January 18, 2015 Matt Landreman

Effect on fluxes of the poloidally varying electrostatic potential

In these notes, we point out a term in the heat flux which is not computed by many neoclassical codes, including DKES, momentum-corrected DKES, or versions of SFINCS prior to 3. This term appears formally as large as the part of the heat flux which is usually computed, even when the species charge is Z=1. Computing this term requires knowledge of the variation of the electrostatic potential Φ on a flux surface.

Suppose we have a solution f_{s1} to the linear drift-kinetic equation in which there is no parallel variation of the electrostatic potential, as in DKES:

$$\upsilon_{\|}\nabla_{\|} f_{s1} - C_s\left[f_{s1}\right] = -\mathbf{v}_{ms}\cdot\nabla\psi \frac{\partial f_{s0}}{\partial\psi}. \tag{1}$$

Here,

$$f_{s0} = n_s(\psi)\left[\frac{m_s}{2\pi T_s(\psi)}\right]^{3/2}\exp\left(-\frac{m_s\upsilon^2}{2T_s(\psi)}\right) \tag{2}$$

is a stationary Maxwellian, s denotes species, and $\mathbf{v}_{ms}$ is the magnetic drift (not including any $\mathbf{E}\times\mathbf{B}$ drift). We neglect the radial electric field in (1) for simplicity. The solution f_{s1} gives rise to a certain particle flux

$$\Gamma_{ms} = \left\langle \int d^3\upsilon\, f_{s1}\mathbf{v}_{ms}\cdot\nabla\psi\right\rangle \tag{3}$$

and an energy flux

$$Q_{ms} = \left\langle \int d^3\upsilon\, f_{s1}\frac{m_s\upsilon^2}{2}\mathbf{v}_{ms}\cdot\nabla\psi\right\rangle. \tag{4}$$

(The subscript m on the fluxes (3)-(4) indicates these fluxes are associated with the radial *magnetic* drift.)

Now consider the "real" drift-kinetic equation, in which parallel variation of the electrostatic potential Φ is retained. We denote the solution to this form of the kinetic equation by $\tilde{f}_{s1}$:

$$\upsilon_{\|}\nabla_{\|}\tilde{f}_{s1} - C_s\left[\tilde{f}_{s1}\right] = -\mathbf{v}_{ms}\cdot\nabla\psi\frac{\partial f_{s0}}{\partial\psi} - \upsilon_{\|}\frac{Z_s e}{T_s}\left(\nabla_{\|}\Phi\right) f_{s0}. \tag{5}$$

The solution to (5) can be written in terms of the solution to (1):

$$\tilde{f}_{s1} = f_{s1} - \frac{Z_s e\Phi}{T_s} f_{s0}. \tag{6}$$

If we evaluate the radial particle and energy fluxes associated with $\tilde{f}_{s1}$ caused by just the magnetic drifts (ignoring the radial $\mathbf{E}\times\mathbf{B}$ drift,) we do not get the same fluxes we got before:

$$\tilde{\Gamma}_{ms} = \left\langle \int d^3\upsilon \, \tilde{f}_{s1} \mathbf{v}_{ms} \cdot \nabla \psi \right\rangle$$
$$= \Gamma_{ms} - \frac{Z_s e}{T_s} \left\langle \Phi \int d^3\upsilon \, f_{s0} \mathbf{v}_{ms} \cdot \nabla \psi \right\rangle \tag{7}$$

and

$$\tilde{Q}_{ms} = \left\langle \int d^3\upsilon \, \tilde{f}_{s1} \frac{m_s \upsilon^2}{2} \mathbf{v}_{ms} \cdot \nabla \psi \right\rangle$$
$$= Q_{ms} - \frac{Z_s e}{T_s} \left\langle \Phi \int d^3\upsilon \, f_{s0} \frac{m_s \upsilon^2}{2} \mathbf{v}_{ms} \cdot \nabla \psi \right\rangle. \tag{8}$$

In other words, including the blue $\nabla_{\parallel}\Phi$ term in (5) causes the radial fluxes to change by the red terms in (7)-(8). Using

$$\mathbf{v}_{ms} \cdot \nabla \psi = \frac{m_s}{2 Z_s e B^3} \left(2\upsilon_{\parallel}^2 + \upsilon_{\perp}^2 \right) \mathbf{B} \times \nabla B \cdot \nabla \psi, \tag{9}$$

we can evaluate the $\int d^3\upsilon$ integrals in the red terms of (7)-(8), giving

$$\tilde{\Gamma}_{ms} = \Gamma_{ms} + n_{s0} \left\langle \Phi \mathbf{B} \times \nabla \left(\frac{1}{B^2} \right) \cdot \nabla \psi \right\rangle \tag{10}$$

and

$$\tilde{Q}_{ms} = Q_{ms} + \frac{5}{2} n_s T_s \left\langle \Phi \mathbf{B} \times \nabla \left(\frac{1}{B^2} \right) \cdot \nabla \psi \right\rangle. \tag{11}$$

Using the MHD equilibrium relation $\nabla \psi \cdot \left(\nabla \times \mathbf{B} \right) = 0$, along with

$$\left\langle \nabla \cdot \mathbf{Q} \right\rangle = \frac{1}{V'} \frac{d}{d\psi} \left(V' \left\langle \mathbf{Q} \cdot \nabla \psi \right\rangle \right), \tag{12}$$

then (10)-(11) can be integrated by parts to find

$$\tilde{\Gamma}_{ms} = \Gamma_{ms} - n_{s0} \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla \Phi \cdot \nabla \psi \right\rangle \tag{13}$$

and

$$\tilde{Q}_{ms} = Q_{ms} - \frac{5}{2} n_s T_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla \Phi \cdot \nabla \psi \right\rangle. \tag{14}$$

When Φ varies on a flux surface, there will also be a radial $\mathbf{E} \times \mathbf{B}$ drift. The particle and heat fluxes associated with this drift are, considering just f_{s0} as opposed to f_{s1},

$$\Gamma_{Es} = \left\langle \int d^3\upsilon \, f_{s0} \frac{1}{B^2} \mathbf{B} \times \nabla \Phi \cdot \nabla \psi \right\rangle = n_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla \Phi \cdot \nabla \psi \right\rangle \tag{15}$$

and

$$Q_{Es} = \left\langle \int d^3\upsilon \, f_{s0} \frac{m_s \upsilon^2}{2} \frac{1}{B^2} \mathbf{B} \times \nabla \Phi \cdot \nabla \psi \right\rangle = \frac{3}{2} n_s T_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla \Phi \cdot \nabla \psi \right\rangle. \tag{16}$$

Combining (13)-(16), we find

$$\tilde{\Gamma}_{ms} + \Gamma_{Es} = \Gamma_{ms} \tag{17}$$

and

$$\tilde{Q}_{ms} + Q_{Es} = Q_{ms} - n_s T_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla\Phi \cdot \nabla\psi \right\rangle . \tag{18}$$

The left-hand sides of (17)-(18) are the "correct" fluxes, since they account for variation of Φ on a flux surface, whereas Γ_{ms} and Q_{ms} do not.

Eq (17) shows that ignoring $\nabla_{\|}\Phi$ in the kinetic equation does not change the particle flux: we get the same particle flux if we do include $\nabla_{\|}\Phi$ and then account for the extra radial flux from the radial $\mathbf{E}\times\mathbf{B}$ drift. However, (18) shows that the heat flux *does* change when $\nabla_{\|}\Phi$ is included in the kinetic equation, and this change cannot be compensated by accounting for the radial heat flux from the radial $\mathbf{E}\times\mathbf{B}$ drift. Since the red term in (18) cannot be computed without knowledge of the variation of Φ on a flux surface, it appears that the "true" radial heat flux $\tilde{Q}_{ms} + Q_{Es}$ cannot be computed from codes that simply report the quantity Q_{ms}, such as DKES, momentum-corrected DKES, or versions of SFINCS prior to 3.

Using the standard estimates $e\Phi_1 / T \sim \rho_*$ (where Φ_1 is the part of Φ which varies on a flux surface), $f_{s1} / f_{s0} \sim \rho_*$, $\mathbf{v}_{ms} \sim \rho_* \sqrt{T_s / m_s}$, and $\nabla\Phi \sim \Phi / L$, where $\rho_* = \sqrt{T_s / m_s} / \left(\Omega_s L\right)$ and L is a macroscopic scale, we find

$$\frac{n_s T_s \left\langle \frac{1}{B^2} \mathbf{B} \times \nabla\Phi \cdot \nabla\psi \right\rangle}{Q_{ms}} \sim 1 . \tag{19}$$

Thus, the red term (18) which is not computed by the aforementioned codes is formally as large in the ρ_* expansion as the term Q_{ms} which is usually computed.