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NTV

Consider the momentum equation summed over species,

$$\boldsymbol{J}\times\boldsymbol{B}-\nabla\cdot\mathbf{P}=\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V})+\frac{\partial(\rho\boldsymbol{V})}{\partial t}. \tag{1}$$

We write

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{B}_0+\boldsymbol{B}_1, &(2)\\ \boldsymbol{J} &= \boldsymbol{J}_0+\boldsymbol{J}_1, &(3)\\ \mathbf{P} &= p_0(\psi)\mathbf{I}+p_1(\psi)\mathbf{I}+\mathbf{\Pi}, &(4)\end{aligned}$$

where

$$\boldsymbol{J}_0\times\boldsymbol{B}_0=p_0'(\psi)\nabla\psi. \tag{5}$$

Subtracting Eq. (5) from Eq. (1), we obtain

$$\frac{\partial\rho\boldsymbol{V}}{\partial t}=\boldsymbol{J}_1\times\boldsymbol{B}_0+\boldsymbol{J}_0\times\boldsymbol{B}_1-\nabla p_1-\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi}). \tag{6}$$

where we have used the vector identity $\nabla B_1^2=2\boldsymbol{B}_1\cdot\nabla\boldsymbol{B}_1+2\boldsymbol{B}_1\times\nabla\times\boldsymbol{B}_1$ to write $\boldsymbol{J}_1\times\boldsymbol{B}_1=-\nabla\cdot\mathbf{M}$, with

$$\mathbf{M}=\frac{1}{\mu_0}\left(\frac{1}{2}B_1^2\mathbf{I}-\boldsymbol{B}_1\boldsymbol{B}_1\right). \tag{7}$$

The two components of Eq. (6) in the directions along $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are particularly interesting as we shall see, because several terms disappear upon taking the flux surface average. Observe that

$$\begin{aligned}\langle\boldsymbol{B}_0\cdot\boldsymbol{J}_0\times\boldsymbol{B}_1\rangle &= \langle\boldsymbol{B}_1\cdot\nabla p_0\rangle=-\langle(\nabla\times\boldsymbol{A}_1)\cdot\nabla p_0\rangle=\langle\nabla\cdot(\nabla p_0\times\boldsymbol{A}_1)\rangle=0, &(8)\\ \langle\boldsymbol{J}_0\cdot\boldsymbol{J}_1\times\boldsymbol{B}_0\rangle &= -\langle\boldsymbol{J}_1\cdot\nabla p_0\rangle=\frac{1}{\mu_0}\langle(\nabla\times\boldsymbol{B}_1)\cdot\nabla p_0\rangle=-\langle\nabla\cdot(\nabla p_0\times\boldsymbol{B}_1)\rangle=0, &(9)\\ \langle\boldsymbol{B}_0\cdot\nabla p_1\rangle &= \langle\nabla\cdot(\boldsymbol{B}_0p_1)\rangle=0, &(10)\\ \langle\boldsymbol{J}_0\cdot\nabla p_1\rangle &= \langle\nabla\cdot(\boldsymbol{J}_0p_1)\rangle=0 &(11)\end{aligned}$$

if we neglect the displacement current. The flux surface average of the scalar product of Eq. (6) with $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ yields

$$\begin{aligned}\frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{B}_0\rangle}{\partial t} &= -\left\langle\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\cdot\boldsymbol{B}_0\right\rangle, &(12)\\ \frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{J}_0\rangle}{\partial t} &= -\left\langle\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\cdot\boldsymbol{J}_0\right\rangle. &(13)\end{aligned}$$

Consequently, the same type of expression also holds for any linear combination $\boldsymbol{L}=\alpha\boldsymbol{B}_0+\beta\boldsymbol{J}_0$ of $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$,

$$\frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{L}\rangle}{\partial t} = -\left\langle\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\cdot\boldsymbol{L}\right\rangle \tag{14}$$

Axisymmetry

In axisymmetry we can express $\boldsymbol{B}$ and $\boldsymbol{J}$ as

$$\begin{aligned} \boldsymbol{B}_0 &= F(\psi)\nabla\phi + \nabla\phi \times \nabla\psi, && (15)\\ \boldsymbol{J}_0 &= -p_0'(\psi)\frac{\partial \boldsymbol{r}}{\partial \phi} + K(\psi)\boldsymbol{B}_0, && (16) \end{aligned}$$

where ϕ is the geometrical toroidal angle. The flux surface average of the parallel component of Eq. (16) gives us $K(\psi) = (\langle \boldsymbol{J}_0 \cdot \boldsymbol{B}_0\rangle + p_0'F)/\langle B^2\rangle$. We are interessed in the toroidal torque on the plasma, so in Eq. (14) we chose

$$\boldsymbol{L} = R\hat{\phi} = \frac{\partial \boldsymbol{r}}{\partial \phi} = -\frac{1}{p_0'}\boldsymbol{J}_0 + \frac{K}{p_0'}\boldsymbol{B}_0 \qquad (17)$$

Non-axisymmetry

For any 3D magnetic configuration, we would now like to define a generalised vector $\boldsymbol{L}$, which in the limit of axisymmetry becomes $R\hat{\phi}$. One way to ensure that this limit is satisfied is to require the streamlines of $\boldsymbol{L}$ to close on themselves toroidally. This also makes sense because we are not interested in any net poloidal torque component.

In Hamada coordinates (V, ϑ, φ), the streamlines of both $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are straight, i.e. $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are linear combinations of $\partial\boldsymbol{r}/\partial\vartheta$ and $\partial\boldsymbol{r}/\partial\varphi$. The sought linear combination of $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ whose streamlines close on themselves toroidally is thus $\boldsymbol{e}_\varphi = \partial\boldsymbol{r}/\partial\varphi$. Because of the above mentioned reasons, $\boldsymbol{e}_\varphi$ becomes parallel to $\hat{\phi}$ in the limit of axisymmetry (note that $\nabla\varphi$ does not become parallel to $\hat{\phi}$). Therefore, since $\nabla \cdot \boldsymbol{e}_\varphi = 0$ and $\nabla \cdot R\hat{\phi} = 0$, we conclude that $\boldsymbol{e}_\varphi \to cR\hat{\phi}$, and the constant $c = 1$ because φ and ϕ are both 2π periodic.

We now want to determine the constants α and β in $\boldsymbol{L} = \boldsymbol{e}_\varphi = \alpha\boldsymbol{B}_0 + \beta\boldsymbol{J}_0$. In Hamada coordinates, we can express $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ as

$$\begin{aligned} \boldsymbol{B}_0 &= \nabla\psi \times \nabla\vartheta + \iota(\psi)\nabla\varphi \times \nabla\psi = I(\psi)\nabla\vartheta + G(\psi)\nabla\varphi + \nabla H(\psi, \vartheta, \varphi), && (18)\\ \boldsymbol{J}_0 &= I'(\psi)\nabla\psi \times \nabla\vartheta - G'(\psi)\nabla\varphi \times \nabla\psi. && (19) \end{aligned}$$

Moreover, the Jacobian is a flux function, so

$$\begin{aligned} V'(\psi) &= \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \frac{1}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} = \frac{4\pi^2}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} && (20)\\ \boldsymbol{e}_\varphi &= \frac{\nabla\psi \times \nabla\vartheta}{\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi} = \frac{V'}{4\pi^2}\nabla\psi \times \nabla\vartheta, && (21) \end{aligned}$$

and equilibrium implies

$$\begin{aligned} \boldsymbol{J}_0 \times \boldsymbol{B}_0 &= -\frac{1}{\mu_0}(\iota I' + G')(\nabla\psi \cdot \nabla\vartheta \times \nabla\varphi)\nabla\psi = p_0'\nabla\psi && (22)\\ \iota I' + G' &= -\frac{V'}{4\pi^2}\mu_0 p_0' && (23) \end{aligned}$$

The equation $\boldsymbol{e}_\varphi = \alpha\boldsymbol{B}_0 + \beta\boldsymbol{J}_0$ becomes

$$\frac{V'}{4\pi^2}\nabla\psi \times \nabla\vartheta = \alpha\left(\nabla\psi \times \nabla\vartheta + \iota\nabla\varphi \times \nabla\psi\right) + \beta\left(I'\nabla\psi \times \nabla\vartheta - G'\nabla\varphi \times \nabla\psi\right) \qquad (24)$$

yielding

$$
\begin{aligned}
\alpha &= \frac{4\pi^2}{V'}\frac{G'}{G'+\iota I'} = -\mu_0 p_0' \frac{G'}{(G'+\iota I')^2}, && (25)\\
\beta &= \frac{4\pi^2}{V'}\frac{\iota}{G'+\iota I'} = -\mu_0 p_0' \frac{\iota}{(G'+\iota I')^2}, && (26)
\end{aligned}
$$

i.e.,

$$
\boldsymbol{e}_\varphi = -\frac{\mu_0 p_0'}{(G'+\iota I')^2}\left(\iota \boldsymbol{J}_0 + G'\boldsymbol{B}_0\right). \tag{27}
$$

Note that in the expressions for α and β, the flux functions $G(\psi)$ and $I(\psi)$ are the same as in Boozer coordinates (ψ, θ, ζ), where

$$
\boldsymbol{B}_0 = \nabla\psi \times \nabla\theta + \iota\nabla\zeta \times \nabla\psi = I(\psi)\nabla\theta + G(\psi)\nabla\zeta + \kappa(\psi,\theta,\zeta)\nabla\psi. \tag{28}
$$

The toroidal viscosity

We now turn our attention to the toroidal viscosity term in Eq. (14),

$$
\langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi} \rangle = \alpha \langle \boldsymbol{B} \cdot \nabla \cdot \boldsymbol{\Pi} \rangle + \beta \langle \boldsymbol{J} \cdot \nabla \cdot \boldsymbol{\Pi} \rangle. \tag{29}
$$

where we have dropped the index 0 on $\boldsymbol{B}_0$, $\boldsymbol{J}_0$ and p_0. First, we note that $\boldsymbol{\Pi} = \tilde{p}(\mathbf{I}/3 - B^{-2}\boldsymbol{B}\boldsymbol{B})$, where $\tilde{p} \equiv p_\perp - p_\parallel$, which implies that

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\Pi} &= \frac{1}{3}\nabla\tilde{p} - \nabla\frac{\tilde{p}}{B^2}\cdot \boldsymbol{B}\boldsymbol{B} - \tilde{p}\left(B^{-1}\nabla B + \frac{\mu_0}{B^2}\nabla p\right) && (30)\\
\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi} &= -\frac{2}{3}\boldsymbol{B}\cdot\nabla\tilde{p} + \frac{\tilde{p}}{B}\nabla B && (31)\\
\langle \boldsymbol{B}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle &= \langle \tilde{p}\nabla_\parallel B\rangle && (32)\\
\langle \boldsymbol{J}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle &= -\left\langle J_\parallel B\boldsymbol{B}\cdot\nabla\frac{\tilde{p}}{B^2}\right\rangle - \left\langle \frac{\tilde{p}}{B}\boldsymbol{J}\cdot\nabla B\right\rangle && (33)
\end{aligned}
$$

In the last expression, we can replace $\boldsymbol{J}$ with $\boldsymbol{J} = J_\parallel B^{-1}\boldsymbol{B} + p'B^{-2}(\boldsymbol{B}\times\nabla\psi)$ where $\nabla\cdot\boldsymbol{J} = 0$ gives $\boldsymbol{B}\cdot\nabla(J_\parallel/B) = 2p'B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B$. If we also use that $\langle a\boldsymbol{B}\cdot\nabla b\rangle = -\langle b\boldsymbol{B}\cdot\nabla a\rangle$, we can write

$$
\begin{aligned}
\langle \boldsymbol{J}\cdot\nabla\cdot\boldsymbol{\Pi}\rangle &= \left\langle \frac{\tilde{p}}{B^2}\boldsymbol{B}\cdot\nabla(J_\parallel B)\right\rangle - \left\langle \frac{\tilde{p}}{B^2}J_\parallel\boldsymbol{B}\cdot\nabla B\right\rangle - p'\left\langle \frac{\tilde{p}}{B^3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle = \\
&= \left\langle \frac{\tilde{p}}{B^2}J_\parallel\boldsymbol{B}\cdot\nabla B\right\rangle + p'\left\langle \frac{\tilde{p}}{B^3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle = \\
&= \frac{1}{2}\left\langle \frac{\tilde{p}}{B^2}\boldsymbol{B}\cdot\nabla(J_\parallel B)\right\rangle. && (34)
\end{aligned}
$$

We obtain

$$
\langle \boldsymbol{e}_\varphi\cdot\nabla\cdot\boldsymbol{\Pi}\rangle = \frac{-\mu_0 p'}{(G'+\iota I')^2}\left(G'\left\langle\tilde{p}\nabla_\parallel B\right\rangle + \frac{\iota}{2}\left\langle\frac{\tilde{p}}{B}\nabla_\parallel(J_\parallel B)\right\rangle\right) \tag{35}
$$

Implementation

Define a normalised torque

$$\tau = \langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi} \rangle \frac{(G' + \iota I')^2}{-\mu_0 p'^2} = \frac{G'}{p'} \left\langle \tilde{p} \nabla_\| B \right\rangle + \frac{\iota}{2p'} \left\langle \frac{\tilde{p}}{B} \nabla_\| (J_\| B) \right\rangle \tag{36}$$

To calculate this, we first need to determine the quantity $u \equiv J_\| / (Bp')$ from the equation

$$\boldsymbol{B} \cdot \nabla u \;\; = \;\; 2B^{-3} \boldsymbol{B} \times \nabla \psi \cdot \nabla B, \tag{37}$$

which in Boozer coordinates corresponds to

$$\left(\iota \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \zeta} \right) = 2 \frac{1}{B^3} \left(G \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \zeta} \right). \tag{38}$$

In SFINCS normalisation, we have $G = \hat{G} \bar{R} \bar{B}$, $B = \hat{B} \bar{B}$, $u = \hat{u} \bar{R} / \bar{B}$ and

$$\left(\iota \frac{\partial \hat{u}}{\partial \theta} + \frac{\partial \hat{u}}{\partial \zeta} \right) = 2 \frac{1}{\hat{B}^3} \left(\hat{G} \frac{\partial \hat{B}}{\partial \theta} - \hat{I} \frac{\partial \hat{B}}{\partial \zeta} \right). \tag{39}$$

The torque becomes (denote $\gamma \equiv G'/p'$)

$$\begin{aligned} \tau \;\; &= \;\; \gamma \left\langle \frac{\tilde{p}}{B\sqrt{g}} \left(\iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) \right\rangle + \frac{\iota}{2} \left\langle \frac{\tilde{p}}{B^2 \sqrt{g}} \left(\iota \frac{\partial (uB^2)}{\partial \theta} + \frac{\partial (uB^2)}{\partial \zeta} \right) \right\rangle = \\ &= \;\; \gamma \int d\theta d\zeta \frac{\tilde{p}}{B} \left(\iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) + \frac{\iota}{2} \int d\theta d\zeta \frac{\tilde{p}}{B^2} \left(\iota \frac{\partial (uB^2)}{\partial \theta} + \frac{\partial (uB^2)}{\partial \zeta} \right) = \\ &= \;\; \int d\theta d\zeta \frac{\tilde{p}}{B^2} \left[\gamma B \left(\iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \zeta} \right) + \frac{\iota}{2} \left(\iota \frac{\partial (uB^2)}{\partial \theta} + \frac{\partial (uB^2)}{\partial \zeta} \right) \right] \end{aligned} \tag{40}$$

In the normalisations used in the SFINCS single species documentation,

$$\begin{aligned} &\tilde{p} = p_\perp - p_\| = m \int d^3 v \; f \left(\frac{v_\perp^2}{2} - v_\|^2 \right) = \frac{2\Delta \hat{T}^{3/2} n}{\sqrt{\pi} \hat{\psi}_a} \int_{-1}^{1} d\xi \int_0^\infty dx \; x^2 \hat{f} m \left(\frac{v_\perp^2}{2} - v_\|^2 \right) = \\ &= \left\{ m \left(\frac{v_\perp^2}{2} - v_\|^2 \right) = \hat{T} \bar{T} \frac{x^2}{2} (1 - 3\xi^2) = -\hat{T} \bar{T} x^2 P_2(\xi) \right\} = \\ &= -\bar{T} n \frac{2\Delta \hat{T}^{5/2}}{\sqrt{\pi} \hat{\psi}_a} \int_{-1}^{1} d\xi \int_0^\infty dx \; x^4 P_2(\xi) \hat{f}. \end{aligned} \tag{41}$$

Note the following about the Legendre polynomial P_2,

$$\int_{-1}^{1} d\xi P_2^2(\xi) = \frac{2}{5}, \tag{42}$$

so that with

$$\hat{f} = \sum_{l=0}^{\infty} f_l P_l(\xi) \tag{43}$$

we obtain

$$\int_{-1}^{1} d\xi P_2(\xi) \hat{f} = \frac{2}{5} f_2. \tag{44}$$

If we define $\gamma = \hat{\gamma}\bar{R}/\bar{B}$ we can write

$$\tau = -\bar{T}n\frac{2\Delta\hat{T}^{5/2}\bar{R}}{\sqrt{\pi}\hat{\psi}_a\bar{B}}\int d\theta d\zeta \frac{\tilde{p}}{\hat{B}^2}\left[\hat{\gamma}B\left(\iota\frac{\partial\hat{B}}{\partial\theta}+\frac{\partial\hat{B}}{\partial\zeta}\right)+\frac{\iota}{2}\left(\iota\frac{\partial(\hat{u}\hat{B}^2)}{\partial\theta}+\frac{\partial(\hat{u}\hat{B}^2)}{\partial\zeta}\right)\right]\frac{2}{5}\int_0^\infty dx\ x^4 f_2 \tag{45}$$

Define $\hat{i} = (I'/p')(\bar{B}/\bar{R})$ and

$$\begin{aligned}\hat{\tau} &= \frac{1}{\bar{n}\bar{T}}\langle \boldsymbol{e}_\varphi\cdot\nabla\cdot\mathbf{\Pi}\rangle = \frac{\tau}{\bar{n}\bar{T}}\frac{-\mu_0 p'^2}{(G'+\iota I')^2} = \frac{\tau}{\bar{n}\bar{T}}\frac{4\pi^2 p'}{V'(G'+\iota I')} = \tau\frac{\bar{B}}{\bar{n}\bar{T}\bar{R}}\frac{4\pi^2}{V'(\hat{\gamma}+\iota\hat{i})} = \\ &= -\hat{n}\frac{2\Delta\hat{T}^{5/2}}{\sqrt{\pi}\hat{\psi}_a(\hat{\gamma}+\iota\hat{i})}\frac{4\pi^2}{V'}\int d\theta d\zeta\frac{\tilde{p}}{\hat{B}^2}\left[\hat{\gamma}B\left(\iota\frac{\partial\hat{B}}{\partial\theta}+\frac{\partial\hat{B}}{\partial\zeta}\right)+\frac{\iota}{2}\left(\iota\frac{\partial(\hat{u}\hat{B}^2)}{\partial\theta}+\frac{\partial(\hat{u}\hat{B}^2)}{\partial\zeta}\right)\right]\frac{2}{5}\int_0^\infty dx\ x^4 f_2\end{aligned} \tag{46}$$