Implementation of Φ_1 in SFINCS

February 10, 2016

EUTERPE old equations vs new equations

We want to modify the implementation of the old EUTERPE equations [1] in SFINCS to the new equations [2].

The old equations for the particle trajectories and the drift-kinetic equation are

$$\dot{\boldsymbol{R}} = v_\parallel \boldsymbol{b} - \frac{\nabla\Phi_0 \times \boldsymbol{b}}{B} \tag{1}$$

$$\dot{v}_\parallel = -\frac{q}{m}\boldsymbol{b}\cdot\nabla\Phi_1 - \mu\boldsymbol{b}\cdot\nabla B - \frac{v_\parallel}{B^2}\left(\boldsymbol{b}\times\nabla B\right)\cdot\nabla\Phi_0 \tag{2}$$

$$\dot{\mu} = 0 \tag{3}$$

and

$$\frac{\partial f_1}{\partial t} + \dot{\boldsymbol{R}}\cdot\nabla f_1 + \dot{v}_\parallel \frac{\partial f_1}{\partial v_\parallel} - C = \\ = -f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right](\boldsymbol{v}_d + \boldsymbol{v}_{E1})\cdot\nabla\psi - \frac{q}{m}\frac{f_M}{v_{\mathrm{th}}^2}\left(v_\parallel\boldsymbol{b} + \boldsymbol{v}_d\right)\cdot(\nabla\Phi_0 + \nabla\Phi_1)\,. \tag{4}$$

The new equations are

$$\dot{\boldsymbol{R}} = v_\parallel \boldsymbol{b} - \frac{\nabla\Phi_0 \times \boldsymbol{b}}{B} \tag{5}$$

$$\dot{v}_\parallel = -\frac{q}{m}\boldsymbol{b}\cdot\nabla\Phi_1 - \mu\boldsymbol{b}\cdot\nabla B - \frac{v_\parallel}{B^2}\left(\boldsymbol{b}\times\nabla B\right)\cdot\nabla\Phi_0 \tag{6}$$

$$\dot{\mu} = 0 \tag{7}$$

and

$$\frac{\partial f_1}{\partial t} + \dot{\boldsymbol{R}}\cdot\nabla f_1 + \dot{v}_\parallel \frac{\partial f_1}{\partial v_\parallel} - C = \\ = -f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right](\boldsymbol{v}_d + \boldsymbol{v}_{E1})\cdot\nabla\psi. \tag{8}$$

Here we have the definitions

$$\Phi(\psi,\theta,\varphi) \equiv \Phi_0(\psi) + \Phi_1(\theta,\varphi), \tag{9}$$

$$\boldsymbol{v}_d = \frac{m}{q}\frac{\mu B + v_\parallel^2}{B^2}\boldsymbol{b}\times\nabla B, \tag{10}$$

$$\boldsymbol{v}_{E1} = -\frac{\nabla\Phi_1 \times \boldsymbol{b}}{B}, \tag{11}$$

$$f_0 = f_M \exp(-q\Phi_1/T) = \frac{n_0(\psi)}{(2\pi)^{3/2} v_{\text{th}}^3}\exp\left[-\frac{\left(v_\parallel^2 + v_\perp^2\right)}{2v_{\text{th}}^2}\right]\exp(-q\Phi_1/T), \tag{12}$$

$q = Ze$ and $v_{\text{th}}^2 = T/m$.

The only differences appear in the RHS:s of Eqs. 4 and 8:
Firstly, f_M has been replaced by f_0 containing the $\exp(-q\Phi_1/T)$ factor.
Secondly, some of the terms have been modified. We rewrite the RHS of 8:

$$\begin{aligned}
\text{RHS}_{\text{NEW}} &= -f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right](\boldsymbol{v}_d + \boldsymbol{v}_{E1})\cdot\nabla\psi = \\
&= -f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right](\boldsymbol{v}_d + \boldsymbol{v}_{E1})\cdot\nabla\psi + \\
&\quad - f_0\frac{q}{T}\Phi_1\frac{1}{T}\frac{\partial T}{\partial\psi}(\boldsymbol{v}_d + \boldsymbol{v}_{E1})\cdot\nabla\psi = \\
&= -f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_d\cdot\nabla\psi + \\
&\quad - f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_{E1}\cdot\nabla\psi - f_0\frac{q}{T}\frac{\partial\Phi_0}{\partial\psi}\boldsymbol{v}_{E1}\cdot\nabla\psi + \\
&\quad - f_0\frac{q}{T}\Phi_1\frac{\nabla T}{T}\cdot(\boldsymbol{v}_d + \boldsymbol{v}_{E1}) = \\
&= -f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_d\cdot\nabla\psi + \\
&\quad -f_0\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_{E1}\cdot\nabla\psi + \\
&\quad -f_0\frac{q}{T}\left[\nabla\Phi_0\cdot\boldsymbol{v}_{E1} + \Phi_1\frac{\nabla T}{T}\cdot\boldsymbol{v}_d + \Phi_1\frac{\nabla T}{T}\cdot\boldsymbol{v}_{E1}\right].
\end{aligned} \tag{13}$$

Similarly, the RHS of 4 is rewritten as:

$$\begin{aligned}\mathrm{RHS}_{\mathrm{OLD}} &= -f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right](\boldsymbol{v}_d + \boldsymbol{v}_{E1})\cdot\nabla\psi - \frac{q}{m}\frac{f_M}{v_{\mathrm{th}}^2}\left(v_\parallel\boldsymbol{b} + \boldsymbol{v}_d\right)\cdot(\nabla\Phi_0 + \nabla\Phi_1) = \\ &= -f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_d\cdot\nabla\psi + \\ &\quad -f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_{E1}\cdot\nabla\psi + \\ &\quad -f_M\frac{q}{T}\left[v_\parallel\boldsymbol{b}\cdot\nabla\Phi_1 + \boldsymbol{v}_d\cdot\nabla\Phi_1\right]. \end{aligned} \tag{14}$$

Comparing $\mathrm{RHS}_{\mathrm{NEW}}$ to $\mathrm{RHS}_{\mathrm{OLD}}$ we see that, apart from $f_M \to f_0$, only the terms in red have changed.

What has to be changed in SFINCS

The only part of the drift-kinetic equation block we need to modify is the RHS, where we need to update the red terms and substitute $f_M \to f_0$. SFINCS had earlier neglected the $\boldsymbol{v}_d\cdot\nabla\Phi_1$-term which is small in the standard ρ_*-expansion. The RHS that was implemented is (see Matt's ISHW poster, also note that $\boldsymbol{v}_E\cdot\nabla\psi = \boldsymbol{v}_{E1}\cdot\nabla\psi$)

$$\begin{aligned}\mathrm{RHS}_{\mathrm{SFINCS,OLD}} &= -f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_d\cdot\nabla\psi + \\ &\quad -f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\boldsymbol{v}_E\cdot\nabla\psi - f_M\frac{q}{T}v_\parallel\boldsymbol{b}\cdot\nabla\Phi_1. \end{aligned} \tag{15}$$

We thus replace

$$v_\parallel\boldsymbol{b}\cdot\nabla\Phi_1 \tag{16}$$

with

$$\nabla\Phi_0\cdot\boldsymbol{v}_E + \Phi_1\frac{\nabla T}{T}\cdot\boldsymbol{v}_d + \Phi_1\frac{\nabla T}{T}\cdot\boldsymbol{v}_E, \tag{17}$$

and make the substitution

$$f_M \to f_0 = f_M\exp\left(-q\Phi_1/T\right). \tag{18}$$

All terms which contain Φ_1 are now nonlinear. It does not make sense to have both switches **includePhi1** and **nonlinear** still available in SFINCS, and consequently we will remove the **nonlinear** switch.

We will also introduce to possibility to run SFINCS with an adiabatic species.

Implementation in SFINCS

Of the equations implemented in SFINCS [4], the only two we need to modify are the kinetic equation

$$R\left(f_1,\Phi_1\right) = K\left\{\theta\right\}\frac{\partial f}{\partial\theta} + K\left\{\zeta\right\}\frac{\partial f}{\partial\zeta} + K\left\{x\right\}\frac{\partial f}{\partial x} + K\left\{\xi\right\}\frac{\partial f}{\partial\xi} + K\left\{\psi\right\}\frac{\partial f_M}{\partial\psi} + \\ - C\left\{f\right\} - S_1 f_M - S_2 f_M x^2 - \frac{Zev}{T}x\xi\frac{\left\langle \boldsymbol{E}\cdot\boldsymbol{B}\right\rangle B}{\left\langle B^2\right\rangle}f_M = 0 \quad (19)$$

and the quasineutrality equation

$$\sum_s Z_s\int d^3v\, f_s + \lambda = 0. \quad (20)$$

Drift-kinetic equation

For the residual $R\left(f_1,\Phi_1\right)$ the only term in the kinetic equation we need to modify is the one in red in Eq. 19. We replace $f_M \to f_0 = f_M\exp\left(-q\Phi_1/T\right)$, and use that $K\left\{\psi\right\} = \boldsymbol{v}_E\cdot\nabla\psi + \boldsymbol{v}_d\cdot\nabla\psi = \boldsymbol{v}_{E1}\cdot\nabla\psi + \boldsymbol{v}_d\cdot\nabla\psi$ to write

$$K\left\{\psi\right\}\frac{\partial f_M}{\partial\psi} = \exp\left(-q\Phi_1/T\right)\frac{\partial f_M}{\partial\psi}\left(\boldsymbol{v}_{E1}\cdot\nabla\psi + \boldsymbol{v}_d\cdot\nabla\psi\right) = \\ = \exp\left(-\frac{q\Phi_1}{T}\right)f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\left(-\frac{\nabla\Phi_1\times\boldsymbol{b}}{B}\cdot\nabla\psi + \boldsymbol{v}_d\cdot\nabla\psi\right) = \\ = \left\|-\frac{\nabla\Phi_1\times\boldsymbol{b}}{B}\cdot\nabla\psi = -\frac{\boldsymbol{B}\times\nabla\psi}{B^2}\cdot\nabla\Phi_1 = \frac{1}{B^2}D\left[B_\theta\frac{\partial\Phi_1}{\partial\zeta} - B_\zeta\frac{\partial\Phi_1}{\partial\theta}\right]\right\| = \\ = \exp\left(-\frac{q\Phi_1}{T}\right)f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\cdot \\ \left(\frac{1}{B^2}D\left[B_\theta\frac{\partial\Phi_1}{\partial\zeta} - B_\zeta\frac{\partial\Phi_1}{\partial\theta}\right] + \boldsymbol{v}_d\cdot\nabla\psi\right) \quad (21)$$

(Here $D = \nabla\psi\cdot\nabla\theta\times\nabla\zeta$.) Written like this we explicitly see the places where Φ_1 appears in $K\left\{\psi\right\}\frac{\partial f_M}{\partial\psi}$. From Eq. 21 we obtain the correspong terms in the Jacobian matrix

$$\frac{\delta}{\delta\Phi_1}\left(K\left\{\psi\right\}\frac{\partial f_M}{\partial\psi}\right) = -\frac{q}{T}\exp\left(-\frac{q\Phi_1}{T}\right)f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\cdot \\ \left(\frac{1}{B^2}D\left[B_\theta\frac{\partial\Phi_1}{\partial\zeta} - B_\zeta\frac{\partial\Phi_1}{\partial\theta}\right] + \boldsymbol{v}_d\cdot\nabla\psi\right) + \\ + \exp\left(-\frac{q\Phi_1}{T}\right)f_M\frac{q}{T}\frac{1}{T}\frac{\partial T}{\partial\psi}\left(\frac{1}{B^2}D\left[B_\theta\frac{\partial\Phi_1}{\partial\zeta} - B_\zeta\frac{\partial\Phi_1}{\partial\theta}\right] + \boldsymbol{v}_d\cdot\nabla\psi\right) + \\ + \exp\left(-\frac{q\Phi_1}{T}\right)f_M\left[\frac{1}{n}\frac{\partial n}{\partial\psi} + \frac{q}{T}\frac{\partial\Phi_0}{\partial\psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial\psi}\right]\cdot \\ \left(\frac{1}{B^2}D\left[B_\theta\frac{\partial}{\partial\zeta} - B_\zeta\frac{\partial}{\partial\theta}\right]\right) \quad (22)$$

Residual

Many of the terms involving $\boldsymbol{v}_d \cdot \nabla\psi$ are almost implemented in SFINCS already except that they now contain the $\exp\left(-\frac{q\Phi_1}{T}\right)$-factor. We therefore rewrite Eq. 21 as

$$K\left\{\psi\right\}\frac{\partial f_M}{\partial \psi} = R_m + R_E \tag{23}$$

where

$$R_m = \exp\left(-\frac{q\Phi_1}{T}\right) f_M \left[\frac{1}{n}\frac{\partial n}{\partial \psi} + \frac{q}{T}\frac{\partial \Phi_0}{\partial \psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial \psi}\right]\boldsymbol{v}_d \cdot \nabla\psi \tag{24}$$

and

$$R_E = \exp\left(-\frac{q\Phi_1}{T}\right) f_M \left[\frac{1}{n}\frac{\partial n}{\partial \psi} + \frac{q}{T}\frac{\partial \Phi_0}{\partial \psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial \psi}\right]\frac{1}{B^2}D\left[B_\theta\frac{\partial \Phi_1}{\partial \zeta} - B_\zeta\frac{\partial \Phi_1}{\partial \theta}\right] \tag{25}$$

$$\begin{aligned}
K\left\{\psi\right\}\frac{\partial f_M}{\partial \psi} &= \exp\left(-\frac{q\Phi_1}{T}\right) f_M \left[\frac{1}{n}\frac{\partial n}{\partial \psi} + \frac{q}{T}\frac{\partial \Phi_0}{\partial \psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial \psi}\right]\boldsymbol{v}_d \cdot \nabla\psi + \\
&+ \exp\left(-\frac{q\Phi_1}{T}\right) f_M \left[\frac{1}{n}\frac{\partial n}{\partial \psi} + \frac{q}{T}\frac{\partial \Phi_0}{\partial \psi} + \left(\frac{mv^2}{2T} - \frac{3}{2} + \frac{q}{T}\Phi_1\right)\frac{1}{T}\frac{\partial T}{\partial \psi}\right]\frac{1}{B^2}D\left[B_\theta\frac{\partial \Phi_1}{\partial \zeta} - B_\zeta\frac{\partial \Phi_1}{\partial \theta}\right] = \\
&= \exp\left(-\frac{q\Phi_1}{T}\right) f_M \left[\frac{1}{n}\frac{\partial n}{\partial \psi} + \frac{q}{T}\frac{\partial \Phi_0}{\partial \psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial \psi}\right]\boldsymbol{v}_d \cdot \nabla\psi + \\
&+ \exp\left(-\frac{q\Phi_1}{T}\right) f_M \left[\frac{1}{n}\frac{\partial n}{\partial \psi} + \left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial \psi}\right]\frac{1}{B^2}D\left[B_\theta\frac{\partial \Phi_1}{\partial \zeta} - B_\zeta\frac{\partial \Phi_1}{\partial \theta}\right] + \\
&+ \exp\left(-\frac{q\Phi_1}{T}\right) f_M \frac{q}{T}\Phi_1\frac{1}{T}\frac{\partial T}{\partial \psi}\boldsymbol{v}_d \cdot \nabla\psi + \\
&+ \exp\left(-\frac{q\Phi_1}{T}\right) f_M \frac{q}{T}\frac{\partial \Phi_0}{\partial \psi}\frac{1}{B^2}D\left[B_\theta\frac{\partial \Phi_1}{\partial \zeta} - B_\zeta\frac{\partial \Phi_1}{\partial \theta}\right] + \\
&+ \exp\left(-\frac{q\Phi_1}{T}\right) f_M \frac{q}{T}\Phi_1\frac{1}{T}\frac{\partial T}{\partial \psi}\frac{1}{B^2}D\left[B_\theta\frac{\partial \Phi_1}{\partial \zeta} - B_\zeta\frac{\partial \Phi_1}{\partial \theta}\right]
\end{aligned} \tag{26}$$

Check of Matt's former implementation of $\dfrac{Ze}{T} f_M v_\parallel \nabla_\parallel \Phi_1$

Looking at Matt's ISHW poster, since Φ_1 is an unknown this term is in the LHS of the square block matrix system. The term is accessed by "rowIndex = BLOCK_F" and "colIndex = BLOCK_QN". We use

$$\nabla_\parallel \Phi_1 = \boldsymbol{b} \cdot \nabla \Phi_1 = \frac{1}{B}\left[B^\theta \frac{\partial \Phi_1}{\partial \theta} + B^\zeta \frac{\partial \Phi_1}{\partial \zeta}\right] = \frac{\bar{\Phi}}{\hat{B}\bar{R}}\left[\hat{B}^\theta \frac{\partial \hat{\Phi}_1}{\partial \theta} + \hat{B}^\zeta \frac{\partial \hat{\Phi}_1}{\partial \zeta}\right],$$

$$f_M = n_0(\psi) \frac{m^{3/2}}{(2\pi T)^{3/2}} \exp\left[-\frac{v^2}{v_s^2}\right] = \hat{n}\bar{n} \frac{\hat{m}^{3/2}}{\left(2\pi \hat{T}\right)^{3/2}} \left(\frac{\bar{m}}{\bar{T}}\right)^{3/2} \exp\left[-x^2\right],$$

$v_\parallel = v_s x \xi = v_s x P_1 = x P_1 \sqrt{2\hat{T}/\hat{m}} \sqrt{\bar{T}/\bar{m}}$ and $x = v/v_s$. With $\alpha = e\bar{\Phi}/\bar{T}$ we obtain

$$\frac{Ze}{T} f_M v_\parallel \nabla_\parallel \Phi_1 = \frac{Ze}{\hat{T}\bar{T}} \hat{n}\bar{n} \frac{\hat{m}^{3/2}}{\left(2\pi \hat{T}\right)^{3/2}} \left(\frac{\bar{m}}{\bar{T}}\right)^{3/2} \exp\left[-x^2\right] x P_1 \sqrt{2\hat{T}/\hat{m}} \sqrt{\bar{T}/\bar{m}} \frac{\bar{\Phi}}{\hat{B}\bar{R}} \left[\hat{B}^\theta \frac{\partial \hat{\Phi}_1}{\partial \theta} + \hat{B}^\zeta \frac{\partial \hat{\Phi}_1}{\partial \zeta}\right] =$$
$$= \frac{Z\alpha}{2\pi^{3/2}} x P_1 \exp\left[-x^2\right] \frac{\hat{n}\hat{m}}{\hat{B}\hat{T}^2} \frac{\bar{n}\bar{m}}{\bar{R}\bar{T}} \left[\hat{B}^\theta \frac{\partial}{\partial \theta} + \hat{B}^\zeta \frac{\partial}{\partial \zeta}\right] \hat{\Phi}_1. \quad (27)$$

In SFINCS the kinetic equation is made dimensionless by multiplying with

$$\frac{\bar{v}^3 \bar{R}}{\bar{n}\bar{v}} = \frac{2\bar{T}\bar{R}}{\bar{m}\bar{n}},$$

which implies that the RHS of Eq. 27 becomes

$$\frac{Z\alpha}{\pi^{3/2}} x P_1 \exp\left[-x^2\right] \frac{\hat{n}\hat{m}}{\hat{B}\hat{T}^2} \left[\hat{B}^\theta \frac{\partial}{\partial \theta} + \hat{B}^\zeta \frac{\partial}{\partial \zeta}\right] \hat{\Phi}_1 \quad (28)$$

in the implementation.

Implementation of $f_0 \dfrac{q}{T} \nabla\Phi_0 \cdot \boldsymbol{v}_{E1}$

$$\left(f_0 \frac{q}{T} \nabla\Phi_0 \cdot \boldsymbol{v}_{E1}\right)_{\text{SFINCS}} = \\ = \frac{Z\alpha^2\Delta}{2\pi^{3/2}} \frac{\hat{n}\hat{m}^{3/2}\hat{D}}{\hat{T}^{5/2}\hat{B}^2\hat{\psi}_a} \frac{\partial\hat{\Phi}_0}{\partial\psi_N} \exp\left(-x^2\right) \exp\left(-\frac{Z\alpha\hat{\Phi}_1}{\hat{T}}\right) \left[\hat{B}_\theta \frac{\partial}{\partial\zeta} - \hat{B}_\zeta \frac{\partial}{\partial\theta}\right] \hat{\Phi}_1 \quad (29)$$

Implementation of $f_0 \dfrac{q}{T} \Phi_1 \dfrac{\nabla T}{T} \cdot \boldsymbol{v}_d$

$$\left(f_0 \frac{q}{T} \Phi_1 \frac{\nabla T}{T} \cdot \boldsymbol{v}_d\right)_{\text{SFINCS}} = \\ = \frac{\alpha\Delta}{3\pi^{3/2}} \frac{\hat{n}\hat{m}^{3/2}\hat{D}}{\hat{T}^{5/2}\hat{B}^3\hat{\psi}_a} \hat{\Phi}_1 \frac{\partial\hat{T}}{\partial\psi_N} x^2 \left(P_2\left(\xi\right) + 2\right) \exp\left(-x^2\right) \exp\left(-\frac{Z\alpha\hat{\Phi}_1}{\hat{T}}\right) \left[\hat{B}_\theta \frac{\partial\hat{B}}{\partial\zeta} - \hat{B}_\zeta \frac{\partial\hat{B}}{\partial\theta}\right] \quad (30)$$

Implementation of $f_0 \dfrac{q}{T} \Phi_1 \dfrac{\nabla T}{T} \cdot \boldsymbol{v}_{E1}$

$$\left(f_0 \frac{q}{T} \Phi_1 \frac{\nabla T}{T} \cdot \boldsymbol{v}_{E1}\right)_{\text{SFINCS}} = \\ = \frac{Z\alpha^2\Delta}{2\pi^{3/2}} \frac{\hat{n}\hat{m}^{3/2}\hat{D}}{\hat{T}^{7/2}\hat{B}^2\hat{\psi}_a} \frac{\partial\hat{T}}{\partial\psi_N} \hat{\Phi}_1 \exp\left(-x^2\right) \exp\left(-\frac{Z\alpha\hat{\Phi}_1}{\hat{T}}\right) \left[\hat{B}_\theta \frac{\partial}{\partial\zeta} - \hat{B}_\zeta \frac{\partial}{\partial\theta}\right] \hat{\Phi}_1 \quad (31)$$

Quasi-neutrality equation

In EUTERPE Φ_1 is calculated from quasi-neutrality by expanding the exponential, assuming adiabatic electrons and neglecting the impurities:

$$n_s = n_{s0}\left(\psi\right)\,\exp\left(-q_s\Phi_1/T_s\right) + n_{s1}, \tag{32}$$

$$\sum_s Z_s n_s = 0, \tag{33}$$

$$\Rightarrow \quad 0 \simeq \sum_s Z_s\left[n_{s0}\left(1 - q_s\Phi_1/T_s\right) + n_{s1}\right] \quad \Leftrightarrow$$

$$\sum_s Z_s\left[n_{s0} + n_{s1}\right] = \sum_s \frac{Z_s^2 e}{T_s}\Phi_1 n_{s0}.$$

Since $n_{s0}\left(\psi\right)$ is obtained by integrating the Maxwellian f_{Ms} over velocity space we must have

$$\sum_s Z_s n_{s0} = 0,$$

which yields

$$\sum_s Z_s n_{s1} - \Phi_1 \sum_s \frac{Z_s^2 e}{T_s} n_{s0} = 0. \tag{34}$$

With kinetic ions, adiabatic electrons ($n_{e1} = 0$) and neglecting impurities we obtain

$$\Phi_1 = \frac{T_e}{e}\left[\frac{T_e}{T_i}n_{i0} + n_{e0}\right]^{-1} n_{i1}. \tag{35}$$

Implementation in SFINCS

For a first benchmark, we want to implement the same equations as EUTERPE in SFINCS.
REMARK: This is not a very generic quasi-neutrality equation so it is possible that we might want to change it in SFINCS later.
In the code we add an adiabatic species which only enters into the quasi-neutrality equation, and neglect its collisional impact on the kinetic species (the effect of ion-electron collisions is small compared to ion-ion collisions). Moreover, we will only consider the first of the kinetic species in quasi-neutrality and neglect the rest. This is implemented by modifying the LHS of the row corresponding to quasi-neutrality in the block-matrix structure of Matt's ISHW poster, adding the adiabatic term to the Φ_1-column and removing all kinetic species except the first.
REMARK: It feels a bit weird to remove species from quasi-neutrality, even if the impurity density is small. Does this mean that we should removed the check that the input densities are quasi-neutral and instead check that $n_{i0}\left(\psi\right) = n_{e0}\left(\psi\right)$ in the input?
The equation we will implement in SFINCS is thus

$$Z_i n_{i1} - \Phi_1\left[\frac{Z_i^2 e}{T_i}n_{i0} + \frac{Z_e^2 e}{T_e}n_{e0}\right] = 0. \tag{36}$$

We note that

$$n_s = n_{s0}(\psi)\exp\left(-q_s\Phi_1/T_s\right) + n_{s1} = \int d^3v f_{Ms}\exp\left(-q_s\Phi_1/T_s\right) + \int d^3v f_{1s} = \\ = d^3v f_{0s} + d^3v f_{1s}. \quad (37)$$

The velocity integration is SFINCS is done in $(x,\xi) = \left(v/v_s, v_\parallel/v\right)$, and

$$\int d^3v = 2\pi v_s^3 \int_0^\infty dx\, x^2 \int_{-1}^1 d\xi \quad (38)$$

(note that $v_s^2 = 2T_s/m_s$ differs from Jose's notation $v_{\text{th}}^2 = T/m$). Using SFINCS normalizations $n_s = \bar{n}\hat{n}_s$, $T_s = \bar{T}\hat{T}_s$, $v_s/\bar{v} = \sqrt{\hat{T}_s/\hat{m}_s}$, $f_s = \bar{n}\hat{f}_s/\bar{v}^3$, we find

$$\hat{n}_s = 2\pi\left(\hat{T}_s/\hat{m}_s\right)^{3/2}\int_0^\infty dx\, x^2 \int_{-1}^1 d\xi \hat{f}_s. \quad (39)$$

Also using $\Phi_1 = \bar{\Phi}\hat{\Phi}_1$ and $\alpha = e\bar{\Phi}/\bar{T}$ we can write Eq. 36

$$Z_i\hat{n}_{i1} - \alpha\hat{\Phi}_1\left[\frac{Z_i^2}{\hat{T}_i}\hat{n}_{i0} + \frac{Z_e^2}{\hat{T}_e}\hat{n}_{e0}\right] = 0 \quad (40)$$

and finally obtain

$$\left[2\pi Z_i\left(\hat{T}_i/\hat{m}_i\right)^{3/2}\int_0^\infty dx\, x^2\int_{-1}^1 d\xi \hat{f}_{i1}\right] - \alpha\hat{\Phi}_1\left[\frac{Z_i^2}{\hat{T}_i}\hat{n}_{i0} + \frac{Z_e^2}{\hat{T}_e}\hat{n}_{e0}\right] = 0. \quad (41)$$

This is the equation we will implement in the code, but adding a λ to make the system square.
REMARK: Is the 2π factor correct in Eq. 41? It is not in the former implementation of quasi-neutrality, but in that situation it could be divided away.

References

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