Kroklinjiga koordinatsystem för tokamaker

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Innehåll

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1 Tensorer i kroklinjiga koordinater

1.1 Vektorer och tensorer

Basvektorer och metriktensorn:

$$\text{kontravarianta basvektorer:}^1\ \ \mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial u^j} \tag{1}$$

$$\text{kovarianta basvektorer:}\ \ \mathbf{e}^j = \nabla u^j \tag{2}$$

$$\mathbf{e}_j \cdot \mathbf{e}^i = \delta^i_j \tag{3}$$

$$h_i = |\mathbf{e}_i| \tag{4}$$

$$\hat{\mathbf{e}}_i = \frac{\mathbf{e}_i}{h_i} \text{ (utan summation över i)} \tag{5}$$

$$\mathbf{A} = A^j \mathbf{e}_j = A_j \mathbf{e}^j \tag{6}$$

$$\text{de kontrav. komp. av vektorn } \mathbf{A} \text{ är:}\ \ A^j = (\mathbf{A})^j = \mathbf{A} \cdot \mathbf{e}^j \tag{7}$$

$$\text{de kov. komp. av vektorn } \mathbf{A} \text{ är:}\ \ A_j = (\mathbf{A})_j = \mathbf{A} \cdot \mathbf{e}_j \tag{8}$$

$$\text{metriktensorn:}\ \ g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{9}$$

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j \tag{10}$$

$$g_{ij} g^{jk} = \delta^k_i \tag{11}$$

$$g = \det\,[g_{ij}] \tag{12}$$

$$A_i = g_{ij} A^j \tag{13}$$

$$A^i = g^{ij} A_j \tag{14}$$

$$\text{fundamentaltensorn :}\ \ \overleftrightarrow{I} = g_{ij}\mathbf{e}^i\mathbf{e}^j = g^{ij}\mathbf{e}_i\mathbf{e}_j = \delta^j_i \mathbf{e}^i \mathbf{e}_j = \delta^i_j \mathbf{e}_i \mathbf{e}^j \tag{15}$$

$$g_i{}^j = g^j{}_i = \delta^j_i \tag{16}$$

Jakobianen:

$$J = \sqrt{g} = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = \tag{17}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial y}{\partial u^1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u^3} & \frac{\partial y}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} \tag{18}$$

$$J^{-1} = \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3 \tag{19}$$

$$d^3\mathbf{R} = J\, du^1 du^2 du^3 \tag{20}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = J\mathbf{e}^3 \tag{21}$$

$$\mathbf{e}^1 \times \mathbf{e}^2 = J^{-1}\mathbf{e}_3 \tag{22}$$

Skalärprodukt och kryssprodukt:

$$\mathbf{A} \cdot \mathbf{B} = A^i B_i = A_i B^i = g_{ij} A^i B^j = g^{ij} A_i B_j \tag{23}$$

$$(\mathbf{A} \times \mathbf{B})_k = \epsilon_{ijk} J A^i B^j \tag{24}$$

$$(\mathbf{A} \times \mathbf{B})^k = \epsilon^{ijk} J^{-1} A_i B_j \tag{25}$$

där ϵ^{ijk} är Levi-Civita-symbolen (som *inte* är en tredje ordningens tensor, vilket däremot permutationstensorn $\overleftrightarrow{\underrightarrow{E}} = \epsilon_{ijk} J \mathbf{e}^i \mathbf{e}^j \mathbf{e}^k = \epsilon^{ijk} J^{-1} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k$ är).

Vektor- och tensortransformation vid koordinatbyte från $\tilde{\mathbf{e}}_j$ till $\mathbf{e}_i$:

$$\text{Transformationstensorn är } T_i^j = \mathbf{e}_i \cdot \tilde{\mathbf{e}}^j \tag{26}$$
$$A_i = T_i^j \tilde{A}_j \tag{27}$$
$$B_{ij} = T_i^k \tilde{B}_{kl} T_j^l \tag{28}$$

1.2 Derivator

Gradient:

$$(\nabla\Phi)_i = \frac{\partial\Phi}{\partial u^i} \tag{29}$$
$$\tag{30}$$

Kovariant derivata:

$$A^j_{,k} = (\frac{\partial \mathbf{A}}{\partial u^k})^j = \frac{\partial A^j}{\partial u^k} + \begin{Bmatrix} & j & \\ i & & k \end{Bmatrix} A^i \tag{31}$$
$$A_{j,k} = (\frac{\partial \mathbf{A}}{\partial u^k})_j = \frac{\partial A_j}{\partial u^k} - \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} A_i \tag{32}$$
$$\text{där Christoffelsymbolen} \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} = \frac{1}{2} g^{ni} \left[\frac{\partial g_{jn}}{\partial u^k} + \frac{\partial g_{kn}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^n} \right] \tag{33}$$

Notera att dimensionen är

$$\dim\left(\begin{Bmatrix} & i & \\ j & & k \end{Bmatrix}\right) = \dim\left(\frac{u^i}{u^j u^k}\right) \tag{34}$$

Divergens och rotation:

$$\nabla \cdot \mathbf{A} = J^{-1} \frac{\partial}{\partial u^i}(J A^i) \tag{35}$$
$$(\nabla \times \mathbf{A})^k = J^{-1}(A_{j,i} - A_{i,j}) = J^{-1} \epsilon^{ijk} \frac{\partial A_j}{\partial u^i} \tag{36}$$

2 Kroklinjiga koordinatsystem

2.1 Clebsch-representation av magnetfält

Clebsch-representation av magnetfältet:

$$\mathbf{B} = \nabla\alpha \times \nabla\beta \tag{37}$$

Magnetiskt flöde genom en yta:

$$\Psi = \int d\alpha d\beta \tag{38}$$

De två koordinaterna α och β är strömningsfunktioner till magnetfältet. Om den tredje koordinaten t ex väljs som båglängden l längs magnetfältet fås

$$\mathbf{B} = B\mathbf{e}_l \tag{39}$$

$$\mathbf{b} \cdot \nabla l = 1 \tag{40}$$

$$\sqrt{g}^{-1} = \nabla l \cdot \nabla\alpha \times \nabla\beta = B \tag{41}$$

2.2 Boozer-Grad-koordinater ρ, ν, χ

Clebsch- och kovariant representation av magnetfältet:

$$\mathbf{B} = \nabla\rho \times \nabla\nu = \lambda\nabla\rho + \nabla\chi \tag{42}$$

Vi har

$$\sqrt{g} = B^{-2} \tag{43}$$

$$\mathbf{b} \cdot \nabla\chi = B \tag{44}$$

$$\mathbf{e}_\chi = \mathbf{b}/B \tag{45}$$

Strömmen:

$$\mu_0\mathbf{J} = \nabla\lambda \times \nabla\rho \tag{46}$$

$$\frac{\partial\lambda}{\partial\nu} = -\mu_0 J_\parallel/B \tag{47}$$

$$\frac{\partial\lambda}{\partial\chi} = \frac{\mu_0}{B^2}\frac{dp}{d\rho} \tag{48}$$

Om vi definierar $\lambda(\rho, \nu = 0, \chi = 0) = 0$ så har vi i vakuum $\mathbf{J} = 0$ och $\mathbf{B} = \nabla\chi$.

2.3 Toroidala flödeskoordinater ψ, θ, ζ

Typiskt val för tokamaker

$$\begin{cases} \psi = \Psi_{\text{pol}}/2\pi & \text{är den sk poloidala flödesfunktionen. (radiell koord. i plasmatvärsnittet)} \\ \theta & \text{är en poloidal koordinat med perioden } 2\pi \\ \zeta = \arctan[x, y] & \text{är den vanliga toroidala koordinaten } (= \varphi \text{ förut}). \end{cases}$$

where

$$\Psi_{\text{pol}} = \frac{1}{2\pi} \int_V \mathbf{B} \cdot \nabla\theta d^3r$$

Magnetfältet är

$$\begin{cases} \mathbf{B}_p = \mathbf{e}^\zeta \times \mathbf{e}^\psi \\ \mathbf{B}_t = B_\zeta \mathbf{e}^\zeta, \text{ där } B_\zeta = I(\psi) \text{ bara beror på } \psi \end{cases} \tag{49}$$

Vi har alltså $|\mathbf{B}_p| = |\mathbf{e}^\psi|/R$, och $|\mathbf{B}_t| = I(\psi)/R$.

$$\begin{array}{lll} \text{Jakobianen} & J = (\mathbf{e}^\zeta \times \mathbf{e}^\psi \cdot \mathbf{e}^\theta)^{-1} = 1/(\mathbf{B} \cdot \mathbf{e}^\theta) \\ \text{flödesytelementet} & dS = J|\mathbf{e}^\psi| d\theta d\zeta \\ \text{poloidala linjeelementet} & dl = \frac{dS}{Rd\zeta} = J|\mathbf{B}_p| d\theta \end{array} \tag{50}$$

Flödesytmedelvärdet av en funktion $Q(\psi, \theta)$ definieras som volymmedelvärdet av Q mellan två närliggande flödesytor ψ och $\psi + d\psi$:

$$\langle Q \rangle(\psi) = \frac{\int Q dV}{\int dV} = \frac{\oint JQ(\psi, \theta) d\theta}{\oint J d\theta} = \oint \frac{Q(\psi, \theta)}{\mathbf{B} \cdot \mathbf{e}^\theta} d\theta \Big/ \oint \frac{d\theta}{\mathbf{B} \cdot \mathbf{e}^\theta} \tag{51}$$

där $\oint$ indikerar att integrationen ska tas ett varv poloidalt runt flödesytan.

Säkerhetsfaktorn $q(\psi)$ definieras som ändringen i ζ dividerat med ändringen i θ längs magnetfältet i medeltal, dvs

$$q(\psi) = \frac{\langle \mathbf{B} \cdot \mathbf{e}^\zeta \rangle}{\langle \mathbf{B} \cdot \mathbf{e}^\theta \rangle} = \frac{I(\psi)}{2\pi} \oint \frac{R^{-2}}{\mathbf{B} \cdot \mathbf{e}^\theta} d\theta \tag{52}$$

Vanligen väljs θ-koordinaten så att medelvärdet ej behövs. Dvs

$$q(\psi) = \frac{B^\zeta}{B^\theta} = \frac{I(\psi)}{R^2 B^\theta} = \frac{I(\psi) J}{R^2} \tag{53}$$

2.4 Flödeskoordinater för $a/R_0 \ll 1$ och cirkulärt tvärsnitt r, θ, ζ

Transformation till cyl. koord. R, α, z:

$$\begin{cases} R = R_0 + r\cos\theta - \Delta(r) - r\eta(r)(1 - \cos 2\theta) \\ \alpha = -\zeta \\ z = r\sin\theta + r\eta(r)\sin 2\theta \end{cases}$$

där $\Delta(r)$ är Shafranov-förskjutningen och

$$\eta(r) \equiv \frac{1}{2}\left(\frac{r}{R_0} + \Delta'\right). \tag{54}$$

$$\mathbf{e}^r = (1-\epsilon\cos\theta)\left[(\cos\theta + 2\eta\cos 2\theta)\hat{R} + (\sin\theta + 2\eta\sin 2\theta)\hat{z}\right] \tag{55}$$

$$\mathbf{e}^\theta = r^{-1}(1-\epsilon\cos\theta)\left[-(\sin\theta + (r\eta)'\sin 2\theta)\hat{R} + (\cos\theta - \Delta' + (r\eta)'(\cos 2\theta - 1))\hat{z}\right] \tag{56}$$

$$\mathbf{e}^\zeta = -R^{-1}\hat{\alpha} \tag{57}$$

Vi antar att $\psi = \psi(r)$ så att $\mathbf{e}^\psi = \psi'\mathbf{e}^r$. Jakonianen blir:

$$J = (\mathbf{e}^\zeta \times \mathbf{e}^r \cdot \mathbf{e}^\theta)^{-1} = (\mathbf{B}_p \cdot \mathbf{e}^\theta)^{-1}\psi' = \psi'/B^\theta \simeq rR_0\left(1 + \frac{2r}{R_0}\cos\theta\right) \simeq \frac{rR^2}{R_0} \tag{58}$$

så att $q = q(r)$ blir oberoende av θ

$$q(r) = \frac{B^\zeta}{B^\theta} = \frac{I(r)}{R^2B^\theta} = \frac{I(r)J}{R^2\psi'(r)} = \frac{I(r)r}{R_0\psi'(r)} \tag{59}$$

Metriktensorn:

$$[g_{ij}] = \begin{pmatrix} 1-2\Delta'\cos\theta & r\sin\theta(\frac{r}{R_0} + (r\Delta')') & 0 \\ r\sin\theta(\frac{r}{R_0} + (r\Delta')') & r^2(1+4\eta\cos\theta) & 0 \\ 0 & 0 & R_0^2(1+\frac{2r}{R_0}\cos\theta) \end{pmatrix} \tag{60}$$

$$\left[g^{ij}\right] = \begin{pmatrix} 1+2\Delta'\cos\theta & -r^{-1}\sin\theta(\frac{r}{R_0} + (r\Delta')') & 0 \\ -r^{-1}\sin\theta(\frac{r}{R_0} + (r\Delta')') & r^{-2}(1-4\eta\cos\theta) & 0 \\ 0 & 0 & R_0^{-2}(1-\frac{2r}{R_0}\cos\theta) \end{pmatrix} \tag{61}$$

där $g_{\zeta\zeta} = (g^{\zeta\zeta})^{-1} \simeq R^2$. Jämviktsmagnetfältet är som förut

$$\mathbf{B} = I(r)\mathbf{e}^\zeta + \psi'\mathbf{e}^\zeta \times \mathbf{e}^r = I(\mathbf{e}^\zeta + Jg^{\zeta\zeta}q^{-1}\mathbf{e}^\zeta \times \mathbf{e}^r) = I(\mathbf{e}^\zeta + g^{\zeta\zeta}q^{-1}\mathbf{e}_\theta) \tag{62}$$

$$B^2 = (I^2 + \psi'^2 g^{rr})g^{\zeta\zeta} = I^2 g^{\zeta\zeta}(1 + \frac{r^2}{R_0^2q^2}g^{rr}) \simeq I^2/R^2. \tag{63}$$

och komponenterna är ($\epsilon = r/R_0$)

$$\begin{array}{lll} B_r = I\epsilon^2 q^{-1}g^{r\theta}, & B_\theta = I\epsilon^2 q^{-1} g^{rr}, & B_\zeta = I, \\ B^r = 0, & B^\theta = I/(qR^2), & B^\zeta = I/R^2. \end{array} \tag{64}$$

Användbara operatorer:

$$\mathbf{B}\cdot\nabla\Phi = \frac{I}{R^2q}\left(\frac{\partial\Phi}{\partial\theta} + q\frac{\partial\Phi}{\partial\zeta}\right) \tag{65}$$

Vågvektorn:

$$\mathbf{k} = n\nabla\zeta - m\nabla\theta, \qquad k_i = (0,\ -m,\ n), \qquad k^i = (-mg^{r\theta},\ -mg^{\theta\theta},\ ng^{\zeta\zeta}) \tag{66}$$

$$\nabla\cdot\mathbf{k} \simeq \frac{m}{r^2}\sin\theta\left(\epsilon - 2\Delta' + (r^2\Delta'')'\right) \tag{67}$$

2.5 Flödeskoordinater för $a/R_0 \ll 1$, $\Delta' = 0$ och cirkulärt tvärsnitt r, θ, ζ

Detta är ett specialfall av det föregående avsnittet. Transformation till cyl. koord. R, α, z:

$$\begin{cases} R = R_0 + r\cos\theta - \frac{1}{2}r\epsilon(1-\cos 2\theta) \\ \alpha = -\zeta \\ z = r\sin\theta + \frac{1}{2}r\epsilon\sin 2\theta \end{cases}$$

Där $\epsilon = r/R_0$. Jakonianen blir:

$$J = (\mathbf{e}^\zeta \times \mathbf{e}^r \cdot \mathbf{e}^\theta)^{-1} \simeq rR_0\,(1 + 2\epsilon\cos\theta) \simeq \frac{rR^2}{R_0} \tag{68}$$

så att $q = q(r)$ blir oberoende av θ. Metriktensorn:

$$[g_{ij}] = \begin{pmatrix} 1 & r\epsilon\sin\theta & 0 \\ r\epsilon\sin\theta & r^2(1+2\epsilon\cos\theta) & 0 \\ 0 & 0 & R_0^2(1+2\epsilon\cos\theta) \end{pmatrix} \tag{69}$$

$$\left[g^{ij}\right] = \begin{pmatrix} 1 & -r^{-1}\epsilon\sin\theta & 0 \\ -r^{-1}\epsilon\sin\theta & r^{-2}(1-2\epsilon\cos\theta) & 0 \\ 0 & 0 & R_0^{-2}(1-2\epsilon\cos\theta) \end{pmatrix} \tag{70}$$

där $g_{\zeta\zeta} = (g^{\zeta\zeta})^{-1} \simeq R^2$. Jämviktsmagnetfältet är som förut

$$\mathbf{B} = I(r)\mathbf{e}^\zeta + \psi'\mathbf{e}^\zeta \times \mathbf{e}^r = I(\mathbf{e}^\zeta + Jg^{\zeta\zeta}q^{-1}\mathbf{e}^\zeta \times \mathbf{e}^r) = I(\mathbf{e}^\zeta + g^{\zeta\zeta}q^{-1}\mathbf{e}_\theta) \tag{71}$$

$$B^2 = (I^2 + \psi'^2 g^{rr})g^{\zeta\zeta} = I^2 g^{\zeta\zeta}(1 + \epsilon^2 q^{-2} g^{rr}) \simeq I^2/R^2. \tag{72}$$

och komponenterna är ($\epsilon = r/R_0$)

$$\begin{array}{lll} B_r = I\epsilon^2 q^{-1} g^{r\theta}, & B_\theta = I\epsilon^2 q^{-1} g^{rr}, & B_\zeta = I, \\ B^r = 0, & B^\theta = I/(qR^2), & B^\zeta = I/R^2. \end{array} \tag{73}$$

Användbara operatorer:

$$\mathbf{B}\cdot\nabla\Phi = \frac{I}{R^2 q}\left(\frac{\partial\Phi}{\partial\theta} + q\frac{\partial\Phi}{\partial\zeta}\right) \tag{74}$$

Vågvektorn:

$$\mathbf{k} = n\nabla\zeta - m\nabla\theta, \qquad k_i = (0,\ -m,\ n), \qquad k^i = (-mg^{r\theta},\ -mg^{\theta\theta},\ ng^{\zeta\zeta}) \tag{75}$$

$$\nabla\cdot\mathbf{k} \simeq \frac{m}{r^2}\epsilon\sin\theta \tag{76}$$

2.6 Elliptisk-toroidala koordinater ρ, ϑ, φ

Detta koordinatsystem användes i Smith *et al.* PoP **10**, 1437 (2003). Det är inte ortogonalt. (ρ är inte "lillradien" och ϑ är inte den poloidala vinkeln.) κ är ellipticiteten hos ett tvärsnitt av torusen.

Transformationer:

$$\begin{cases} x = (R_0 + \rho\cos\vartheta)\sin\varphi \\ y = (R_0 + \rho\cos\vartheta)\cos\varphi \\ z = \kappa\rho\sin\vartheta \end{cases} \tag{77}$$

$$\begin{cases} \rho = \sqrt{(\sqrt{x^2+y^2}-R_0)^2 + (\frac{z}{\kappa})^2} \\ \vartheta = \arctan\,[z/\kappa, \sqrt{x^2+y^2}-R_0] \\ \varphi = \arctan\,[x, y] \end{cases} \tag{78}$$

där $\arctan\,[a,b] = \arctan\frac{a}{b} + \pi(1-H(b))$ och $H(x)$ är Heavisidefunktionen. Notera att $\varphi = -\phi$ i cylindriska koordinater (R, ϕ, z)

Basvektorer:

$$\mathbf{e}_\rho = (\cos\vartheta\sin\varphi, \cos\vartheta\cos\varphi, \kappa\sin\vartheta) \tag{79}$$
$$\mathbf{e}_\vartheta = \rho(-\sin\vartheta\sin\varphi, -\sin\vartheta\cos\varphi, \kappa\cos\vartheta) \tag{80}$$
$$\mathbf{e}_\varphi = R(\cos\varphi, -\sin\varphi, 0) \tag{81}$$
$$\mathbf{e}^\rho = (\cos\vartheta\sin\varphi, \cos\vartheta\cos\varphi, \kappa^{-1}\sin\vartheta) \tag{82}$$
$$\mathbf{e}^\vartheta = \frac{1}{\rho}(-\sin\vartheta\sin\varphi, -\sin\vartheta\cos\varphi, \kappa^{-1}\cos\vartheta) \tag{83}$$
$$\mathbf{e}^\varphi = \frac{1}{R}(\cos\varphi, -\sin\varphi, 0) \tag{84}$$
$$\text{där } R = R_0 + \rho\cos\vartheta \tag{85}$$

Metriktensorn

$$[g_{ij}] = \begin{pmatrix} \cos^2\vartheta + \kappa^2\sin^2\vartheta & \rho\sin\vartheta\cos\vartheta(\kappa^2-1) & 0 \\ \rho\sin\vartheta\cos\vartheta(\kappa^2-1) & \rho^2(\kappa^2\cos^2\vartheta + \sin^2\vartheta) & 0 \\ 0 & 0 & R^2 \end{pmatrix} \tag{86}$$

$$\left[g^{ij}\right] = \rho^{-2}\kappa^{-2} \begin{pmatrix} \rho^2(\kappa^2\cos^2\vartheta + \sin^2\vartheta) & -\rho\sin\vartheta\cos\vartheta(\kappa^2-1) & 0 \\ -\rho\sin\vartheta\cos\vartheta(\kappa^2-1) & \cos^2\vartheta + \kappa^2\sin^2\vartheta & 0 \\ 0 & 0 & \rho^2\kappa^2R^{-2} \end{pmatrix} \tag{87}$$

$$g = \det[g_{ij}] = \rho^2R^2\kappa^2 \tag{88}$$
$$J = \sqrt{g} = \rho R\kappa \tag{89}$$

Transformationstensorn från elliptisk-toroidala koordinater till ordinära ($\kappa = 1$) toroidala koordinater med basvektorer $\tilde{\mathbf{e}}_i = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$:

$$\begin{aligned} T_i^j = \tilde{\mathbf{e}}_i \cdot \mathbf{e}^j &= \{ \text{ matris med i som rader, j som kolonner } \} = \\ &= \begin{pmatrix} (\cos^2\vartheta + \kappa^2\sin^2\vartheta)^{-\frac{1}{2}} & 0 & 0 \\ \rho\sin\vartheta\cos\vartheta(\kappa^{-1}-\kappa) & \kappa\sin^2\vartheta + \kappa^{-1}\cos^2\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{90}$$

3 Vektoridentiteter

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \tag{91}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{92}$$

$$\epsilon_{ijk}\epsilon^{lmi} = \delta_j^l \delta_k^m - \delta_j^m \delta_k^l \tag{93}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{94}$$

$$\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi \tag{95}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{96}$$

$$\nabla \cdot (\Phi\mathbf{A}) = \Phi\nabla \cdot \mathbf{A} + (\nabla\Phi) \cdot \mathbf{A} \tag{97}$$

$$\nabla \times (\Phi\mathbf{A}) = \Phi\nabla \times \mathbf{A} + (\nabla\Phi) \times \mathbf{A} \tag{98}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{99}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \tag{100}$$

$$\nabla^2\Phi \equiv \nabla \cdot \nabla\Phi \tag{101}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{102}$$

$$\nabla \times (\nabla\Phi) = 0 \tag{103}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{104}$$

$$\nabla^2(\Phi\mathbf{A}) = \mathbf{A}\nabla^2\Phi + 2(\nabla\Phi \cdot \nabla)\mathbf{A} + \Phi\nabla^2\mathbf{A} \tag{105}$$

$$(\mathbf{C} \cdot \nabla)(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot [(\mathbf{C} \cdot \nabla)\mathbf{B}] + \mathbf{B} \cdot [(\mathbf{C} \cdot \nabla)\mathbf{A}] \tag{106}$$

$$(\mathbf{C} \cdot \nabla)|\mathbf{A}|^2 = 2\mathbf{A} \cdot [(\mathbf{C} \cdot \nabla)\mathbf{A}] \tag{107}$$

$$(\mathbf{C} \cdot \nabla)(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times [(\mathbf{C} \cdot \nabla)\mathbf{B}] + [(\mathbf{C} \cdot \nabla)\mathbf{A}] \times \mathbf{B} \tag{108}$$

$$\mathbf{A} \cdot (\mathbf{B} \cdot \nabla)\nabla = \mathbf{B} \cdot (\mathbf{A} \cdot \nabla)\nabla \tag{109}$$

$$\nabla \cdot \mathbf{r} = 3 \tag{110}$$

$$\nabla \times \mathbf{r} = 0 \tag{111}$$

$$\nabla r = \mathbf{r}/r \tag{112}$$

$$\nabla(1/r) = -\mathbf{r}/r^3 \tag{113}$$

$$\nabla \cdot (\mathbf{r}/r^3) = 4\pi\delta(\mathbf{r}) \tag{114}$$

$$\nabla\mathbf{r} = \overleftrightarrow{I} \tag{115}$$

Dyader

$$\overleftrightarrow{I}\cdot\mathbf{A}=\mathbf{A}\cdot\overleftrightarrow{I}=\mathbf{A} \tag{116}$$

$$\overleftrightarrow{I}\cdot\mathbf{AB}=\mathbf{AB} \tag{117}$$

$$\overleftrightarrow{I}\cdot\nabla\mathbf{B}=\nabla\mathbf{B} \tag{118}$$

$$\mathbf{AB}:\mathbf{CD}=(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})=\mathbf{D}\cdot\mathbf{AB}\cdot\mathbf{C} \tag{119}$$

$$\overleftrightarrow{F}:\mathbf{AB}=(\overleftrightarrow{F}\cdot\mathbf{A})\cdot\mathbf{B}=\mathbf{B}\cdot\overleftrightarrow{F}\cdot\mathbf{A} \tag{120}$$

$$\overleftrightarrow{I}:\mathbf{AB}=\mathbf{A}\cdot\mathbf{B} \tag{121}$$

$$\overleftrightarrow{I}:\nabla\mathbf{B}=\nabla\cdot\mathbf{B} \tag{122}$$

$$\overleftrightarrow{I}:\overleftrightarrow{F}=F_i^i=\mathrm{tr}\overleftrightarrow{F} \tag{123}$$

$$\nabla\cdot(a\overleftrightarrow{I})=\nabla a \tag{124}$$

$$\nabla\cdot\mathbf{AB}=\mathbf{A}\cdot\nabla\mathbf{B}+(\nabla\cdot\mathbf{A})\mathbf{B} \tag{125}$$

$$\overleftrightarrow{I}\times\mathbf{A}=\mathbf{A}\times\overleftrightarrow{I} \tag{126}$$

$$(\mathbf{A}\times\mathbf{B})\times\overleftrightarrow{I}=\overleftrightarrow{I}\times(\mathbf{A}\times\mathbf{B})=\mathbf{BA}-\mathbf{AB} \tag{127}$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=\mathbf{C}\times(\mathbf{B}\times\mathbf{A})=\mathbf{B}(\mathbf{C}\cdot\mathbf{A})-\mathbf{A}(\mathbf{C}\cdot\mathbf{B})=\mathbf{C}\cdot(\mathbf{AB}-\mathbf{BA}) \tag{128}$$

$$\mathbf{A}\times(\mathbf{BC})=(\mathbf{A}\times\mathbf{B})\mathbf{C} \tag{129}$$

$$\nabla\cdot(\overleftrightarrow{I}\times\mathbf{C})=\nabla\times\mathbf{C} \tag{130}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=\nabla\cdot(\mathbf{BA}-\mathbf{AB}) \tag{131}$$

$$\mathbf{A}\times(\nabla\times\mathbf{B})=(\nabla\mathbf{B})\cdot\mathbf{A}-\mathbf{A}\cdot\nabla\mathbf{B} \tag{132}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+\mathbf{A}\cdot\nabla\mathbf{B}+\mathbf{B}\cdot\nabla\mathbf{A}= \tag{133}$$

$$=(\nabla\mathbf{A})\cdot\mathbf{B}+(\nabla\mathbf{B})\cdot\mathbf{A} \tag{134}$$

$$\nabla(\mathbf{A}\times\mathbf{B})=\nabla\mathbf{A}\times\mathbf{B}-\nabla\mathbf{B}\times\mathbf{A} \tag{135}$$

$$\nabla\times(\mathbf{AB})=(\nabla\times\mathbf{A})\mathbf{B}-\mathbf{A}\times\nabla\mathbf{B} \tag{136}$$

$$\mathbf{A}\times\nabla\mathbf{B}-(\mathbf{A}\times\nabla\mathbf{B})^{\mathrm{T}}=[(\nabla\cdot\mathbf{B})\mathbf{A}-\nabla\mathbf{B}\cdot\mathbf{A}]\times\overleftrightarrow{I} \tag{137}$$

$$\nabla\mathbf{B}\times\mathbf{A}-(\nabla\mathbf{B}\times\mathbf{A})^{\mathrm{T}}=[(\nabla\cdot\mathbf{B})\mathbf{A}-\mathbf{A}\cdot\nabla\mathbf{B}]\times\overleftrightarrow{I} \tag{138}$$

$$(\nabla\cdot\mathbf{B})\mathbf{A}-\nabla\mathbf{B}\cdot\mathbf{A}=-(\mathbf{A}\times\nabla)\times\mathbf{B} \tag{139}$$

$$(\nabla\cdot\mathbf{B})\mathbf{A}-\mathbf{A}\cdot\nabla\mathbf{B}=\mathbf{A}\times(\nabla\times\mathbf{B})-(\mathbf{A}\times\nabla)\times\mathbf{B} \tag{140}$$

$$(\mathbf{A}\times\nabla)\cdot\mathbf{B}=\mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{141}$$

$$\mathbf{A}\times\nabla\mathbf{B}+(\mathbf{A}\times\nabla\mathbf{B})^{\mathrm{T}}=[\mathbf{A}\cdot(\nabla\times\mathbf{B})]\overleftrightarrow{I}-\mathbf{A}(\nabla\times\mathbf{B}) \tag{142}$$

$$\nabla\mathbf{B}\times\mathbf{A}+(\nabla\mathbf{B}\times\mathbf{A})^{\mathrm{T}}=[\mathbf{A}\cdot(\nabla\times\mathbf{B})]\overleftrightarrow{I}-(\nabla\times\mathbf{B})\mathbf{A} \tag{143}$$

$$\mathbf{A}\times\nabla\mathbf{B}+\nabla\mathbf{B}\times\mathbf{A}=\overleftrightarrow{I}\times[(\nabla\cdot\mathbf{B})\mathbf{A}-\nabla\mathbf{B}\cdot\mathbf{A}]+ \tag{144}$$

$$+[\mathbf{A}\cdot(\nabla\times\mathbf{B})]\overleftrightarrow{I}-\mathbf{A}(\nabla\times\mathbf{B})= \tag{145}$$

$$=\overleftrightarrow{I}\times[(\nabla\cdot\mathbf{B})\mathbf{A}-\nabla\mathbf{A}\cdot\mathbf{B}]+ \tag{146}$$

$$+[\mathbf{A}\cdot(\nabla\times\mathbf{B})]\overleftrightarrow{I}-\mathbf{B}(\nabla\times\mathbf{A}) \tag{147}$$

$$(\mathbf{A}\times\mathbf{B})\cdot\overleftrightarrow{F}=\mathbf{A}\cdot(\mathbf{B}\times\overleftrightarrow{F})=-\mathbf{B}\cdot(\mathbf{A}\times\overleftrightarrow{F}) \tag{148}$$

$$\overleftrightarrow{F}\cdot(\mathbf{A}\times\mathbf{B})=(\overleftrightarrow{F}\times\mathbf{A})\cdot\mathbf{B}=-(\overleftrightarrow{F}\times\mathbf{B})\cdot\mathbf{A} \tag{149}$$

$$\mathbf{A}\cdot\overleftrightarrow{F}\times\mathbf{B}=(\mathbf{A}\cdot\overleftrightarrow{F})\times\mathbf{B}=\mathbf{A}\cdot(\overleftrightarrow{F}\times\mathbf{B})=-\mathbf{B}\times(\mathbf{A}\cdot\overleftrightarrow{F}) \tag{150}$$

$$\mathbf{A}\times\overleftrightarrow{F}\cdot\mathbf{B}=\mathbf{A}\times(\overleftrightarrow{F}\cdot\mathbf{B})=(\mathbf{A}\times\overleftrightarrow{F})\cdot\mathbf{B}=-(\overleftrightarrow{F}\cdot\mathbf{B})\times\mathbf{A} \tag{151}$$

$$\mathbf{A}\cdot\nabla\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\nabla\mathbf{B}\cdot\mathbf{A}=\mathbf{C}\times[\mathbf{A}\times(\nabla\times\mathbf{B})] \tag{152}$$

$$\mathbf{A}\cdot\nabla\mathbf{B}\times\mathbf{C}-\mathbf{C}\times\nabla\mathbf{B}\cdot\mathbf{A}=(\mathbf{A}\times\mathbf{C})\cdot[(\nabla\cdot\mathbf{B})\overleftrightarrow{I}-\nabla\mathbf{B}] \tag{153}$$

$$\mathbf{A}\cdot\nabla\mathbf{B}\times\mathbf{C}-\mathbf{C}\cdot\nabla\mathbf{B}\times\mathbf{A}=[(\nabla\cdot\mathbf{B})\overleftrightarrow{I}-\nabla\mathbf{B}]\cdot(\mathbf{A}\times\mathbf{C}) \tag{154}$$

$$\mathbf{A}\cdot\nabla\mathbf{B}\cdot\mathbf{C}-\mathbf{C}\cdot\nabla\mathbf{B}\cdot\mathbf{A}=(\mathbf{A}\times\mathbf{C})\cdot(\nabla\times\mathbf{B}) \tag{155}$$

$$\int_V dV \nabla f = \int_S dS f \tag{156}$$

$$\int_V dV \nabla \cdot \mathbf{A} = \int_S dS \cdot \mathbf{A} \tag{157}$$

$$\int_V dV \nabla \times \mathbf{A} = \int_S dS \times \mathbf{A} \tag{158}$$

$$\int_V dV (f \nabla^2 g + \nabla g \cdot \nabla f) = \int_S dS \cdot f \nabla g \tag{159}$$

$$\int_V dV (\mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A}) = \int_S dS \cdot (\mathbf{B} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \mathbf{B}) \tag{160}$$

$$\int_S dS \times \nabla f = \oint_C d\mathbf{l} f \tag{161}$$

$$\int_S dS \cdot \nabla \times \mathbf{A} = \oint_C d\mathbf{l} \cdot \mathbf{A} \tag{162}$$

$$\int_S (dS \times \nabla) \times \mathbf{A} = \oint_C d\mathbf{l} \times \mathbf{A} \tag{163}$$

$$\int_S dS \cdot (\nabla f \times \nabla g) = \oint_C f dg = -\oint_C g\, df \tag{164}$$

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