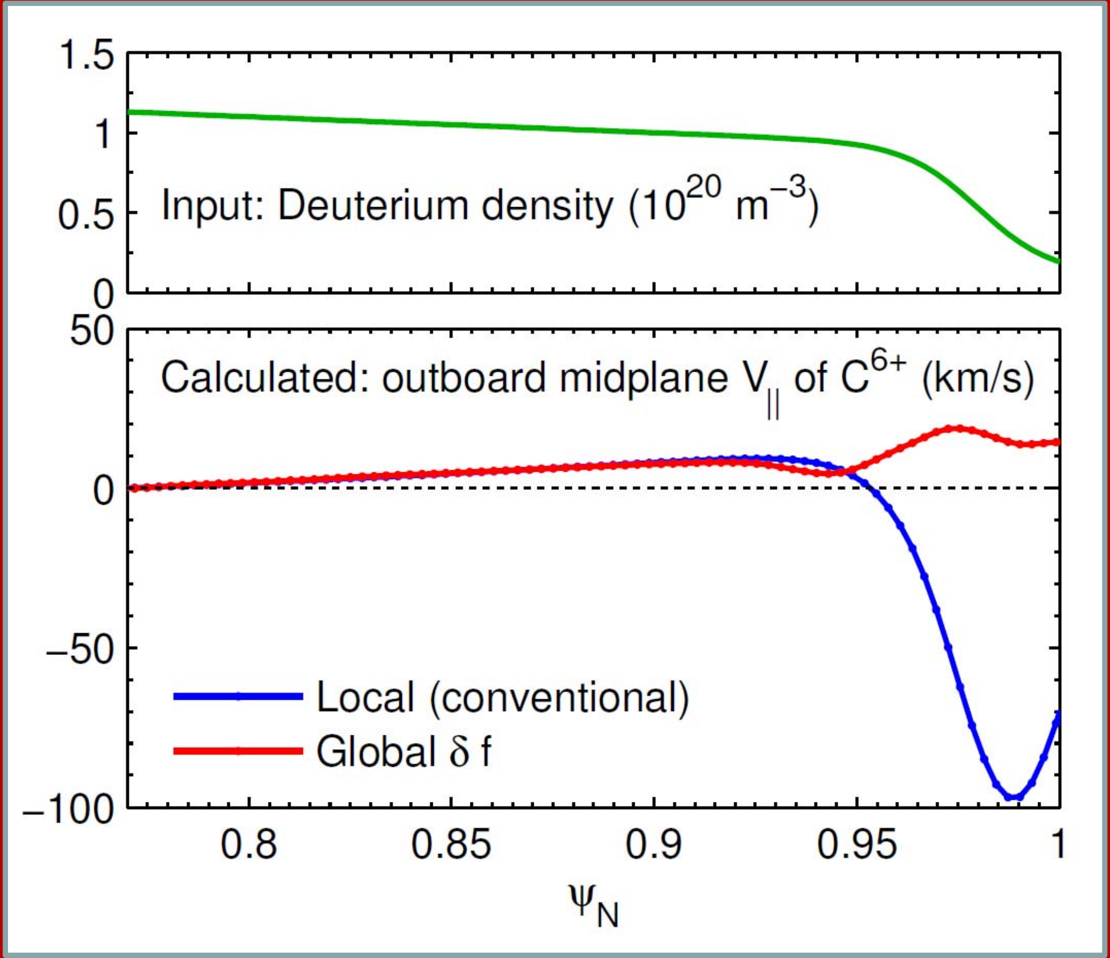
4D Fokker-Planck calculations of neoclassical phenomena in tokamak pedestals and stellarators



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PPCF **54**, 115006 (2012)

J Comp. Phys. **243**, 130 (2013)

4D Fokker-Planck solvers for tokamak pedestals and stellarators

1. Finite-orbit-width effects on neoclassical physics in tokamak pedestals
   - Motivation & the global δf model
   - Need for source/sink
   - New code PERFECT (Pedestal & Edge Radially-global Fokker-Planck Evaluation of Collisional Transport)
     - Velocity-space discretization & collision operator
     - Comparison with analytic theory for $a/R \ll 1$

2. Stellarator applications
   - Comparison of collision operators
   - Comparison of E_r terms

Improved neoclassical calculations are needed for the pedestal.

Neoclassical effects are significant in the pedestal:

- Neoclassical flow, bootstrap current, and heat flux will be large due to strong gradients.
- Current affects stability (e.g. ELMs).
- Ion heat flux could be at the neoclassical level.

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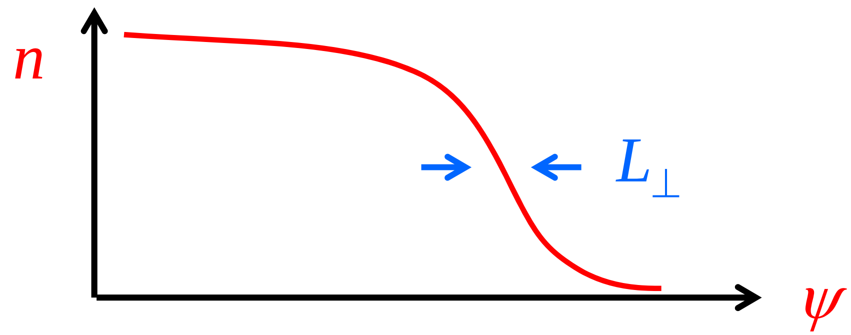


But, conventional local neoclassical theory & codes assume

$$L_\perp \gg \rho_\theta \text{ (poloidal ion gyroradius)}.$$

When $L_\perp \sim \rho_\theta$, the conventional neoclassical ordering breaks down.

- Physics: ρ_θ is ~ ion orbit width $\left(\times \sqrt{R/a}\right)$.

Neoclassical physics is contained in the drift-kinetic equation

Average the Fokker-Planck equation over Larmor gyration and over turbulence:

At constant μ and total energy

$$\underbrace{\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_{d}\right)\cdot\nabla f}_{df/dt \text{ along guiding-center trajectory}} = \underbrace{C\{f\}}_{\text{collisions}}$$

1. Solve for f,
2. Take moments:

$$\mathbf{V}=\frac{1}{n}\int d^{3}\upsilon\ \mathbf{v}f,\qquad \mathbf{j}=e\left(n_{i}\mathbf{V}_{i}-n_{e}\mathbf{V}_{e}\right),\qquad \mathbf{q}=\int d^{3}\upsilon\frac{m\upsilon^{2}}{2}\mathbf{v}f,\quad \text{etc.}$$

Neoclassical phenomena can be computed from several versions of the drift-kinetic equation

$$f = f_M(\psi)\left[1 - \frac{Ze}{T}\left(\Phi - \langle\Phi\rangle\right)\right] + f_1$$

Local δf:

$$\upsilon_\parallel \mathbf{b}\cdot\nabla f_1 + \mathbf{v}_d \cdot\nabla f_M = C\{f_1\}$$

NEO, NCLASS, and most neoclassical theory are based on this version.

3D: $f_1(\theta,\mu,\upsilon)$

Global δf:

$$\left(\upsilon_\parallel \mathbf{b} + \mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_m \cdot\nabla f_M = C\{f_1\}$$

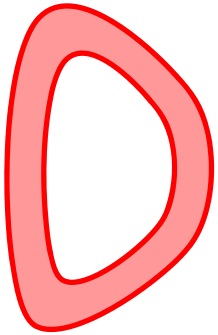
Allows stronger radial gradients.

4D: $f_1(\psi,\theta,\mu,\upsilon)$

For general input profiles, no time-independent solution of the global δf kinetic equation exists without a source/sink.

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f_1+\mathbf{v}_d\cdot\nabla f_M=C\left\{f_1\right\}+S$$

$$\text{Apply}\ \int_{\psi_{\min}}^{\psi_{\max}}d\psi\, V'\left\langle\int d^3\upsilon\,\frac{m\upsilon^2}{2}\left(\quad\ldots\quad\right)\right\rangle$$



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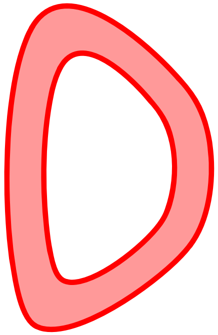
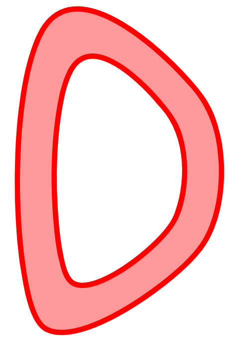


$$\text{Apply}\quad \int_{\psi_{\min}}^{\psi_{\max}} d\psi\, V'\left\langle \int d^3\upsilon \frac{m\upsilon^2}{2}\left(\quad\ldots\quad\right)\right\rangle$$

$$\underbrace{\left[V'\left\langle \int d^3\upsilon\ f_1\frac{m\upsilon^2}{2}\mathbf{v}_d\cdot\nabla\psi\right\rangle\right]_{\psi=\psi_{\min}}^{\psi_{\max}}}_{\text{Heat out } - \text{ heat in}} = \underbrace{\int_{\psi_{\min}}^{\psi_{\max}} d\psi\, V'\left\langle \int d^3\upsilon \frac{m\upsilon^2}{2} S\right\rangle}_{\text{Total heat source in the volume}}$$

For general input profiles, no time-independent solution of the global δf kinetic equation exists without a source/sink.

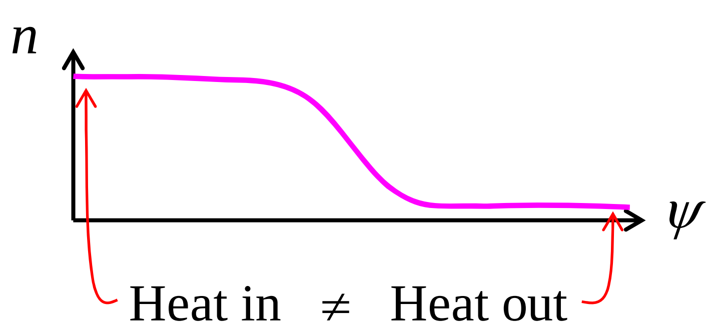
$$\left(v_\parallel \mathbf{b} + \mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C\{f_1\} + S$$



$$\text{Apply} \quad \int_{\psi_{\min}}^{\psi_{\max}} d\psi\, V' \left\langle \int d^3 v \frac{m v^2}{2} \left(\quad \ldots \quad \right) \right\rangle$$

$$\underbrace{\left[V' \left\langle \int d^3 v \; f_1 \frac{m v^2}{2} \mathbf{v}_d \cdot \nabla \psi \right\rangle \right]_{\psi=\psi_{\min}}^{\psi_{\max}}}_{\text{Heat out } - \text{ heat in}} = \underbrace{\int_{\psi_{\min}}^{\psi_{\max}} d\psi\, V' \left\langle \int d^3 v \frac{m v^2}{2} S \right\rangle}_{\text{Total heat source in the volume}}$$

$$\langle \mathbf{q}\cdot\nabla\psi\rangle \sim \frac{n^2}{\sqrt{T}} \frac{dT_i}{d\psi}$$



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$$\left(\upsilon_\parallel \mathbf{b} + \mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C\{f_1\} + S$$

~ divergence of turbulent fluxes

Apply $\int_{\psi_{\min}}^{\psi_{\max}} d\psi\, V' \left\langle \int d^3\upsilon \frac{m\upsilon^2}{2}\left(\quad \ldots \quad \right) \right\rangle$

$$\underbrace{\left[V' \left\langle \int d^3\upsilon\, f_1 \frac{m\upsilon^2}{2} \mathbf{v}_d \cdot \nabla\psi \right\rangle \right]_{\psi=\psi_{\min}}^{\psi_{\max}}}_{\text{Heat out } - \text{ heat in}} = \underbrace{\int_{\psi_{\min}}^{\psi_{\max}} d\psi\, V' \left\langle \int d^3\upsilon \frac{m\upsilon^2}{2} S \right\rangle}_{\text{Total heat source in the volume}}$$

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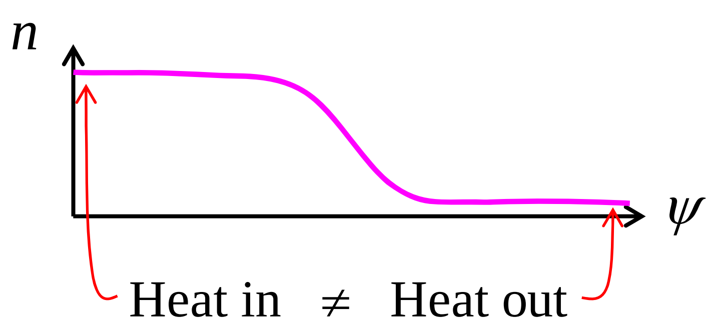

Very similar issue faced by global δf turbulence codes.

Unlike local case, sources cannot be ignored.

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_{d}\right)\cdot\nabla f_{1}-C_{\ell}\left\{f_{1}\right\}-\underbrace{f_{M}\left[S_{1}\left(\psi\right)+\upsilon^{2}S_{2}\left(\psi\right)\right]y\left(\theta\right)}_{S}=-\mathbf{v}_{d}\cdot\nabla f_{M}$$

$$E.g.\quad y\left(\theta\right)=1\quad\text{or}\quad y\left(\theta\right)=1+\cos\theta$$

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$$\begin{array}{r} \text{Kinetic equation } \{ \\ \left\langle \int d^3\upsilon\ f_1 \right\rangle_\psi = 0\ \{ \\ \left\langle \int d^3\upsilon\ f_1\upsilon^2 \right\rangle_\psi = 0\ \{ \end{array} \begin{pmatrix} \ddots & \ddots & \ddots \\ \ddots & 0 & 0 \\ \ddots & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} f_1 \\ S_1 \\ S_2 \end{pmatrix}}_{\text{Vector of unknowns}} = \begin{pmatrix} -\mathbf{v}_d\cdot\nabla f_M \\ 0 \\ 0 \end{pmatrix}$$

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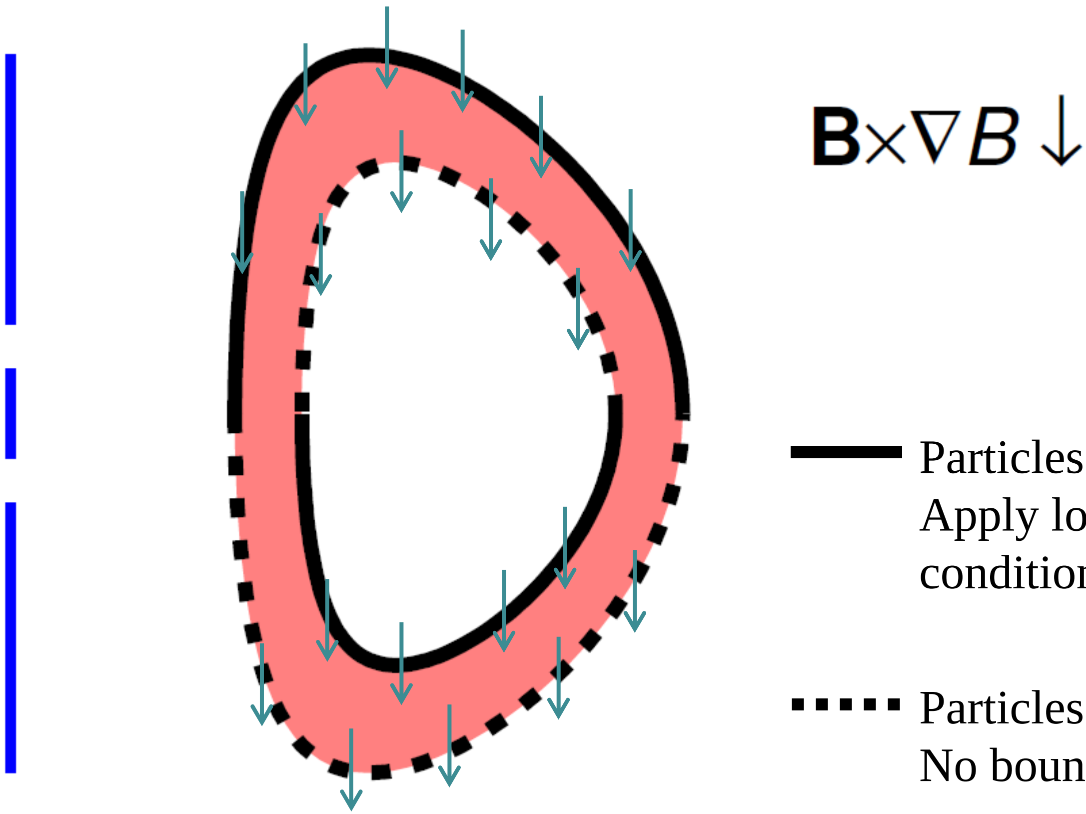
- Linear system solved using preconditioned GMRES/BiCGStab(l) using PETSc library.

Unlike local case, radial boundary conditions are required.

$$\left[\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right]\cdot\nabla\theta\frac{\partial f_1}{\partial\theta}+\mathbf{v}_d\cdot\nabla\psi\frac{\partial f_1}{\partial\psi}-C_\ell\{f_1\}-S=-\mathbf{v}_d\cdot\nabla f_M. \qquad \text{4D: } f_1=f_1(\psi,\theta,\upsilon,\xi)$$



Particles drift **into** domain:
Apply local neoclassical f_1 as Dirichlet condition.

Particles drift **out of** domain:
No boundary condition applied.

PERFECT uses the full linearized Fokker-Planck collsion operator.

$$C_i\{f_1\} = \underbrace{\begin{pmatrix}\text{pitch-angle \&} \\ \text{energy scattering}\end{pmatrix}}_{\text{test particle part}} + \underbrace{\nu_{ii} 3 e^{-\upsilon^2/\upsilon_{th,i}^2} \left[f_1 - \frac{H}{2\pi\upsilon_{th,i}^2} + \frac{\upsilon^2}{2\pi\upsilon_{th,i}^4} \frac{\partial^2 G}{\partial \upsilon^2} \right]}_{\text{field particle part}}$$

$$\nabla_{\mathbf{v}}^2 H + 4\pi f_1 = 0$$

$$\nabla_{\mathbf{v}}^2 G - 2H = 0$$

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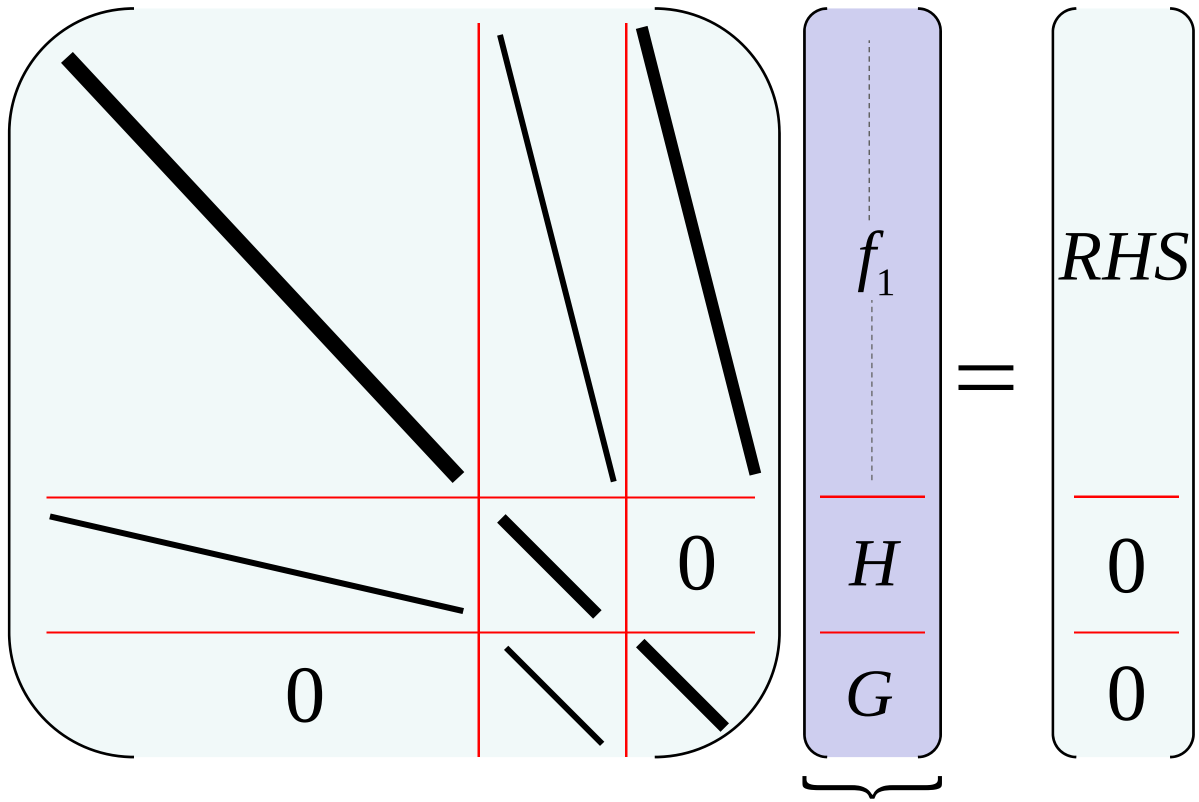
Kinetic equation:

$\nabla_{\mathbf{v}}^2 H + 4\pi f_1 = 0$

$\nabla_{\mathbf{v}}^2 G - 2H = 0$



Vector of unknowns

Similar to independent work in Lyons et al, *PoP* **19**, 082515 (2012)

We have developed a new spectral discretization for v using non-classical orthogonal polynomials.

Laguerre polynomials: $\int_0^\infty dy\, L_i(y) L_j(y) e^{-y} \propto \delta_{i,j}, \qquad y = \frac{m\upsilon^2}{2T}$

Loses accuracy because of nonanalytic $\sqrt{\ }$ in Jacobian at $y = 0$.

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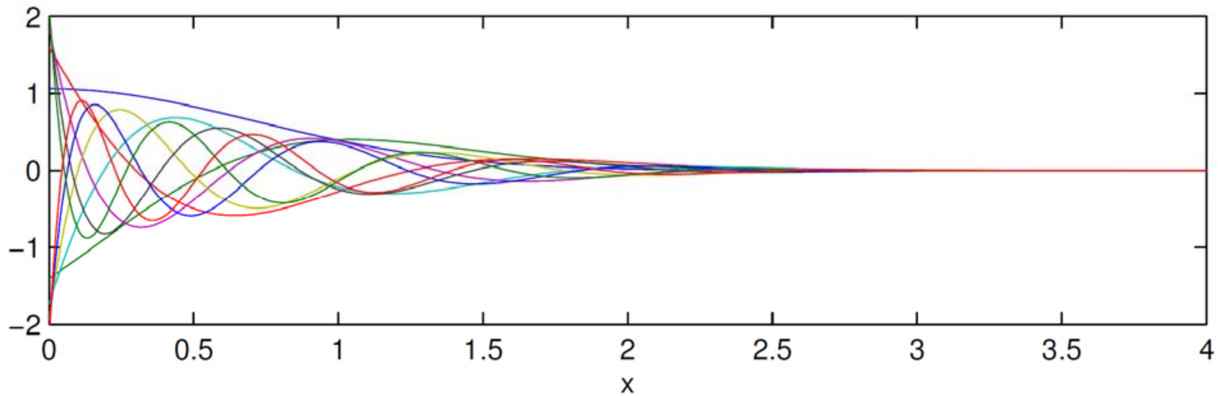
New polynomials: $\int_0^\infty dx\, P_i(x) P_j(x) e^{-x^2} \propto \delta_{i,j}, \qquad x = \upsilon\sqrt{\frac{m}{2T}}$

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x - \frac{1}{\sqrt{\pi}} \\
P_2(x) &= x^2 - \frac{\sqrt{\pi}}{\pi - 2}x + \frac{4-\pi}{2(\pi-2)} \\
P_3(x) &= x^3 - \frac{3\pi - 8}{2\sqrt{\pi}(\pi-3)}x^2 + \frac{10-3\pi}{2(\pi-3)}x - \frac{16-5\pi}{4\sqrt{\pi}(\pi-3)}
\end{aligned}
$$

First 10 modes:

$P_j(x) e^{-x^2}$



New speed discretization can be very efficient

Gaussian integration, spectral differentiation

(*Weideman & Reddy, ACM Trans. Math. Software 26 (2000) 465–519.*)

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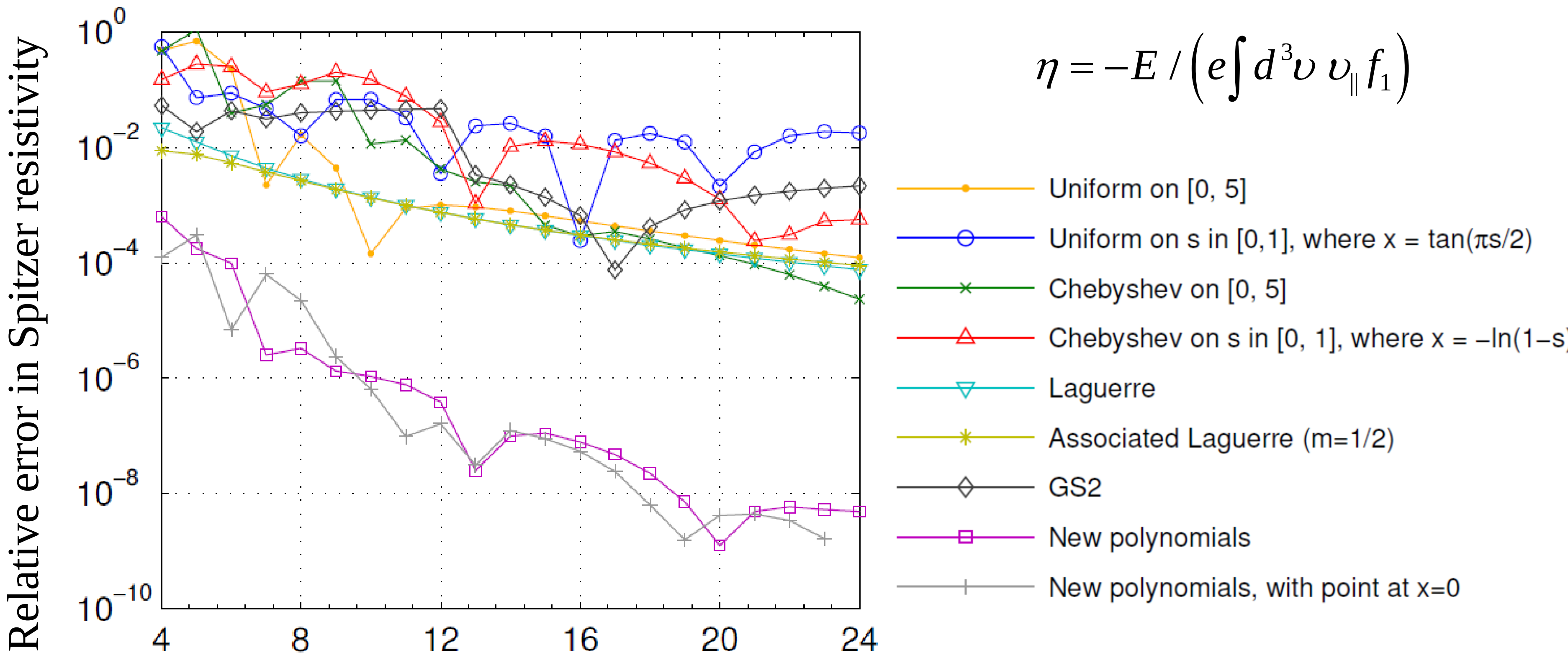
1D problem: Spitzer resistivity

$$C\{f_1\} = \frac{eE}{T} \upsilon_{\|} f_M$$

$$\eta = -E \Big/ \left(e \int d^3\upsilon \, \upsilon_{\|} f_1 \right)$$



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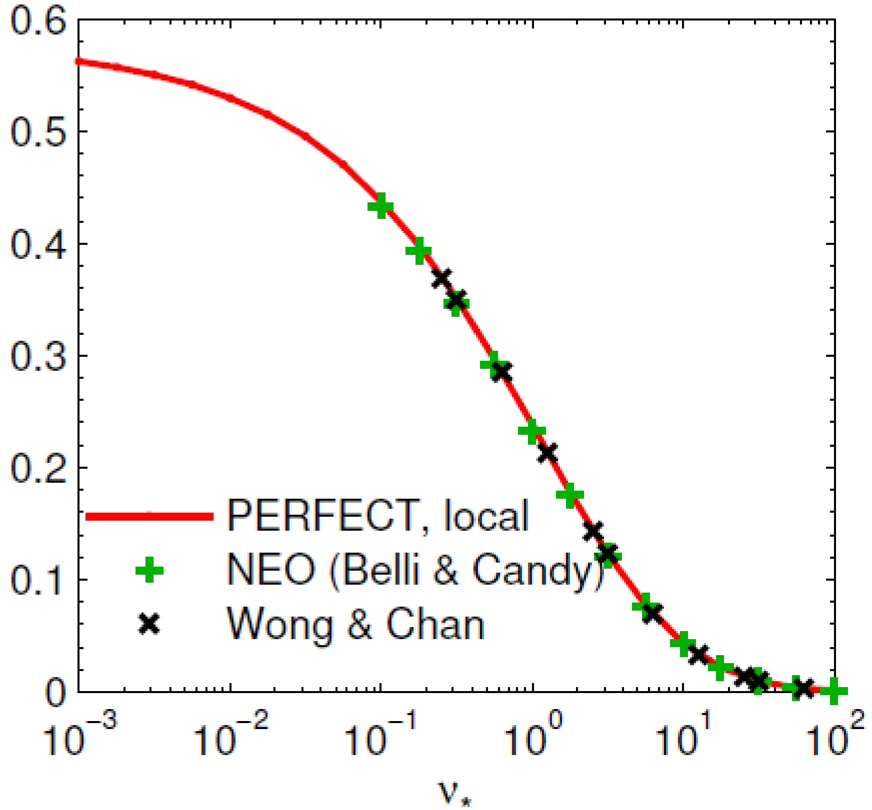
With $v_d \cdot \nabla f_1$ term turned off, new code agrees exactly with other Fokker-Planck codes

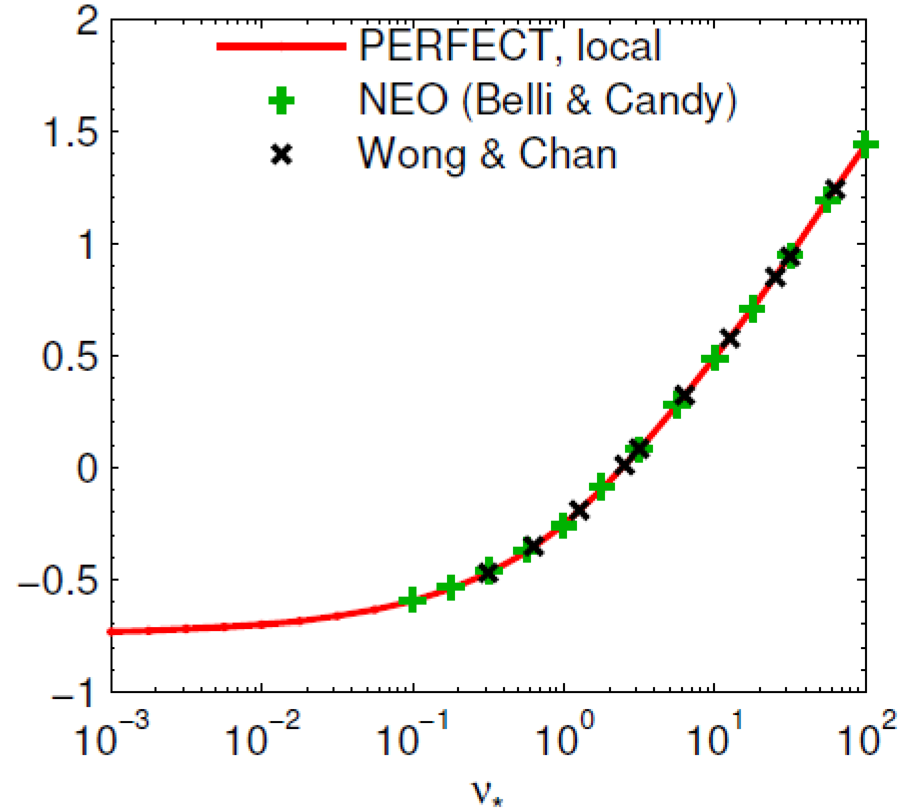
Wong & Chan, PPCF 53, 095005 (2011).

Belli & Candy, PPCF 54, 015015 (2012).

∇p contribution to j_{bs} (normalized)



∇T_i contribution to j_{bs} (normalized)



Finite-orbit-width analytic neoclassical theory has recently been developed.

Based on expansions in aspect ratio (circular geometry) and collisionality.
Example of analytic results:

Ion heat flux in plateau collisionality regime.

$$q_{\mathrm{local}} = -\frac{3\sqrt{\pi}}{4}\frac{\varepsilon^2 n \upsilon_{th} \rho_\theta^2}{qR}\frac{dT}{dr}$$

$$q_{\mathrm{global}} = q_{\mathrm{local}} \frac{1}{3}\left(\frac{4U^8 + 16U^6 + 24U^4 + 12U^2 + 3}{2U^4 + 2U^2 + 1}\right) e^{-U^2}$$

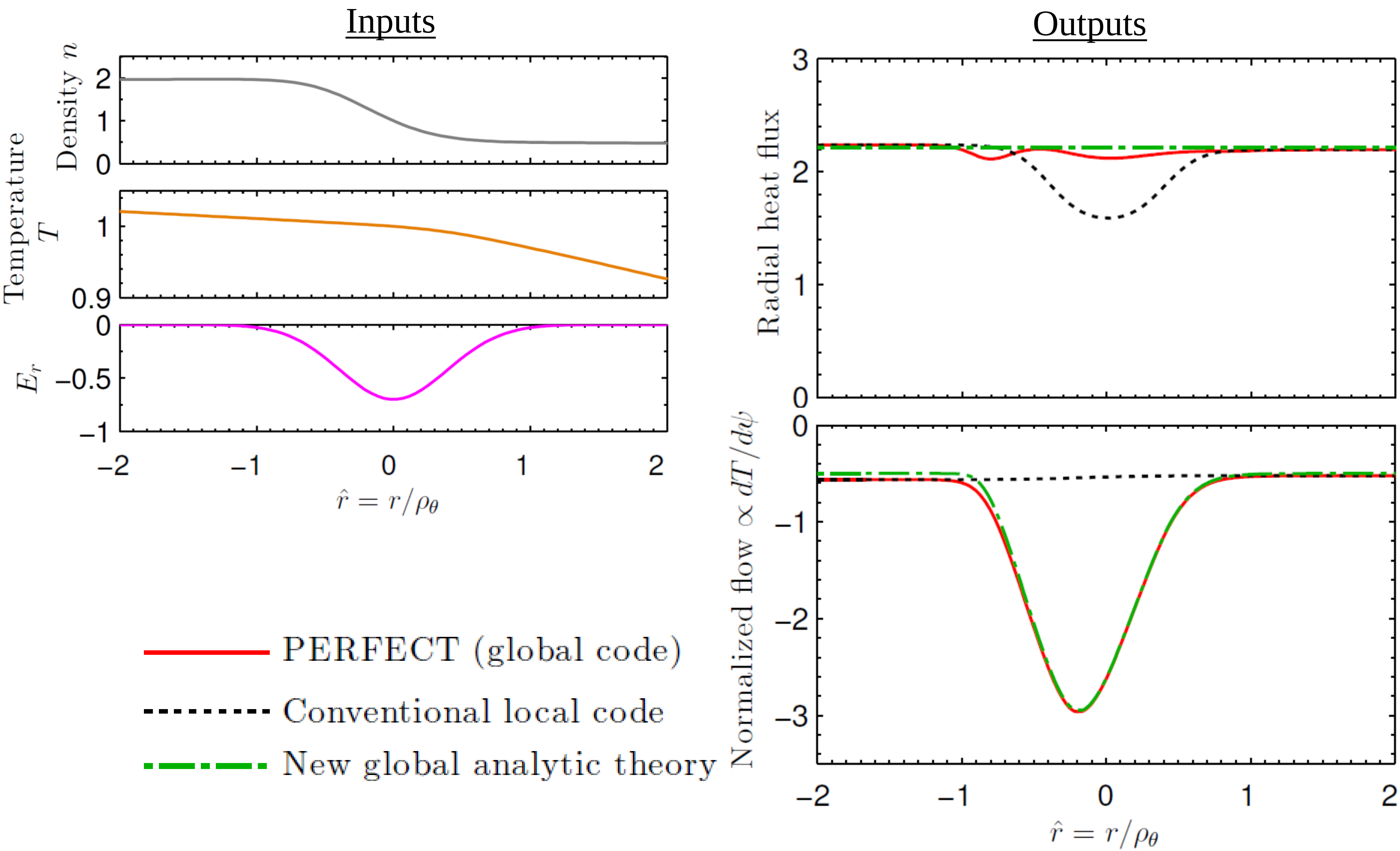
where $U = \rho_\theta / (2L_n)$.

Pusztai & Catto, PPCF 52, 075016 (2010)

More analytic results, including calculation of flow, in preparation by F. Parra et al.

Pedestal code agrees with our new finite-orbit-width analytic theory

$r/R = 0.001$, plateau regime of collisionality



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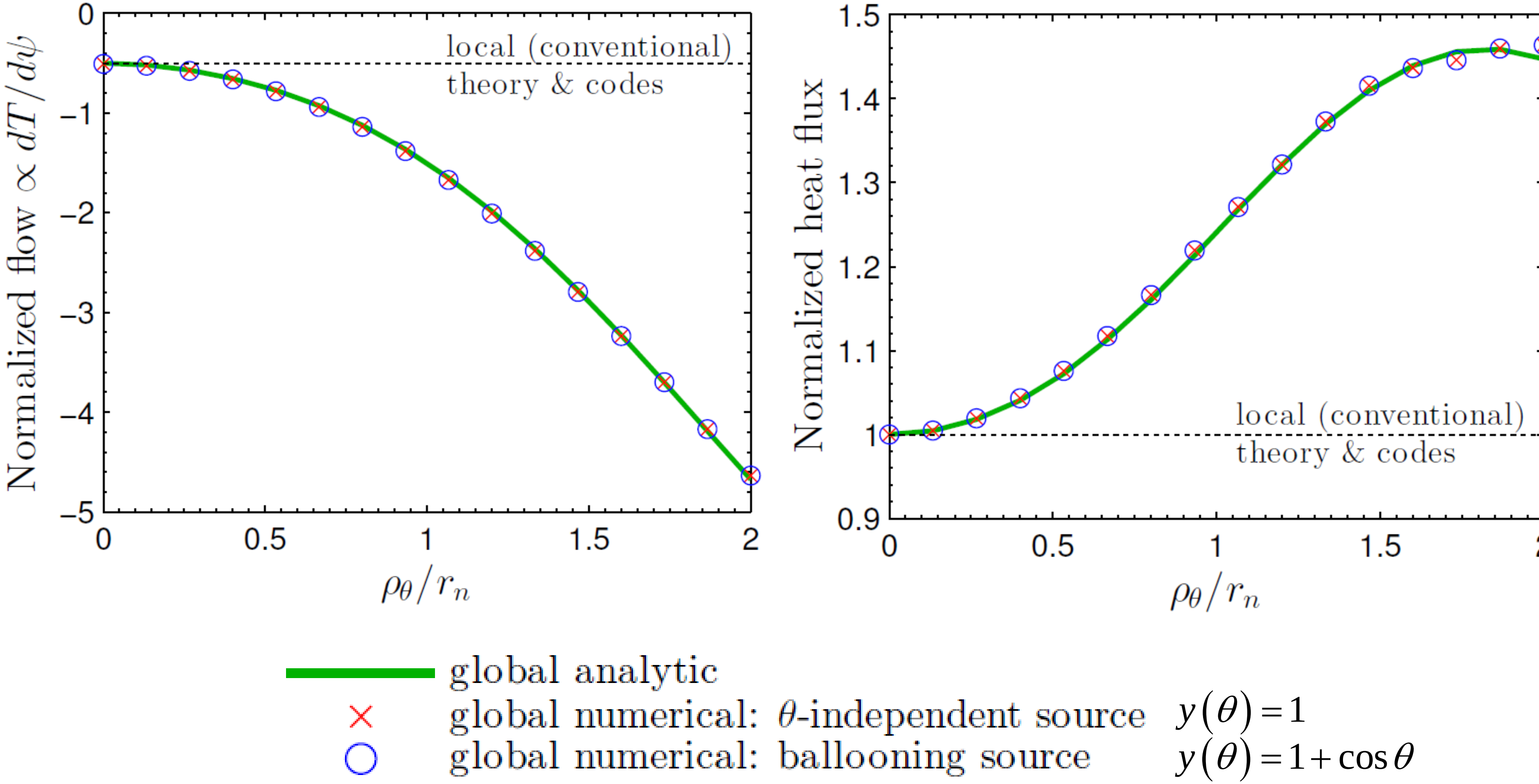
Mid-pedestal flow and heat flux, varying pedestal steepness:



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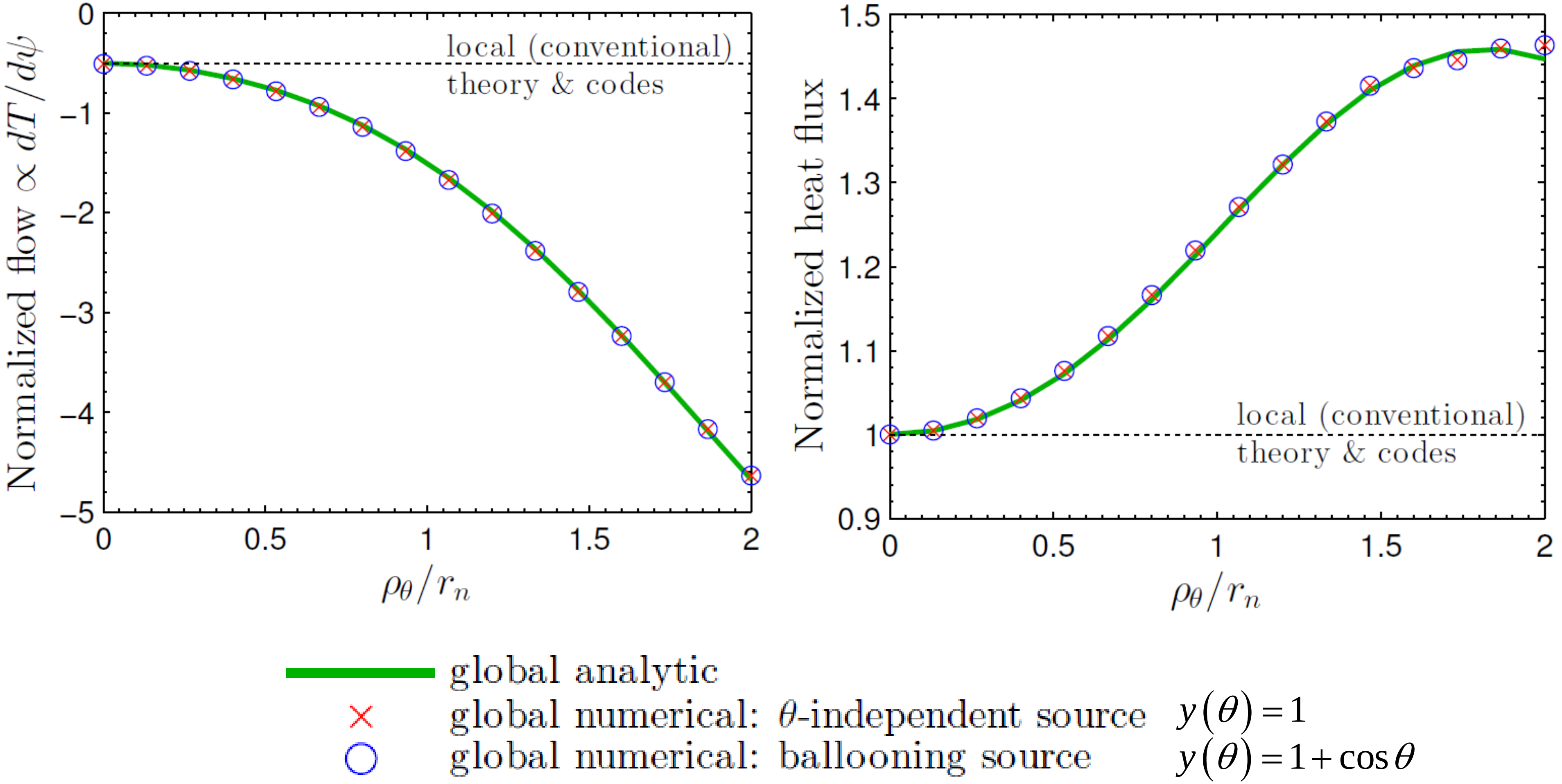


$y(\theta) = 1$

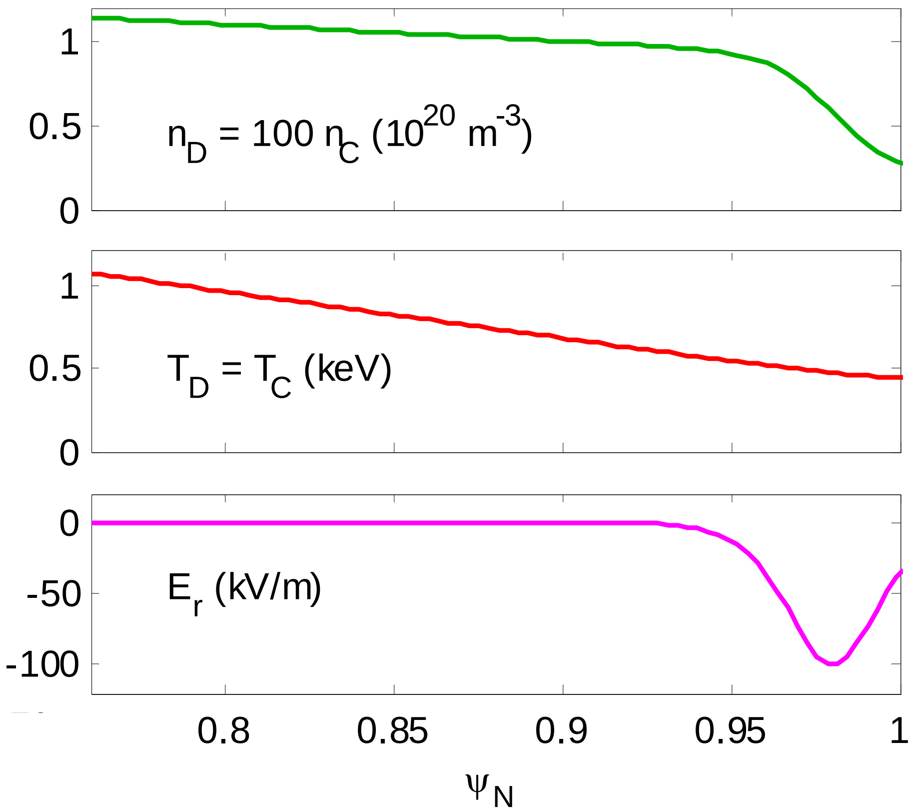
$y(\theta) = 1 + \cos\theta$

Each simulation ~ 2 minutes on 1 node (24 cores) of hopper (NERSC).

Global code sometimes predicts substantial changes to pedestal flows compared to local approach.

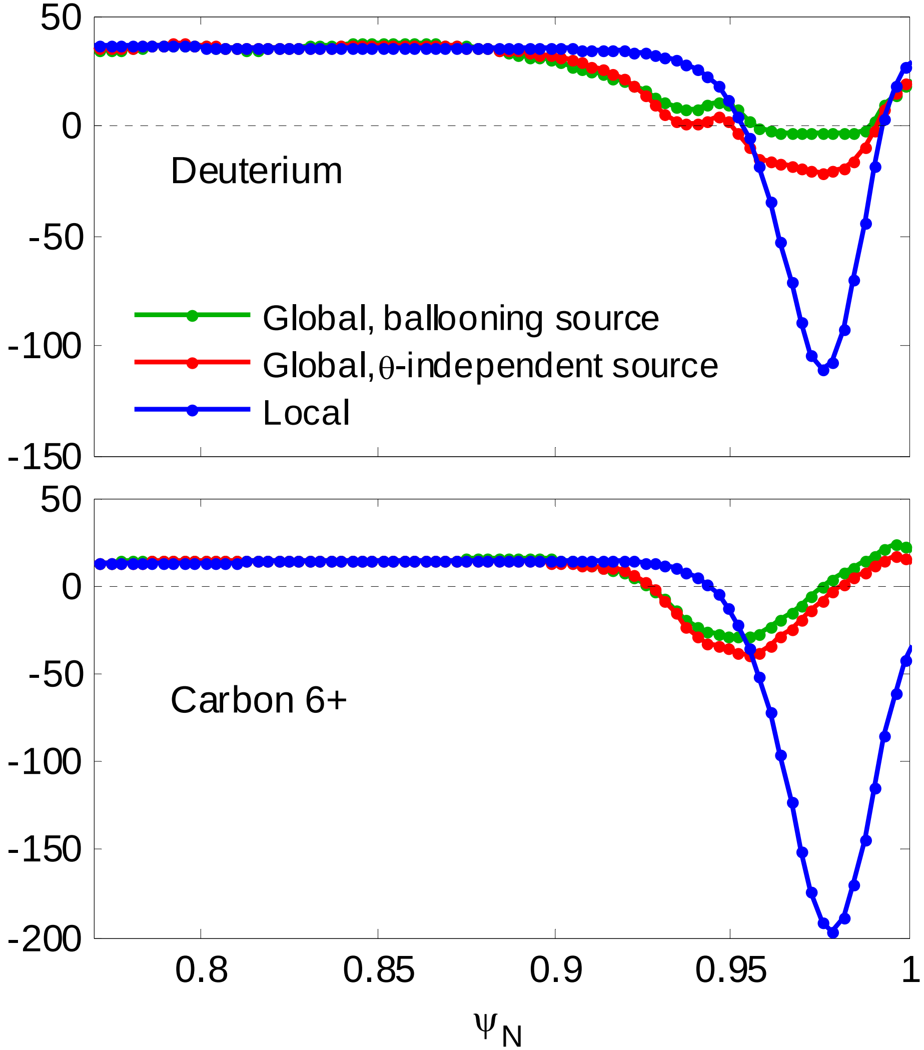
Inputs:

DIII-D-like parameters, 1% C^{6+} impurity



Outputs:

Parallel flow at outboard midplane (km/s)



Future work for pedestal code

- Comparison to experiments
- Include nonlinearities
- Open field lines, separatrix
  - Examine scrape-off layer width
- Other treatments of sources/sinks
  - Iterate to find consistent profiles with $S = 0$?
  - Can S be made more representative of turbulence?

Part II: Stellarators

$$f_1(\psi, \theta, \upsilon, \xi) \quad \rightarrow \quad f_1(\zeta, \theta, \upsilon, \xi)$$

SFINCS:

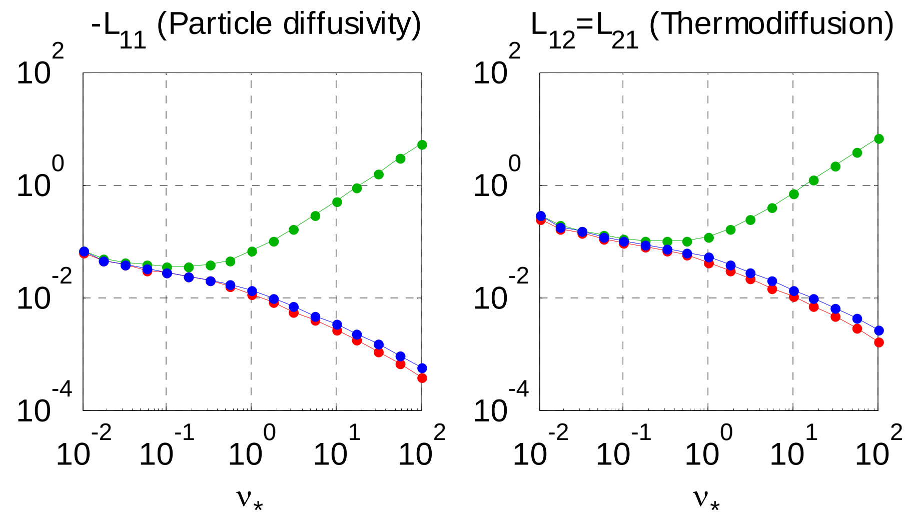
Stellarator Fokker-Planck Iterative Neoclassical Conservative Solver

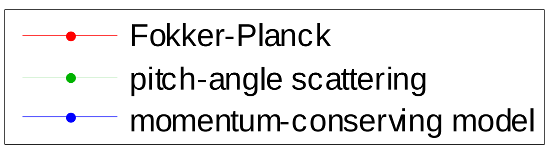
The ion transport matrix elements were computed for the LHD standard configuration.

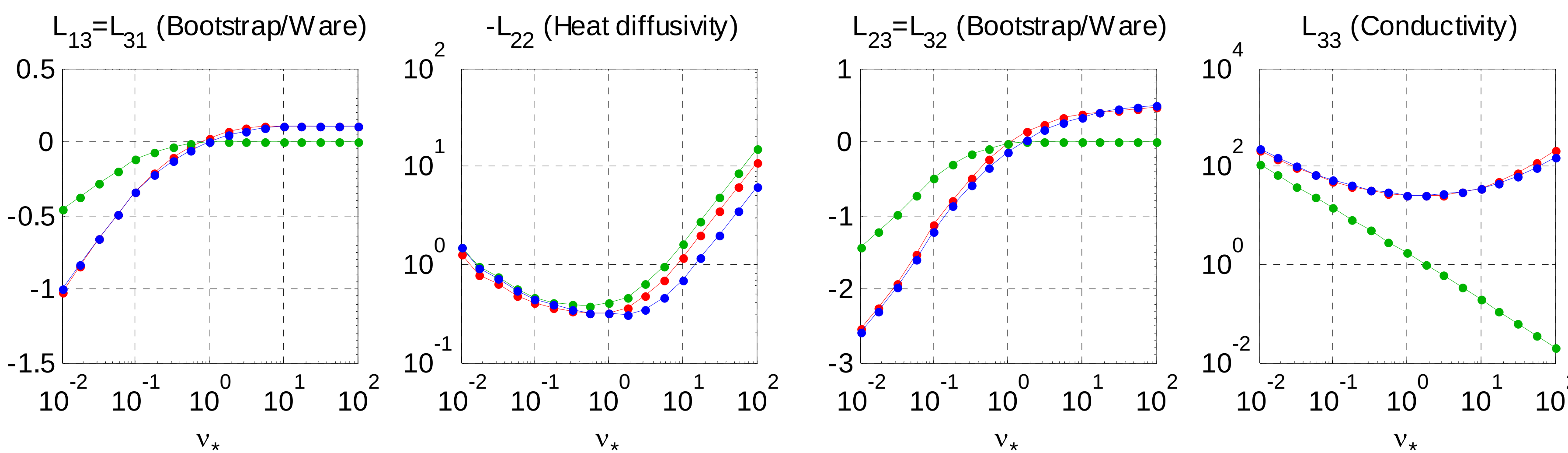
$$\begin{pmatrix} \langle \Gamma \cdot \nabla \psi \rangle \\ \langle \mathbf{q} \cdot \nabla \psi \rangle \\ \langle V_{\parallel} B \rangle \end{pmatrix} \propto \underbrace{\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}}_{\text{Transport matrix}} \begin{pmatrix} \dfrac{d \ln p}{d\psi} + \dfrac{e}{T}\dfrac{d\Phi}{d\psi} + \dfrac{d \ln T}{d\psi} \\ \dfrac{d \ln T}{d\psi} \\ \langle E_{\parallel} B \rangle \end{pmatrix}$$

For ion neoclassical physics in LHD, momentum-conserving model collision operator compares well to full Fokker-Planck operator.

$$\begin{pmatrix}\langle \Gamma\cdot\nabla\psi\rangle\\ \langle \mathbf{q}\cdot\nabla\psi\rangle \\ \langle V_\parallel B\rangle\end{pmatrix} \propto \underbrace{\begin{pmatrix}L_{11}&L_{12}&L_{13}\\ L_{21}&L_{22}&L_{23}\\ L_{31}&L_{32}&L_{33}\end{pmatrix}}_{\text{Transport matrix}} \begin{pmatrix}\dfrac{d\ln p}{d\psi}+\dfrac{e}{T}\dfrac{d\Phi}{d\psi}+\dfrac{d\ln T}{d\psi}\\ \dfrac{d\ln T}{d\psi}\\ \langle E_\parallel B\rangle\end{pmatrix}$$







$$\nu_* = \nu_{ii} R / \upsilon_{th,i}, \qquad E_r = 0$$

Several choices are available for the E_r terms

1. "Incompressible" ExB drift, used e.g. in DKES:

$$\left(\upsilon_{\|}\mathbf{b}+\frac{c}{\left\langle B^2\right\rangle}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi\right)\cdot\nabla f_1-\frac{\left(1-\xi^2\right)}{2B}\upsilon\left(\nabla_{\|}B\right)\frac{\partial f_1}{\partial\xi}-C\{f_1\}=-\mathbf{v}_m\cdot\nabla\psi\frac{\partial f_M}{\partial\psi}$$

$$\xi=\upsilon_{\|}/\upsilon$$

2. Correct ExB drift:

$$\left(\upsilon_{\|}\mathbf{b}+\frac{c}{B^2}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi\right)\cdot\nabla f_1-\frac{\left(1-\xi^2\right)}{2B}\upsilon\left(\nabla_{\|}B\right)\frac{\partial f_1}{\partial\xi}-C\{f_1\}=-\mathbf{v}_m\cdot\nabla\psi\frac{\partial f_M}{\partial\psi}$$

3. Including other terms required to conserve μ:

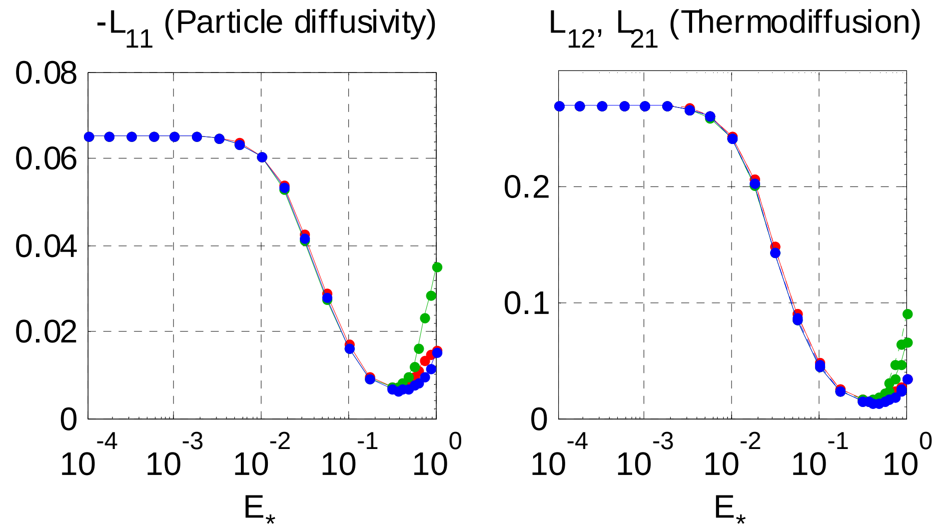
$$\left(\upsilon_{\|}\mathbf{b}+\frac{c}{B^2}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi\right)\cdot\nabla f_1+\left[-\frac{\left(1-\xi^2\right)}{2B}\upsilon\left(\nabla_{\|}B\right)+\frac{c\xi\left(1-\xi^2\right)}{2B^3}\frac{d\Phi}{d\psi}\mathbf{B}\times\nabla\psi\cdot\nabla B\right]\frac{\partial f_1}{\partial\xi}$$

$$+\frac{c\upsilon}{2B^3}\left(1+\xi^2\right)\frac{d\Phi}{d\psi}\left(\mathbf{B}\times\nabla\psi\cdot\nabla B\right)\frac{\partial f_1}{\partial\upsilon}-C\{f_1\}=-\mathbf{v}_m\cdot\nabla\psi\frac{\partial f_M}{\partial\psi}$$

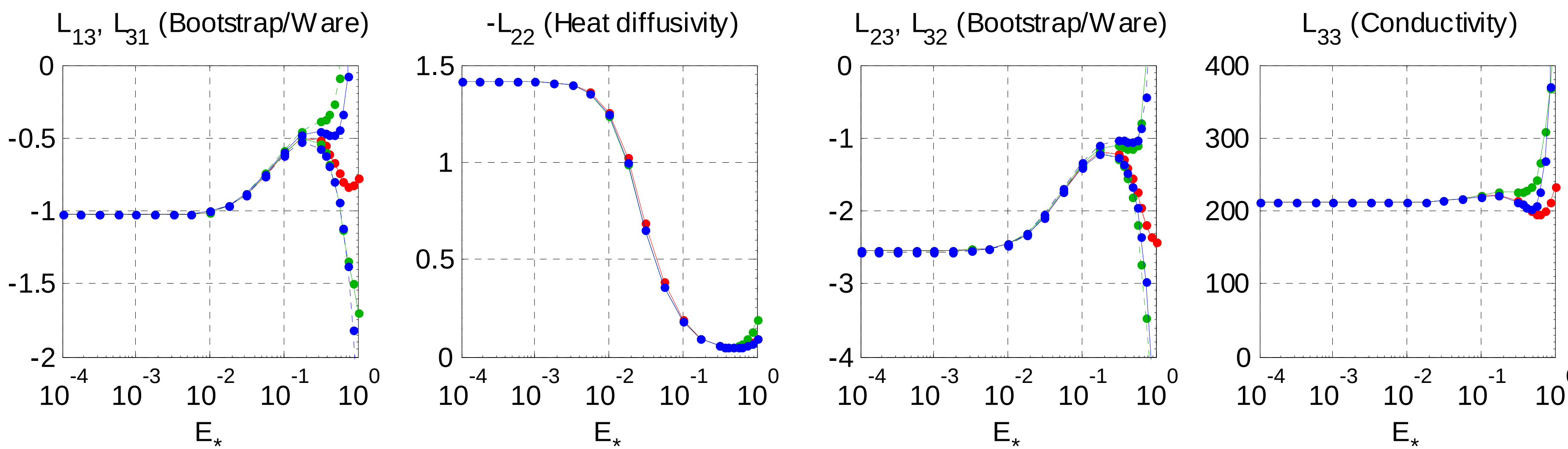
SFINCS allows comparisons between the options for E_r terms.

$$\begin{pmatrix} \langle \Gamma \cdot \nabla \psi \rangle \\ \langle \mathbf{q} \cdot \nabla \psi \rangle \\ \langle V_\parallel B \rangle \end{pmatrix} \propto \underbrace{\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}}_{\text{Transport matrix}} \begin{pmatrix} \dfrac{d \ln p}{d\psi} + \dfrac{e}{T}\dfrac{d\Phi}{d\psi} + \dfrac{d \ln T}{d\psi} \\ \dfrac{d \ln T}{d\psi} \\ \langle E_\parallel B \rangle \end{pmatrix}$$

(Results for $\nu_{ii} R / \upsilon_{th} = 0.01$.)



Incompressible ExB, no xDot or xiDot
True ExB, no xDot or xiDot
True ExB, with xDot and xiDot



$$E_* = E_r c R / (\iota \upsilon R B)$$

Future possibilities for stellarator code

- Impurities
- Include non-Boltzmann effect of poloidal electric field.
- Other suggestions?

Conclusions

- New global δf tokamak code PERFECT is a unique tool.
  - Captures some of the strong-gradient physics missing from conventional local δf theory/codes.
  - Directly solves time-independent equation.
  - Formulation finds self-consistent sources.
  - Agrees with finite-orbit-width theory in appropriate limits.
- The closely related stellarator code SFINCS…
  - Computes the neoclassical distribution function & moments over a full stellarator flux surface, including full linearized Fokker-Planck collisions and E_r.
  - Allows comparison of different models for collisions and for E_r.
- Ideas for applications & collaborations are welcome

Extra slides

Neoclassical phenomena can be computed from several versions of the drift-kinetic equation

Global full-f:

$$\left(v_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f = C_{\text{nonlinear}}\left\{f, f\right\}$$

$$f = f_M + f_1, \qquad \mathbf{b}\cdot\nabla f_M = 0, \qquad f_1 \ll f_M, \qquad \Phi \approx \langle\Phi\rangle$$

f_M and $\langle\Phi\rangle$ specified

Global δf:

$$\left(v_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\left\{f_1\right\}$$

PERFECT

$\mathbf{v}_d\cdot\nabla f_1$ dropped.

Local δf:

$$v_{\|}\mathbf{b}\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\left\{f_1\right\}$$

NEO, NCLASS, and most neoclassical theory are based on this version.

Completeness, complexity

Stronger radial gradients allowed

The 3 versions of the drift-kinetic equation differ in complexity

Global full-f:

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f = C_{\text{nonlinear}}\{f,f\}$$

4D: $\left(\psi,\ \theta,\ \mu,\ m\upsilon^2/2+e\Phi\right)$

Nonlinear in unknowns:

$C \propto nf,\quad B^{-2}\mathbf{B}\times\nabla\Phi\cdot\nabla f.$

$$f = f_M + f_1,\qquad \mathbf{b}\cdot\nabla f_M = 0,\qquad f_1 \ll f_M,\qquad \Phi \approx \langle\Phi\rangle$$

f_M and $\langle\Phi\rangle$ specified

Global δf:

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\{f_1\}$$

4D: $\left(\psi,\ \theta,\ \mu,\ m\upsilon^2/2+e\langle\Phi\rangle\right)$

Linear in f_1.

$\mathbf{v}_d\cdot\nabla f_1$ dropped.

Local δf:

$$\upsilon_{\|}\mathbf{b}\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\{f_1\}$$

3D: $(\theta,\ \mu,\ \upsilon)$

Linear in f_1.

Completeness, complexity

Stronger radial gradients allowed

The 3 versions of the drift-kinetic equation allow different maximum gradients

Global full-f:

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f = C_{\text{nonlinear}}\{f,f\}$$

$L_{Ti} \sim \rho_\theta$ and $L_n \sim \rho_\theta$ can be allowed.

$$f = f_M + f_1, \quad \mathbf{b}\cdot\nabla f_M = 0, \quad f_1 \ll f_M, \quad \Phi \approx \langle \Phi \rangle$$

f_M and $\langle \Phi \rangle$ specified

Global δf:

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\{f_1\}$$

Must have $L_{Ti} > \rho_\theta$, but $L_n \sim \rho_\theta$ allowed: electrostatic ion confinement $(ne\mathbf{E} \sim \nabla p)$ does not drive f_1.

$\mathbf{v}_d\cdot\nabla f_1$ dropped.

Local δf:

$$\upsilon_{\|}\mathbf{b}\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\{f_1\}$$

Must have $L_{Ti} > \rho_\theta$ and $L_n > \rho_\theta$.

Completeness, complexity

Stronger radial gradients allowed

In all 3 models, L_{Te} only needs to be $> \rho_{\theta e}$, which is always satisfied.

The global δf model includes some edge physics without the full complexity of the full-f model

Global full-f:

$$\left(\upsilon_{\parallel}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f = C_{\text{nonlinear}}\left\{f,f\right\}$$

$$f = f_M + f_1, \qquad \mathbf{b}\cdot\nabla f_M = 0, \qquad f_1 \ll f_M, \qquad \Phi \approx \left\langle\Phi\right\rangle$$

f_M and $\left\langle\Phi\right\rangle$ specified

Global δf:

$$\left(\upsilon_{\parallel}\mathbf{b}+\mathbf{v}_d\right)\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\left\{f_1\right\}$$

$\mathbf{v}_d\cdot\nabla f_1$ dropped.

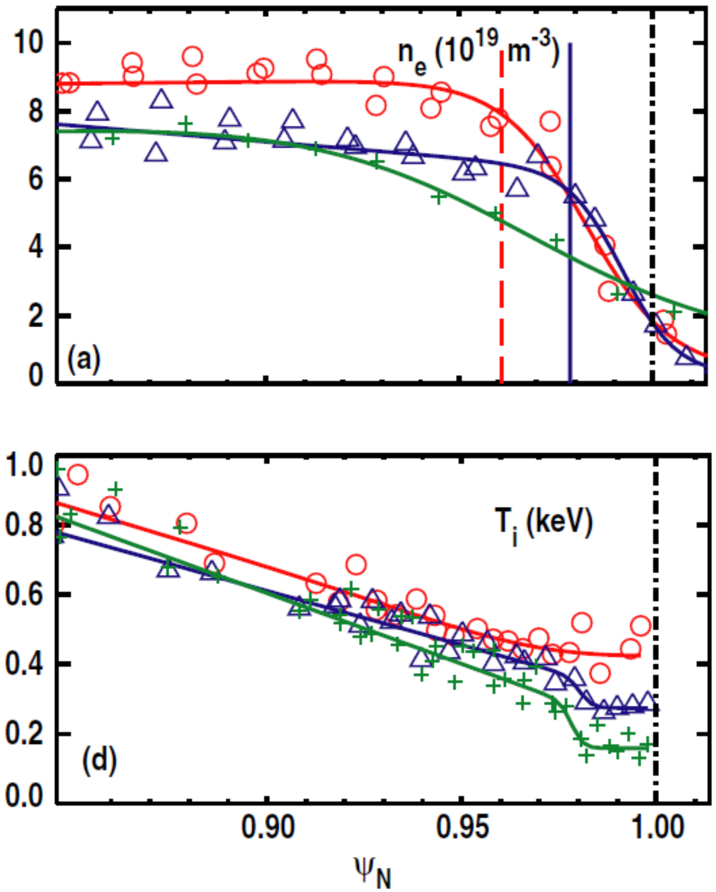
Local δf:

$$\upsilon_{\parallel}\mathbf{b}\cdot\nabla f_1 + \mathbf{v}_d\cdot\nabla f_M = C_{\text{linear}}\left\{f_1\right\}$$

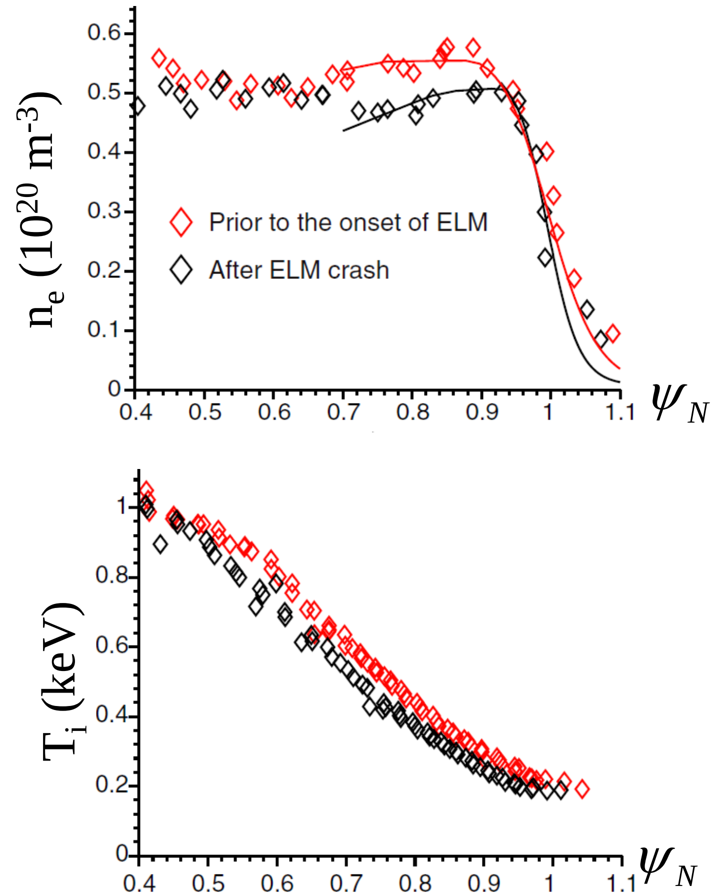
Global δf operator is also related to the Jacobian of the full-f equation, useful for iterative solution of this nonlinear equation.

Some experiments see $L_n < L_{Ti}$ in the pedestal

DIII-D: Groebner et al, Nucl. Fusion 49, 045013 (2009)



NSTX: Diallo et al, Nucl. Fusion 51, 103031 (2011)



MAST: Morgan et al, 37th EPS, P5.222; Meyer et al, Nucl. Fusion 51, 113011 (2011).

ASDEX: T. Pütterich et al, Nucl. Fusion 52, 083013 (2012).

JET: Y. Corre et al, PPCF 50, 115012 (2008).

Spatial pattern of flow is fundamentally different in local vs global δf models.

The $\int d^3 \upsilon (...)$ moment of the drift-kinetic equation gives a constraint on the flow:

<u>Local δf:</u>

$$V_{\parallel} = -\frac{cRB_{\phi}}{eB}\left(e\frac{d\Phi}{d\psi} + \frac{1}{n_i}\frac{dp_i}{d\psi} - k\frac{B^2}{\left\langle B^2 \right\rangle}\frac{dT_i}{d\psi} \right)$$

where $k = k\left(\psi\right)$

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where $k = k(\psi)$

Global δf:

Still true,

but now k varies on a flux surface

Possibly related to observations in C-Mod and ASDEX-U:

Marr et al, PPCF 52, 055010 (2010)
Pütterich et al, Nucl Fusion 52, 083013 (2012).

New speed discretization is highly efficient

Spectral collocation method based on non-standard orthogonal polynomials in v, not v^2:

$$\int_0^\infty dx\, P_i(x) P_j(x) e^{-x^2} \quad \propto \quad \delta_{ij}$$

Laguerre/Sonine polynomials lose accuracy because of nonanalytic √ in Jacobian at $x = 0$.

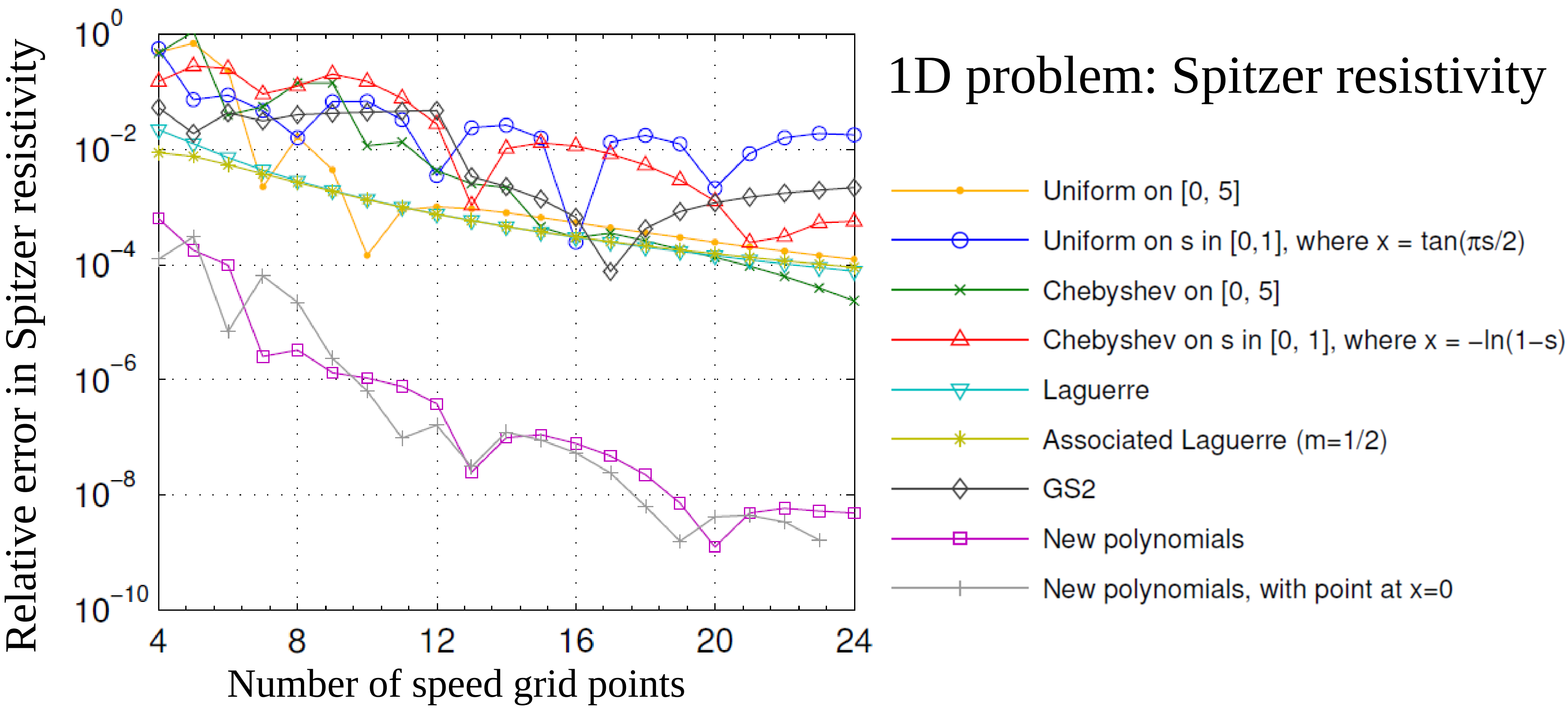
New speed discretization is highly efficient

Spectral collocation method based on non-standard orthogonal polynomials in v, not v²:

$$\int_0^\infty dx\, P_i(x)\, P_j(x)\, e^{-x^2} \propto \delta_{ij}$$

Laguerre/Sonine polynomials lose accuracy because of nonanalytic √ in Jacobian at $x = 0$.

1D problem: Spitzer resistivity



J Comp. Phys. ***243**, 130 (2013)*

Spatial pattern of flow is fundamentally different in local vs global δf models.

The $\left\langle \int d^3\upsilon (\ldots) \right\rangle$ moment of the drift-kinetic equation gives a constraint on the flow:

Local δf:

$$V_\parallel = -\frac{cRB_\phi}{eB}\left(e\frac{d\Phi}{d\psi} + \frac{1}{n_i}\frac{dp_i}{d\psi} - k\frac{B^2}{\langle B^2 \rangle}\frac{dT_i}{d\psi} \right)$$

where $k = k(\psi)$

Global δf:

$\longrightarrow$ Still true,

but now k varies on a flux surface

This effect can be understood from a fluid perspective:

$$0 = \nabla \cdot (n\mathbf{V}), \qquad \mathbf{V} = V_\parallel \mathbf{b} + \mathbf{V}_\perp, \qquad \mathbf{V}_\perp \approx \frac{c}{B^2}\mathbf{B}\times\nabla\Phi + \frac{c}{neB^2}\mathbf{B}\times\nabla\cdot\overleftrightarrow{\Pi}$$

$\nabla \cdot (n\mathbf{V}_\perp) \neq 0,$ so a $V_\parallel$ "return flow" must arise.

In global case, $\nabla \cdot \left(\frac{c}{eB^2}\mathbf{B}\times\nabla\cdot\overleftrightarrow{\Pi}_{anisotr} \right)$

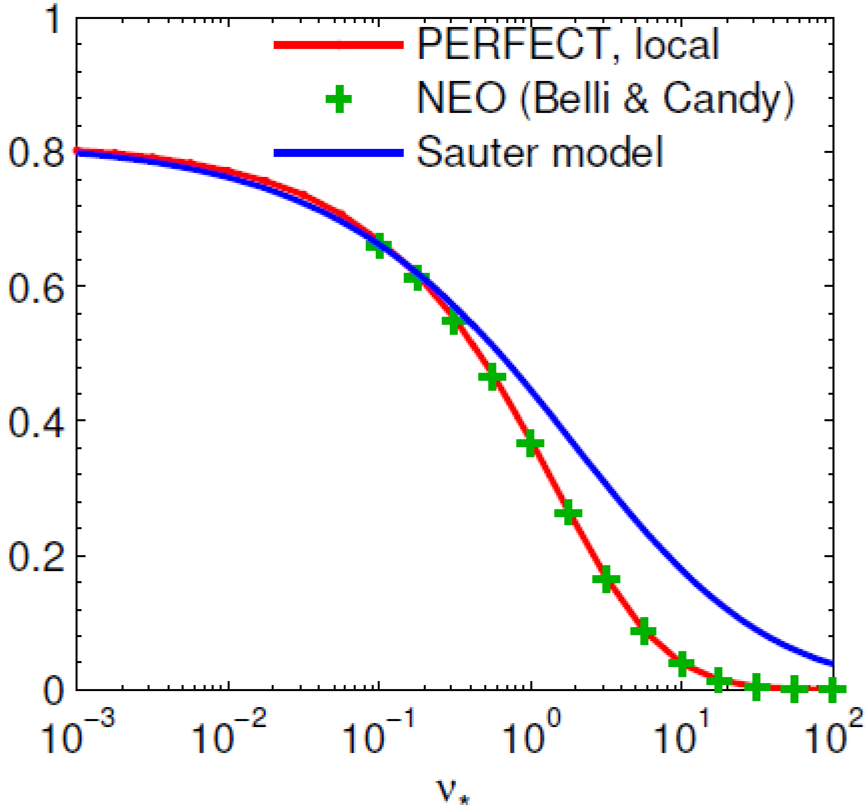
affects $\nabla \cdot (n\mathbf{V})$ to leading order

With $v_d \cdot \nabla f_1$ term turned off, PERFECT agrees exactly with a local Fokker-Planck code.

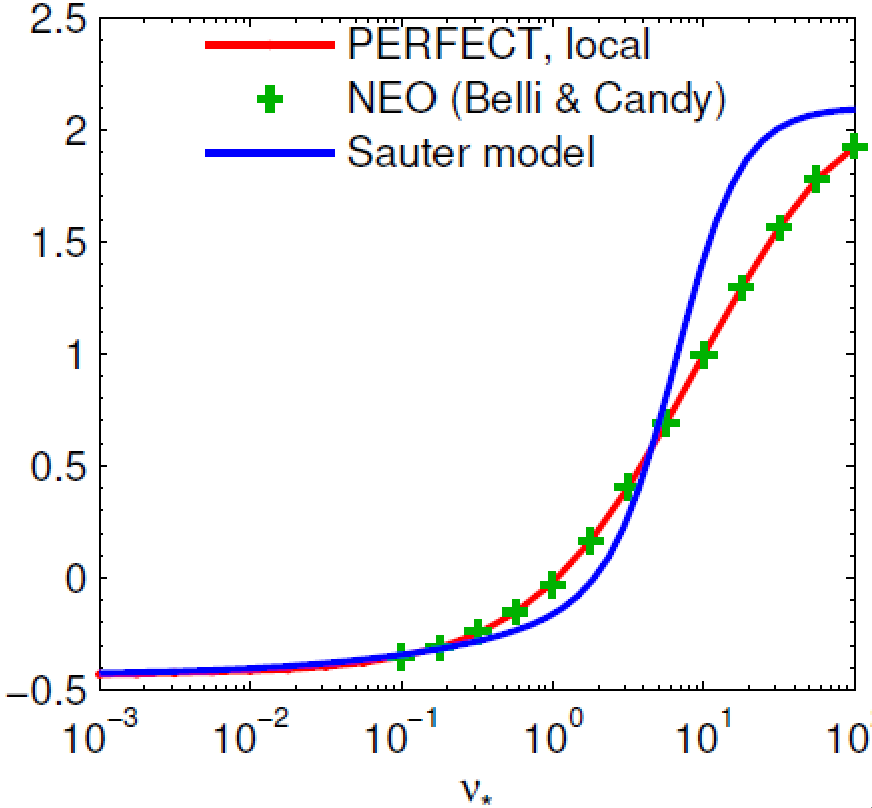
Local Fokker-Planck: Belli & Candy, PPCF 54, 015015 (2012).

Fit to local Fokker-Planck: Sauter, Angioni & Lin-Liu, PoP 6, 2834 (1999).

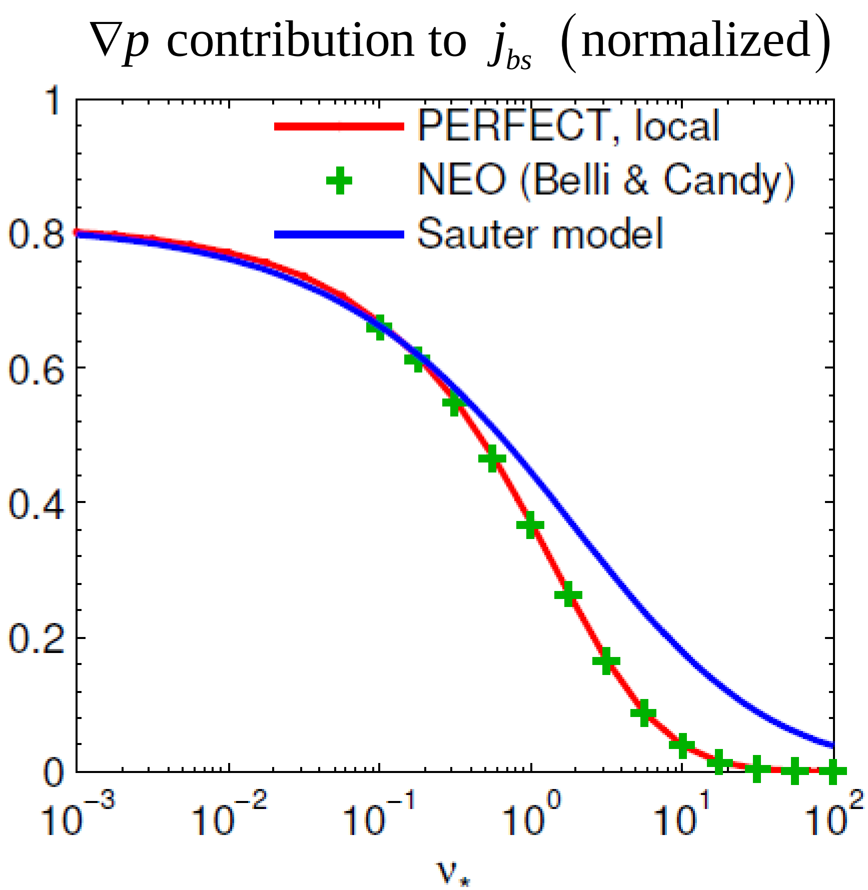
∇p contribution to j_{bs} (normalized)

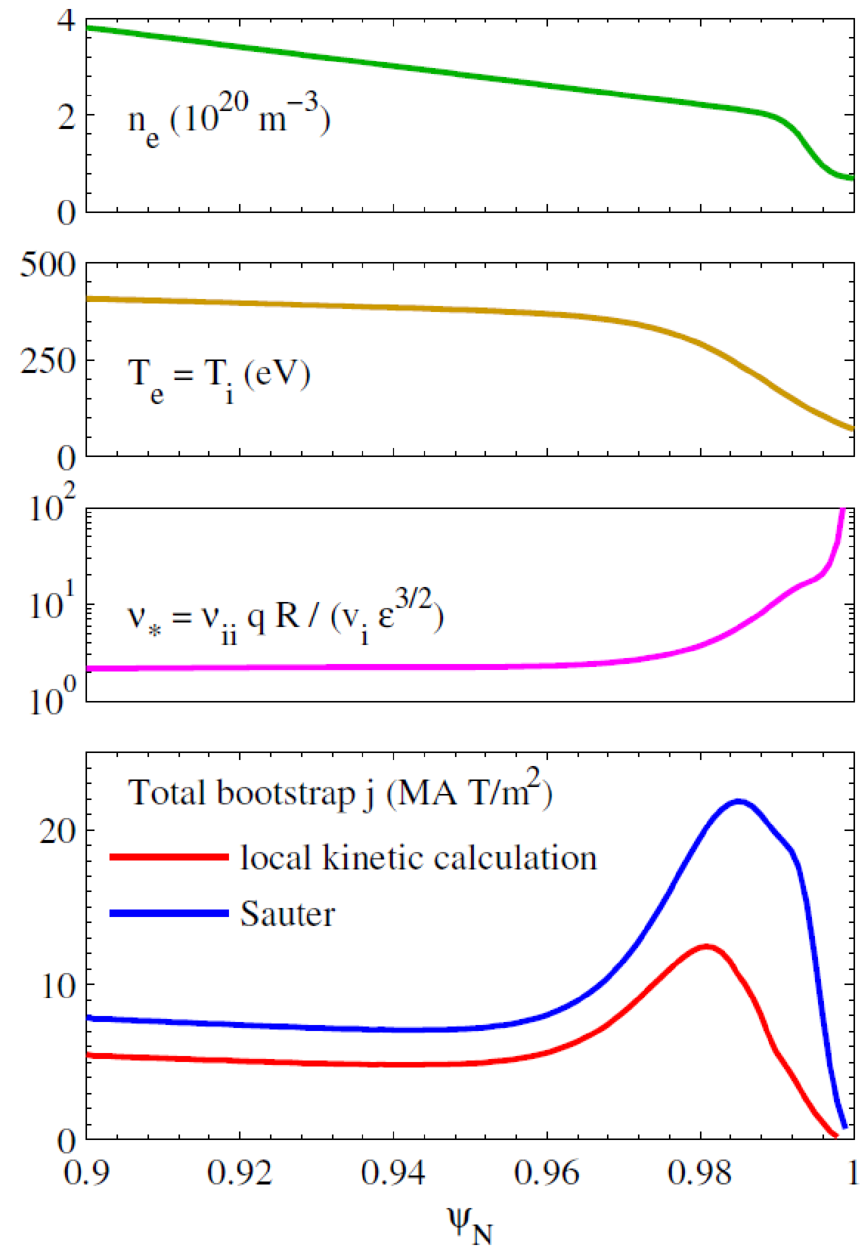


∇T_i contribution to j_{bs} (normalized)



For realistic high-ν_* profiles, Sauter over-predicts j_{bs} even within local theory

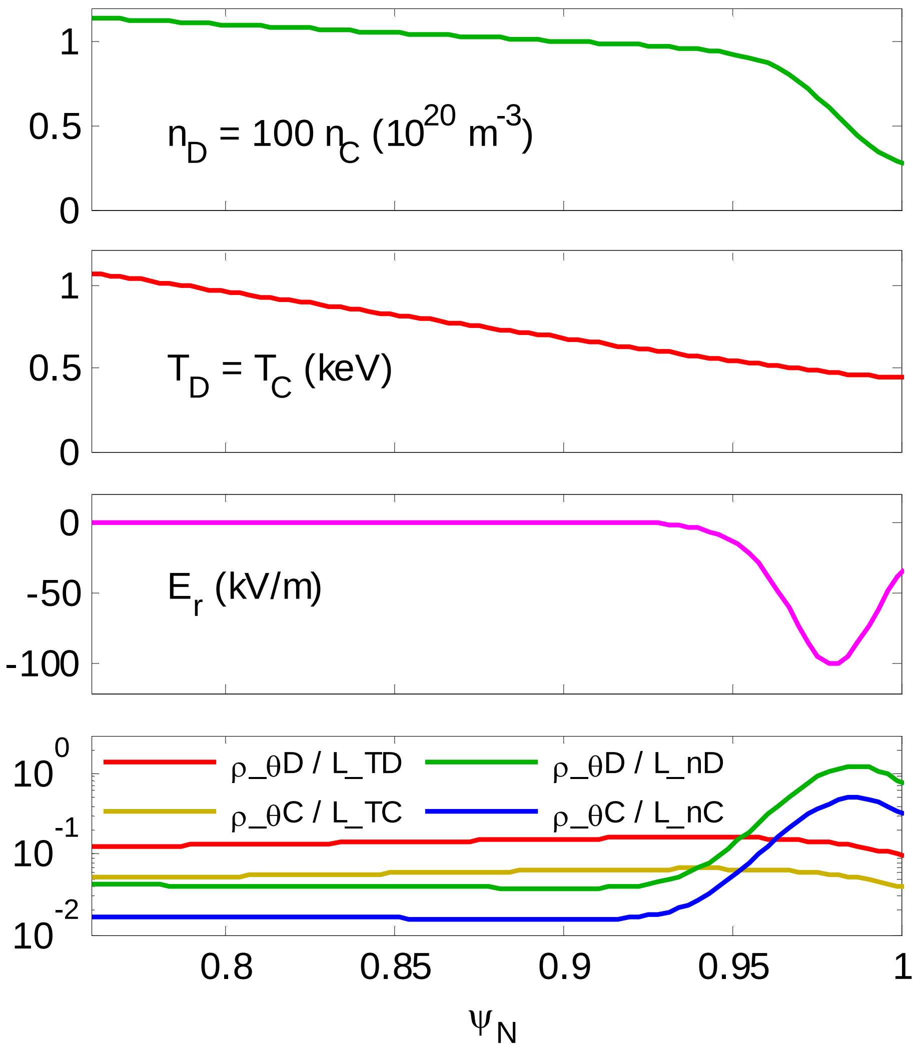




Global code predicts substantial changes to pedestal flows compared to local approach.

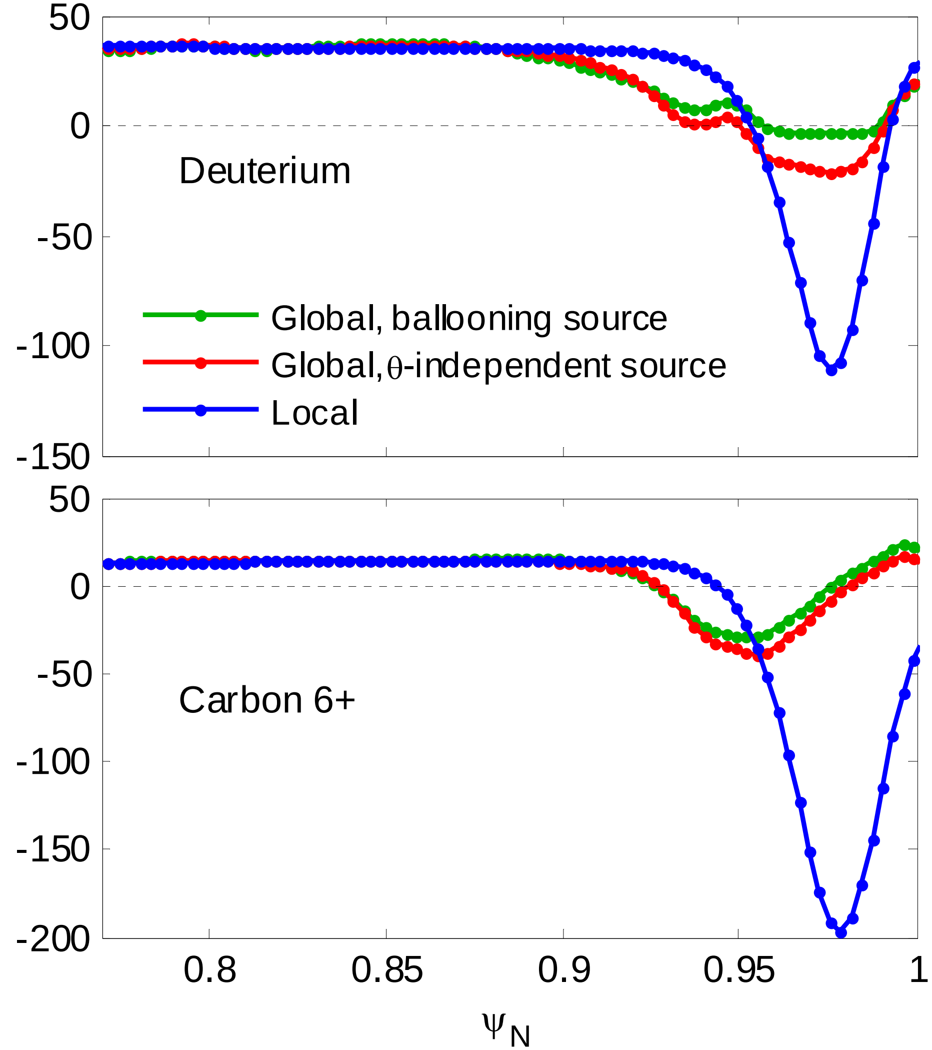
Inputs:

DIII-D-like parameters, 1% C^{6+} impurity



Outputs:

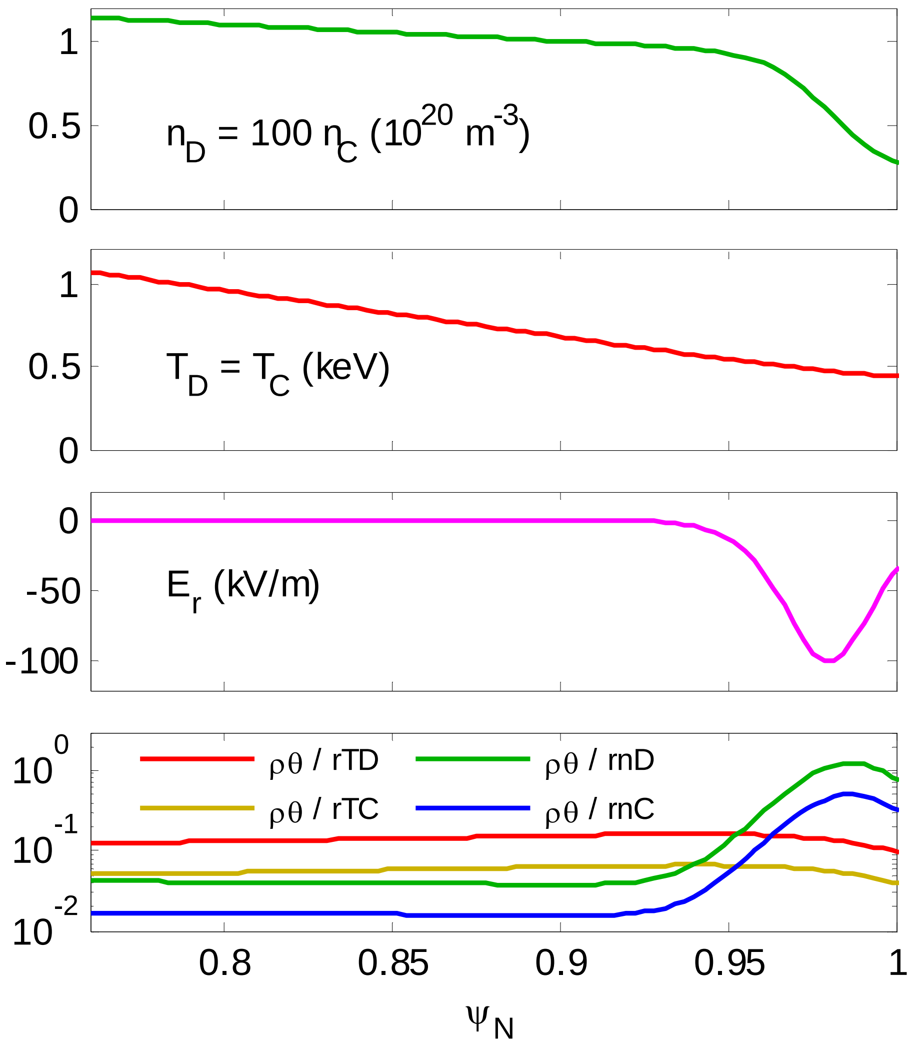
Parallel flow at outboard midplane (km/s)



Unlike local approach, global approach gives in-out asymmetries in *n* & *T*.

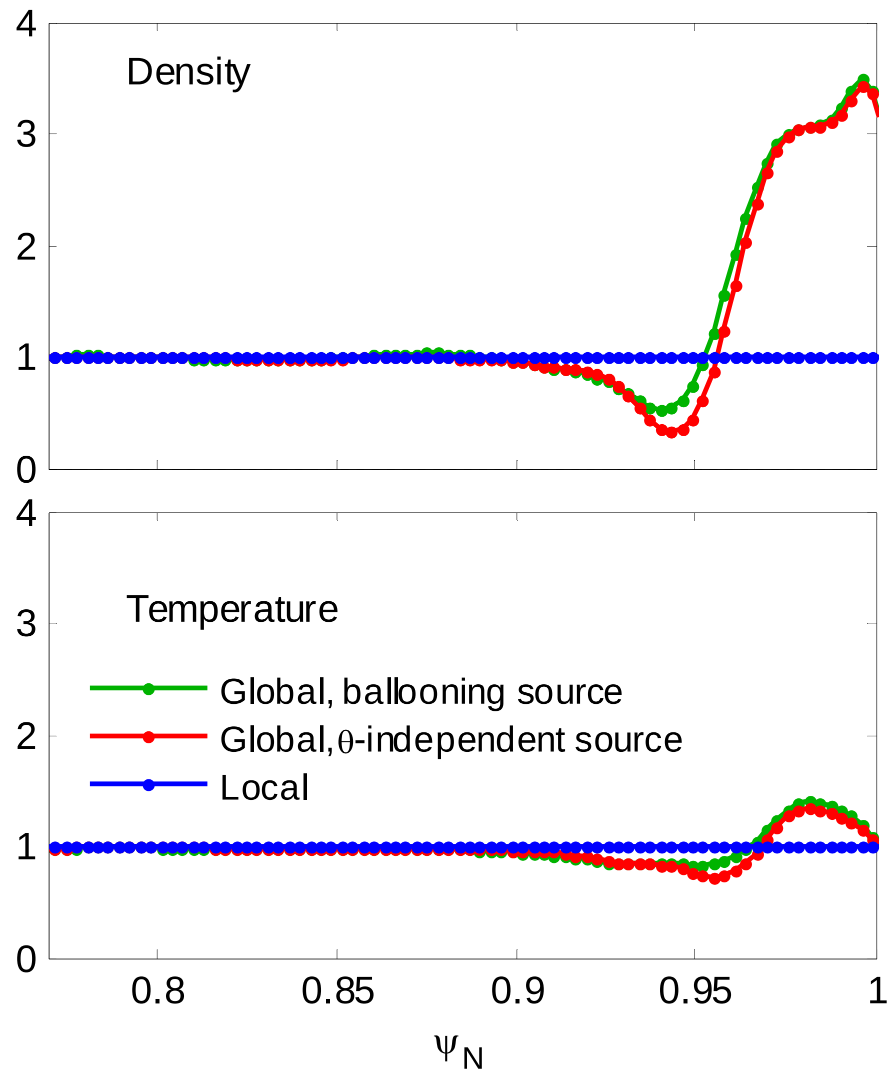
Inputs:

DIII-D-like parameters, 1% C^{6+} impurity



Outputs:

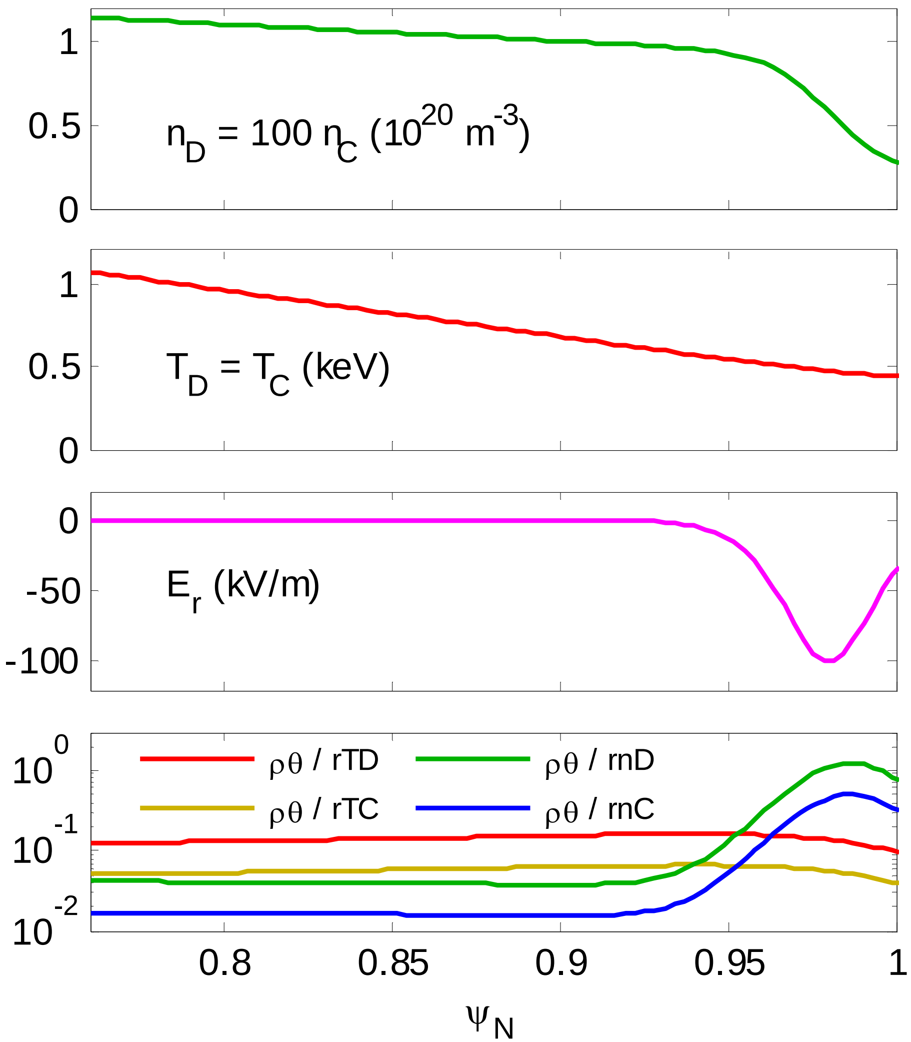
C^{6+} asymmetry (inboard/outboard)



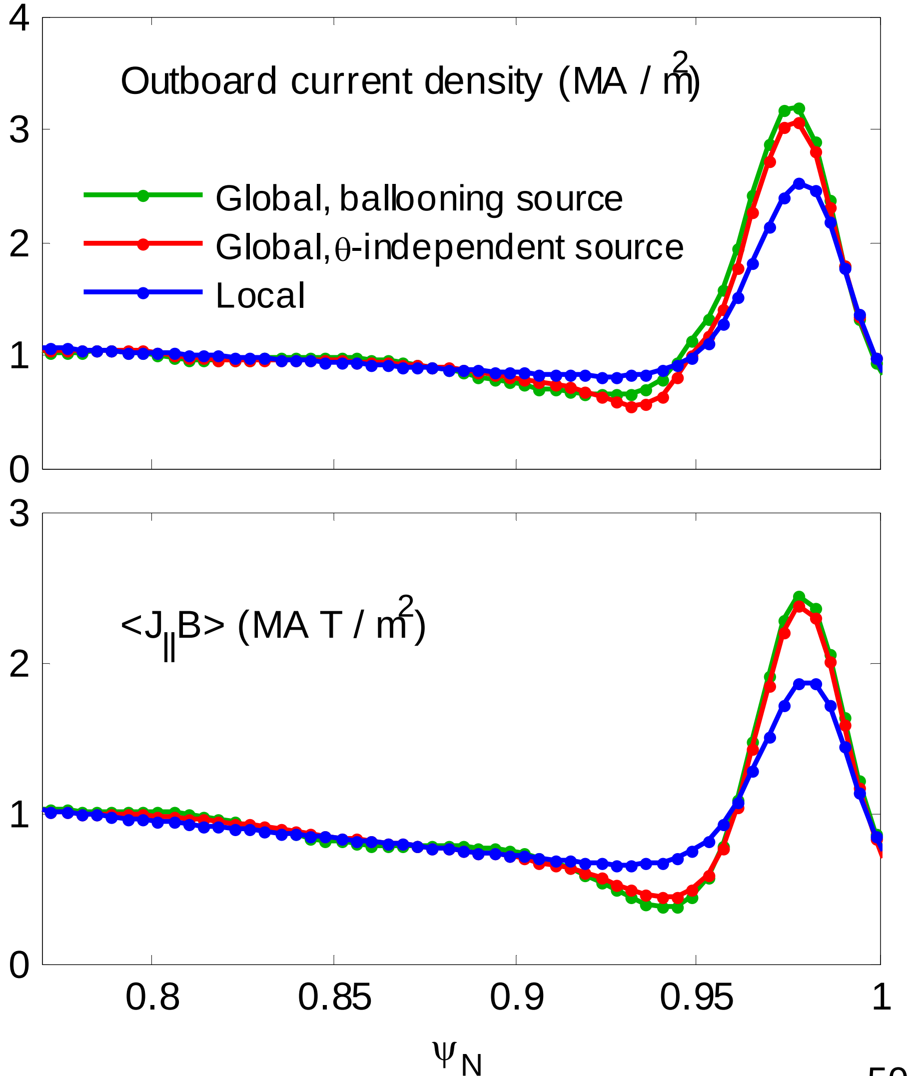
Local and global δf models predict different current

Inputs:

DIII-D-like parameters, 1% C^{6+} impurity



Outputs:



How large can the gradients be in each model?

Local approximation:

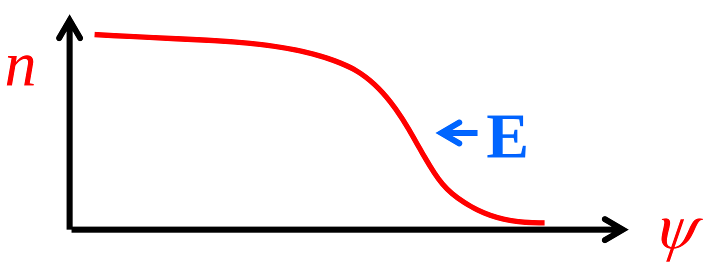
Dropping $\mathbf{v}_d \cdot \nabla f_1$ compared to other terms requires $r_n \gg \rho_\theta$, $r_{Ti} \gg \rho_\theta$.

$\Rightarrow$ Local = zero orbit width compared to equilibrium scales,
Global = finite orbit width.

δf approximation: $f_1 \ll f_\mathrm{M}$ $\quad$ f_1 ~ Departure from thermodynamic equilibrium.

$$f_1 \propto \mathbf{v}_d \cdot \nabla\psi \left(\frac{\partial f_M}{\partial \psi}\right)_W \quad \text{so} \quad \left(\frac{\partial f_M}{\partial \psi}\right)_W \quad \text{cannot be too large.}$$

A strong n gradient balanced by strong inward E_r (electrostatic ion confinement) does not drive f_1, so it is permitted!

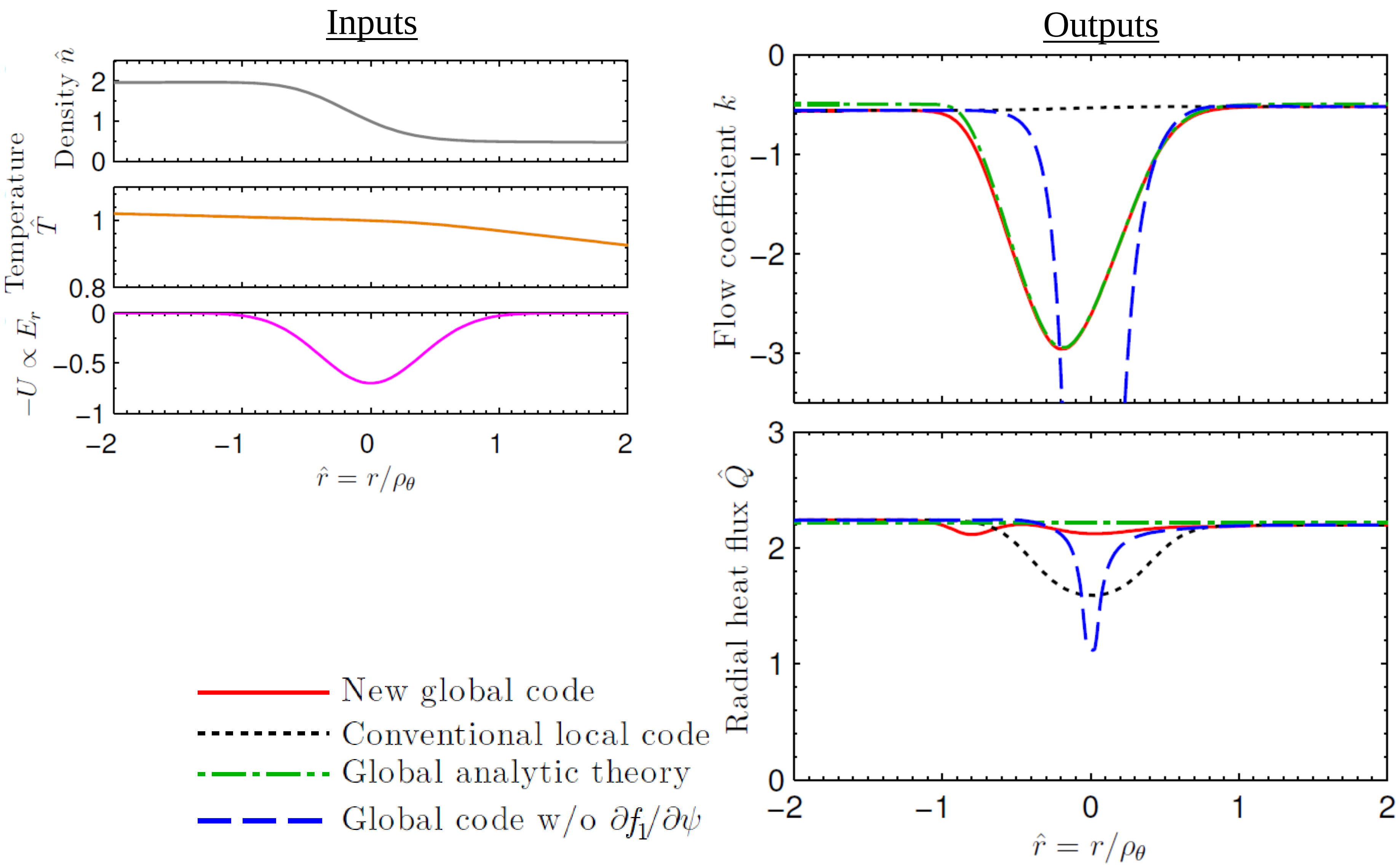


$$f_M = \eta\left[\frac{m}{2\pi T}\right]^{3/2} \exp\left(-\frac{W}{T}\right), \qquad n = \eta \exp\left(-\frac{e\Phi}{T}\right).$$

$$\left(\frac{\partial f_M}{\partial \psi}\right)_W \propto \frac{dT}{d\psi} \text{ and } \frac{d\eta}{d\psi}, \text{ but not } \frac{dn}{d\psi}.$$

No restriction on r_n, but still need $r_{Ti} \gg \rho_\theta$ and $r_\eta \gg \rho_\theta$.

It is not a good approximation to drop the $df_1/d\psi$ term



Analytic results for the global δf model have been derived recently in asymptotic limits

- Recent analytic work found changes to flows, heat flux, and j_{bs} in the pedestal:
  - Kagan & Catto, PPCF 52, 055004 (2010)
  - Pusztai & Catto, PPCF 52, 075016 (2010)
  - Kagan & Catto, PRL 105, 045002 (2010)
  - Catto et al, PPCF 55, 045009 (2013)
- But these analytic calculations required assuming $\sqrt{\varepsilon} \ll 1$ (large aspect ratio), $\nu qR/v \ll 1$, & circular flux sufaces.
- What can be said if we relax these approximations?

Neoclassical phenomena can be computed from several versions of the drift-kinetic equation

Global full-f:

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_{m}+\mathbf{v}_{E}\right)\cdot\left(\nabla f\right)_{\mu,W}=C_{\text{nonlinear}}\left\{f,f\right\}$$

$$\mathbf{v}_{E}=\frac{1}{B^{2}}\mathbf{B}\times\nabla\Phi$$

$$W=\frac{m\upsilon^{2}}{2}+e\Phi$$

$$f=f_{M}-\frac{Ze\left(\Phi-\langle\Phi\rangle\right)}{T}f_{M}+f_{1},\qquad \mathbf{b}\cdot\nabla f_{M}=0,\qquad f_{1}\ll f_{M},\qquad \Phi-\langle\Phi\rangle\ll\langle\Phi\rangle$$

f_M and $\langle\Phi\rangle$ specified.

Global δf:

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_{m}+\mathbf{v}_{E0}\right)\cdot\left(\nabla f_{1}\right)_{\mu,W_{0}}+\mathbf{v}_{m}\cdot\left(\nabla f_{M}\right)_{W_{0}}=C_{\text{linear}}\left\{f_{1}\right\}$$

$$\mathbf{v}_{E0}=\frac{1}{B^{2}}\mathbf{B}\times\nabla\langle\Phi\rangle$$

$$W_{0}=\frac{m\upsilon^{2}}{2}+e\langle\Phi\rangle$$

$\mathbf{v}_{d}\cdot\nabla f_{1}$ dropped.

Local δf:

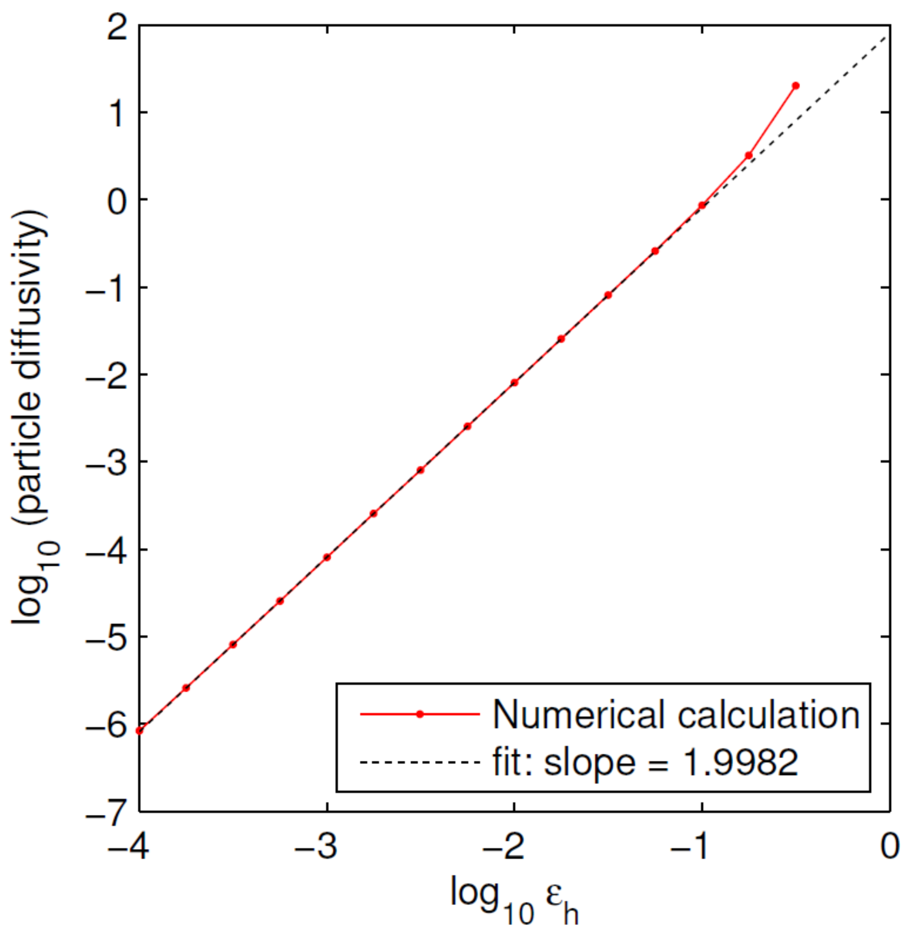
$$\upsilon_{\|}\mathbf{b}\cdot\left(\nabla f_{1}\right)_{\mu,W_{0}}+\mathbf{v}_{d}\cdot\left(\nabla f_{M}\right)_{W_{0}}=C_{\text{linear}}\left\{f_{1}\right\}$$

Transport can be computed when symmetry-breaking ripple is added to a tokamak or quasisymmetric stellarator.

When collisionality is not very small, nonambipolar radial current scales as ε_h^2, as predicted by Ivan Calvo et al.

$$B(\theta,\zeta) = B_0\left[1+\varepsilon_t\cos\theta+\varepsilon_h\cos(M\theta-N\zeta)\right]$$

$$\varepsilon_t = 1/3, \quad M=0, \quad N=1, \qquad \nu_{ii}R/\upsilon_{th}=1$$

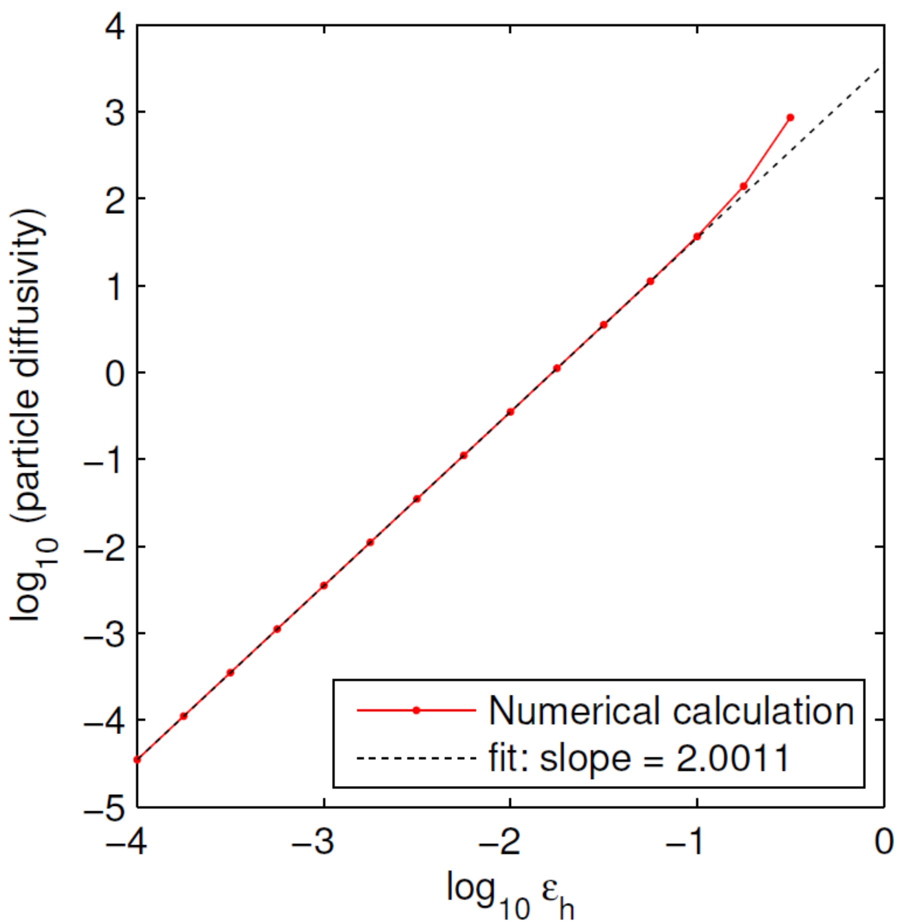


Transport can be computed when symmetry-breaking ripple is added to a tokamak or quasisymmetric stellarator.

When collisionality *is* very small but toroidal mode number is small, nonambipolar current appears to still scale as ε_h^2 in the code. But 1/ν transport should scale as $\varepsilon_h^{3/2}$. What is going on?

$$B(\theta,\zeta) = B_0\left[1+\varepsilon_t\cos\theta+\varepsilon_h\cos(M\theta-N\zeta)\right]$$

$$\varepsilon_t = 1/3,\quad M=0,\quad N=1,\qquad \nu_{ii}R/\upsilon_{th}=0.01$$

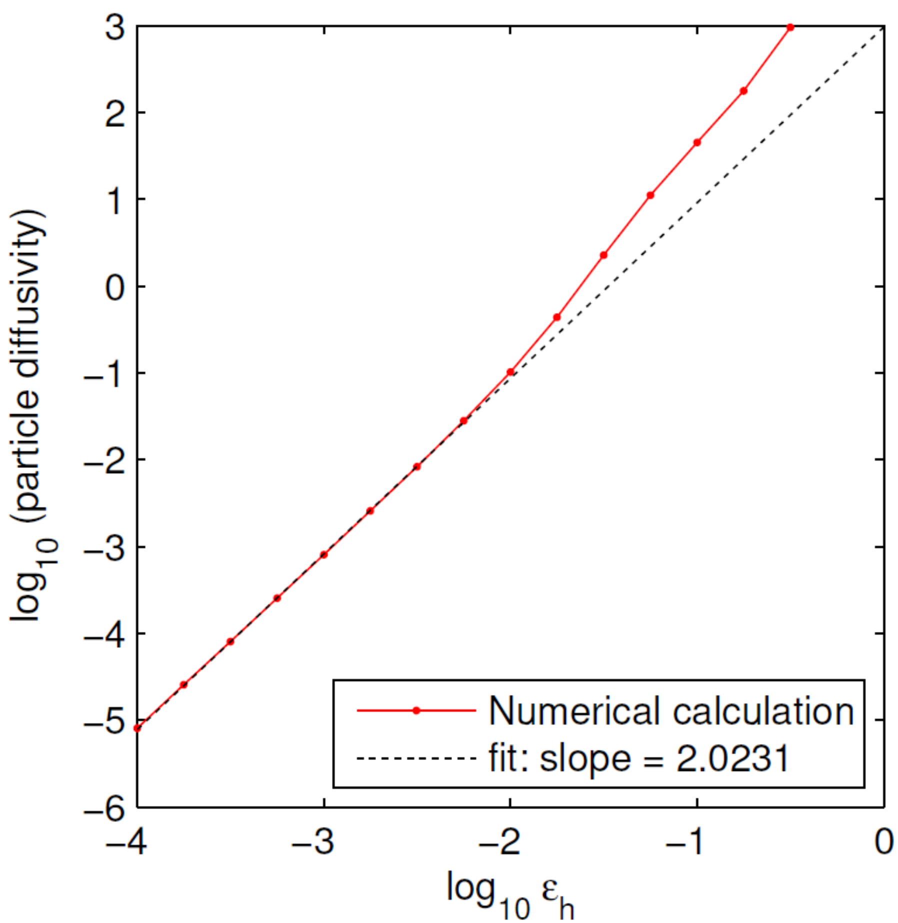


Transport can be computed when symmetry-breaking ripple is added to a tokamak or quasisymmetric stellarator.

When collisionality *is* very small and toroidal mode number is large, nonambipolar current appears to still scale *faster than* ε_h^2 in the code. But $1/\nu$ transport should scale as $\varepsilon_h^{3/2}$. What is going on?

$$B(\theta,\zeta) = B_0\left[1+\varepsilon_t\cos\theta+\varepsilon_h\cos(M\theta-N\zeta)\right]$$

$$\varepsilon_t = 1/3, \quad M = 0, \quad N = 8, \qquad \nu_{ii}R/\upsilon_{th} = 0.01$$



$$B(\theta,\zeta) = B_0\left[1+\varepsilon_t \cos(\theta) + \varepsilon_h \cos(N\zeta)\right]$$

$$E_r = 0, \quad \varepsilon_t = 1/3, \quad \iota = 1.6, \quad N = 100, \quad \nu_{ii} R / \upsilon_{th} = 0.01$$

$$\mathbf{B}\cdot\nabla B = -B_0\left(\mathbf{B}\cdot\nabla\zeta\right)\left[\varepsilon_t \iota \sin(\theta) + \varepsilon_h N \sin(N\zeta)\right]$$

To create new wells $(1/\nu \text{ transport})$, $\mathbf{B}\cdot\nabla B$ must vanish, so you need $\varepsilon_h \geq \iota\varepsilon_t / N$.