq. (25)

$$\frac{\partial p}{\partial\psi}\frac{\partial}{\partial\alpha}\oint\frac{dl}{B} = 0, \quad (26)$$

which means that if there exists a pressure gradient (which we can assume) then $\oint\frac{dl}{B}$ is independent of α and therefore Eq. 25 yields

$$h_{N,M}(\psi) = 0. \quad (27)$$

The singularity in $u_{N,M}(\psi)$ in Eq. 19 from $\iota = N/M$ is cancelled by $h_{N,M}(\psi) = 0$.

Numerical details

The magnetic field in a stellarator can have a φ-period T_φ which is shorter than 2π, i.e. the slowest oscillating Fourier component is $N_\varphi = 2\pi/T_\varphi$ and all components are multiples of N_φ. Numerically it is thus better to evaluate the coefficients in Eq. 21 from

$$h_{N_\varphi\cdot N,M}(\psi) = \frac{1}{2\pi^2}\frac{2\pi}{T_\varphi}\int_0^{T_\varphi}\int_0^{2\pi}\cos(M\theta - N_\varphi\cdot N\varphi)\,B^{-2}d\theta d\varphi, \quad (28)$$

since all other coefficients will be 0.
Fixing ψ, (and consequently $I(\psi)$, $J(\psi)$) the magnetic field can be evaluated on a grid in θ–φ space

$$B(\psi,\theta,\varphi) = B_{i,j}. \quad (29)$$

Letting the grid consist of P_θ points in θ-space and P_φ points in φ-space, and defining the step lengths $l_\theta = 2\pi/P_\theta$ and $l_\varphi = T_\varphi/P_\varphi$, the grid points $i = 1, 2, \ldots, P_\theta$ and $j =$

$1, 2, \ldots, P_\varphi$ corresponds to $\theta_i = 0, l_\theta, 2l_\theta, \ldots, 2\pi - l_\theta$ and $\varphi_j = 0, l_\varphi, 2l_\varphi, \ldots, T_\varphi - l_\varphi$. The integral in Eq. 28 is thus approximated by a Riemann sum, according to

$$h_{N_\varphi \cdot N,M}(\psi) = \frac{1}{2\pi^2}\frac{2\pi}{T_\varphi} l_\theta l_\varphi \sum_{j=0}^{P_\varphi}\sum_{i=0}^{P_\theta} \cos\left(M\theta_i - N_\varphi \cdot N\varphi_j\right) B_{i,j}^{-2} = \frac{2}{P_\theta P_\varphi}\sum_{j=0}^{P_\varphi}\sum_{i=0}^{P_\theta} \cos\left(M\theta_i - N_\varphi \cdot N\varphi_j\right) B_{i,j}^{-2}. \tag{30}$$

Note that due to numerical reasons the number of Fourier harmonics used in the expansion should not be larger than the number of grid points, i.e. if $M > P_\theta$ and $N > P_\varphi$ then $h_{N,M}(\psi) = 0$.

Fluxes

For any quantity X, the flux surface average is computed from [2] Eq. (16)

$$\langle X \rangle = \frac{1}{V'}\int_0^{2\pi}\int_0^{2\pi} X g(\psi)^{1/2}\, d\theta d\varphi, \tag{31}$$

where

$$V'(\psi) = \int_0^{2\pi}\int_0^{2\pi} g(\psi)^{1/2}\, d\theta d\varphi \tag{32}$$

and from Eq. 4 $g(\psi)^{1/2} = (\iota I(\psi) + J(\psi))/B^2$ in Boozer coordinates. Thus we can write

$$\langle X \rangle = \int_0^{2\pi}\int_0^{2\pi} \frac{X}{\hat{B}^2} d\theta d\varphi \Big/ \int_0^{2\pi}\int_0^{2\pi} \frac{1}{\hat{B}^2} d\theta d\varphi. \tag{33}$$

where $\hat{B} = B/\bar{B}$ is the magnetic field normalized to some dimension $\bar{B}$.

The flux-surface averaged radial ion heat flux is in the Pfirsch-Schlüter regime given by Eq. (14) in [1]

$$\langle \mathbf{q} \cdot \nabla\chi \rangle = \frac{8}{5}\frac{\nu p}{M}\left(\frac{B}{\Omega}\right)^2 \frac{\partial T}{\partial \chi}\left(\frac{\langle uB^2\rangle^2}{\langle B^2\rangle} - \langle u^2 B^2\rangle\right) = \frac{8}{5}\frac{\nu n T}{M}\left(\frac{Mc}{e}\right)^2 \frac{\partial T}{\partial \chi}\left(\frac{\langle uB^2\rangle^2}{\langle B^2\rangle} - \langle u^2 B^2\rangle\right), \tag{34}$$

where $\nu = 4\pi^{1/2} n e^4 \ln\Lambda / \left(3M^{1/2}T^{3/2}\right)$ while M and e are the ion mass and magnitude of the electron charge, and χ is the poloidal flux. Since

$$\nabla\chi = \frac{\partial\chi}{\partial\psi}\nabla\psi = \iota\nabla\psi \tag{35}$$

and

$$\frac{\partial}{\partial\chi} = \frac{\partial}{\partial\psi}\frac{\partial\psi}{\partial\chi} = \frac{1}{\iota}\frac{\partial}{\partial\psi}, \tag{36}$$

we can rewrite Eq. 34 as

$$\langle \mathbf{q} \cdot \nabla\psi \rangle = \frac{8}{5}\frac{\nu n T}{M}\left(\frac{Mc}{e}\right)^2 \frac{1}{\iota^2}\frac{\partial T}{\partial \psi}\left(\frac{\langle uB^2\rangle^2}{\langle B^2\rangle} - \langle u^2 B^2\rangle\right). \tag{37}$$

Particle flux

The coefficient L_{11} is obtained by putting $\frac{\partial T}{\psi} = 0$ and $\frac{\partial \Phi}{\psi} = 0$. From Eqs. (131) and (96) in the SFINCS manual together with

$$\Delta = \frac{mc\bar{v}}{e\bar{B}\bar{R}} \tag{38}$$

we get

$$\begin{aligned} L_{11} &= (\text{particleFlux})\frac{\bar{R}}{V'\bar{B}}\frac{4\left(\hat{J}+\iota\hat{I}\right)}{\hat{J}}\frac{\hat{n}}{\hat{J}\hat{T}^{3/2}}\frac{B_0}{\bar{B}}\left(\frac{d\hat{n}}{d\psi_N}\right)^{-1} = \\ &= \frac{\hat{\psi}_a V'\left\langle\Gamma_i\cdot\nabla\psi\right\rangle}{\Delta^2 n\bar{v}\bar{R}^2}\frac{\bar{R}}{V'\bar{B}}\frac{4\left(\hat{J}+\iota\hat{I}\right)}{\hat{J}}\frac{\hat{n}}{\hat{J}\hat{T}^{3/2}}\frac{B_0}{\bar{B}}\left(\frac{d\hat{n}}{d\psi_N}\right)^{-1} = \frac{e^2\hat{\psi}_a\bar{R}}{m^2c^2\bar{v}^3}\frac{4\left(\hat{J}+\iota\hat{I}\right)}{\hat{J}^2}\frac{B_0}{\bar{n}\hat{T}^{3/2}}\left(\frac{d\hat{n}}{d\psi_N}\right)^{-1}\left\langle\Gamma_i\cdot\nabla\psi\right\rangle . \end{aligned} \tag{39}$$

From [1] Eq. (26) with $\eta = 0.96nT/\nu$, $\mathbf{J} = e\Gamma_i$ and using Eqs. 35, 36, the particle flux is obtained as

$$\left\langle\Gamma_i\cdot\nabla\psi\right\rangle = \frac{1}{\iota^2}\frac{3}{4}c^2\frac{0.96nT}{\nu}G_1(\psi)\frac{T}{e^2n}\frac{\partial n}{\partial\psi} = \frac{3c^2}{4\iota^2e^2}\frac{0.96T^2}{\nu}G_1(\psi)\frac{\partial n}{\partial\psi}. \tag{40}$$

Using

$$\frac{\partial n}{\partial\psi} = \frac{\bar{n}}{\psi_a}\frac{\partial\hat{n}}{\partial\psi_N} = \frac{\bar{n}}{\hat{\psi}_a\bar{R}^2\bar{B}}\frac{\partial\hat{n}}{\partial\psi_N}, \tag{41}$$

and

$$\nu = \left(\frac{T}{m}\right)^{1/2}\frac{B_0}{J+\iota I}\nu' = \left(\frac{\hat{T}}{2}\right)^{1/2}\bar{v}\frac{B_0}{\bar{R}\bar{B}\left(\hat{J}+\iota\hat{I}\right)}\nu' \tag{42}$$

with Eq. 40 into Eq. 39 yields

$$\begin{aligned} L_{11} &= \frac{e^2\hat{\psi}_a\bar{R}}{m^2c^2\bar{v}^3}\frac{4\left(\hat{J}+\iota\hat{I}\right)}{\hat{J}^2}\frac{B_0}{\bar{n}\hat{T}^{3/2}}\left(\frac{d\hat{n}}{d\psi_N}\right)^{-1}\frac{3c^2}{4\iota^2e^2}0.96T^2\frac{\bar{R}\bar{B}\left(\hat{J}+\iota\hat{I}\right)}{\left(\frac{\hat{T}}{2}\right)^{1/2}\bar{v}B_0\nu'}G_1(\psi)\frac{\bar{n}}{\hat{\psi}_a\bar{R}^2\bar{B}}\frac{\partial\hat{n}}{\partial\psi_N} = \\ &= \frac{0.96\cdot 3\cdot 2^{1/2}\left(\hat{J}+\iota\hat{I}\right)^2}{\iota^2\hat{J}^2}\frac{1}{m^2\bar{v}^4}\frac{T^2}{\hat{T}^2}G_1(\psi)\frac{1}{\nu'} = \frac{0.96\cdot 3\cdot 2^{1/2}\left(\hat{J}+\iota\hat{I}\right)^2}{4\iota^2\hat{J}^2}G_1(\psi)\frac{1}{\nu'}. \end{aligned} \tag{43}$$

Before we proceed we define a normalized parallel gradient from Eq. 7

$$\hat{\nabla}_\parallel = \bar{R}\,\nabla_\parallel = \bar{R}\,\mathbf{b}\cdot\nabla = \frac{\bar{R}B}{(\iota I+J)}\left(\iota\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\varphi}\right) = \frac{\hat{B}}{\left(\iota\hat{I}+\hat{J}\right)}\left(\iota\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\varphi}\right). \tag{44}$$

$G_1(\psi)$ we obtain from Eq. (27) which, written in terms of SFINCS normalization with $\hat{u}$ defined according to

$$\hat{u} \equiv u\bar{B}/\bar{R} \quad\Leftrightarrow\quad u = \hat{u}\bar{R}/\bar{B} \tag{45}$$

, is

$$G_1(\psi) = \frac{\left\langle \left(\frac{1}{\bar{R}}\hat{\nabla}_\| \ln \hat{B}\right) \frac{1}{\bar{R}}\hat{\nabla}_\| \left(\hat{u}\bar{R}/\bar{B}\cdot \bar{B}^2\hat{B}^2\right)\right\rangle^2}{\left\langle \left(\frac{1}{\bar{R}}\hat{\nabla}_\| \bar{B}\hat{B}\right)^2\right\rangle} - \left\langle \left[\frac{\frac{1}{\bar{R}}\hat{\nabla}_\| \left(\hat{u}\bar{R}/\bar{B}\cdot \bar{B}^2\hat{B}^2\right)}{\bar{B}\hat{B}}\right]^2\right\rangle =$$

$$= \frac{\left\langle \left(\hat{\nabla}_\| \ln \hat{B}\right) \hat{\nabla}_\| \left(\hat{u}\hat{B}^2\right)\right\rangle^2}{\left\langle \left(\hat{\nabla}_\| \hat{B}\right)^2\right\rangle} - \left\langle \left[\frac{\hat{\nabla}_\| \left(\hat{u}\hat{B}^2\right)}{\hat{B}}\right]^2\right\rangle. \quad (46)$$

The parallel gradients of interest are

$$\hat{\nabla}_\| \ln \hat{B} = \frac{1}{\left(\iota\hat{I}+\hat{J}\right)}\left(\iota\frac{\partial \hat{B}}{\partial\theta}+\frac{\partial \hat{B}}{\partial\varphi}\right), \quad (47)$$

$$\hat{\nabla}_\| \hat{B} = \hat{B}\,\hat{\nabla}_\| \ln \hat{B} \quad (48)$$

and

$$\hat{\nabla}_\| \left(\hat{u}\hat{B}^2\right) = \frac{\hat{B}}{\left(\iota\hat{I}+\hat{J}\right)}\left(\iota\hat{B}^2\frac{\partial \hat{u}}{\partial\theta}+2\iota\hat{B}\hat{u}\frac{\partial \hat{B}}{\partial\theta}+\hat{B}^2\frac{\partial \hat{u}}{\partial\varphi}+2\hat{B}\hat{u}\frac{\partial \hat{B}}{\partial\varphi}\right). \quad (49)$$

SFINCS

TODO: Fix this section, contains error

In SFINCS

$$\text{nu\_ii} = \frac{4\left(2\pi\right)^{1/2} nZ^4e^4\ln\Lambda}{3M^{1/2}T^{3/2}} = \sqrt{2}\nu, \quad (50)$$

as compared to ν in [1]. Furthermore

$$\text{nu\_ii} = \frac{\bar{v}}{\bar{R}}\text{nuN} = \frac{\bar{v}}{\bar{R}}\frac{\hat{n}}{\hat{T}^{3/2}}\text{nu\_nbar}, \quad (51)$$

and therefore

$$\nu = \frac{1}{\sqrt{2}}\frac{\bar{v}}{\bar{R}}\frac{\hat{n}}{\hat{T}^{3/2}}\text{nu\_nbar}. \quad (52)$$

Also in SFINCS

$$\frac{Mc}{e} = \frac{\bar{B}\bar{R}}{\bar{v}}\text{Delta} \quad (53)$$

in Gaussian units, $n = \hat{n}\bar{n}$, $T = \hat{T}\bar{T}$, $\psi = \psi_N\psi_a$ where ψ_a is the toroidal flux of the last closed flux surface, and

$$\frac{\partial T}{\partial\psi} = \frac{\bar{T}}{\psi_a}\frac{\partial \hat{T}}{\partial\psi_N}, \quad (54)$$

where $\psi_a = \hat{\psi}_a\bar{B}\bar{R}^2$.

From $I = \hat{I}\bar{B}\bar{R}$, $J = \hat{J}\bar{B}\bar{R}$, $h = B^{-2} = \hat{B}^{-2}\bar{B}^{-2} \equiv \hat{h}\bar{B}^{-2}$ and Eq. 19 we see that

$$u_{n,m}(\psi) = \frac{\hat{J}\bar{B}\bar{R}m + n\hat{I}\bar{B}\bar{R}}{n - \iota m}\hat{h}_{n,m}(\psi)\,\bar{B}^{-2}. \tag{55}$$

Hence it is convenient to define

$$\hat{u} \equiv u\bar{B}/\bar{R} \quad \Leftrightarrow \quad u = \hat{u}\bar{R}/\bar{B} \tag{56}$$

with

$$\hat{u}_{n,m}(\psi) = \frac{\hat{J}m + n\hat{I}}{n - \iota m}\hat{h}_{n,m}(\psi). \tag{57}$$

Then Eq. 37 is

$$\langle \mathbf{q}\cdot\nabla\psi\rangle = \frac{8}{5}\frac{1}{\sqrt{2}}\frac{\bar{v}}{\bar{R}}\frac{\hat{n}}{\hat{T}^{3/2}}\text{nu\_nbar}\frac{\hat{n}\bar{n}\hat{T}\bar{T}}{M}\left(\frac{\bar{B}\bar{R}}{\bar{v}}\text{Delta}\right)^2\frac{1}{\iota^2}\frac{\bar{T}}{\hat{\psi}_a\bar{B}\bar{R}^2}\frac{\partial\hat{T}}{\partial\psi_N}\left(\frac{\left\langle \hat{u}\bar{R}/\bar{B}\left(\hat{B}\bar{B}\right)^2\right\rangle^2}{\left\langle\left(\hat{B}\bar{B}\right)^2\right\rangle} - \left\langle\left(\hat{u}\bar{R}/\bar{B}\right)^2\left(\hat{B}\bar{B}\right)^2\right\rangle\right) =$$

$$= \frac{8}{5\sqrt{2}}\frac{\bar{n}\bar{T}^2\bar{B}\bar{R}}{\bar{v}}\frac{\hat{n}^2}{M\iota^2\hat{\psi}_a\hat{T}^{1/2}}\frac{\partial\hat{T}}{\partial\psi_N}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\text{nu\_nbar}\cdot\text{Delta}^2 =$$

$$= \frac{\sqrt{2}}{5}\text{Delta}^2\frac{n\bar{v}^3M\bar{B}\bar{R}}{\hat{\psi}_a}\frac{\hat{n}}{\iota^2\hat{T}^{1/2}}\frac{\partial\hat{T}}{\partial\psi_N}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\text{nu\_nbar}. \tag{58}$$

Furthermore

$$V' = \frac{\partial V}{\partial\psi} = \int_0^{2\pi}\int_0^{2\pi} g^{1/2}d\theta d\varphi = \int_0^{2\pi}\int_0^{2\pi}\frac{\iota I + J}{B^2}d\theta d\varphi = \int_0^{2\pi}\int_0^{2\pi}\frac{\iota\bar{B}\bar{R}\hat{I} + \bar{B}\bar{R}\hat{J}}{\left(\bar{B}\hat{B}\right)^2}d\theta d\varphi = \frac{\bar{R}}{\bar{B}}\left(\iota\hat{I} + \hat{J}\right)\hat{V}'. \tag{59}$$

Combining Eqs. 58 and 59 the heat flux is

$$V'\langle\mathbf{q}\cdot\nabla\psi\rangle = \text{Delta}^2\frac{n\bar{v}^3M\bar{R}^2}{\hat{\psi}_a}\frac{\sqrt{2}}{5}\frac{\hat{n}\left(\iota\hat{I} + \hat{J}\right)\hat{V}'}{\iota^2\hat{T}^{1/2}}\frac{\partial\hat{T}}{\partial\psi_N}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\text{nu\_nbar}, \tag{60}$$

to be compared with Eq. (104) in the SFINCS Technical Documentation.

The quantity in SFINCS called heatFlux is

$$\text{heatFlux} = \frac{\sqrt{2}}{5}\frac{\hat{n}\left(\iota\hat{I} + \hat{J}\right)\hat{V}'}{\iota^2\hat{T}^{1/2}}\frac{\partial\hat{T}}{\partial\psi_N}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\text{nu\_nbar}. \tag{61}$$

Furthermore

$$\nu' = \frac{1}{\hat{T}^{1/2}}\frac{\bar{B}}{B_0}\left(\iota\hat{I} + \hat{J}\right)\text{nuN} = \frac{1}{\hat{T}^{1/2}}\frac{\bar{B}}{B_0}\left(\iota\hat{I} + \hat{J}\right)\frac{\hat{n}}{\hat{T}^{3/2}}\text{nu\_nbar} \tag{62}$$

i.e.

$$\text{nu\_nbar} = \frac{\hat{T}^2 B_0}{\hat{n}\bar{B}\left(\iota\hat{I}+\hat{J}\right)}\nu'. \tag{63}$$

Consequently

$$\text{heatFlux} = \frac{\sqrt{2}}{5}\frac{\hat{V}'\hat{T}^{3/2}}{\iota^2}\frac{\partial\hat{T}}{\partial\psi_N}\frac{B_0}{\bar{B}}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\nu'. \tag{64}$$

SFINCS Technical Documentation Eq. (135) gives the coefficient $L_{2,2}$

$$L_{2,2} = \text{heatFlux}\frac{\bar{R}}{V'\bar{B}}\frac{8\left(\iota\hat{I}+\hat{J}\right)}{\hat{J}}\frac{1}{\hat{T}^{3/2}\hat{J}}\frac{B_0}{\bar{B}}\left(\frac{\partial\hat{T}}{\partial\psi_N}\right)^{-1} = \frac{\sqrt{2}}{5}\left(\frac{B_0}{\bar{B}}\right)^2\frac{\bar{R}}{\bar{B}}\frac{8\left(\iota\hat{I}+\hat{J}\right)\hat{V}'}{\iota^2 V'\hat{J}^2}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\nu' =$$

$$= \frac{\sqrt{2}}{5}\left(\frac{B_0}{\bar{B}}\right)^2\frac{8}{\iota^2\hat{J}^2}\left(\frac{\left\langle\hat{u}\hat{B}^2\right\rangle^2}{\left\langle\hat{B}^2\right\rangle} - \left\langle\hat{u}^2\hat{B}^2\right\rangle\right)\nu'. \tag{65}$$

Axisymmetric analytical model

In the axisymmetric limit u is given by [1] Eq. (15), apart from a flux function

$$u_{\text{as}} = -\frac{J}{B^2} = -\frac{\hat{J}\bar{B}\bar{R}}{\left(\bar{B}\hat{B}\right)^2} = -\frac{\bar{R}}{\bar{B}}\frac{\hat{J}}{\hat{B}^2} \;\Rightarrow\; \hat{u}_{\text{as}} = -\frac{\hat{J}}{\hat{B}^2}. \tag{66}$$

TODO: Compare axisymmetric model to Simakov
TODO: Compare pure helical model to analytical, $\epsilon_t = 0$, $\epsilon_h \neq 0$

Pfirsch-Schlüter regime test cases

The Pfirsch-Schlüter regime is define from

$$\nu^* \equiv \frac{\nu_{ii}R}{\iota\, v_T} \gg 1, \tag{67}$$

which means that in SFINCS

$$\nu_n \equiv \frac{\nu_{ii}\bar{R}}{\bar{v}} \gg 1. \tag{68}$$

Since

$$\nu_n = \text{nu\_nbar}\frac{\hat{n}}{\hat{T}^{3/2}}, \tag{69}$$

we should choose a large $\hat{n}$ and/or a small $\hat{T}$.

Large-aspect ratio tokamak

Input

$B_0/\bar{B}$	1.55
ϵ_t	-0.178539
ϵ_h	0
helicity$_l$	0
helicity$_n$	0
ι	0.397543
$\hat{G}$	6.2
$\hat{I}$	0.0785669
$\hat{\psi}_a$	0.323266
$\hat{T}$	0.0001
$\hat{n}$	1.0
$\frac{\partial \hat{\phi}}{\partial \psi}$	0
$\frac{\partial \hat{T}}{\partial \psi}$	-0.2
$\frac{\partial \hat{n}}{\partial \psi}$	-0.2
$\hat{E}$	0
collisionOperator	0

Output

```
Sources: -0.00140448,  0.000612761
FSADensityPerturbation:  -1.1166e-15
FSAFlow:                 -0.000103857
FSAPressurePerturbation: 0
particleFlux:            -3.84474e-05
momentumFlux:            -4.3654e-12
heatFlux:                0.000161097
```

References

[1] A. N. Simakov, P. Helander, *Phys. Plasmas* **16**, 042503 (2009).

[2] P. Helander, Theory of plasma confinement in non-axisymmetric magnetic fields (2013).