Master ISEFAR 1
English for probability and statistics

Assignment 3

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Write an algorithm to simulate a Galton Watson process with offspring distribution $NB(1, p)$, $p \in [1/2, 1)$ and immigration with a Poisson distribution with mean $\vartheta > 0$. Estimate the stationary distribution by simulation. Simulate a large number of long paths of the process and draw a histogram of the size of the population of the last generation. Estimate the mean and variance of the stationary distribution, as well as the long term probability that the population is zero.

Travail correct, mais le code n'est pas excellent et la rédaction succincte

$\frac{8}{10}$

Solution

Let's write the algorithm that will allow us to simulate the Galton Watson process with immigration:

Algorithm 1: Algorithm that simulates a Galton Watson process with immigration

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Input : n the number of generations,
X_0 the initial population,
p the probability of the negative binomial distrubution,
ϑ the mean of the poison distribution

Output: The size of the population after the end of the Galton Watson process

```
Let X_0 = 1 and X_i be the size of the population at the i-th generation
for i in [1, n] do
    if X_i equal 0 then
        Return X_i and stop the algorithm.
    end
    for j in [1, X_i] do
        X_i = Z_j + X_i
        Where Z_j, the number of offspring of the individual j of the generation i that
          follows a negative binomial distribution NB(1, p)
    end
    X_i = W_i + X_i where W_i is the number of people migrating, that follows a Poisson
      distribution with a mean of ϑ
    if i equals n then
        Return X_i and stop the algorithm.
    end
end
```

pourquoi stopper ici ? il y a toujours l'immigration, même si l'accroissement naturel est nul.

To estimate the stationary distribution we must simulate a large number of long paths of Galton Watson process. We will from now on, let $p = 0.6$ and $\vartheta = 10$.

Let $W_n \sim P(\vartheta)$ and $Z_n \sim NB(1, p)$, we know that the expectation of the negative binomial,

le GW est sous-critique

why? $p > 1/2$

$$E[Z_n] = \frac{1-p}{p} = \frac{1-0.6}{0.6} = \frac{2}{3} < 1.$$

Therefore we can say that a stationary distribution of our Galton Watson process with immigration exist. And has an expectation μ that is solution to the following equation :

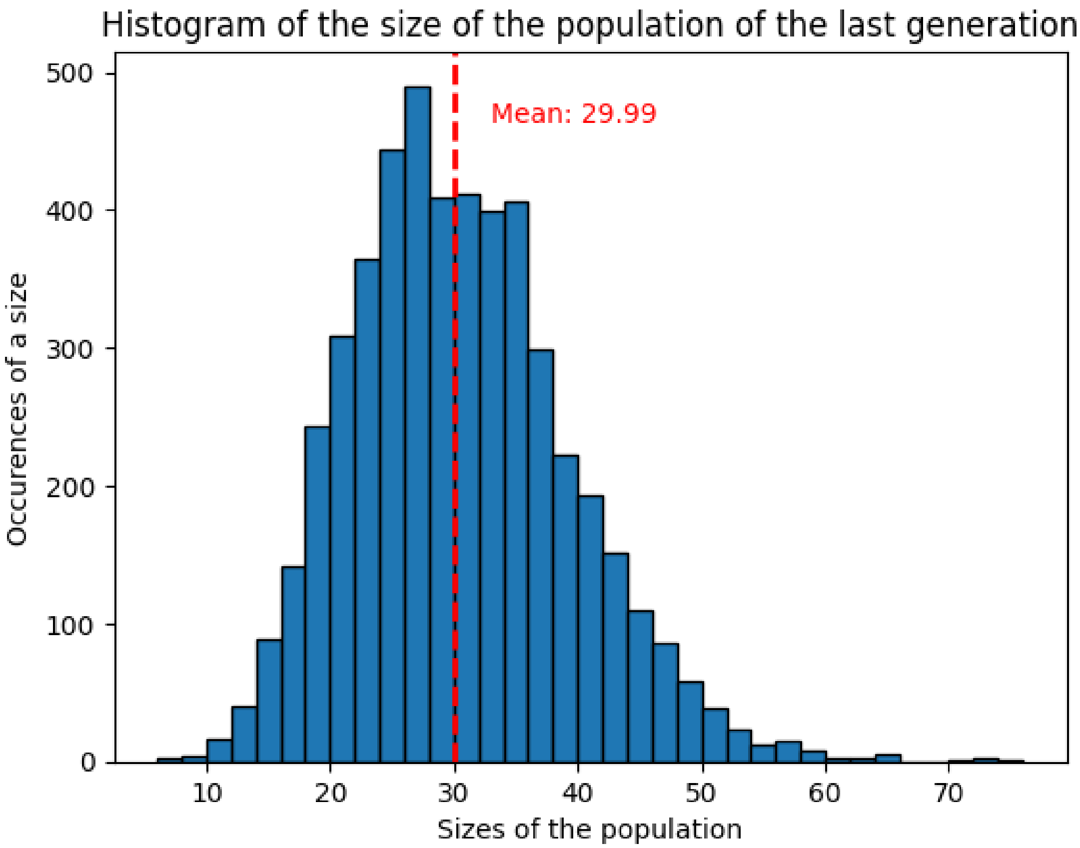
$$\mu = E[Z_n] \times \mu + E[W_n]$$

$\mu = \frac{\theta}{1 - \frac{1-p}{p}} = \frac{p\theta}{2p-1}$

formule pour μ ? sous la loi stationnaire.

We can now draw an histogram of the population of the last generation after simulating a large number of paths of the Galton Watson process:

$p = \frac{3}{5}$ $\theta = 10$ $\to$ $\mu = 30$.



After computing μ with the our parameters, we have that, $\mu = 30$. ok

But we can also estimate the mean value as well as the value of the variance of the stationary distribution:

We have that:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \approx 30$$

Where X_n is the sample resulting from our multiple Galton Watson process with immigration, that we call a stationary distribution. Let S^2 be variance of the stationary distribution, it can be estimated with:

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{X_n})^2 \approx 77,7$$

Finally we estimated the probability of extinction and found that it's 0.

We know that because $p \in [1/2, 1)$ there is certain extinction of the population, however because there is immigration, it stops the population from ever going extinct. We can conclude that the long term probability that the population is zero is null.

Peut être la question était mal posée mais vous avez répondu à côté. L'accroissement "naturel" est nul avec une certaine probabilité à estimer.

Appendix: python code

```
import random
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import poisson
from scipy.stats import nbinom

def recurrence(X,nodeTorode,proba,theta):
    if nodeTorode == 0: ###Condition to stop the code
        return(X[-1])
    elif X[-1] == 0:
        return(X[-1])
    else:
        nbPopulation = 0
        ###Simulation for each current offspring of their future offspring
        for loop in range(X[-1]):
            ###We update the number of total offsprings in the future
            nbPopulation = nbinom.rvs(n = 1, p = proba) + nbPopulation
        X[len(X):] = [nbPopulation + poisson.rvs( mu = theta)] ###We add the immigration
        ###We then reduce the amount of times we repeat this code
        nodeTorode-=1
        ###Finally we test future generations with recurrence
        recurrence(X,nodeTorode,proba,theta)
    return X[-1]

###This simulates the Galton Watson Walk that gives us the size of the population
after n generations

def gw_walk(start,nbPath,SzPath,proba,theta):
    walk = []
    for loop in range(nbPath):
        walk.append(recurrence(start,SzPath,proba,theta))
    return walk

###We can also write a function that estimates the probability of the extinction
of the population

def extinctionProba(randomWalk):
    zeros = 0
    for simu in range(len(randomWalk)):
        if randomWalk[simu] == 0:
            zeros = 1 + zeros
    return (zeros/len(randomWalk))
```

```
###We have here our simulation and estimation of extinction
X = gw_walk([1],5000,1000,0.6,10)
#it's a long process and take a long time to compute
#213 min for this one
print(X)
extinctionProba(X)
```

peut être que votre code n'est pas très optimal et vous devriez simuler des trajectoires moins longues.

```
###Estimation of the mean and variance of our stationary distribution
np.mean(X)
np.var(X)

###Finally here is the code to draw the histogram of our stationary distribution

plt.hist(X,  bins=35,histtype='bar',  ec='black')
plt.title("Histogram of the size of the population of the last generation")
min_ylim, max_ylim = plt.ylim()
plt.axvline(np.mean(X), color='red', linestyle='dashed', linewidth=2)
plt.text(np.mean(X)*1.1, max_ylim*0.9, 'Mean: {:.2f}'.format(np.mean(X)), color='red')
plt.xlabel("Sizes of the population")
plt.ylabel("Occurences of a size")

plt.show()
```