Simulation of Gaussian random fields

May 28, 2019

Gaussian Random Fields

Definition

$\{Z(x)\}_{x \in \mathbb{R}^d}$ is called a (real) Gaussian random field if $\forall n \in \mathbb{N} \backslash \{0\}, \forall x_1, ..., x_n \in \mathbb{R}^d$, the random vector $(Z(x_1), ..., Z(x_n))$ is Gaussian.

Definition

(Second order stationarity: usual definition)
A random field $Z(\cdot)$ on $\mathbb{R}^d$ is second order stationary if the first two moments exist and verify:

$$E(Z(x)) = \mu \tag{1}$$

$$Cov(Z(x), Z(x+h)) = C(h), \tag{2}$$

for any x and $h \in \mathbb{R}^d$.

Variogram

Remark

A 2nd order stationary Gaussian field Z is entirely determined by its mean and its covariance function $C(h)$.

Definition

Let $Z(x)$ a second order stationary random field on $\mathbb{R}^d$. The variogram of $Z(x)$ is the function

$$\gamma(h) = \frac{1}{2}E[(Z(x+h) - Z(x))^2] = C(0) - C(h) \tag{3}$$

Definition (Isotropy)

A covariance function $C(h)$ is said isotrope if it depends only on the distance $r = \|h\|$ and not the direction of the vector h. We will therefore note $C(h) = C(\|h\|) = C(r)$.

Table: Examples of covariance functions

Name	**Function**
Pure nugget	$\gamma(h;\sigma^2) = \begin{cases} \sigma^2 & \text{, if } h > 0 \\ 0 & \text{, if } h = 0 \end{cases}$
Exponential	$\gamma(h;a,\sigma^2) = \sigma^2(1 - exp(-\|h\|/a))$
Spherical	$\gamma(h;a,\sigma^2) = \begin{cases} \sigma^2(\frac{3\|h\|}{2a} - (\frac{\|h\|}{2a})^3) & \text{, if } \|h\| \leq a \\ \sigma^2 & \text{, if } \|h\| > a \end{cases}$
Gaussian	$\gamma(h;a,\sigma^2) = \sigma^2(1 - exp(-(\|h\|/a)^2))$

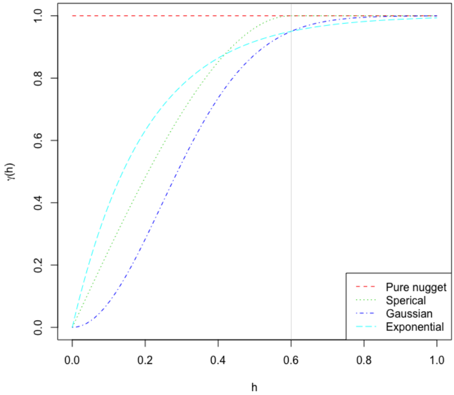


Figure: Theoretical variograms for the real range of the spherical variogram and the practical range of the exponential and gaussian variograms; with nugget = 0, sill = 1 and range = 0.6.

Suppose we have n points to simulate with mean 0 and covariance $C(r)$. Let K be the covariance matrix. We decompose this matrix as $K = LU$, with $L = {}^tU$ (Cholesky decomposition). We draw n independant random values y_i, $i = 1...n$ with distribution $\mathcal{N}(0,1)$. Set z=Ly; z is then a realization of Z having the right distribution law.

Proof.

$Cov[LY, LY] = \mathbb{E}[LY^tY^tL] = L\mathbb{E}[Y^tY]^tL = L^tL = LU = K_{n\times n}$ □

We choose an arbitrary origin 0 in $\mathbb{R}^3$ and generate L lines i such that the corresponding direction vectors u_i are uniformly distributed on the unit sphere.
Let x_k be the points to be simulated where $k \in \{1, ..., n\}$. We denote $x_{ki} = x_k \cdot u_i$ the projection of the point x_k on the line i.
Along each line i, we generate a second-order stationary unidimensional process having a covariance function

$$C_1(r) = \frac{d}{dr} rC(r).$$

Finally we assign to the point x_k the value $z_s(x_k)$ given by

$$z_s(x_k) = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} z_i(x_{ki})$$

as the realization of the three-dimensional random field.

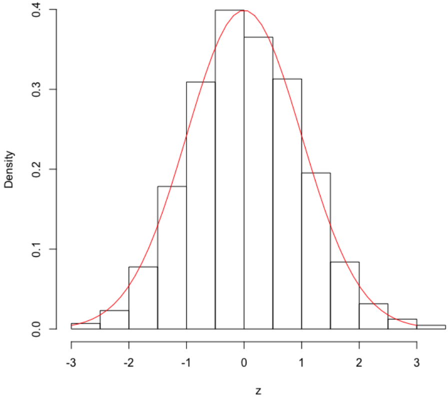
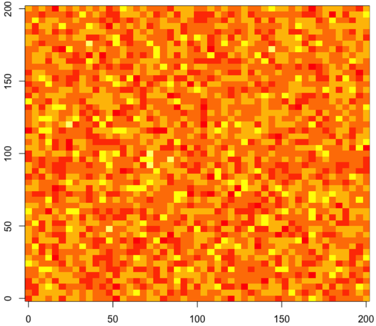
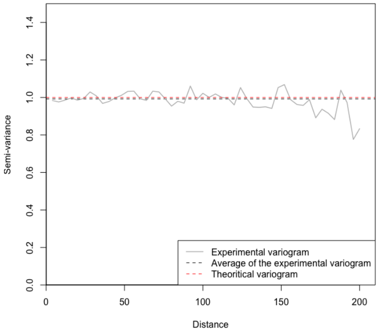


Figure: Histogram of $Z(\cdot)$

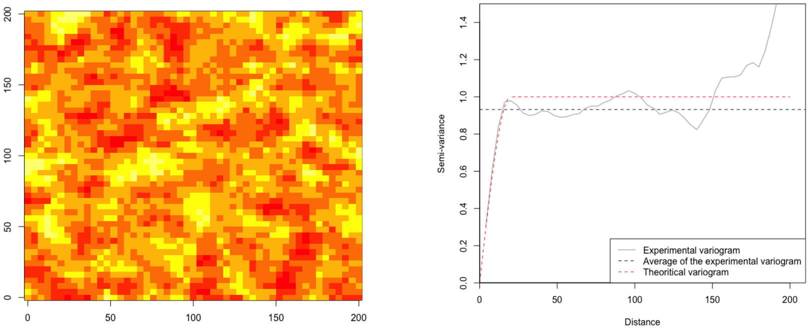


(a)



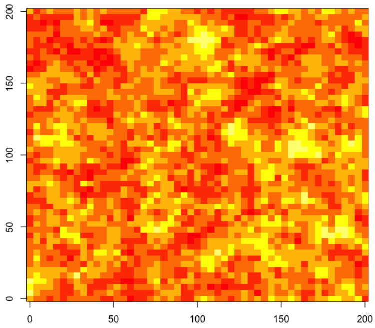
(b)

Figure: Pure nugget effect (a) gaussian random field and (b) variogram with sill = 1.

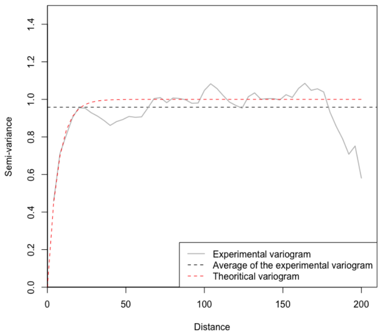


(a) (b)

Figure: (a) Gaussian random field and (b) variogram of spherical covariance with nugget $= 0$, sill $= 1$ and range $= 20$.

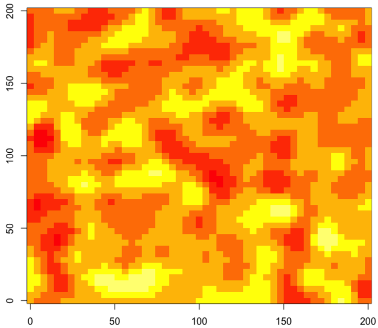


(a)

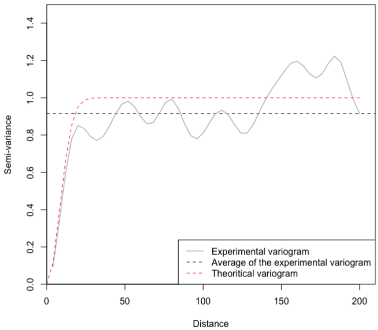


(b)

Figure: (a) Gaussian random field and (b) variogram of exponential covariance with nugget = 0, sill = 1 and range = 20.



(a)



(b)

Figure: (a) Gaussian random field and (b) variogram of gaussian covariance with nugget = 0, sill = 1 and range = 20.

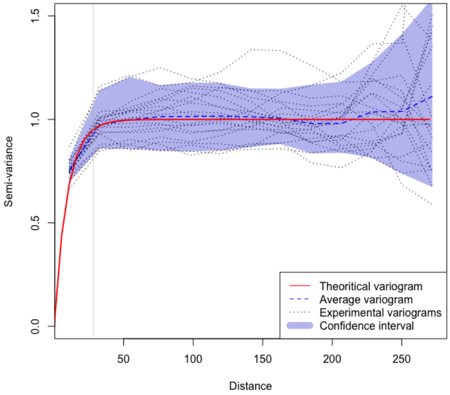
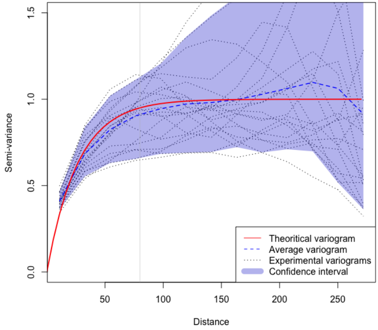
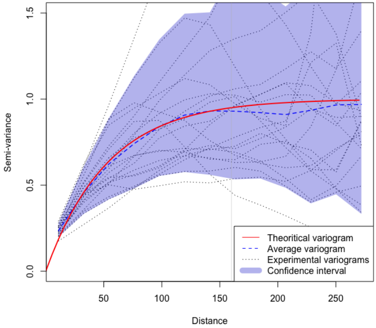


Figure: Averages of 20 exponential variograms with a practical ranges of 28.



(b)



(c)

Figure: Averages of 20 exponential variograms with a practical range of (a) 80 and (b) 160.

Thank you for your attention

Q/A