

STOAT

SKEIN
LASAGNA
MODULES
AND
HANDLE
ATTACHMENTS

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LINDBLAD

ROUGH OUTLINE

Hello! I'm Ayo and today I'll be telling you about handle attachments in the study of skein lasagna modules

The lovely talks these past few days have given a great introduction to skein lasagna modules, so we just need to start by introducing handles

A 4-D k-handle is a copy of $D^k \times D^{4-k}$, attached to a 4-mfld X^4 via an embedding $S^{k-1} \times D^{4-k} \hookrightarrow \partial X$

TODAY, we'll discuss how

$$(I) S^{\bullet}_o(X; \emptyset) \cong S^{\bullet}_o(X \cup \text{4-handle}; \emptyset)$$

$$(II) S^{\bullet}_o(X; \emptyset) \rightarrow S^{\bullet}_o(X \cup \text{3-handle}; \emptyset)$$

$$(III) S^{\bullet}_o(B^4 \cup_L 2\text{-handle}; \emptyset) \cong \text{a gadget we'll define called the}$$

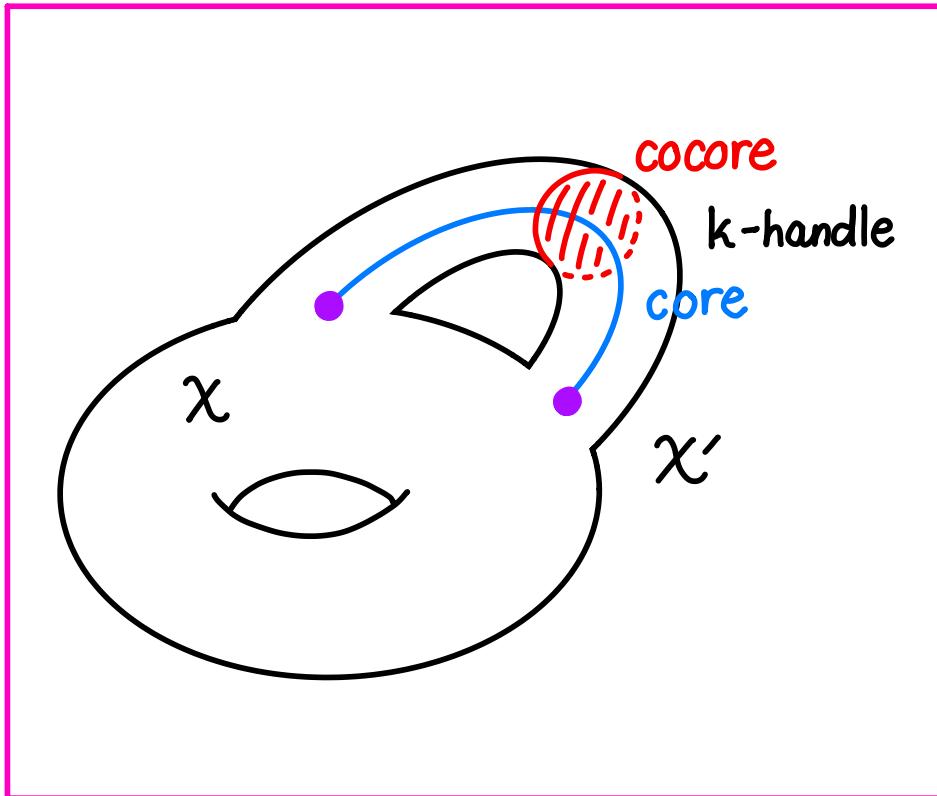
cabled Khovanov-Rozansky homology of L $\text{KhR}_o(L)$

which is a certain quotient of an infinite direct sum of Khovanov-Rozansky homologies of cables of L .

Again, we'll formally define that later.

Oh, and this is All due to Manolescu-Neithalath '22

Now, consider a 4-mfld $X' = X \cup k\text{-handle}$. Let's draw a dimensionally reduced picture: 2



Here, we'll mark out the core and co-core of the handle. The core then has $\boxed{\dim. k}$, while the co-core has $\boxed{\dim. 4-k}$.

Observe that the inclusion $i: X \hookrightarrow X'$ induces \rightsquigarrow a mapping $i_*: S^{\circ}_0(X; \emptyset) \rightarrow S^{\circ}_0(X'; \emptyset)$ given by simply considering a lasagna filling in X as a lasagna filling in X' .

Now, taking $k \in \{3, 4\}$, we have $\Rightarrow 2 + \dim \text{cocore}$

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$$= 2 + (4 - k) < 4, \text{ so}$$

by \Rightarrow_{P} any $\Sigma^2 \subset X'$ is isotopic to one in $X' \setminus \text{cocore}$, which we may observe deformation retracts onto $\cong X$.

\Rightarrow any lasagna filling in X' can be isotoped to one in X

$\Rightarrow i_*$ is surj.



\Rightarrow (II) is proved \square

Now, we can Consider an isotopy F between

$\Sigma^2, \tilde{\Sigma}^2 \subset X'$ as a 3-submfd. of $X' \times I$.

As just noted before, we may isotope Σ and $\tilde{\Sigma}$ to only lie in X .

Then, taking $k = 4$, we have $\Rightarrow 3 + (4 - k + 1) < 4 + 1$,

so by \Rightarrow_{P} F is isotopic to a 3-submfd. of $(X' \setminus \text{cocore}) \times I$; and we can arrange that it corr. to an isotopy between $\Sigma, \tilde{\Sigma}$.

\Rightarrow any surface isotopy in X' can be done in $X' \setminus \text{cocore} \cong X$.

\Rightarrow any isotopy of lasagna fillings in X' can be done in X

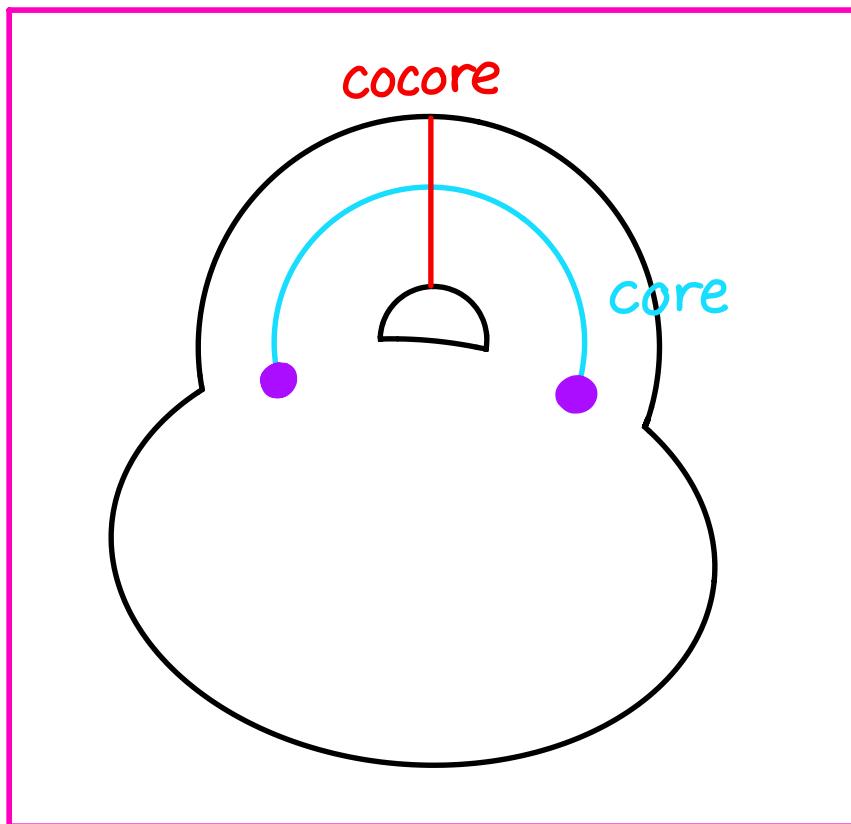
$\Rightarrow i_*$ is inj. (and thus an iso.)

\Rightarrow (I) is proved \square



So, we Now just have (III) left to show. To this end, take $k=2$, $X = B^4$.

Let's draw a dimensionally reduced picture:

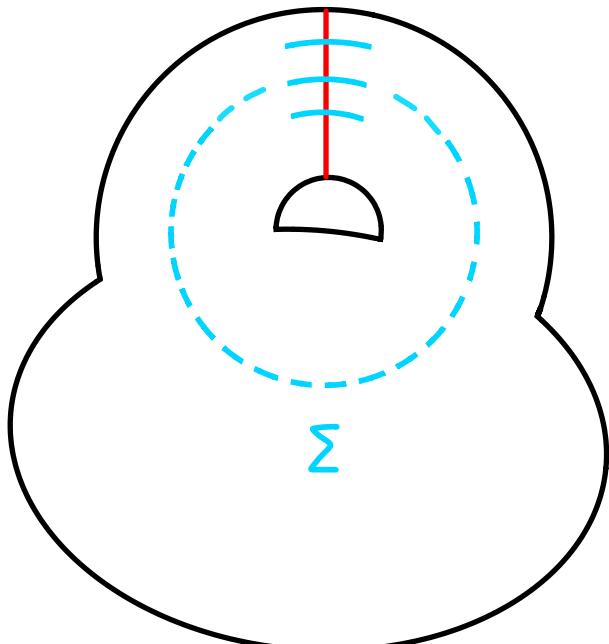


$\partial(\text{core})$ is then = some knot K K in S^3 , and the pushoff of K into the core has some linking # we call n w/ K .

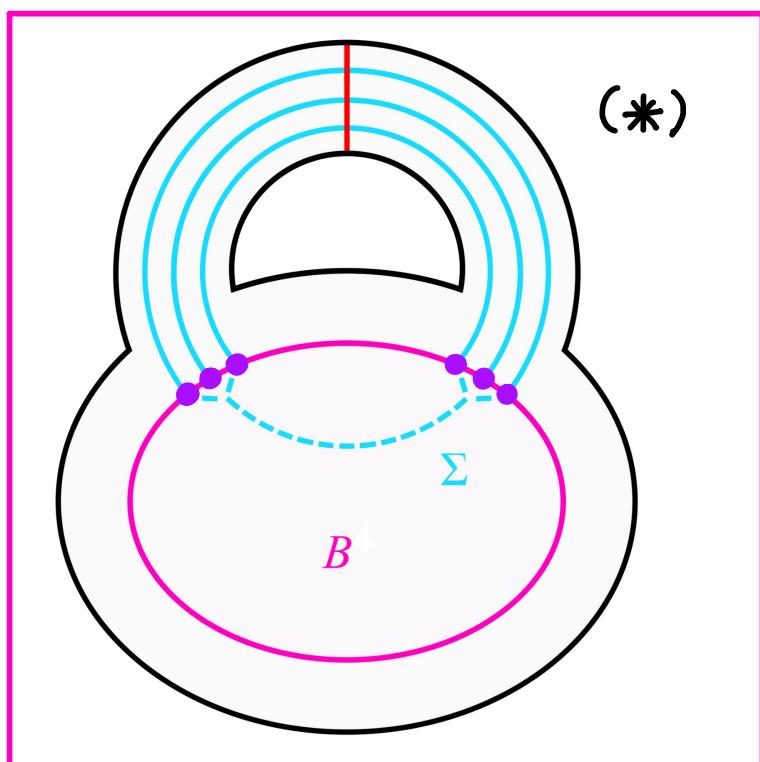
X' is then called := the n -trace of K , denoted by $X_n(K)$.

Now, let's consider a surface $\Sigma \subset X_n(K)$.

We can arrange Σ in cocore of the 2-handle. Near the cocore, Σ will then look like some number of parallel copies of the core ; let's draw what that looks like



We can then continue to isotope Σ until it looks like parallel sheets outside a $B^4 =: B$



$\Sigma \cap \partial B$ will then be = some # k_- of -ori. copies of K 6
 + some # k_+ of +ori. copies of K

We call this link $=: K(k_-, k_+)$. As an unoriented link, this is an example of a cable of K .

Considering any lasagna filling $F := (\Sigma, \{\mathcal{E}(B_i, L_i, v_i)\})$ of X w/ surface $\Sigma \pitchfork$ core, input balls B_i , links L_i , and Khovanov-Rozansky labelings v_i ,

we can then \rightarrow isotope B_i to lie in B and Σ to look like (*), looking like parallel copies of the core in the 2-handle.

Σ gives rise to \rightarrow a cob. from $\cup L_i$ to $K(k_-, k_+)$, and considering the induced action of this cobordism on the Khovanov-Rozansky labelings v_i then gives

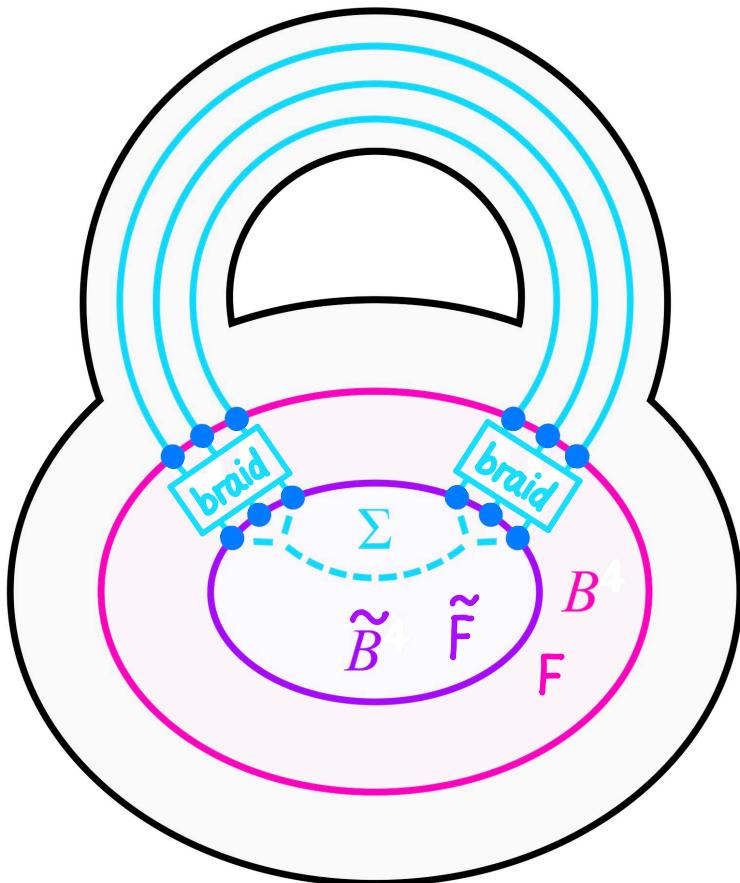
\rightarrow an elt. we'll call $\Phi(F)$ of $\text{KhR.}(K(k_-, k_+))$

Now, we're on our way to getting an isomorphism between the skein lasagna module and some object built from Khovanov-Rozansky homologies of cables of K .

However, this process is Not yet well-defined on $S^{\bullet}_0(X_n(K); \emptyset)$ - for example, we could perform an operation we call

(a) — we Could braid together sheets of Σ to change $\Phi(F)$ by the induced cob. map on $\text{KhR.}(K(k_-, k_+))$

Let's draw what this could look like...



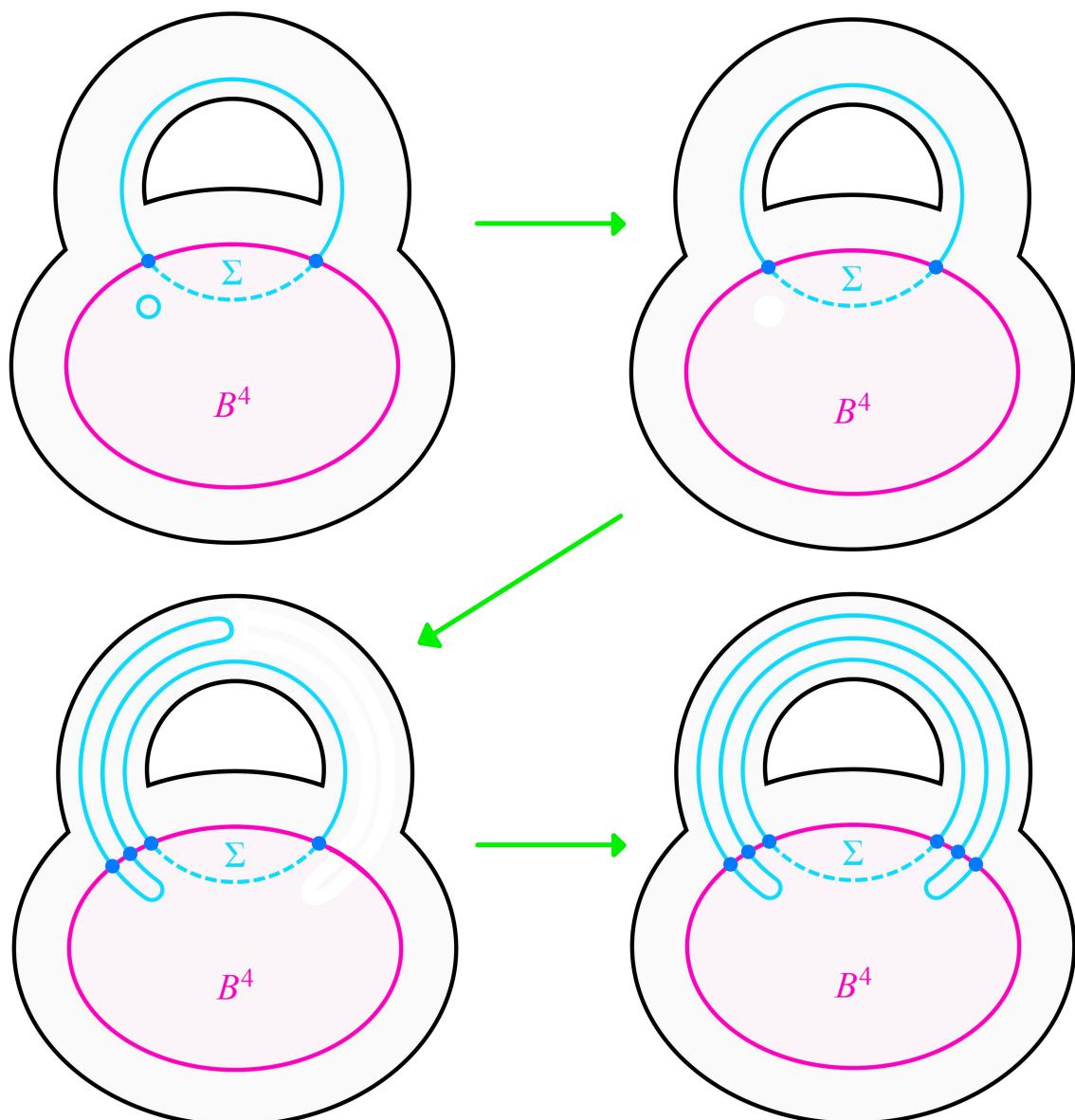
Then, $\varphi(\tilde{F}) = (\text{braid cob.}) * \varphi(F)$, so these two equivalent lasagna fillings can, under this map φ , give us different Khovanov-Rozansky labelings.

We'll denote by

(b) another way to produce such fillings showing φ is not yet well-defined on the skein lasagna module. Specifically, we can

Add a pair of opp. ori. sheets to Σ to change $\varphi(F)$ by a map $\text{KhR.}(K(k_-, k_+)) \rightarrow \text{KhR.}(K(k_- + 1, k_+ + 1))$

Informally, we can do this by introducing a disjoint 2-sphere to Σ in the input ball, then stretching it through the 2-handle to create 2 new parallel sheets. Let's draw this procedure:



Of course, we could also take this \mathbb{Q} -sphere to be dotted; this would add a dot to one of the two sheets and change the way this operation alters the Khovanov-Rozansky labeling we end up with

So, we've described operations showing that our map Ψ we defined on lasagna fillings (whose surfaces are arranged to be transverse to the cocore)

is not well-defined on equivalence classes of lasagna fillings. However, note that neither of these two operations changed the difference $k_+ - k_-$ between the numbers of positively and negatively oriented sheets : (a) keeps both the same, while (b) adds one of each.

There is a homological reason for this. Specifically,

$$k_+ - k_- = \text{sheets of } \Sigma \cap \text{cores} - \text{-sheets of } \Sigma \cap \text{cores} = [\Sigma]$$

$\in H_2(X_n(K); \mathbb{Z})$ and $[\Sigma]$ is the same for any elt. of $[F] \in S_o^*(X_n(K); \emptyset)$, as all surfaces of elements of this equivalence class are, by definition, homologous rel some collection of balls.

Then, writing $\alpha := k_+ - k_- = [\Sigma]$,

any KhR labeling resulting from this process Ψ applied to $(\tilde{F} \in [F])$ lies in $\bigoplus_{r \in \mathbb{N}} \text{KhR}_o(K(r - \min(\alpha, 0), r + \max(\alpha, 0)))$

Let's write $\ddot{\alpha}^-$ and $\ddot{\alpha}^+$ for the min and max of α and 0 respectively.

Now, Pick $v = \Psi(\tilde{F} \in [F])$, so v is a Khovanov-Rozansky labeling we can get from applying this procedure Ψ to some lasagna filling equivalent to F . Then there exist \tilde{k}_{\pm} s.t. $v \in \text{KhR}_o(K(\tilde{k}_-, \tilde{k}_+))$, and by construction, we then have that the lasagna filling F_v w/ input ball B slightly smaller than the 0-handle, surface $\sum \tilde{k}_{\pm} \tilde{k}_{\pm} \pm\text{-ori. cores outside } B$, and KhR labeling v will then satisfy $\Rightarrow [F_v] = [F] \in S_o^*(X_n(K); \emptyset, \alpha)$.

\Rightarrow can recover $[F]$ from any $v \in \Psi([F])$

\Rightarrow the subsets $\Psi([F])$ are disjoint

\Rightarrow they define an equiv. rel. \sim_φ on $\bigoplus_{r \in \mathbb{N}} \text{KhR}_*(K(r-\alpha^-, r+\alpha^+))$

Therefore, Defining the Cabled KhR homology \triangleright

$$\underline{\text{KhR}}_{*,\alpha}(K) := \bigoplus_{r \in \mathbb{N}} \text{KhR}_*(K(r-\alpha^-, r+\alpha^+)) \{ (1 - \bullet)(2r + 1d1) \}_{/\sim_\varphi}$$

with a grading shift we won't think too hard about now,
 we get an isomorphism $S_0^*(X_n(K); \emptyset, \alpha) \cong \underline{\text{KhR}}_{*,\alpha}(K)$,
 exactly proving our last point (III) \square .

Now, note that we're not quite done if we want to use this in practice because we haven't described this equivalence relation \sim_φ very explicitly.

Manolescu and Neithalath do this formally in their paper

In specific, recall the operations (a)

and (b) we discussed which braid the sheets of a lasagna filling and add oppositely-oriented sheets. Manolescu and Neithalath discussed how the relations these induce on the infinite direct sum of Khovanov-Rozansky homologies exactly generate the desired equivalence relation whose associated quotient gives the cabled Khovanov-Rozansky homology. They also describe these induced relations very explicitly, making everything nicely computable!

But we don't have time for all that - that's all I have for today, so thanks for listening!

