STOAT

SKEIN LASAGNA MODULES AND HANDLE ATTACHMENTS

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ROUGH OUTLINE

Hello! I'm Ayo and today I'll be telling you about handle attachments in the study of skein lasagna modules

The lovely talks these past few days have given a great introduction to skein lasagna modules, so we just need to start by introducing handles

A 4-D k-handle is a copy of $D^k \times D^{4-k}$, attached to a 4-mfld X^4 via an embedding $S^{k-1} \times D^{4-k} \hookrightarrow \partial X$

TODAY, we'll discuss how

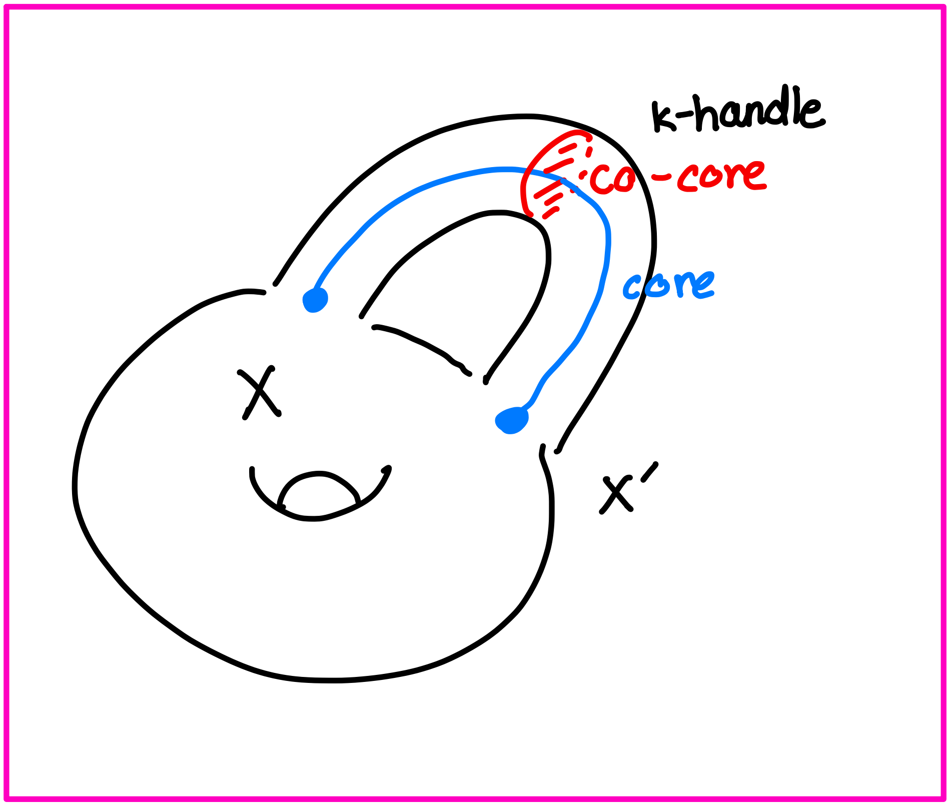
(I) $S_0^{\bullet}(X; \emptyset) \cong S_0^{\bullet}(X \cup \text{4-handle}; \emptyset)$

(II) $S_0^{\bullet}(X; \emptyset) \twoheadrightarrow S_0^{\bullet}(X \cup \text{3-handle}; \emptyset)$

(III) $S_0^{\bullet}(B^4 \cup_L \text{2-handle}; \emptyset) \cong$ a gadget we'll define called the cabled Khovanov-Rozansky homology of L $KhR_{\bullet}(L)$ which is a certain quotient of an infinite direct sum of Khovanov-Rozansky homologies of cables of L. Again, we'll formally define that later.

Oh, and this is All due to Manolescu-Neithalath '22

Now, consider a 4-mfld $X' = X \cup k\text{-handle}$. Let's draw a dimensionally reduced picture:



Here, we'll mark out the core and co-core of the handle. The core then has $\boxed{\text{dim. } k}$, while the co-core has $\boxed{\text{dim. } 4-k}$.

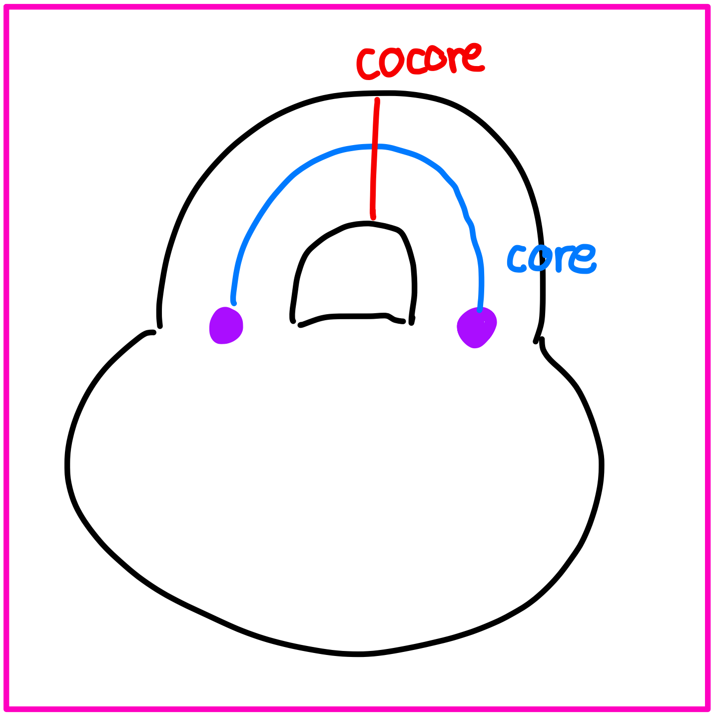
Observe that the inclusion $i: X \hookrightarrow X'$ induces $\rightsquigarrow$ a mapping $i_*: S_0^{\bullet}(X;\emptyset) \to S_0^{\bullet}(X';\emptyset)$ given by simply considering a lasagna filling in X as a lasagna filling in X'.

Now, taking $k \in \{3,4\}$, we have $\Rightarrow$ $2 + \dim \text{cocore}$
$= 2 + (4-k) < 4$, so
by $\Rightarrow_{\pitchfork}$ any $\Sigma^2 \subset X'$ is isotopic to one in $X' \setminus \text{cocore}$, which
we may observe deformation retracts onto $\cong X$.
$\Rightarrow$ any lasagna filling in X' can be isotoped to one in X
$\Rightarrow i_*$ is surjective
$\Rightarrow$ (II) is proved $\square$ $\boxed{\checkmark}$

Now, we can Consider an isotopy F between
$\Sigma^2, \tilde{\Sigma}^2 \subset X'$ as a 3-submfld. of $X' \times I$.
As just noted before, we may isotope Σ and $\tilde{\Sigma}$ to only lie in X.
Then, taking $k=4$, we have $\Rightarrow$ $3 + (4-k+1) < 4+1$,
so by $\Rightarrow_{\pitchfork}$ F is isotopic to a 3-submfld. of
$(X' \setminus \text{cocore}) \times I$; and we can arrange that it corr. to an
isotopy between $\Sigma, \tilde{\Sigma}$.
$\Rightarrow$ any surface isotopy in X' can be done in $X' \setminus \text{cocore} \cong X$.
$\Rightarrow$ any isotopy of lasagna fillings in X' can be done in X
$\Rightarrow i_*$ is injective (and thus an iso.)
$\Rightarrow$ (I) is proved $\square$ $\boxed{\checkmark}$

So, we Now just have (III) left to show. To this end, take $k = 2$, $X = B^4$.

Let's draw a dimensionally reduced picture:

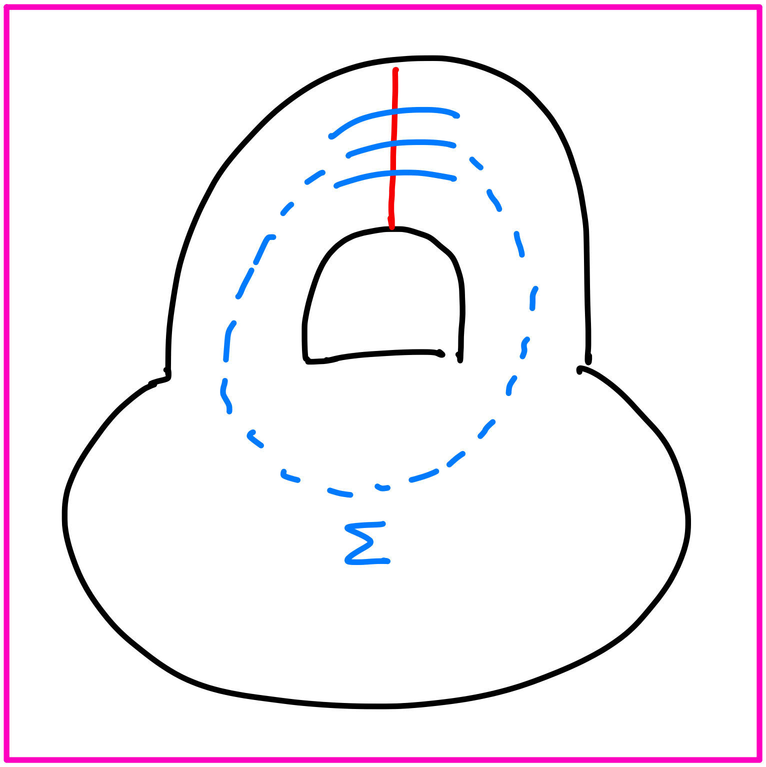


∂(core) is then = some knot K $\boxed{K}$ in S^3, and the pushoff of K into the core has some linking # we call n w/ K.

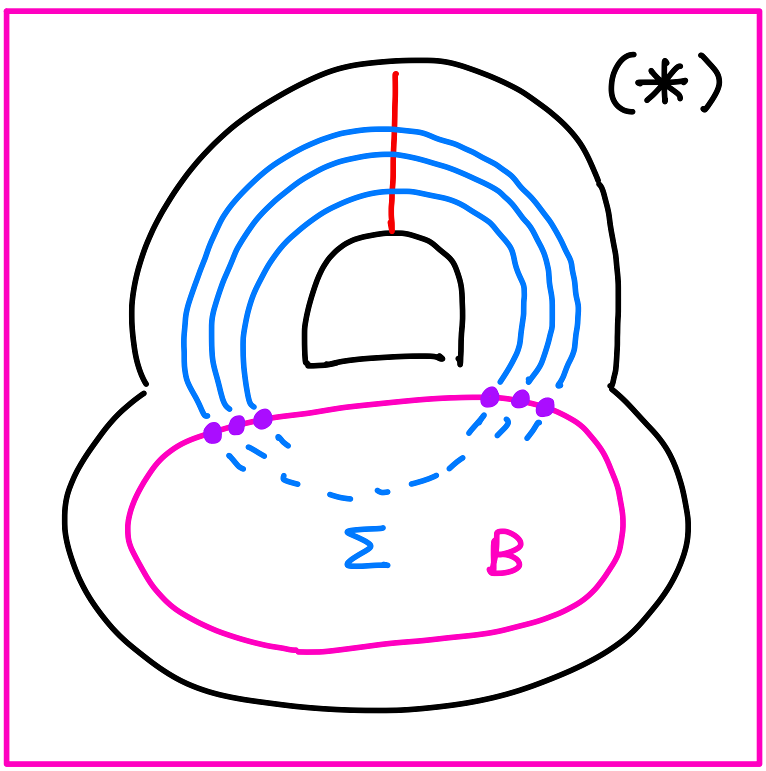
X' is then called =: the n-trace of K, denoted by $X_n(K)$.

Now, let's consider a surface $\Sigma \subset X_n(K)$.

We can arrange $\Sigma \pitchfork$ cocore of the 2-handle. Near the cocore, Σ will then look like some number of parallel copies of the core; let's draw what that looks like



We can then continue to isotope Σ until it looks like parallel sheets outside a $B^4 =: B$



$\Sigma \cap \partial B$ will then be $=$ some # k_- of $-$ori. copies of K

$\sqcup$ some # k_+ of $+$ori. copies of K

We call this link $=: K(k_-, k_+)$. As an unoriented link, this is an example of a cable of K.

Considering any lasagna filling $F := (\Sigma, \{(B_i, L_i, v_i)\})$ of X w/ surface $\Sigma \pitchfork$ cocore, input balls B_i, links L_i, and Khovanov-Rozansky labelings v_i,

we can then $\longrightarrow$ isotope B_i to lie in B and Σ to look like ($*$), looking like parallel copies of the core in the 2-handle.

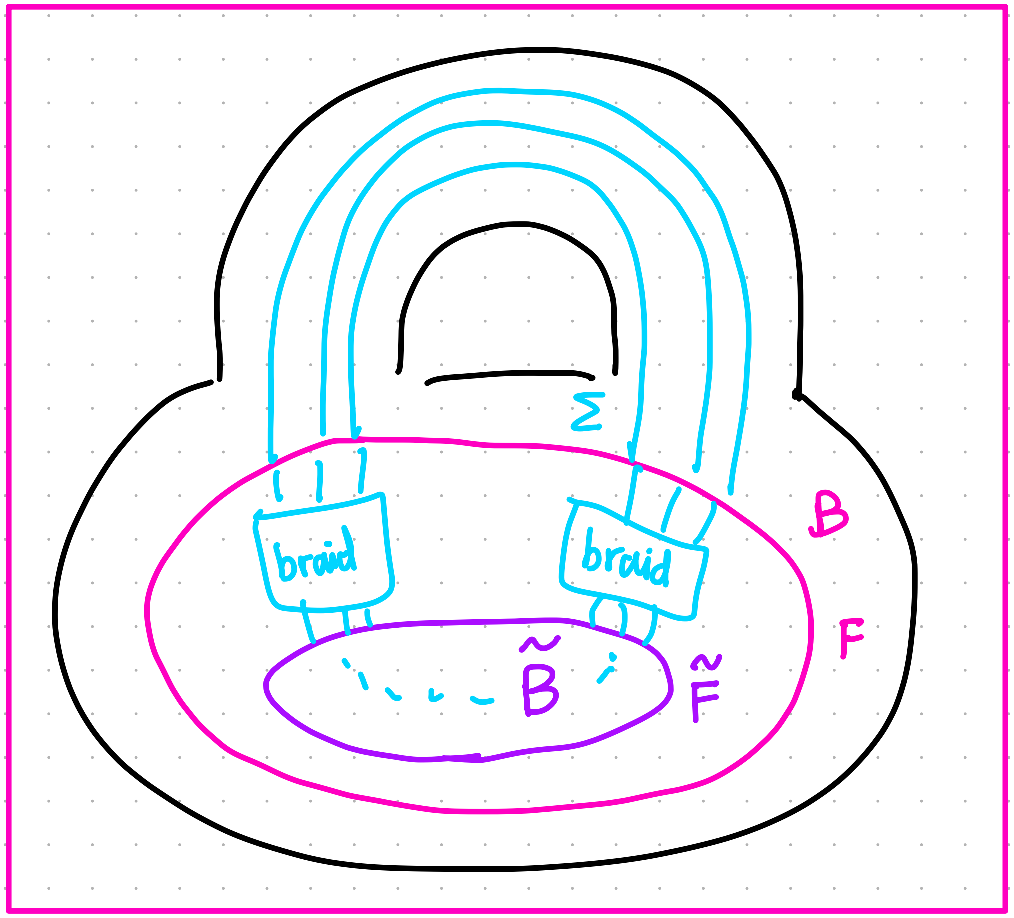
Σ gives rise to $\longrightarrow$ a cob. from $\cup L_i$ to $K(k_-, k_+)$, and considering the induced action of this cobordism on the Khovanov-Rozansky labelings v_i then gives

$\longrightarrow$ an elt. we'll call $\varphi(F)$ of $KhR_\bullet(K(k_-, k_+))$

However, this process is not yet well-defined on $S_0^\bullet(X_n(K); \emptyset)$
- for example, we could perform an operation we call

(a) — we Could braid together sheets of Σ to change $\varphi(F)$ by the induced cob. map on $KhR_\bullet(K(k_-, k_+))$

Let's draw what this could look like...

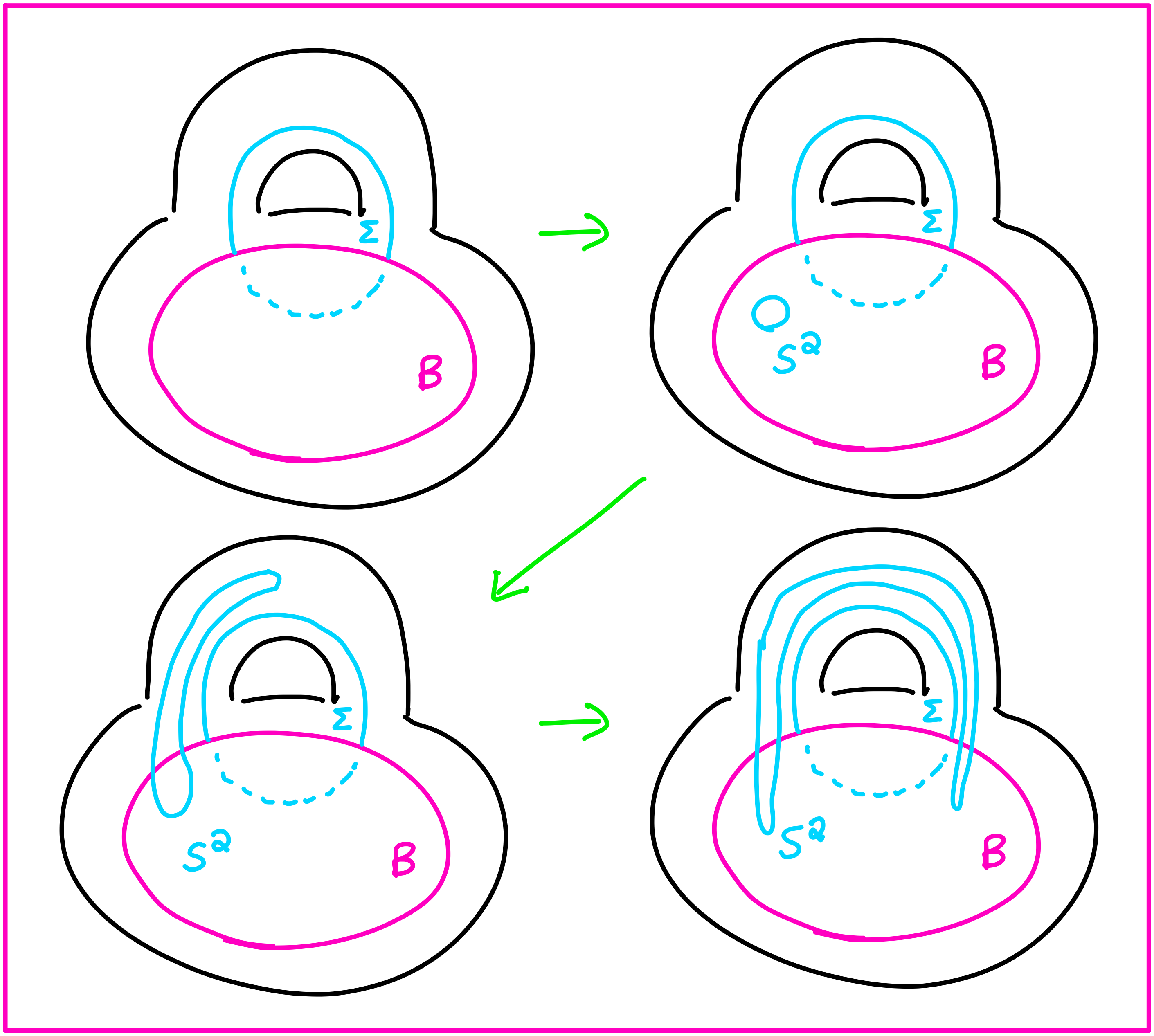


Then, $\varphi(\tilde{F}) = (\text{braid cob.})_* \varphi(F)$, so these two equivalent lasagna fillings can, under this map φ, give us different Khovanov-Rozansky labelings.

We'll denote by

(b) another way to produce such fillings showing φ is not yet well-defined on the skein lasagna module. Specifically, we can Add a pair of opp. ori. sheets to Σ to change $\varphi(F)$ by a map $KhR_\bullet(K(k_-, k_+)) \to KhR_\bullet(K(k_- + 1, k_+ + 1))$

Informally, we can do this by introducing a disjoint 2-sphere to Σ in the input ball, then stretching it through the 2-handle to create 2 new parallel sheets. Let's draw this procedure:



Of course, we could also take this 2-sphere to be dotted; this would add a dot to one of the two sheets and change the way this operation alters the Khovanov-Rozansky labeling we end up with.

So, we've described operations showing that our map φ we defined on lasagna fillings (whose surfaces are arranged to be transverse to the cocore) is not well-defined on equivalence classes of lasagna fillings. However, note that neither of these two operations changed the difference $k_+ - k_-$ between the numbers of positively and negatively oriented sheets: (a) keeps both the same, while (b) adds one of each.

There is a homological reason for this. Specifically,

$k_+ - k_- = \begin{smallmatrix}+\text{ sheets}\\ \text{of } \Sigma\end{smallmatrix} \pitchfork \text{core} - \begin{smallmatrix}-\text{ sheets}\\ \text{of } \Sigma\end{smallmatrix} \pitchfork \text{core} = [\Sigma]$

$\in H_2(X_n(K); \partial)$ and $[\Sigma]$ is the same for any elt. of $[F] \in S_0^\bullet(X_n(K); \emptyset)$, as all surfaces of elements of this equivalence class are, by definition, homologous rel some collection of balls.

Then, writing $\alpha := k_+ - k_- = [\Sigma]$,
any KhR labeling resulting from this process φ applied to $(\tilde{F} \in [F])$ lies in $\bigoplus_{r \in \mathbb{N}} KhR_\bullet(K(r - \min(\alpha, 0), r + \max(\alpha, 0))$

Let's write $\boxed{\overset{!!}{\alpha_-}}$ and $\boxed{\overset{!!}{\alpha_+}}$ for the min and max of α and 0 respectively.

Now, Pick $v = \varphi(\tilde{F} \in [F])$, so v is a Khovanov-Rozansky labeling we can get from applying this procedure φ to some lasagna filling equivalent to F. There then $\exists\ \tilde{k}_\pm$ s.t. $v \in KhR_\bullet(K(\tilde{k}_-, \tilde{k}_+))$, and by construction, we then have that the lasagna filling F_v w/ input ball B slightly smaller than the 0-handle, surface $\Sigma_{\tilde{k}_\pm}$ $\tilde{k}_\pm$ $\pm$-ori. cores outside B, and KhR labeling v will then satisfy $\Rightarrow [F_v] = [F] \in S_0^\bullet(X_n(K); \emptyset, \alpha)$.

$\Rightarrow$ can recover $[F]$ from any $v \in \varphi([F])$

$\Rightarrow$ the subsets $\varphi([F])$ are disjoint

$\Rightarrow$ they define an equiv. rel. $\sim_\varphi$ on $\bigoplus_{r \in \mathbb{N}} KhR_\bullet(K(r-\alpha^-, r+\alpha^+))$

Therefore, Defining the (cabled KhR homology)

$$\underline{KhR}_{\bullet,\alpha}(K) := \bigoplus_{r\in\mathbb{N}} KhR_\bullet(K(r-\alpha^-, r+\alpha^+))\{(1-\bullet)(2r+|\alpha|)\} /{\sim_\varphi}$$

with a grading shift we won't think too hard about now,

we get an isomorphism $S_0^\bullet(X_n(K); \emptyset, \alpha) \cong \underline{KhR}_{\bullet,\alpha}(K)$,

exactly proving our last point (III) $\square$.

Now, note that we're not quite done if we want to use this in practice because we haven't described this equivalence relation $\sim_\varphi$ very explicitly.

Mandolescu and Neithalath do this formally in their paper.

In specific, recall the operations (a) and (b) we discussed which braid the sheets of a lasagna filling and add oppositely-oriented sheets. Manolescu and Neithalath discussed how the relations these induce on the infinite direct sum of Khovanov-Rozansky homologies exactly generate the desired equivalence relation whose associated quotient gives the cabled Khovanov-Rozansky homology. They also describe these induced relations very explicitly, making everything nicely computable!

But we don't have time for all that - that's all I have for today, so thanks for listening!