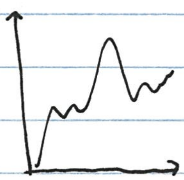
Stochastic Processes I

Def: Collection of Random Variables indexed by time

$X_0 \quad X_1 \quad X_2 \; . \; . \; . \; .$ discrete time

$\{X_t\}_{t \geq 0}$ continuous time

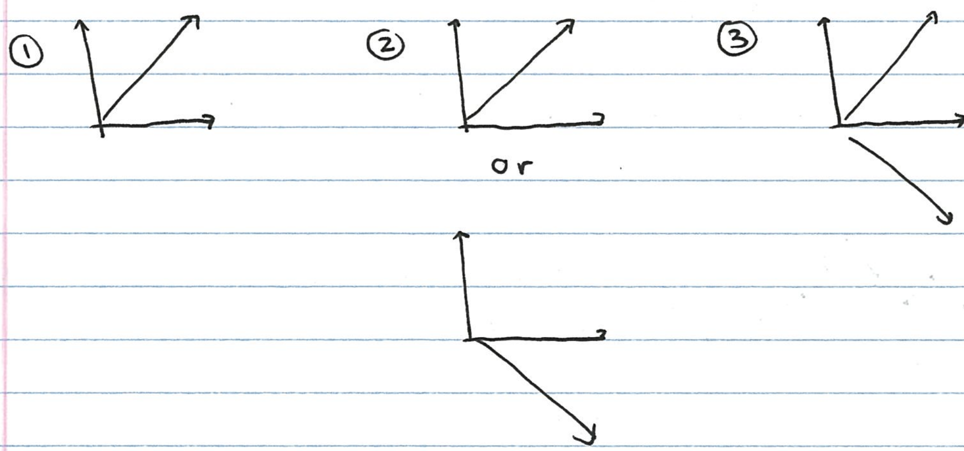
Alternative Def: Probability distribution over a space of paths



- one realization of many

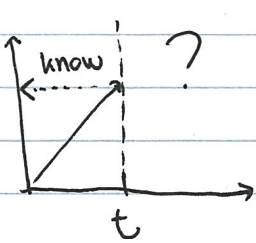
Example

1. $f(t) = t$ with prob 1
2. $f(t) = t$ or $-t$ with prob 1/2
3. For each t, $f(t) = \begin{cases} t & 1/2 \\ -t & 1/2 \end{cases}$



Note

When we want to model event



what happens next?

- some cases we know, others we don't

FIVE STAR.

When given a stocnastic process standing at time t we have control of determining $t+1$ from

- probability distribution

3 types of questions we study

a) what are the dependencies in the sequence of values

b) What is the long term behavior of the sequence?

c) What are boundary events?

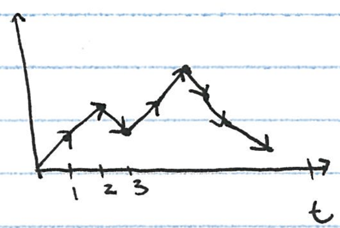
- extreme events

Simple Random Walk

Y_i : iid RV $\begin{cases} +1 & P(y) = 1/2 \\ -1 & P(y) = 1/2 \end{cases}$

for each t, $X_t = \sum Y_i$, $Y_0 = 0$

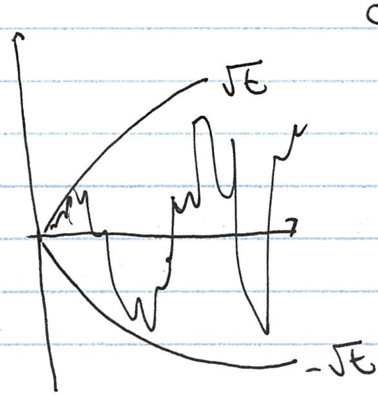
$X_0 \; X_1 \; \ldots \; X_n$ - simple random walk



what if we apply central limit theorem to the sequence?

$\sigma^2 = t$

$\sigma = \sqrt{t}$



- Stochastic process will be bounded by this interval, you can calculate confidence intervals for it

Properties

i) $E X_k = 0$

ii) (Independent Increments)

$0 = t_0 \leq t_1 \leq \dots \leq t_k$

Then $X_{t_{i+1}} - X_{t_i}$ are mutually independent

iii) (stationary)

for all $n \geq 1$, $t \geq 0$

distribution of $X_{t+n} - X_t$

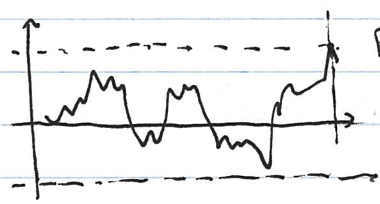
is the same as X_n

Problem Example

$\begin{cases} \text{head} : +1\ \$ \\ \text{tail} : -1\ \$ \end{cases}$ coin toss game

My balance = Simple Random Walk

Play until $\underset{\text{prob } 1/2}{\underline{\text{win } \$100}}$ or $\underset{\text{prob } 1/2}{\underline{\text{lose } \$100}}$



prob: $\frac{B}{A+B}$

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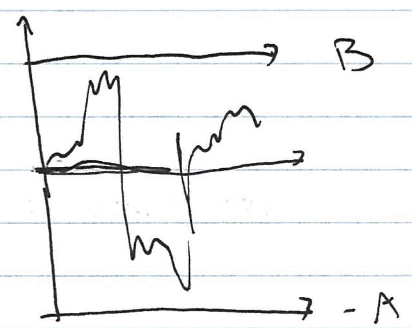


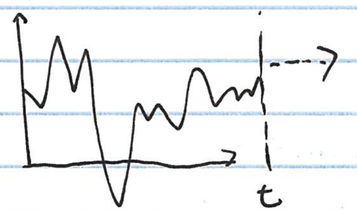
For each $-A \leq k \leq B$ define

$f(k) = P(\text{hits } B \text{ from } k)$

$$f(k) = \frac{1}{2} f(k+1) + \frac{1}{2} f(k-1)$$

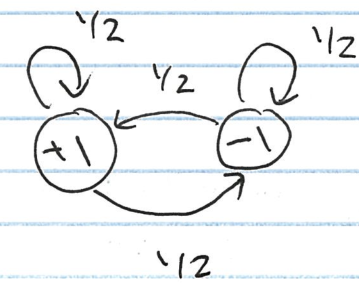
Markov Chain

Def: stochastic processes whose affect of the past on the future is summarized only by current state



t+1 depends on t

Simple Random Walk



Formal Def: Discrete-time stochastic processes is a markov chain if

$$P(X_{t+1} = s \mid X_1 \dots X_n) = P(X_{t+1} = s \mid X_n)$$
$\forall\, n \geq t$

If X_i have values in S (finite set)

- sum of probability equal to 1

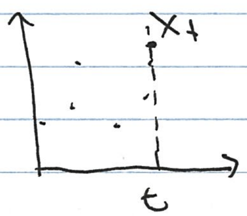
transition matrix $A = \begin{pmatrix} P_{11} & P_{12} \dots \\ \vdots & \\ P_{1m} & \dots P_m \end{pmatrix}$ · has eigenvalue connection

$\lim\limits_{n \to \infty} A^n$ = Stationary distribution

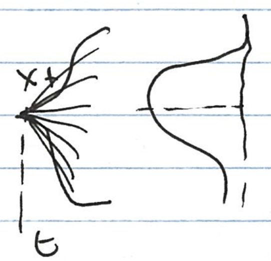
Martingale

Def: Stochastic process which are fair game

Formal Def: a stochastic process is a martingale if
$X_t = E[X_{t+1} | \mathcal{F}_t]$
for all $t \geq 0$ $\quad T_1 = \{X_0, X_1, \dots X_t\}$



at time $t+1$ the probability distriution is designed so that all next possible point X_{t+1} has mean X_t



Example

① Random Walk is a martingale
② Balance of Ralette player is not a martingale
③ $Y_1, Y_2, \dots$ iid random variables

$$Y_i = \begin{cases} 2 & \text{prob } 1/3 \\ 1/2 & \text{prob } 2/3 \end{cases}$$

Let $X_0 = 1$ $\quad X_K = \Pi$

FIVE STAR.

Optional Stopping Theorem

- theorem abat martingale

if you play a martingale game, your expectation won't be positive or negative, it will stay fixed

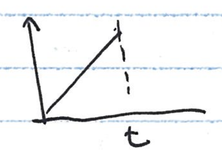
~~Def (Stoppin~~

Def (Stopping time)

Given a stochastic process $\{X_0, X_1 \dots\}$ a non-negative integer RV τ is called a stopping time if for all integer k, $k \geq 0$, τ depends only on $X_1 \dots X_k$, $\tau \leq k$

τ - some strategy you want to use, look at k rounds outcome & stop playing
- strategy only depends on the outcome of the stochastic process up to right now then it is a stopping time

What is ~~NOT~~ NOT a stopping time?



Stop
- at peak

Theorem

Suppose $X_0, X_1 \dots$ is a martingale
τ is a stopping time
Furthermore $\exists$ constant T s.t $\tau \leq T$

then $E[X_\tau] = X_0$