Notes on NTV

From Lewandowski et al. (2001) we have that

$$\langle \mathbf{e}_\zeta \cdot (\nabla \cdot \mathbf{P}) \rangle = \frac{1}{2}\left\langle \frac{\partial}{\partial \zeta}(p_\parallel + p_\perp) \right\rangle = \frac{1}{2}\int \frac{\partial}{\partial \zeta}(p_\parallel + p_\perp) J d\zeta d\theta =$$
$$= -\frac{1}{2}\int (p_\parallel + p_\perp)\frac{\partial \ln J}{\partial \zeta} J d\zeta d\theta = \int (p_\parallel + p_\perp)\frac{\partial \ln B}{\partial \zeta} J d\zeta d\theta = \left\langle \frac{p_\parallel + p_\perp}{B}\frac{\partial B}{\partial \zeta} \right\rangle, \quad (1)$$

where we use the the contravariant toroidal basis vector $\mathbf{e}_\zeta = \partial \mathbf{r}/\partial \zeta = J \nabla \Psi \times \nabla \theta$ in Boozer coordinates, which are the coordinates used in SFINCS. Note that $\mathbf{e}_\zeta$ depends on the choice of coordinates, i.e., it is for instance different in Hamada coordinates, which are often used in NTV calculations (Shaing & Callen 1983).

In the normalisations used in the SFINCS single species documentation,

$$p_\parallel + p_\perp = m \int d^3v\, f \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) = \frac{2\Delta \hat{T}^{3/2} n}{\sqrt{\pi}\hat{\psi}_a} \int_{-1}^{1} d\xi \int_0^\infty dx\, x^2 \hat{f} m \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) =$$
$$= \left\{ m\left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) = \hat{T}\bar{T} x^2 (1+\xi^2) \right\} = \bar{T} n \frac{2\Delta \hat{T}^{5/2}}{\sqrt{\pi}\hat{\psi}_a} \int_{-1}^{1} d\xi \int_0^\infty dx\, x^4 (1+\xi^2) \hat{f}. \quad (2)$$

Note the following about the Legendre polynomials P_l,

$$1 + \xi^2 = \frac{4}{3}P_0(\xi) + \frac{2}{3}P_2(\xi) \quad (3)$$

$$\int_{-1}^{1} d\xi P_0^2(\xi) = 2 \quad (4)$$

$$\int_{-1}^{1} d\xi P_2^2(\xi) = \frac{2}{5}, \quad (5)$$

so that with

$$\hat{f} = \sum_{l=0}^{\infty} f_l P_l(\xi) \quad (6)$$

we obtain

$$\int_{-1}^{1} d\xi (1+\xi^2)\hat{f} = \frac{8}{3} f_0 + \frac{4}{15} f_2. \quad (7)$$

We can write

$$\langle \mathbf{e}_\zeta \cdot (\nabla \cdot \mathbf{P}) \rangle \equiv \frac{2\Delta \bar{T}}{\hat{\psi}_a}\frac{n}{\hat{V}'} \cdot \mathrm{NTV}, \quad (8)$$

where the normalised torque NTV is

$$\mathrm{NTV} = \frac{\hat{T}^{5/2}}{\sqrt{\pi}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^\infty dx\, x^4 (1+\xi^2) \hat{f} \frac{1}{\hat{B}^3}\frac{\partial \hat{B}}{\partial \zeta} =$$

$$= \frac{\hat{T}^{5/2}}{\sqrt{\pi}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_0^{\infty} dx\ x^4 \left(\frac{8}{3} f_0 + \frac{4}{15} f_2\right) \frac{1}{\hat{B}^3} \frac{\partial \hat{B}}{\partial \zeta}. \tag{9}$$

Compare with normalised particle flux, which is a similar quantity, defined in the single species code as

$$\text{particleFlux} = \frac{\hat{\Psi}_a V'}{\Delta^2 n \sqrt{2\bar{T}/m} \bar{R}^2} \left\langle \int d^3 v f \mathbf{v}_d \cdot \nabla\Psi \right\rangle =$$
$$-\frac{\hat{T}^{5/2}}{\sqrt{\pi}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4 (1+\xi^2) \hat{f} \frac{1}{\hat{B}^3} \left[\hat{G} \frac{\partial \hat{B}}{\partial \theta} - \hat{I} \frac{\partial B}{\partial \zeta}\right] \tag{10}$$

and in the multi species code as

$$\text{particleFlux}_s = \frac{1}{\bar{n}\bar{v}\bar{R}\bar{B}} \left\langle \int d^3 v f_s \mathbf{v}_d \cdot \nabla\Psi \right\rangle$$
$$= -\frac{\pi \tilde{\Delta} \hat{T}_s^{\ 5/2}}{Z_s \hat{m}_s \hat{V}'} \frac{1}{\hat{G} + \iota \hat{I}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4 (1+\xi^2) \hat{f}_s \frac{1}{\hat{B}^3} \left[\hat{G} \frac{\partial \hat{B}}{\partial \theta} - \hat{I} \frac{\partial B}{\partial \zeta}\right] \tag{11}$$

where the definitions of $\hat{f}$ and Δ in the two codes differ in the following way,

$$\hat{f}_s = \frac{\tilde{\Delta} \hat{m}^2 \hat{n}}{Z \pi^{3/2} \hat{\psi}_a} \hat{f} = \frac{\bar{v}^3}{\bar{n}} f, \tag{12}$$

$$\tilde{\Delta} = \frac{Z_s}{\sqrt{\hat{m}}} \Delta = \bar{m}\bar{v}/(e\bar{B}\bar{R}). \tag{13}$$

For the NTV torque, we may define

$$\langle \mathbf{e}_\zeta \cdot (\nabla \cdot \mathbf{P}_s) \rangle \equiv \bar{T}\bar{n} \cdot \text{NTV}_s. \tag{14}$$

Comparing with the previous definition (8) we have

$$\text{NTV}_s = \frac{2\tilde{\Delta}\sqrt{\hat{m}_s}\hat{n}}{Z_s \hat{\psi}_a \hat{V}'} \cdot \text{NTV} =$$
$$= \frac{2\tilde{\Delta}\sqrt{\hat{m}_s}\hat{n}}{Z_s \hat{\psi}_a \hat{V}'} \frac{\hat{T}_s^{5/2}}{\sqrt{\pi}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4 (1+\xi^2) \hat{f} \frac{1}{\hat{B}^3} \frac{\partial \hat{B}}{\partial \zeta} =$$
$$= \frac{2\pi \hat{T}_s^{5/2}}{\hat{m}_s^{3/2}} \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_{-1}^{1} d\xi \int_0^{\infty} dx\ x^4 (1+\xi^2) \hat{f}_s \frac{1}{\hat{B}^3} \frac{\partial \hat{B}}{\partial \zeta} \tag{15}$$

REFERENCES

Lewandowski, J. L. V., Williams, J., Boozer, A. H., & Lin, Z. 2001, Phys. Plasmas, 8, 2849

Shaing, K. C., & Callen, J. D. 1983, Phys. Fluids, 26, 3315