Deutsch-Jozsa Algorithm

Problem Statement

Given a function f from $\{0, 1\}^n$ to $\{0, 1\}$, the function is either:

1. **Constant**: $f(x)$ returns the same value (either 0 or 1) for all inputs x.
2. **Balanced**: $f(x)$ returns 0 for exactly half of the inputs and 1 for the other half.

The goal is to determine whether f is constant or balanced using the fewest possible evaluations of f.

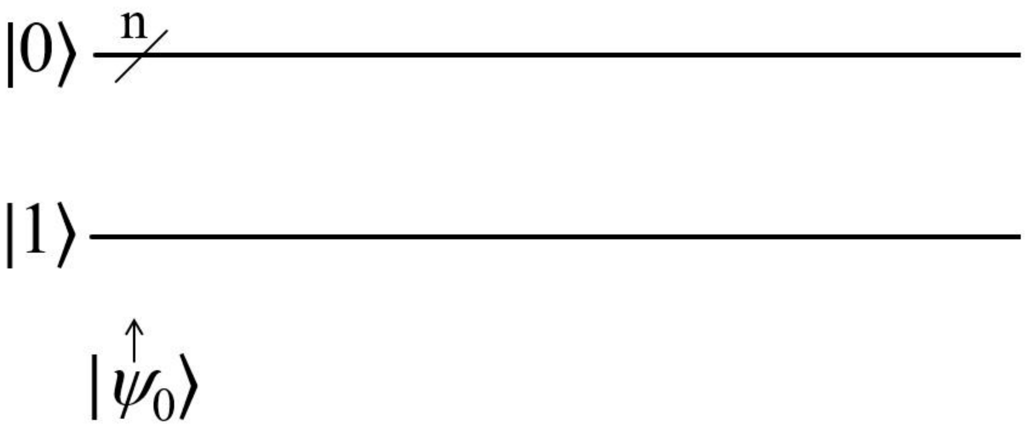
Classical Approach: In the worst case, a classical deterministic algorithm would need to evaluate the function f $2^{(n-1)} + 1$ times to be certain whether f is constant or balanced.

Quantum Approach (Deutsch-Jozsa Algorithm): The quantum algorithm can determine whether f is constant or balanced with just one evaluation of f.

Deutsch-Jozsa Algorithm Steps

1. **Initialization**:

   - Start with two quantum registers: one n-qubit register initialized to $|0\rangle^{\otimes n}$ and one single-qubit register initialized to $|1\rangle$.
   - The overall state is $|0\rangle^{\otimes n} \otimes |1\rangle$.

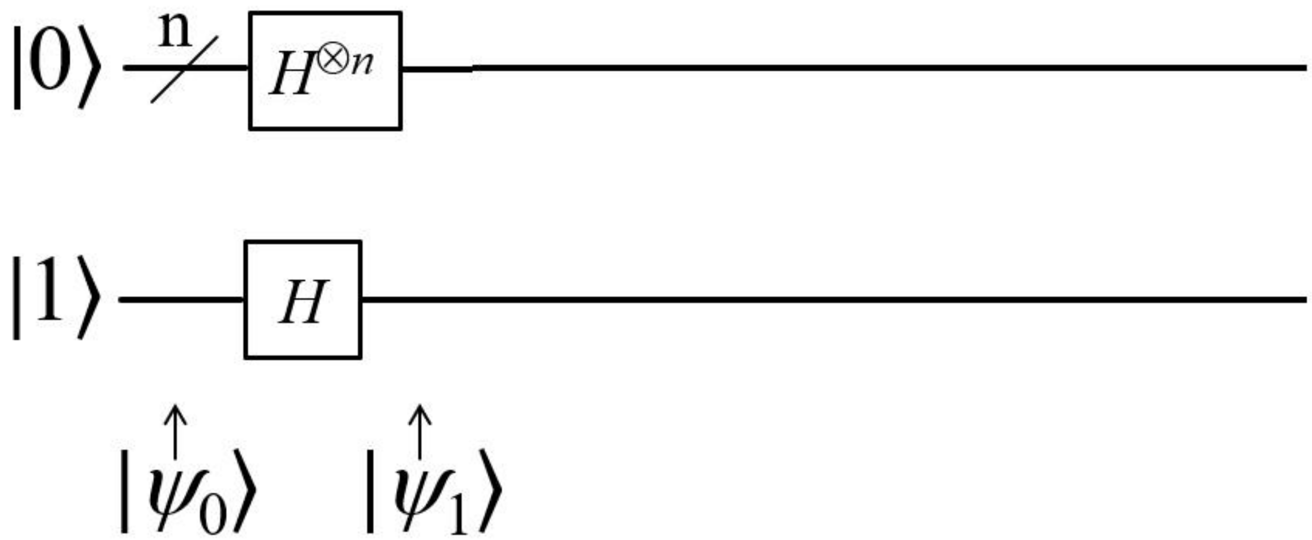


2. **Apply Hadamard Gates**:

   - Apply the Hadamard transform H to each qubit.
   - The n-qubit register goes from $|0\rangle^{\otimes n}$ to $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$. *
   - The single-qubit register goes from $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
   - The combined state becomes:

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

* $$H^{\otimes n}|0\rangle^{\otimes n} = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \cdots \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$


$|0\rangle$
n
$H^{\otimes n}$
$|1\rangle$
H
$|\psi_0\rangle$
$|\psi_1\rangle$

3. **Oracle Query**:

- Apply the oracle U_f which maps $|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$.
- After applying the oracle, the state is:

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle)$$

- This simplifies to:

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle)$$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{1}{\sqrt{2}}(|f(x)\rangle - |1 \oplus f(x)\rangle)$$

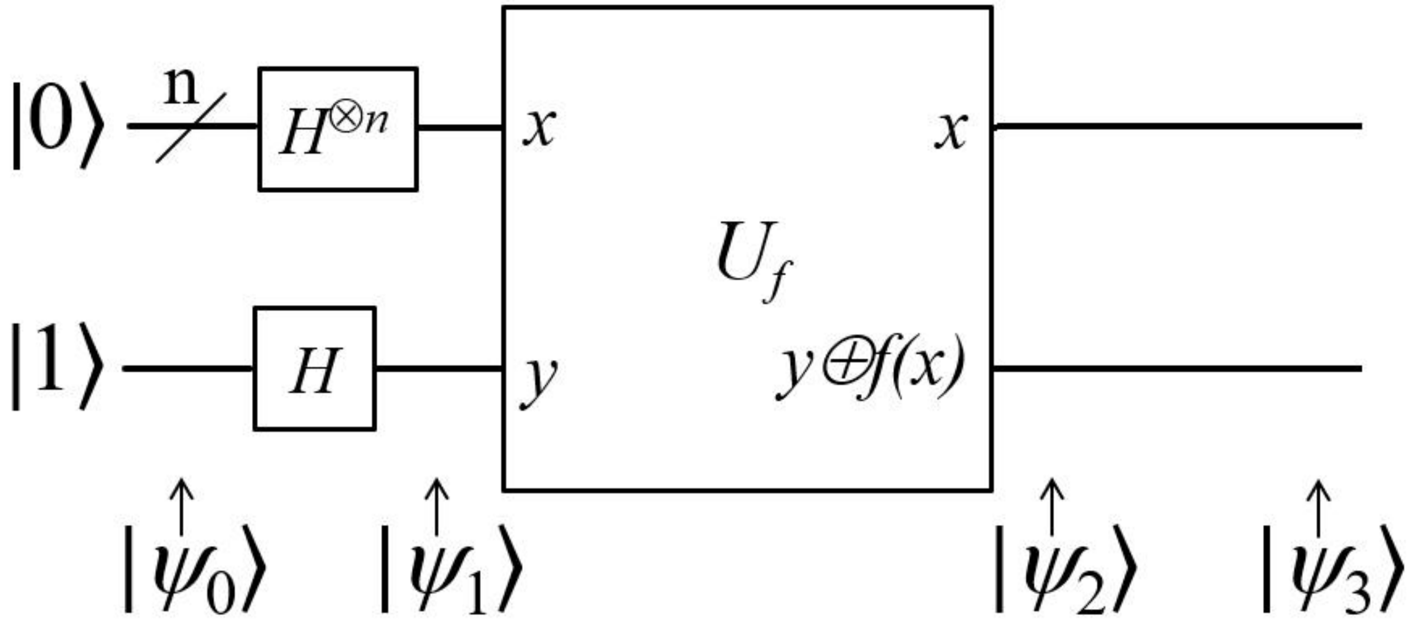
- If $f(x) = 0$, then $|f(x)\rangle - |1 \oplus f(x)\rangle = |0\rangle - |1\rangle$
- If $f(x) = 1$, then $|f(x)\rangle - |1 \oplus f(x)\rangle = |1\rangle - |0\rangle = -(|0\rangle - |1\rangle)$

This can be expressed compactly as:

$$(-1)^{f(x)}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \frac{1}{\sqrt{2}}(-1)^{f(x)}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)}|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



|0⟩
n
H⊗n
x
x
Uf
|1⟩
H
y
y⊕f(x)
|ψ0⟩
|ψ1⟩
|ψ2⟩
|ψ3⟩

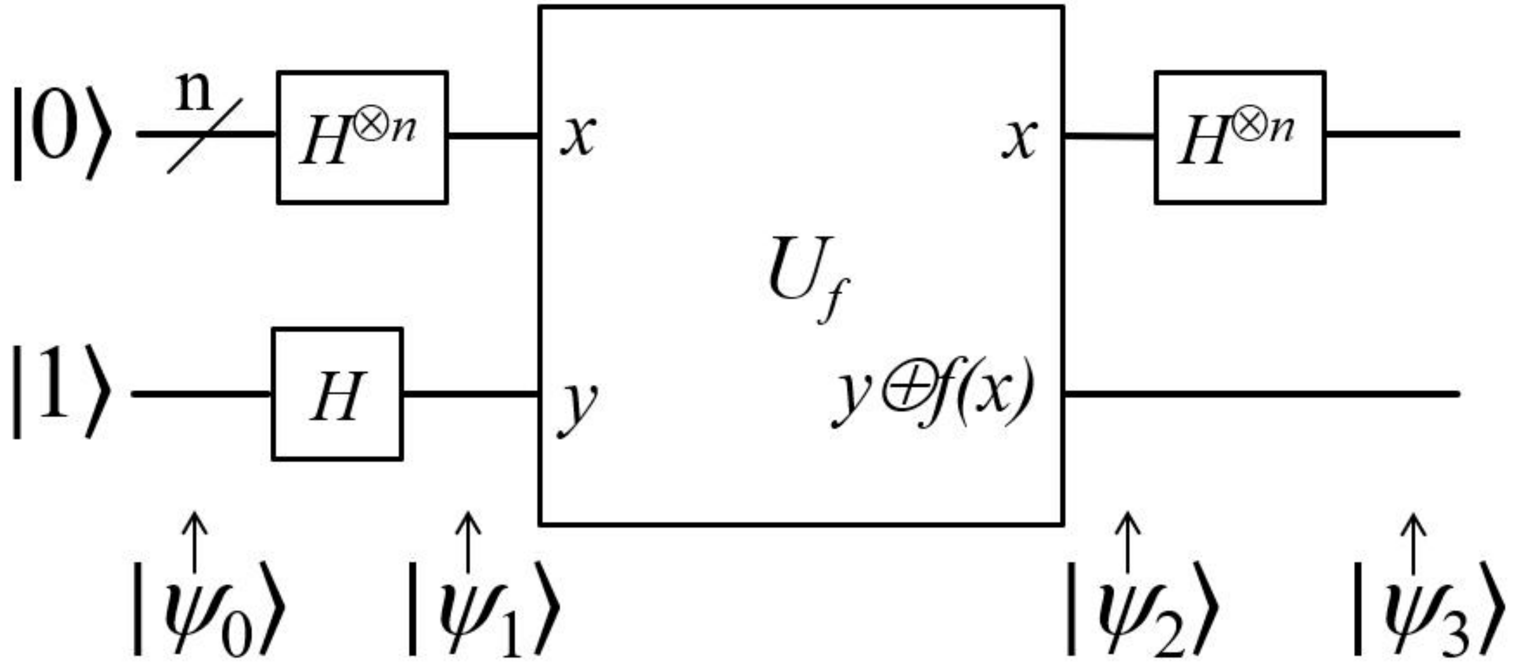
4. **Apply Hadamard Gates Again**:

- Apply the Hadamard transform to the n-qubit register again.
- This results in:

$$\frac{1}{2^n}\sum_{z=0}^{2^n-1}\left[\sum_{x=0}^{2^n-1}(-1)^{f(x)\oplus x\cdot z}\right]|z\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

* $$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{z=0}^{2^n-1}(-1)^{x\cdot z}|z\rangle$$

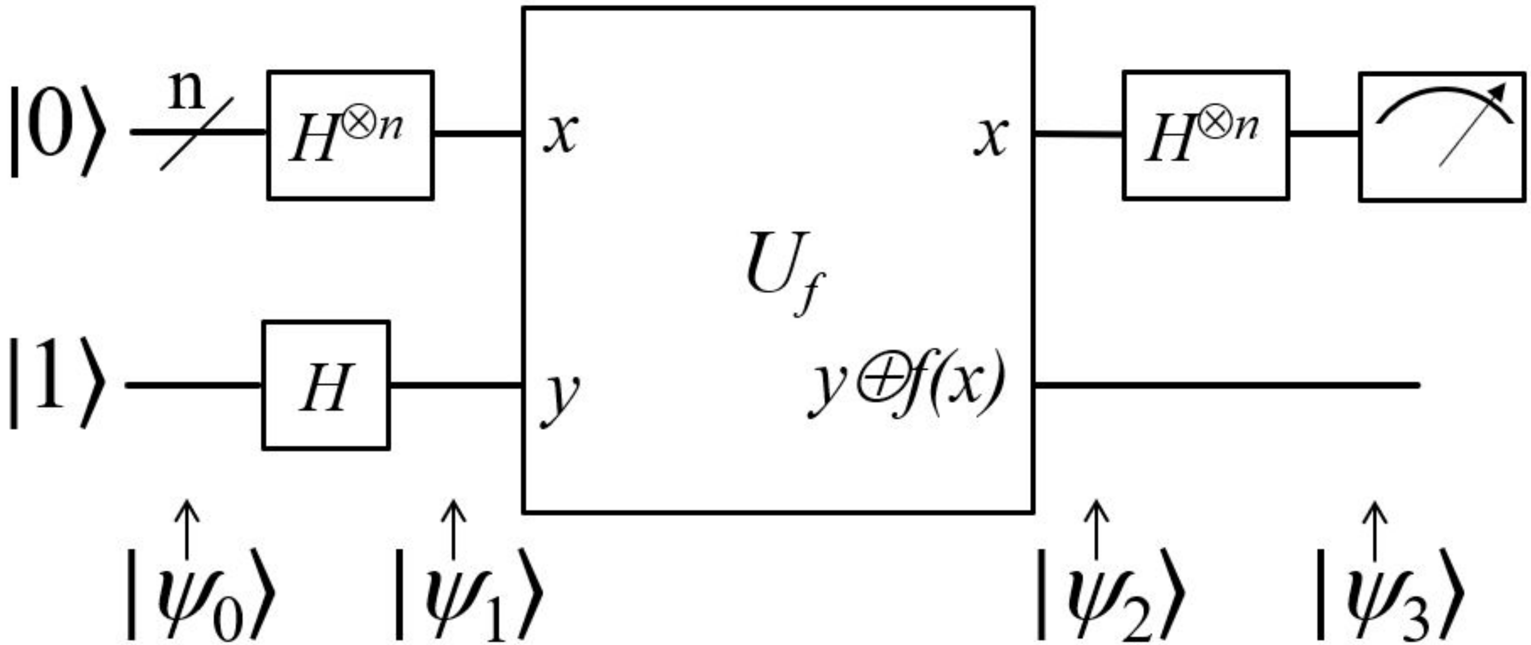
* $$x \cdot z = \bigoplus_{i=1}^{n}(x_i \cdot z_i)$$



|0⟩
n
H⊗n
x
x
H⊗n
|1⟩
H
y
U_f
y⊕f(x)
|ψ0⟩
|ψ1⟩
|ψ2⟩
|ψ3⟩

5. **Measurement**:

- Measure the n-qubit register.
- If the outcome is $|0\rangle^{\otimes n}$, then f is constant.
- If the outcome is anything else, f is balanced.



https://en.wikipedia.org/wiki/Deutsch%E2%80%93Jozsa_algorithm#/media/File:Deutsch-Jozsa-algorithm-quantum-circuit.png