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CONTINUOUS LOGIC AND ALGORITHMS FOR SOLUTION OF SOME COMBINATORIAL PROBLEMS

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The class of combinatorial problems equivalent to the problem of determination of relative positions of n interval sequences is formulated. It is shown that an adequate mathematical model of solving a stated problem is a finite dynamic memoryless automaton and that the adequate mathematical apparatus is continuous logic. Algorithms that solve the problem are constructed.

Keywords: *continuous logic, combinatorial problem, automaton model.*

INTRODUCTION

Problems of data processing, job scheduling, design of object control systems, and many others are mathematically reduced to the solution of the combinatorial problem of computating suitable indicators of relative positions of n sequences of (time or spatial) intervals and to the determination of conditions under which these relative positions has some qualitative character or other. We give several examples of such problems.

1. Let there be a sequence A of time intervals in which some main technical device is operable and a sequence B of time intervals in which a backup device is operable. The system "main and backup devices" is efficient if at least one of its two devices, i.e., the main or backup device, is operable. Thus, to establish a sequence of operability intervals of the system, one should determine relative positions of sequences of operability intervals of the main device (the sequence A) and backup device (the sequence B) and find the time intervals in which there are intervals of at least one of sequences A or B; it is exactly these intervals that are the operability intervals of the system. In order to determine, for example, the operability of the system over an arbitrary given time interval T, it is necessary to find the conditions under which intervals of the sequence B within this time interval cover interspaces between intervals of the sequence A.

2. Let us consider a sequence A of time intervals during which some organization (a store, a bank, a repair shop, etc.) serves clients and a sequence B of time intervals during which some client can visit his service organization. To determine the sequence of possible time intervals of serving the client by his organization, it is necessary to determine relative positions of sequences of the intervals A and B and to reveal the time intervals that contain intervals of both sequences A and B; it is exactly these intervals that will be time intervals of possible client servicing. To determine when the organization can serve its client visiting it at any time instant suitable for him, one should find conditions under which the intervals of the sequence A cover all the intervals of the sequence B.

3. Let we have a sequence A of time intervals in which the chairman of a board can held a meeting and sequences $B_1,\ldots,B_{10}$ of time intervals during which board members $1,\ldots,10$ can participate in this meeting. A board meeting can be hold only if the chairman and no less than any five members of the board participate in it. In order to find a sequence of time intervals during which a board meeting can be hold, it is necessary to determine relative positions of interval sequences $A, B_1,\ldots,B_{10}$ and to find the time intervals that simultaneously contain intervals of the sequence A and intervals of any five or more sequences out of 10 sequences $B_1,\ldots,B_{10}$; it is these intervals that will be the time intervals of holding a possible board meeting. In order to determine when a board meeting can be held in an arbitrary given time interval T, it is necessary

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to find the conditions under which the interval T is covered by some interval of the sequence A and any five or more intervals taken out of five or more corresponding sequences $B_1,\ldots,B_{10}$.

The problem of determining indicators of relative positions of n interval sequences is a combinatorial problem whose examples are presented above. In solving problems of this type, various exhaustive methods can be used. However, essential drawbacks of these methods (a fast growth in the complexity of computations with increasing the problem size, a nonanalytic (search-based) character of solution algorithms, and the impossibility of an observable representation of a solution algorithm in the case of high-dimensional problems) force one to search for other ways of solution. A possible approach consists of finding a reusable mathematical model that is deeply and minutely developed on the basis of some convenient analytical apparatus and applying it to the problem being solved. In the present article, it is shown that such a model can be a dynamic finite automaton [1–4] and such an apparatus can be continuous logic [1, 2, 5, 6].

1. PROBLEM STATEMENT

Let n finite sequences of nonintersecting intervals be given,

$$\begin{aligned} A_1 &= (a_{11}, b_{11}), (a_{12}, b_{12}), \ldots, (a_{1m_1}, b_{1m_1}); \\ A_2 &= (a_{21}, b_{21}), (a_{22}, b_{22}), \ldots, (a_{2m_2}, b_{2m_2}); \\ &\text{-----------------------------} \\ A_n &= (a_{n1}, b_{n1}), (a_{n2}, b_{n2}), \ldots, (a_{nm_n}, b_{nm_n}). \end{aligned} \tag{1}$$

It is required (1) to find relative positions of the available system of interval sequences (1); (2) to find conditions under which these relative positions have some qualitative character or other. A relative position of intervals is understood to be a situation determined by the intersection of some intervals and nonintersection of other intervals. The first problem is aimed at computing relative positions of any combinations (in twos, threes, etc.) of any subsequences of sequences (1) from given positions of all intervals of all sequences (1), and the second problem is aimed at finding the conditions imposed on the positions of all intervals of all subsequences (1) under which the mentioned relative positions have some required form or other. Thus, Problem 1 is the problem of analysis of the system of interval sequences (1), and Problem 2 is the problem of synthesis of such a system. We will solve problems of analysis and synthesis of a system of interval sequences of the form (1) using a mathematical model of a dynamic memoryless finite automaton that is adequate to system (1) and the mathematical apparatus of continuous logic necessary for the adequate description of the mentioned automaton. It is proposed to find the solution in the analytical form of superposition of continuous-logical operations that simultaneously gives a logical algorithm for obtaining a numerical solution.

2. DYNAMIC FINITE AUTOMATA AND CONTINUOUS LOGIC

A dynamic finite memoryless automaton (DA) [1, 2, 4] is a mathematical model in the form of an $(n,1)$-pole unit (Fig. 1) that realizes some Boolean function of its inputs $x_1,\ldots,x_n$ at its output y,

$$y = f(x_1,\ldots,x_n),\ \ x_1,\ldots,x_n,\ \ y \in \{0,1\}. \tag{2}$$

The inputs of the DA of Fig. 1 receive input binary dynamic processes

$$\begin{aligned} x_1(t) &= 1(a_{11},\, b_{11})0(-,-)1(a_{12},\, b_{12})\ldots1(a_{1m_1},\, b_{1m_1}); \\ &\text{---------------------------------} \\ x_n(t) &= 1(a_{n1},\, b_{n1})0(-,-)1(a_{n2},\, b_{n2})\ldots1(a_{nm_n},\, b_{nm_n}), \end{aligned} \tag{3}$$

in which $1(a,b)$ denote time intervals of unit values of a process (impulses) and $0(-,-)$ denote intermediate time intervals of zero values of the process (pauses). An output binary dynamic process

$$y(t) = 1(c_1, d_1)0(-,-)1(c_2, d_2)\ldots1(c_m, d_m), \tag{4}$$

is obtained at the output of the DA and corresponds to applied input processes (3) of the DA and Boolean function (2) realized by the DA. The main problem for such a memoryless DA is the problem of obtaining the output dynamic

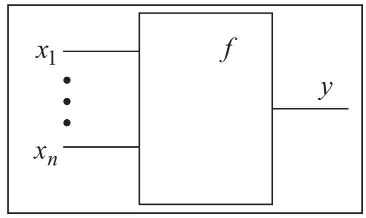

Fig. 1. Mathematical model
of a finite DA.

process $y(t)$ from its known input dynamic processes $x_1(t),\ldots,x_n(t)$ and the Boolean function f being realized. In 1971–1972, the author established that this problem can be solved in the analytical form for any memoryless DA that has any number of inputs, input processes, and a function realized with the help of the mathematical apparatus of continuous logic (CL) [1, 2, 4, 5]. Continuous logic is defined as follows. Let $C=[A,B]$ be an arbitrary segment of the real axis. Then, for any numbers $a,b,e\in C$, one can introduce the following logical operations:

$$a\vee b=\max(a,b) \text{ (disjunction)}, \tag{5}$$

$$a\wedge b=\min(a,b) \text{ (conjunction)}, \tag{6}$$

$$\bar{e}=A+B-e \text{ (negation)}. \tag{7}$$

CL operations (5)–(7) are similar to the corresponding two-valued logic operations (where the set $C=\{0,1\}$) and generalize them to the case of a continuous supporting set. In CL, the following laws of two-valued logic remain valid:

$$a\vee a=a,\ \ a\wedge a=a \text{ (tautologies)}; \tag{8}$$

$$a\vee b=b\vee a,\ \ a\wedge b=b\wedge a \text{ (the commutative law)}; \tag{9}$$

$$(a\vee b)\vee c=a\vee(b\vee c),\ \ (a\wedge b)\wedge c=a\wedge(b\wedge c) \text{ (the associative law)}; \tag{10}$$

$$a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c),\ \ a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c) \text{ (the distributive law)}; \tag{11}$$

$$\overline{a\vee b}=\bar{a}\wedge\bar{b},\ \ \overline{a\wedge b}=\bar{a}\vee\bar{b} \text{ (De Morgan's law)}; \tag{12}$$

$$a\vee(a\wedge b)=a,\ \ a\wedge(a\vee b)=a \text{ (the absorption law)}. \tag{13}$$

In addition to them, CL contains some important specific laws, for example, the following laws of estimation and simplification of logical expressions:

$$a\vee b\ge a,b;\ \ a\wedge b\le a,b, \tag{14}$$

$$a_1\vee\ldots\vee a_{i-1}\vee a_i\vee a_{i+1}\vee\ldots\vee a_m=a_1\vee\ldots\vee a_{i-1}\vee a_{i+1}\vee\ldots\vee a_m \text{ when } a_i\le a_k\ (k\ne i), \tag{15}$$

$$a_1\wedge\ldots\wedge a_{i-1}\wedge a_i\wedge a_{i+1}\wedge\ldots\wedge a_m=a_1\wedge\ldots\wedge a_{i-1}\wedge a_{i+1}\wedge\ldots\wedge a_m \text{ when } a_i\ge a_k\ (k\ne i). \tag{16}$$

The idea of determination of the output process of a memoryless DA from its given input processes and its logical function is simple. We introduce the following denotations: 1 (a binary dynamic process assuming the constant value 1), 0 (a binary dynamic process assuming the constant value 0), $1'$ (the change $0\to1$ in the value of a process), $0'$ (the change $1\to0$ in the value of a process), $1'_a$ (the change $1'$ at an instant of time a), $0'_b$ (the change $0'$ at an instant of time b), $1'_a0'_b$ (the impulse $1(a,b)$ within a time interval (a,b)), and $0'_a1'_b$ (the pause $0(a,b)$ within a time interval (a,b)). Any binary dynamic process can be written in the form of a sequence of impulses and pauses (as in expression (3)) or as a sequence of changes in process values. For example, the process presented in Fig. 2 can be written in the form

$$x(t)=1(-\infty,a)0(-,-)1(b,e)0(-,\infty) \text{ or } x(t)=0'_a1'_b0'_e.$$

The number of changes in the value of a binary process is called the process depth. For example, the depth of the process depicted in Fig. 2 is equal to 3. The corresponding concept for a system (a vector) of binary processes is its vector depth. For example, the vector depth of the system of the two processes $x_1(t)=1(a,b)$ and $x_2(t)=1(c,d)$ is equal to $(2,2)$.

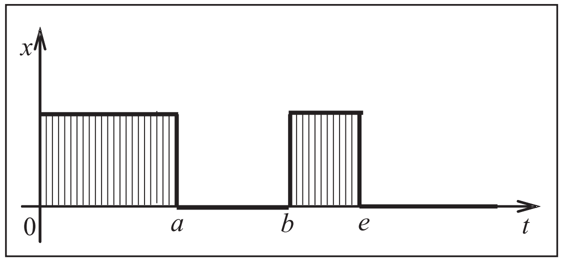

Fig. 2. Output process of a memoryless DA.

The determination of the output process of a memoryless DA from its input processes and its logical function is illustrated below by examples. We will restrict ourselves to the simplest DAs, namely, to the following two-input disjunctors and conjunctors realizing the Boolean functions of disjunction $\vee$ and conjunction $\wedge$:

$$y = x_1 \vee x_2 = \begin{cases} 0 \text{ when } x_1 = x_2 = 0, \\ 1 \text{ otherwise}; \end{cases} \qquad y = x_1 \wedge x_2 = \begin{cases} 1 \text{ when } x_1 = x_2 = 1, \\ 0 \text{ otherwise}, \end{cases} \tag{17}$$

and to simplest input processes whose depth does not exceed 1. Let the output process of a conjunctor should be find that corresponds to the input processes $x_1(t) = 1'_a$ and $x_2(t) = 0'_b$. The sought-for process is obviously equal to the single impulse $1(a, b)$ or to the null equation depending on whether b or a is larger. Therefore, interpreting the null equation as a single impulse with its superposed beginning and end, we can write the sought-for process in the form

$$y(t) = x_1(t) \wedge x_2(t) = 1'_a \wedge 0'_b = \begin{cases} 1(a, b) & \text{when } b > a; \\ 0 = 1(a, a) & \text{when } b \le a. \end{cases} \tag{18}$$

From this, by means of the CL disjunction operation $\vee$, we finally find

$$1'_a \wedge 0'_b = 1(a, a \vee b). \tag{19}$$

The output processes of a disjunctor and a conjunctor for all other input processes whose depth does not exceed 1 are similarly obtained in the form

$$0 \wedge 0'_a = 0 \wedge 1'_b = 0;\ 1 \wedge x'_a = x'_a;\ 0'_a \wedge 0'_b = 0'_{a \wedge b};\ 1'_a \wedge 1'_b = 1'_{a \vee b};\ 1'_a \wedge 0'_b = 1(a, a \vee b);$$

$$1 \vee 0'_a = 1 \vee 1'_b = 1;\ \ 0 \vee x'_a = x'_a;\ \ 0'_a \vee 0'_b = 0'_{a \vee b};\ \ 1'_a \vee 1'_b = 1'_{a \wedge b};\ \ 1'_a \vee 0'_b = 0\,(b, a \vee b). \tag{20}$$

Formulas (20) demonstrably illustrate the convenience and adequacy of the CL apparatus as a means for the determination of output processes of memoryless DAs from their input processes and logical functions being realized.

If the depth of input processes of a disjunctor or a conjunctor exceeds 1, then the finding of their output processes requires the use of formal methods. Among them, the basic methods are the direct method, decomposition method, and inversion method. The direct method is based on a complete exhaustive search for all possible cases of relative positions of the input processes of an element. For each case, the output process of an element is separately written in explicit form. A general expression for this process is obtained from individual processes with the use of CL. The decomposition method is as follows: one of two input processes of an element $x_1(t)$ or $x_2(t)$, for example, $x_1(t)$ is divided into two sequential subprocesses $x_{11}(t)$ and $x_{12}(t)$. Then the output component processes $y_1(t)$ and $y_2(t)$, i.e., the reactions of the element to the input component processes $\{x_{11}(t), x_2(t)\}$ and $\{x_{12}(t), x_2(t)\}$ are found. If $y_1(t)$ and $y_2(t)$ do not intersect in the time domain by fragments containing all changes in the process value, then the sought-for output process $y(t)$ is determined as the sequence of the processes $y_1(t)$ and $y_2(t)$. The inversion method is based on the formulas

$$\overline{x_1(t) \vee x_2(t)} = \overline{x_1(t)} \wedge \overline{x_2(t)},\ \ \overline{x_1(t) \wedge x_2(t)} = \overline{x_1(t)} \vee \overline{x_2(t)} \tag{21}$$

that follow from De Morgan's law of two-valued (Boolean) logic and allow one, based on the already known reaction of a disjunctor (a conjunctor) to the input processes $x_1(t)$ and $x_2(t)$, to easily determine the reaction of a conjunctor (a disjunctor) to the input processes $\overline{x_1(t)}$ and $\overline{x_2(t)}$. Using these methods, it is easy to obtain formulas for the output processes of a disjunctor and a conjunctor for various input processes of depth (1,2) in the form

$$0'_a \vee 1(b,c) = 0(a, a\vee b)1(-, a\vee c);\;\; 1'_a \vee 1(b,c) = 1(a\wedge b, c)0(-, a\vee c);$$

$$0'_a \vee 0(b,c) = 0(a\vee b, a\vee c);\;\; 1'_a \vee 0(b,c) = 0(a\wedge b, a\wedge c);$$

$$0'_a \wedge 1(b,c) = 1(a\wedge b, a\wedge c);\;\; 1'_a \wedge 1(b,c) = 1(a\vee b, a\vee c); \tag{22}$$

$$0'_a \wedge 0(b,c) = 0(a\wedge b, c)1(-, a\vee c);\;\; 1'_a \wedge 0(b,c) = 1(a, a\vee b)0(-, a\vee c),$$

for input processes of depth (2,2) in the form

$$1(a,b)\vee 1(c,d) = 1[a\wedge c, (a\wedge d)\vee (b\wedge c)]\,0(-,-)1(a\vee c, b\vee d);$$

$$1(a,b)\wedge 1(c,d) = 1[a\vee c, a\vee c\vee (b\wedge d)];\;\; 0(a,b)\vee 0(c,d) = 0[(a\wedge d)\vee (b\wedge c), b\wedge d];$$

$$0(a,b)\wedge 0(c,d) = 0[a\wedge c, (a\wedge d)\vee (b\wedge c)]\,1(-, a\vee c)0(-, b\vee d); \tag{23}$$

$$0(a,b)\vee 1(c,d) = 0(a\wedge c, b\wedge c)1(-, a\vee d)0(-, b\vee d);$$

$$0(a,b)\wedge 1(c,d) = 1(a\wedge c, a\wedge d)0(-, b\vee c)1(-, b\vee d),$$

etc. CL expressions for output processes of multiinput disjunctors and conjunctors realizing many-placed Boolean functions of disjunction and conjunction similar to their two-place prototypes (17) are similarly found in the form

$$0'_a \wedge 0'_b \wedge\ldots\wedge 0'_d = 0'_{a\wedge b\wedge\ldots\wedge d};\;\; 1'_a \wedge 1'_b \wedge\ldots\wedge 1'_d = 1'_{a\vee b\vee\ldots\vee d};$$

$$1'_a \wedge 1'_b \wedge\ldots\wedge 1'_d \wedge 0'_e \wedge 0'_g \wedge\ldots\wedge 0'_f = 1[a\vee b\vee\ldots\vee d; a\vee b\vee\ldots\vee d\vee (e\wedge g\wedge\ldots\wedge f)];$$

$$0'_a \vee 0'_b \vee\ldots\vee 0'_d = 0'_{a\vee b\vee\ldots\vee d};\;\; 1'_a \vee 1'_b \vee\ldots\vee 1'_d = 1'_{a\wedge b\wedge\ldots\wedge d}; \tag{24}$$

$$1'_a \vee 1'_b \vee\ldots\vee 1'_d \vee 0'_e \vee 0'_g \vee\ldots\vee 0'_f = 0[e\vee g\vee\ldots\vee f; e\vee g\vee\ldots\vee f\vee (a\wedge b\wedge\ldots\wedge d)].$$

3. THE IDEA AND A METHOD OF SOLUTION

We interpret intervals in a system of interval sequences (1) as time intervals. Then a collection of binary dynamic processes $x_i(t)$ (3) can be biuniquely associated with system (1). Namely, the ith process of this collection corresponds to the ith sequence of the system $(i = \overline{1,n})$ and, at the same time, the kth impulse of the process corresponds to the kth interval of the sequence $(k = \overline{1,m_i})$. In other words, a binary variable x_i $(i = \overline{1,n})$ is the indicator of the presence of some interval of the ith sequence of intervals (1), i.e., $x_i = 1$ indicates the presence and $x_i = 0$ indicates the absence of an interval. We apply the collection of binary dynamic processes $x_i(t)$ (3) defined above to the inputs $x_1,\ldots,x_n$ of a memoryless DA (Fig. 1) that realizes some chosen Boolean function of inputs $y = f(x_1,\ldots,x_n)$ of the form (2). Then the DA will produce some binary dynamic process $y(t)$ of the form (4) at its output y. What characterizes this process? As is well known, a Boolean function is specified by a given set of unit collections of argument values for which the function assumes the value 1. Thus, a definite chosen Boolean function f intended for the implementation in a memoryless DA (Fig. 1) forces this DA to produce a definite binary dynamic process $y(t)$ of the form (4) at its output whose impulses correspond to time intervals at which input processes $x_i(t)$ (3) of the DA assume a collection of values coinciding with one of unit collections of the function f. This means that, in applying a collection of binary processes $x_i(t)$ (3) to the inputs of the memoryless DA that biuniquely correspond to the system of interval sequences (1), the choice of some Boolean function $y = f(x_1,\ldots,x_n)$ for implementation in this DA means the choice of the corresponding partial indicator of relative positions of the system of interval sequences (1), and binary process $y(t)$ (4) realized at the output of the DA is the numerical value of this indicator. For example, if a many-placed Boolean conjunction is chosen in the capacity of the function f, then this implies (since such a function has only one unit collection $(1,1,\ldots,1)$) the choice of a partial indicator of relative positions of the system of interval sequences (1) in the form of separation of all the cases when intervals of all n sequences (1) intersect. In this case, the numerical value of this indicator is of the form of binary process $y(t)$ (4) whose impulses correspond to time intervals in which intervals of all n sequences (1) intersect.

Thus, as an adequate mathematical model for solving the problem of analysis of the system of interval sequences (1), a memoryless DA presented in Fig. 1 can be chosen. Input binary dynamic processes of this DA consist of a collection of processes (3) that biuniquely corresponds to the system of interval sequences (1), i.e., the collection of processes modelling this system. The DA realizes some chosen Boolean function $y = f(x_1,\ldots,x_n)$ that is a partial indicator of relative positions of the system of interval sequences. Then binary dynamic process $y(t)$ (4) is produced at the output of the DA and yields the numerical value of the chosen partial indicator of relative positions of intervals (more precisely, separates time intervals in which intervals of system (1) are in given relative positions). In other words, output process (4) of the DA model given in Fig. 1 models the numerical value of some partial indicator of relative positions of the system of interval sequences (1) or other that corresponds to the chosen Boolean function f realized by the DA.

According to the stated idea, the algorithm of solving the problem of analysis of a system of interval sequences (1) is as follows.

Begin

Step 1. Choose some partial indicator Π that describes relative positions of intervals of system (1) (if an indicator Π is already specified by the conditions of the problem, step 1 is omitted).

Step 2. Construct the Boolean function $y = f(x_1,\ldots,x_n)$ that corresponds to the indicator Π.

Step 3. Construct a mathematical model of the problem, i.e., the scheme (Fig. 1) of a memoryless DA that implements the function f and, at n its inputs, receives binary dynamic processes $x_i(t)$ (3) whose collection biuniquely corresponds to the given system of interval sequences (1). Output binary process $y(t)$ (4) of the DA models the indicator of relative positions of the system of intervals (1) (selects the periods during which intervals are at these relative positions).

Step 4. Using methods of the theory of DAs [1–4], determine output process $y(t)$ (4) of the DA from the input processes of the DA model $x_1(t),\ldots,x_n(t)$ and the function f realized by it. The parameters (the moments of changing values) of this process are expressed through similar parameters of the input processes of the DA model in analytical form with the help of the CL operations of disjunction $\vee$ and conjunction $\wedge$.

Step 5. Expanding the analytical expressions for the parameters of the output process $y(t)$ of the DA model that are found at step 4, obtain algorithms of computation of these parameters in terms of the CL operations $\vee$ and $\wedge$.

Step 6. Computing the parameters of the output process $y(t)$ of the DA model with the help of the algorithms found at step 5, obtain the numerical value of this process that is the numerical value of the selected partial indicator Π (or f) of relative positions of the system of interval sequences.

End

Note that, in the general case, the solution of the problem of analysis of a system of interval sequences (1) can require the use not one but several partial indicators of relative positions of intervals of system (1). In this case, we obtain a general analysis problem that is decomposed into several partial problems corresponding to the mentioned partial indicators. To solve the general analysis problem, one should solve all partial problems with the help of the described algorithm and integrate the solutions obtained.

The algorithm for solution of the problem of synthesizing a system of interval sequences (1) under given requirements imposed on relative positions of intervals is constructed using the described algorithm of solving the problem of analysis of system (1). It consists of the following steps.

Begin

Step 1. Solve the partial problem of analysis (by executing steps 1–4 of the analysis algorithm) for the system of interval sequences (1) that should be synthesized under the assumption that an indicator of relative positions of intervals is given and all the parameters of system (1) (the coordinates of the beginning and end of all intervals) are given in alphabetical form. As a result, a binary process $y(t)$ is obtained that models the given indicator of relative positions of intervals (1) (more exactly, that contains impulses-periods during which intervals are at given relative positions).

Step 2. Formulate a system of equations and inequalities expressing the given requirements on relative positions of intervals (1) in mathematical form. This system is obtained by writing required relationships (>, =) between the coordinates of the beginning and end of the corresponding impulses of the process $y(t)$. Since these coordinates are expressed through the parameters of intervals (1) with the help of CL operations, the obtained system is a system of CL equations and inequalities.

Step 3. Solve the system of CL equations and inequalities obtained at step 2 with the help of special methods [1, 2, 5, 6] based on the principle of sequential decomposition of a CL separate equation (inequality) into several simpler equations (inequalities). As a result of solution of the mentioned system of CL equations (inequalities), the conditions imposed on the parameters of individual intervals (1) are found under which relative positions of these intervals satisfy the prescribed requirements.

End

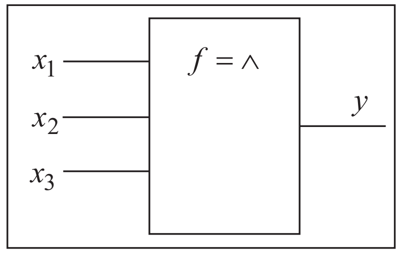

Fig. 3. A memoryless DA implementing a 3-place conjunction.

4. EXAMPLE

A shop is open daily from a_{11} to b_{11} and from a_{12} to b_{12} ($b_{11} < a_{12}$). Two employees plan to simultaneously visit it. The first of them is free and can visit it within the time interval from a_{21} to b_{21} and, similarly, the second employee can visit it within the interval from a_{31} to b_{31}. It is required to determine time periods in each of which they can realize their plan of visiting the shop and to establish the conditions under which such a visit is possible, i.e., the mentioned periods exist (are not degenerated).

The employees can realize their plan of visiting the shop in such and only such time periods during which it is open and both employees are free. Hence, to answer the first question, it is necessary to solve the problem of analysis of the system of interval sequences

$$A_1 = (a_{11}, b_{11}), (a_{12}, b_{12});\ A_2 = (a_{21}, b_{21});\ A_3 = (a_{31}, b_{31}),$$

i.e., to determine the necessary relative positions of intervals of this system. More exactly, the time periods are of interest during which relative positions are such that the intervals of all three sequences A_1, A_2, and A_3 are present. To solve the problem, we use the analysis algorithm from Sec. 3.

Begin

Step 1. The indicator Π describing necessary relative positions of intervals of the system A_1, A_2, and A_3 is already informally specified by the conditions of the problem.

Step 2. The Boolean function $y = f(x_1, x_2, x_3)$ corresponding to the indicator Π is the three-place conjunction $y = x_1 \wedge x_2 \wedge x_3$.

Step 3. The mathematical model of the problem is a memoryless DA with 3 inputs and 1 output that realizes the mentioned function f of its inputs at its output (Fig. 3). Binary processes

$$x_1(t) = 1(a_{11}, b_{11})0(-,-)1(a_{12}, b_{12}),\ x_2(t) = 1(a_{21}, b_{21}),\ x_3(t) = 1(a_{31}, b_{31})$$

biuniquely corresponding to the given system of intervals (A_1, A_2, A_3) arrive at the inputs of the DA model. The binary process $y(t)$ of the form (4) that models the indicator Π of relative positions of the system of intervals (A_1, A_2, A_3) is obtained at the output of the DA.

Step 4. Based on the input processes $x_1(t)$, $x_2(t)$, and $x_3(t)$ of the DA model and the function $f = \wedge$ being realized, find the output process $y(t)$ of the DA [1, 2, 4] using the ready-made formula $1(a,b) \wedge 1(c,d) = 1[a \vee c, a \vee c \vee (b \wedge d)]$,

$$y(t) = x_1(t) \wedge x_2(t) \wedge x_3(t) = x_1(t) \wedge [x_2(t) \wedge x_3(t)] = [1(a_{11}, b_{11})0(-,-)1(a_{12}, b_{12})]$$

$$\wedge [1(a_{21}, b_{21}) \wedge 1(a_{31,} b_{31})] = [1(a_{11}, b_{11})0(-,-)1(a_{12}, b_{12})]$$

$$\wedge 1[a_{21} \vee a_{31}, a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31})] = \{1(a_{11}, b_{11}) \wedge 1[\cdot]\}0(-,-)\{1(a_{12}, b_{12}) \wedge 1[\cdot]\}$$

$$= 1\{a_{11} \vee a_{21} \vee a_{31}, a_{11} \vee a_{21} \vee a_{31} \vee [b_{11} \wedge (a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31}))]\}0(-,-)1\{a_{12}$$

$$\vee a_{21} \vee a_{31}, a_{12} \vee a_{21} \vee a_{31} \vee [b_{12} \wedge (a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31}))]\}.$$

Step 5. Analytical expressions of parameters A, B, C, and D of the output process $y(t) = 1(A,B)0(-,-)1(C,D)$ of the DA model yield algorithms of computation of this process in terms of the CL operations $\vee$ (max) and $\wedge$ (min). For example, we have $A = \max(a_{11}, a_{21}, a_{31})$, $B = \max(A, \min(b_{11}, \max(a_{21}, a_{31}, \min(b_{21}, b_{31}))))$, etc.

Step 6. Using the algorithms found at step 5, determine the parameters of the output process $y(t)$ of the DA model that correspond to concrete values of parameters a_{ij} and b_{ij} of the input processes $x_1(t), x_2(t)$, and $x_3(t)$. For example, when $a_{11} = 9$, $a_{12} = 14$, $b_{11} = 13$, $b_{12} = 20$, $a_{21} = 12$, $b_{21} = 16$, $a_{31} = 11$, and $b_{31} = 15$, we obtain $A = 12$, $B = 13$, $C = 14$, and $D = 15$. Thus, there are two time periods during which both employees can simultaneously visit the shop, namely, $(12,13)$ and $(14,15)$.

End

To establish common conditions under which both employees can simultaneously visit the shop, it is necessary to determine when there are time periods during which they can simultaneously visit the shop. Thus, to answer the second question, it is necessary to solve the problem of synthesis of the system of interval sequences (A_1, A_2, A_3), i.e., to find the conditions under which the relative position of intervals of this system has the necessary qualitative character owing to which the existence of the mentioned time periods is provided. To solve the problem, we will use the synthesis algorithm from Sec. 3.

Begin

Step 1. This step is already made and is contained in steps 1–4 of the algorithm of analysis of the system (A_1, A_2, A_3) that are performed above.

Step 2. Obtain the system of CL equations and inequalities expressing requirements on relative positions of intervals of the system (A_1, A_2, A_3) that provide necessary time periods after requiring that the output process $y(t) = 1(A,B)0(-,-)1(C,D)$ of the DA model that is found during the analysis of the system (A_1, A_2, A_3) and models the prescribed indicator of relative positions of intervals have nondegenerate impulses modelling required time periods $B > A$ or $D > C$. Substituting the expressions for A, B, C, and D from the expanded expression of the process $y(t)$ presented above, obtain

$$a_{11} \vee a_{21} \vee a_{31} \vee [b_{11} \wedge (a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31}))] > a_{11} \vee a_{21} \vee a_{31}$$

or

$$a_{12} \vee a_{21} \vee a_{31} \vee [b_{12} \wedge (a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31}))] > a_{12} \vee a_{21} \vee a_{31}.$$

Step 3. Simplifying the CL inequalities obtained at step 2, obtain

$$b_{11} \wedge [a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31})] > a_{11} \vee a_{21} \vee a_{31}$$

or

$$b_{12} \wedge [a_{21} \vee a_{31} \vee (b_{21} \wedge b_{31})] > a_{12} \vee a_{21} \vee a_{31}.$$

End

Relative positions of intervals of the system (A_1, A_2, A_3) that correspond to at least one of the written inequalities provide the presence of time periods during which the employees can simultaneously visit the shop. We again pay attention to the fact that both sides of the written inequalities (conditions) represent the expressions constructed from the parameters of intervals of the system (A_1, A_2, A_3) with the help of the CL operations of disjunction $\vee$ and conjunction $\wedge$.

CONCLUSIONS

In this paper, it is shown that the investigation of the class of combinatorial problems equivalent to the combinatorial problem of determination of relative positions of *n* interval sequences can be performed with the help of a mathematical model of a dynamic finite memoryless automaton and the mathematical apparatus of continuous logic. This approach allows one to formally find algorithms for solution of the mentioned problems and also to formally analyze these solutions, for example, to find necessary and sufficient conditions of their existence. Another advantage of the proposed approach is its applicability to the solution of problems of arbitrarily high dimensions.

REFERENCES

1. V. I. Levin, Introduction to the Dynamic Theory of Finite Automata [in Russian], Zinatne, Riga (1975).
2. D. Bochmann, V. N. Roginskij, und V. I. Levin, Dinamische Processe in Automaten, Technik, Berlin (1977).
3. V. I. Levin, Dynamics of Logical Devices and Systems [in Russian], Energiya, Moscow (1980).
4. V. I. Levin, Theory of Dynamic Automata [in Russian], Izd. PGTU, Penza (1995).
5. V. I. Levin, Infinite-Valued Logic in Problems of Cybernetics [in Russian], Radio i Svyaz ’, Moscow (1982).
6. V. I. Levin, Structural-Logical Methods of Investigation of Complex Systems [in Russian], Nauka, Moscow (1987).