Prediction of Individual sequences 2020
Homework

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Part 1 : Link between online learning and game theory

1 Question 1

For Rock-Paper-Scissors the loss matrix is :

$$L = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

2 Question 2

See the attached jupyter-notebook for the implementation.

3 Question 3 : EWA vs fixed-strategy adversary

a) + b) + c) + d)

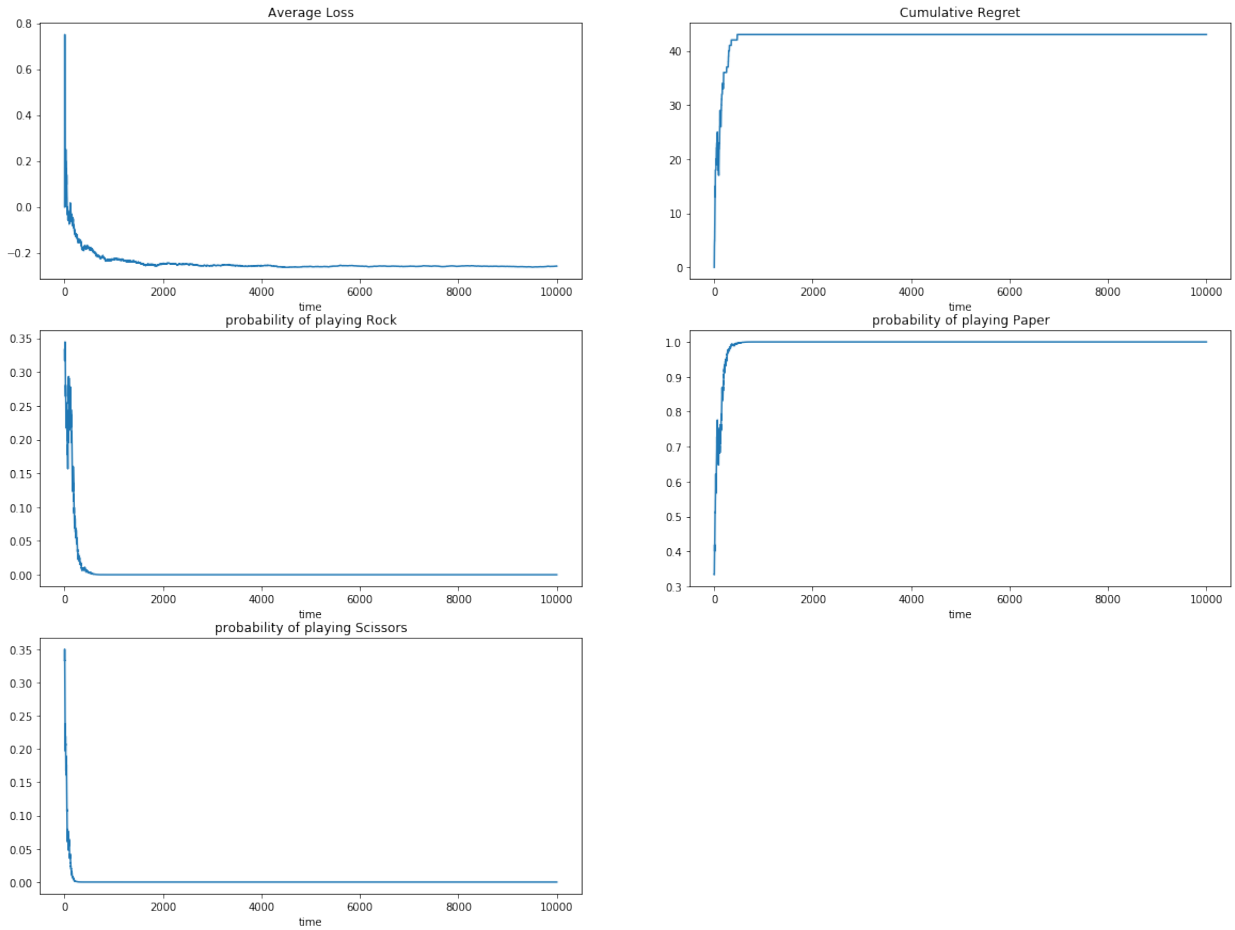

Figure 1: Results

e)

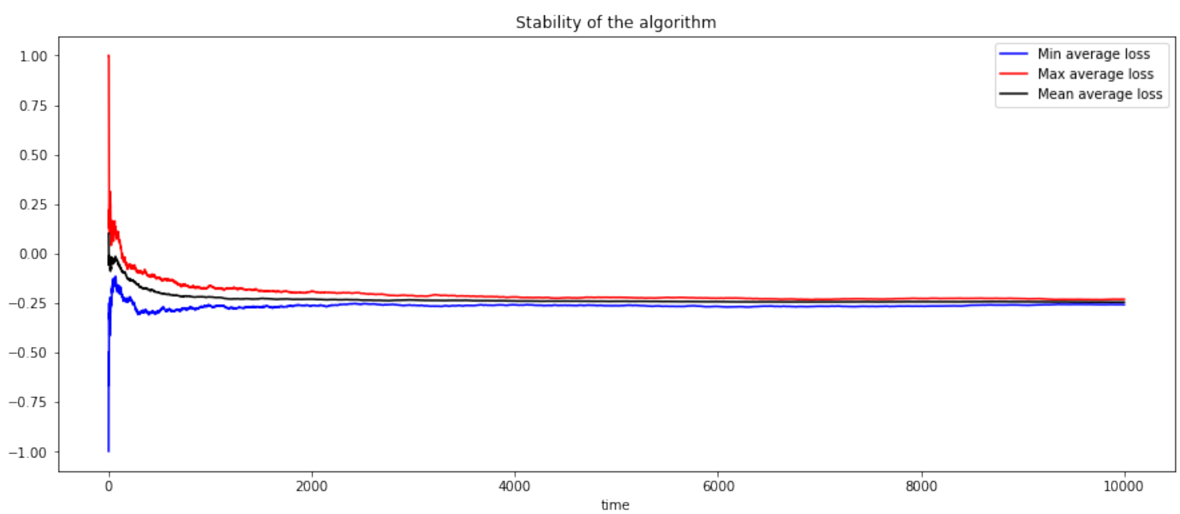

Figure 2: Variance of EWA

We note that EWA vs a fixed-strategy adversary has little variance. f)

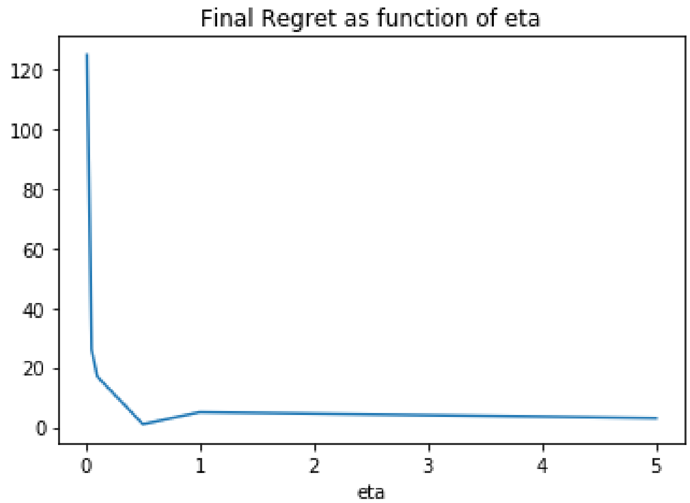

Figure 3: dependance of EWA on eta

4 Question 4: EWA player vs on EWA adversary

a)

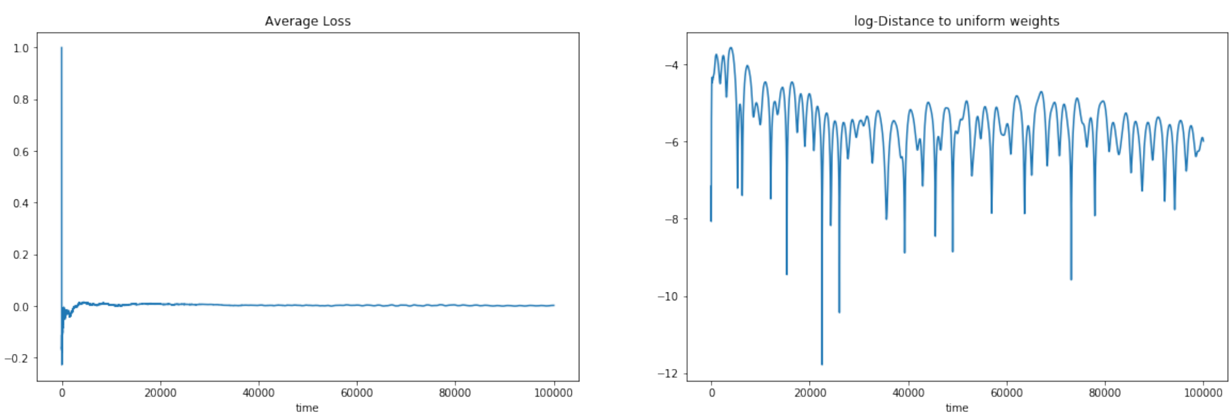

Figure 4: convergence to Nash equilibrium with loss equal value of the game (zero)

5 Question 5:

See the jupyter notebook for the implementation.

6 Question 6: EXP3 player vs fixed strategy adversary

a)

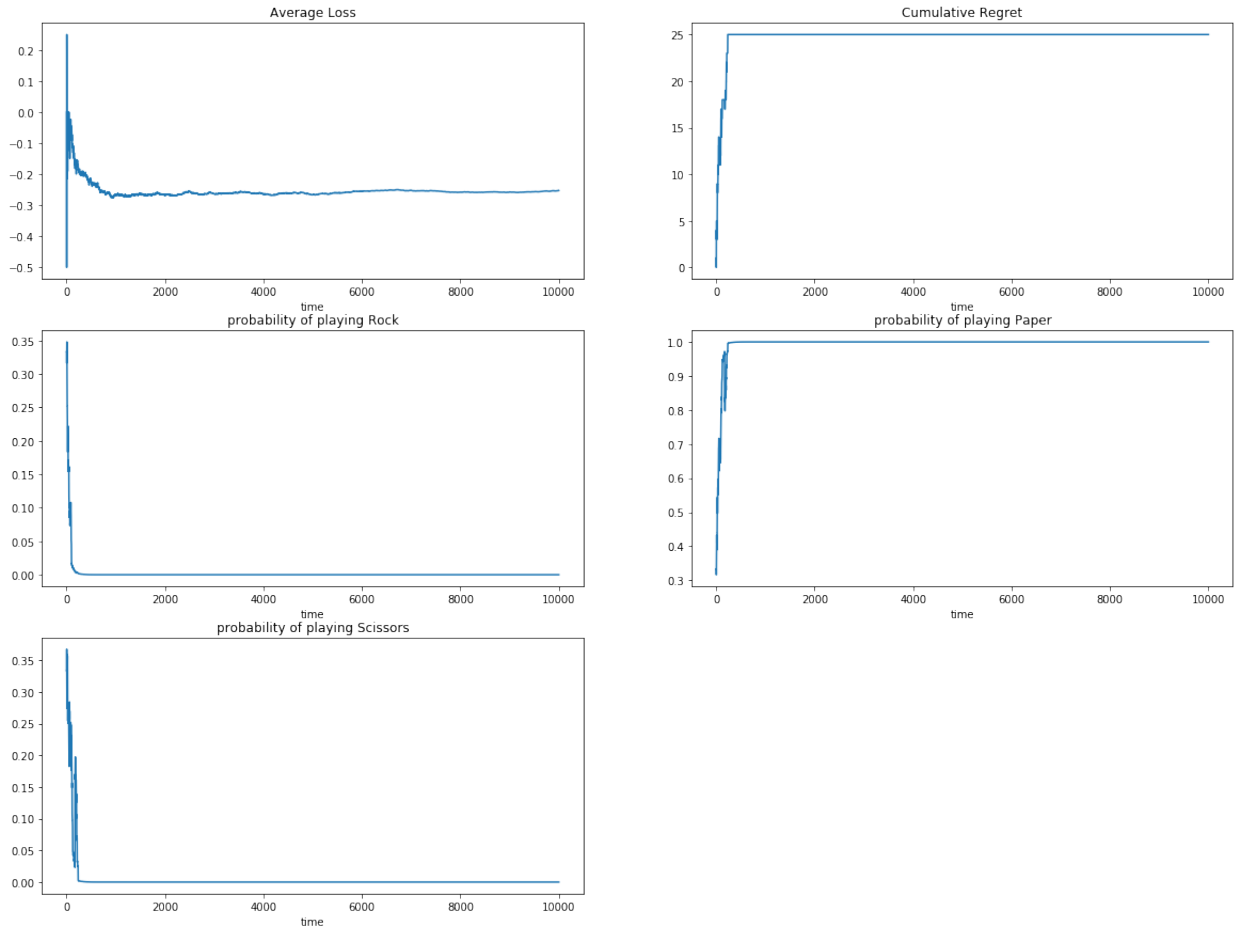

Figure 5: Losses and weights of actions

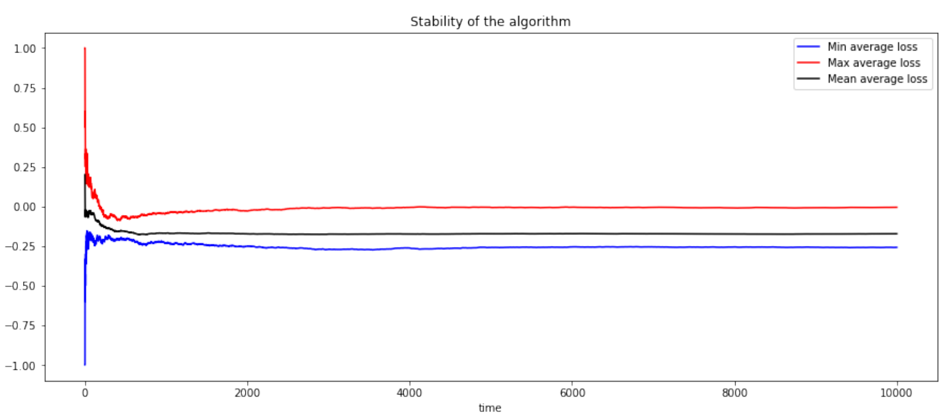

Figure 6: Variance of EXP3

We note that EXP3 has considerably higher variance than EWA.

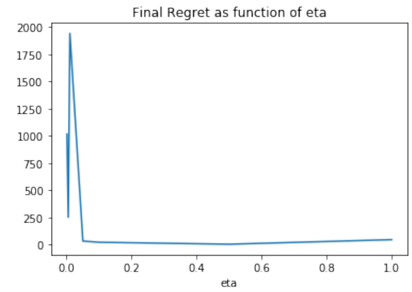

Figure 7: dependance of EXP3 on eta

7 Question 7: EXP3 player vs on EXP3 adversary

a)

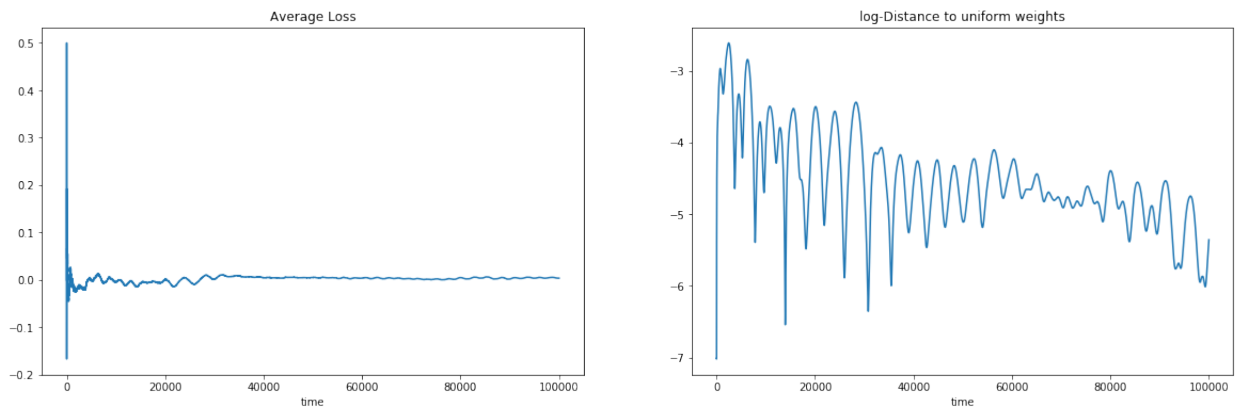

Figure 8: convergence to Nash equilibrium with loss equal value of the game (zero) (EXP3 case)

8 Question 8: UCB vs EXP3

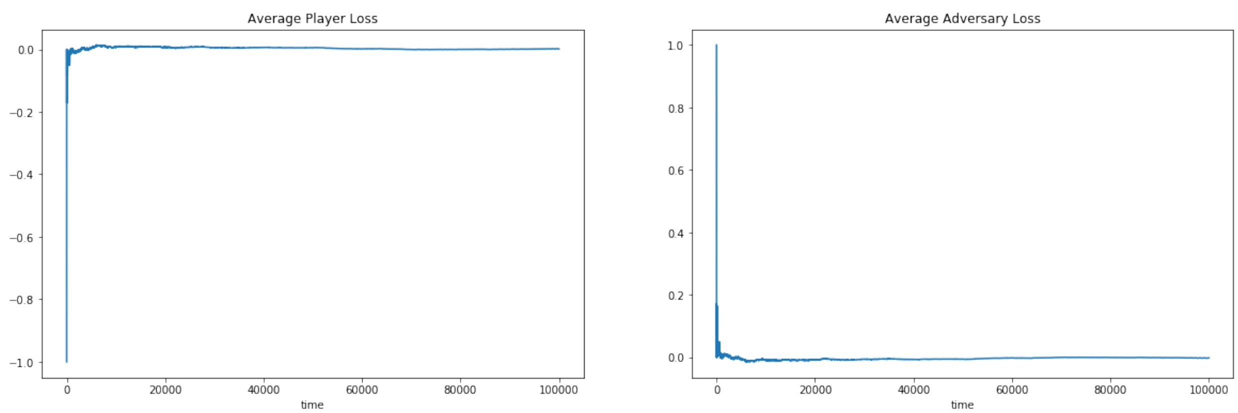

Figure 9: UCB vs EXP3

None of the algorithms beats the other, both have equal performance.

9 Question 10: Non-zero-sum game : Prisonner's dilemma

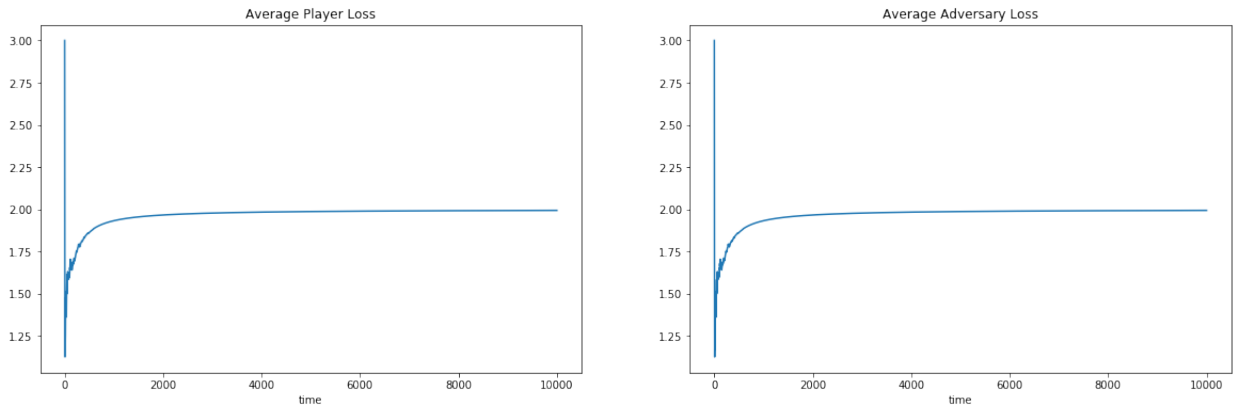

Figure 10: Prisoner's dilemma : EWA player and EWA adversary

Both losses converge to the value of 2, which is the value when both prisoners choose to betray each other. We clearly see that it's ineffective as they can both cooperate and have a loss of only 1.

Part 2 : Theory - Sleeping experts

10 Question 11

a) Let $\Delta(x) = \log(1+x) - (x - x^2)$, then $\Delta'(x) = \frac{1}{1+x} - (1-2x) = \frac{x(x+2)}{1+x}$. Therefore $\Delta(x)$ is decreasing on $[-1, 0]$, and increasing on $[0, \infty[$, which implies that $\Delta(0) = 0$ is a minimum of *Delta* over $[-\frac{1}{2}, \infty[$. In conclusion, $\forall x \geq -\frac{1}{2} \quad \log(1+x) \geq x - x^2$.

b) Since $(w_t(k))_k$ are positive and $x \mapsto \log(x)$ is increasing, we have :

$$\forall k \in [K] : \quad \log(W_{T+1}) \geq \log(w_t(k)) = \sum_{t=1}^{T} \log[1 + \eta(k)(p_t \cdot l_t - l_t(k)]$$

$$= \sum_{t=1}^{T} \log[1 + \eta(k)(p_t \cdot l_t - l_t(k)]$$

$$\geq_{(a)} \sum_{t=1}^{T} \eta(k)[(p_t \cdot l_t - l_t(k)) - [\eta(k)(p_t \cdot l_t - l_t(k))]^2$$

$$= \eta(k) \sum_{t=1}^{T} (p_t \cdot l_t - l_t(k)) - (\eta(k))^2 \sum_{t=1}^{T} ((p_t \cdot l_t - l_t(k))^2$$

c) We have :

$$W_{t+1} = \sum_{k=1}^{K} w_{t+1}(k)$$

$$= \sum_{k=1}^{K} w_t(k)[1 + \eta(k)(p_t \cdot l_t - l_t(k))]$$

$$= W_t + \sum_{k=1}^{K} w_t(k)\eta(k)[p_t \cdot l_t - l_t(k)]$$

$$= W_t + (p_t \cdot l_t) \left[\sum_{k=1}^{K} w_t(k)\eta(k)\right] - \sum_{k=1}^{K} w_t(k)\eta(k)l_t(k)$$

$$= W_t + \sum_{j=1}^{K} \left[\sum_{k=1}^{K} w_t(k)\eta(k)\right] p_t(j)l_t(j) - \sum_{k=1}^{K} w_t(k)\eta(k)l_t(k)$$

But, by definition of $p_t(j)$ we have :

$$\left[\sum_{k=1}^{K} w_t(k)\eta(k)\right] p_t(j) = \eta(j)w_t(j)$$

so that :

$$W_{t+1} = W_t + \sum_{j=1}^{K} \eta(j) w_t(j) l_t(j) - \sum_{k=1}^{K} w_t(k)\eta(k) l_t(k) = W_t$$

d) Let us note $S_k = \sum_{t=1}^{T}(p_t \cdot l_t - l_t(k))$ and $R_k = \sum_{t=1}^{T}((p_t \cdot l_t - l_t(k))^2$ then we have by using (b) and (c) that :

$$\eta S_k - \eta^2 R_k \leq \log(K) \Rightarrow S_k \leq \sqrt{\frac{\log(K)}{\eta} + \eta R_k}$$

By optimizing $\eta \mapsto \sqrt{\frac{\log(K)}{\eta} + \eta R_k}$, we get :

$$\sum_{t=1}^{T}(p_t \cdot l_t - l_t(k)) \leq 2\sqrt{\log(K)\sum_{t=1}^{T}((p_t \cdot l_t - l_t(k))^2} \tag{1}$$

for a value of $\eta = \sqrt{\frac{\log(K)}{R_k}}$.

11 Question 12

a) Using definition of $\tilde{l}_t(k)$ $k \in A_t$ then :

$$\tilde{p_t} \cdot \tilde{l_t} - \tilde{l_t}(k) = (p_t \cdot l_t - l_t(k))1_{k \in A_t} \Leftrightarrow \ \tilde{p_t} \cdot \tilde{l_t} = p_t \cdot l_t$$

The same goes when $k \notin A_t$. So we will prove that $\tilde{p_t} \cdot \tilde{l_t} = p_t \cdot l_t$:
We have :

$$p_t \cdot l_t = \sum_{k=1}^{K} p_t(k) l_t(k)$$

, which by definition of $p_t(k)$ gives :

$$p_t \cdot l_t = \sum_{k=1}^{K} \frac{\tilde{p_t}(k)1_{k \in A_t} l_t(k)}{\sum_{j=1}^{K} \tilde{p_t}(j)1_{j \in A_t}}$$

Let us note $S_t = \sum_{j=1}^{K} \tilde{p_t}(j)1_{j\in A_t} = \sum_{j \in A_t} \tilde{p_t}(j)$ then :

$$p_t \cdot l_t = \frac{1}{S_t}\sum_{k=1}^{K} \tilde{p_t}(k)1_{k \in A_t} l_t(k)$$

On the other hand, by definition of $\tilde{l_t}(k)$ we have :

$$l_t(k)1_{k \in A_t} = \tilde{l_t}(k) - (p_t \cdot l_t)1_{k \notin A_t}$$

so that :

$$p_t \cdot l_t = \frac{1}{S_t}\sum_{k=1}^{K} \tilde{p_t}(k)[\tilde{l_t}(k) - (p_t \cdot l_t)1_{k \notin A_t}]$$

$$= \frac{\tilde{p_t} \cdot \tilde{l_t} - (p_t \cdot l_t) \left[\sum_{k=1}^{K} \tilde{p_t}(k) 1_{k \notin A_t}\right]}{S_t}$$

But, since $\tilde{p_t}$ is a probability vector :

$$\sum_{j=1}^{K} \tilde{p_t}(k) 1_{k \notin A_t} = \sum_{j=k}^{K} \tilde{p_t}(k) - \sum_{j=1}^{K} \tilde{p_t}(j) 1_{j \notin A_t} \ = 1 - S_t$$

In conclusion :

$$(p_t \cdot l_t) = \frac{\tilde{p_t} \cdot \tilde{l_t} - (p_t \cdot l_t)(1 - S_t)}{S_t}$$

which implies : $p_t \cdot l_t = \tilde{p_t} \cdot \tilde{l_t}$ CQFD.

b) Using question 11 applied to $\tilde{p}$ and $\tilde{l}$ we have :

$$\sum_{t=1}^{T} (\tilde{p_t} \cdot \tilde{l_t} - \tilde{l}_t(k)) \leq 2 \sqrt{\log(K) \sum_{t=1}^{T} ((\tilde{p_t} \cdot \tilde{l_t} - \tilde{l}_t(k))^2}$$

Then replacing by the result of question 12.a, we get :

$$R_t(k) = \sum_{t=1}^{T} (p_t \cdot l_t - l_t(k)) \leq 2 \sqrt{\log(K) \sum_{t=1}^{T} 1_{k \in A_t} ((p_t \cdot l_t - l_t(k))^2}$$

But since $p_t(k) \in [0, 1]$ and $l_t(k) \in [0, 1]$, we have :

$$((p_t \cdot l_t - l_t(k))^2 \leq 1$$

which finally implies :

$$R_t(k) \leq \sum_{t=1}^{T} (p_t \cdot l_t - l_t(k)) \leq 2 \sqrt{\log(K) \sum_{t=1}^{T} 1_{k \in A_t}} = 2 \sqrt{\log(K) T_k}$$