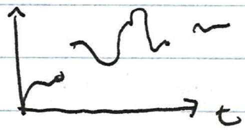
Stochastic Processes II

Continuous Time

$\{X_t\}_{t \geq 0}$



Discrete Time

- Markov Chain
- Martingale

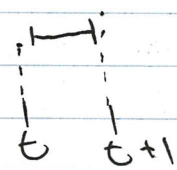
Time Series

Regression analysis

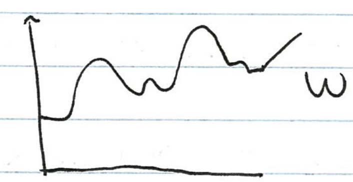
How to describe the probability distribution?

Simple Random Walk

$$X_t - X_{t-1} = \begin{cases} 1 & P(1/2) \\ -1 & P(1/2) \end{cases}$$



for continuous time you can't just do $t - t+1$



$P(w) = ?$ probability density function?

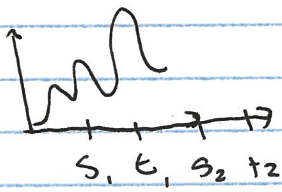
Brownian Motion

Theorem: there exists a probability distribution over the set of continuous function $B: \mathbb{R}_{\geq 0} \to \mathbb{R}$

i) $P(B(0) = 0) = 1$

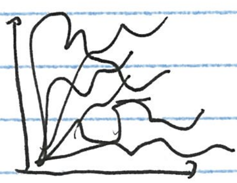
ii) (stationary) $B(t) - B(s) \sim N(0, t-s)$

iii) (Independent increment) If intervals $[s_i, t_i]$ are not non overlapping then $B(t) - B(s)$



- there exists some distribution
- $s - t \sim N(0, t-s)$
- s_1 & t_1 & s_2 & t_2 are indep

Def: The probability distribution given by the theorem is called Brownian motion aka Wiener Process

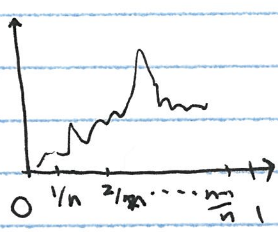


- space of all possible function

as your time scales, what happens between intervals is normally distributed

Philosophy

Brownian motion is the limit of simple random walks



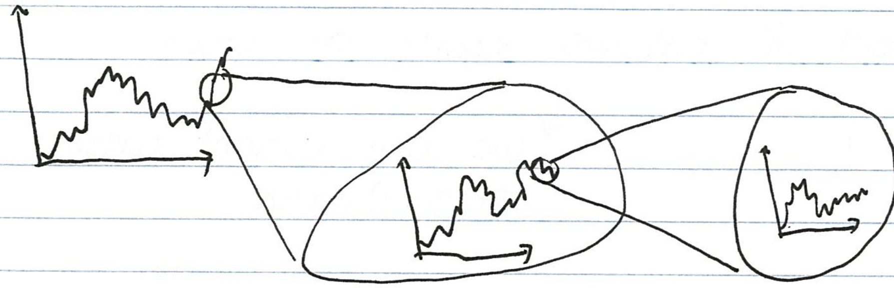
Let $Y_0, Y_1 \ldots Y_n$ be a simple random walk

$$Z\left(\frac{t}{n}\right) = Y_t$$

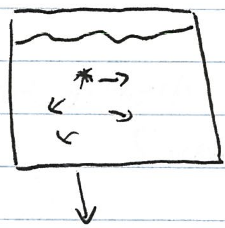
Linearly extended in intermediate values (connecting the dots)

Take $n \to \infty$

Then resulting distribution is the Brownian motion



Example



gravity

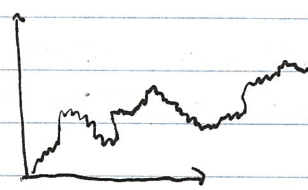
Observed pollen particle in water

Moves acording to Brownian Motion

Action of pollen determined by "infinitesimal" interactions with water molecules

Stock Prices

Stock price S



Brownian Motion is not a bad model

- price is determined by

Properties

① Crosses the t-axis infinitely often
② Does not deviate much from $t = y^2$
③ Is nowhere differentiable !!! → Calculus X

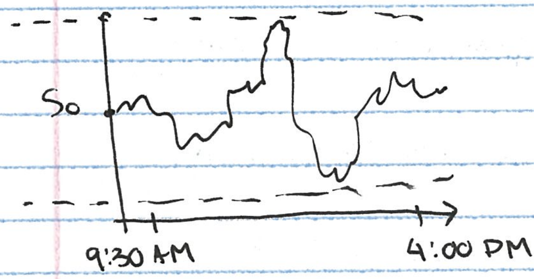
- continuous path but nowhere differentiable
(can't use calculus anymore)

FIVE STAR.

Standard tool of calculus can't be used

↓ we use variant called Ito's Calculus

Using Brownian Motion to Model Daily Stock Price



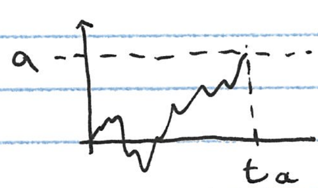
What is the distribution of the min & the max?

$$M(t) = \max_{s: s \le t} B(s)$$

theorem: For all $t > 0$, & $a > 0$

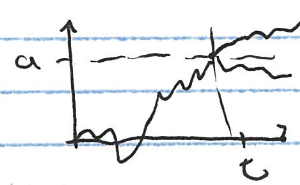
$$P(M(t) > a) = 2P(B(t) > a)$$

proof: $\tau_a = \min_t \{ B(t) = a \}$



$\cancel{P(B(t) - B(\tau_a)}$ $P(B(t) - B(\tau_a) \mid \tau_a < t)$

$= P(B(t) - B(\tau_a) < 0 \mid \tau_a < t)$



(symmetry)

$P(M(t) > a)$

$= P(\tau_a < t)$

$= P(B(t) - B(\tau_a) \geq 0 \mid \tau_a < t)$

$+ P(B(t) - B(\tau_a) < 0) \mid \tau_a < t)$

Property

For each $t \geq 0$, the Brownian motion is differentiable at t with prob 1

Suppose $\frac{dB}{dt}(t) = A$

$|B(t+\varepsilon) - B(t)| \leq \varepsilon A$

for all ε

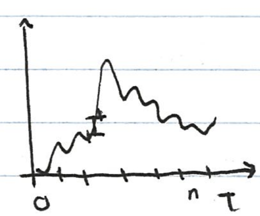
$= 2P(N(0,1) \leq \sqrt{\varepsilon} A)$

theorem (Quadratic variation)

for all $T > 0$

$$\lim_{n \to \infty} \sum_{t=1}^{n} \left(B(t/n) - B(t-1/n)\right)^2 = T$$

function



-take each square, sum it & it goes to t

Assume f is continuously differentiable

Then $\sum^n (f(t_i) - f(t_i))^2 < (f'(s_i)(t - t_i))^2$

$$\max_{0 \le s < t} f'(s)^2 \sum^n (t_i - t_i)^2$$

$$\boxed{(dB)^2 = dt}$$

For stock prices

we want the percentile difference to be normally distributed

$\frac{dS_t}{S_t} = dB_t$ (differential equation)

$\uparrow$ not differentiable

Is $\cancel{S_t = e^{B_t}}$? WRONG!!!

$$\cancel{dS_t = e^{B_t} \cdot dB_t = S_t \, dB_t}$$

Ito's Calculus

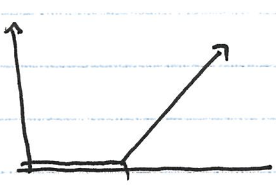
Suppose we want to compute

$$f(B_t)$$

for some smooth function f

you put Brownian motion as input & estimate infitesimal difference

$f =$



S_0

$f(B_t)$ = value of option at expiration

$f(B_t)$

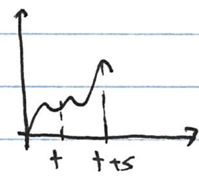
function of an underlying stock: financial derivative

If B_t was differentiable

$$df = \left(\frac{dB_t}{dt}\right) dt$$ cannot do this

What about

$$df = f'(B_t)\, dB_t$$



Reason it is not valid is b/c of the fact

$$(dB)^2 = dt$$

$$f(t+x) = f(t) + f'(t)x + \frac{f''(t)}{2} x^2 + \dots$$

$$f(t+x) - f(t) = f(t) = f'(t) + \frac{f''(t)}{2} x^2 + \dots$$

$$df = f'(B_t)\, dB_t + \frac{f''(B_t)}{2} dt$$