
Ayo
TALK

KEY:

Only spoken

Only written

Spoken + written

Instruction

Draw this, describe while drawing

Draw this, don't describe while drawing

LENGTH: 1HR

HELLO! I'm Ayo and today I'll be telling you about Boundary Dehn twists after a process called abelianization. Specifically, we'll be focusing on these in dimension 4, where there's been a great increase of activity in the study of diffeomorphism groups in the past decade. Let's start by introducing our notation for these groups:

Consider a SM oriented mfld. X.

We denote by $\mathrm{Diff}(X) := \{\text{ori.-pres. diffeo. } X \circlearrowleft\}$.

This is a group with group operation given by composition, so we call it the diffeomorphism group of X.

Now, when X has boundary, we denote by

$\mathrm{Diff}(X,\partial) := \{f \in \mathrm{Diff}(X) \mid f|_{\partial X} = id_{\partial X}\}$;

we call this the diffeomorphism group of X relative to the boundary, or rel ∂.

Now, we want a notion of equivalence on these groups because, as is, they're just way too big! Isotopy gives us a nice one. We say $f, g \in \mathrm{Diff}(X,\partial)$ are SM/TOP isotopic rel ∂ if $\exists$ SM/TOP map $h : [0,1] \to \mathrm{Diff}(X,\partial)$ w/ $h(0) = f$, $h(1) = g$

So, by definition, diffeomorphisms f and g of X are topologically isotopic exactly when they lie in the same path component of the group of orientation-preserving homeomorphisms rel ∂ of X,

and smoothly
isotopic exactly when they lie in the same path component of the diffeomorphism group rel ∂ of X. Writing this out, we have

$\pi_0(\mathrm{Diff}(X, \partial)) = \{$SM isotopy classes of $\mathrm{Diff}(X,\partial)\}$,

and we call this the $=:$ SM mapping class group rel ∂ of X.

Analogously, the topological mapping class group rel ∂ is the group of topological isotopy classes of orientation-preserving homeomorphisms rel ∂ of X.

Now, let's try to cook up a diffeomorphism which isn't trivial in this mapping class group of some manifold rel boundary.

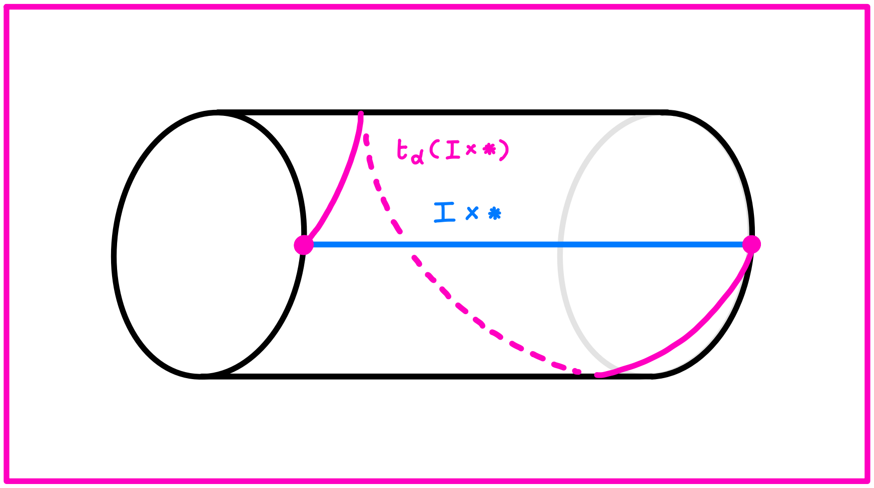
Let's start by working in the super simple setting when $X = (I := [0,1]) \times S^{n-1}$. Now, what's a super natural way to make a diffeomorphism of $I \times S^{n-1}$? Well, we can consider a diffeomorphism $t_\alpha : I \times S^{n-1} \circlearrowleft, (s,z) \mapsto (s, \alpha(s)z)$ for $\alpha \in \pi_1(SO(n))$, so α is a loop of rotations based at the identity. Of course, if α is isotopic to the constant loop, t_α will be isotopic to the identity, meaning that

$\alpha = 1 \in \pi_1(SO(n)) \Rightarrow t_\alpha = 1 \in \pi_0(\mathrm{Diff}(I \times S^{n-1}, \partial))$;

in fact, it may be intuitive that the converse is also true $\boxed{\Leftrightarrow}$;

to visualize this, let's take $n=2$, and $\alpha : s \mapsto e^{2\pi i s}$ to be the generator of $\in \pi_1(SO(2)) \cong \mathbb{Z}$.

Let's draw the resulting diffeomorphism t_α:



The intuition is then that any isotopy of this diffeomorphism can't unwind this twist you see here, so this diffeomorphism t_α is non-trivial in the smooth mapping class group rel ∂ of $I \times S^{n-1}$. In fact, this obstruction is a topological one, so this diffeomorphism isn't even trivial in the topological mapping class group rel ∂.

t_α is then called the $=:$ Dehn twist t_2 on $I \times S^1$.

More generally, for $n > 2$, and α gen. of $\pi_1(SO(n)) \cong \mathbb{Z}/2$

$\rightsquigarrow$ t_α is called the $=:$ Dehn twist t_n on $I \times S^{n-1}$.

So, great! We have some nontrivial elements of smooth mapping class groups rel ∂, ones which aren't even topologically isotopic to the identity. Let's use these to build more natural diffeomorphisms. At this point, we'll restrict ourselves to dimension 4, our main focus in this talk and, to my knowledge, the dimension in which the natural type of diffeomorphisms we're about to introduce have recieved the most attention.

Now, for a SM ori.ented closed 4-mfld X, which let's assume is simply-connected $\pi_1 X \cong 1$, we denote by X° the manifold with boundary we get removing the interior of a 4-ball from $X := X \setminus \mathrm{int} B^4$.

Then, $\partial X^\circ \cong S^3 \Rightarrow \nu(\partial X^\circ) \cong I \times S^3$, so we may Consider the diffeo. $t_X : X^\circ \circlearrowleft$, given by an n-dimensional Dehn twist in a neighborhood of the boundary $t_X|_{\nu(\partial X^\circ)} = t_4$, and the identity outside this neighborhood $t_X|_{X^\circ \setminus \nu(\partial X^\circ)} = id$.

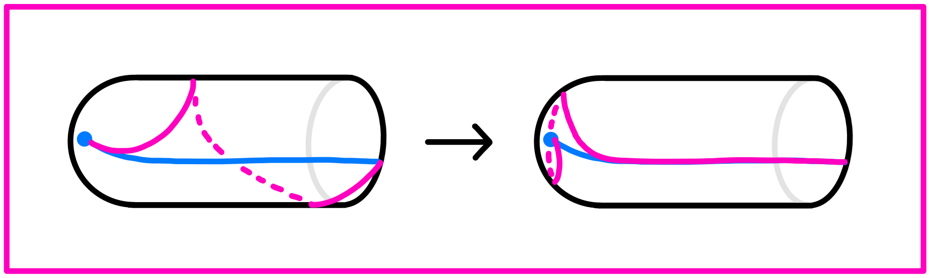
t_X is then called the =: Boundary Dehn twist on X° and is a notably important diffeomorphism of X° - for one thing, a Thm (of Giansiracusa from '08) states that

t_X gens. $\mathrm{Ker}(\pi_0(\mathrm{Diff}(X^\circ, \partial)) \twoheadrightarrow \pi_0(\mathrm{Diff}(X)))$

given by extending a diffeomorphism on punctured X to the identity on the 4-ball we removed from X to get the puncture, so this kernel is trivial if the boundary Dehn twist is trivial and $\mathbb{Z}/2$ otherwise.

The importance of these diffeomorphisms has motivated great interest in the Question: when is t_X trivial in the smooth or topological mapping class groups rel ∂?

Let's first visualize a simple case. Taking $X = S^4$, we can easily draw a dimensionally reduced picture:



So, we can see that $\Rightarrow$ t_{S^4} is trivial in the smooth and topological categories. In fact, in the topological category, this is the norm: a Thm (of Orson, and Powell from '25) says that $\forall$ ori. closed X^4 w/ $\pi_1 X \cong 1$, t_X is trivial in the topological mapping class group rel ∂ of X° $\pi_0(\mathrm{Homeo}(X^\circ, \partial))$.

This behavior is also seen in the smooth category when X is not spin: formally, another Thm (of Orson, and Powell from '25) says that If X is not spin ($w_2(X) \neq 0$), t_X is trivial in $\pi_0(\mathrm{Diff}(X^\circ, \partial))$.

However, Much less is known when X is spin: I'll survey the short list of what are, as far as I know, all known results in this setting:

In terms of results known when t_X is trivial in $\pi_0(\mathrm{Diff}(X^\circ, \partial))$: the only cases I know are when X is some $\#(S^2 \times S^2)$s.

And as for cases where it's known that it's Nontrivial:

- This is known for X the K3 surface (by Kronheimer, and Mrowka in '20)

- the once-stabilized K3 surface $K3 \# (S^2 \times S^2)$ (by J.ianfeng Lin in '23)
- $K3 \# K3$ (by Tilton very recently '25)
- and $\forall X$ satisfying a technical condition (by Baraglia, and Konno in '25). Principal examples of manifolds satisfying this condition are

↳ Many double log transforms of the elliptic surfaces $E(4n-2)_{i,j}$ and

↳ all Complete intersections $w/ c_1 \equiv 0, \sigma \equiv 16 \pmod{32}$.

We'll talk more about these later.

Being topologically isotopic but not smoothly isotopic to the identity means by definition that These are what we call exotic diffeos. The study of exotica is one of the main focuses of 4-manifold topology, and if you don't care about exotic diffeomorphisms, it's worth noting that they're deeply connected to exotic bundles through the association between smooth X-bundles and the diffeomorphism group of X.

In specific, for a Bundle $\begin{array}{c} X \to E \\ \downarrow \\ B \end{array}$, $\rightsquigarrow$ we have that the Bundle of fiberwise diffeos. of this bundle is a principal Diff(X)-bundle over B

And vice versa, for a principal Bundle $\begin{array}{c} \mathrm{Diff}(X) \to P \\ \downarrow \\ B \end{array}$,

$\rightsquigarrow$ we get the Associated bundle $\begin{array}{c} X \to P \times_{\mathrm{Diff}(X)} X \\ \downarrow \\ B \end{array}$.

These operations give a bijection between

$\{$ X-bundles over B $\} / \cong$ and

$\leftrightarrow$ $\{$ principal Diff(X) bundles over B $\} / \cong$.

Now, Diff(X) is a topological group, so $\exists$ a space called the classifying space BDiff(X) s.t. Diff(X)-bundles over B are in bijection with the space of maps of B into the classifying space $\mathrm{Map}(B, \mathrm{BDiff}(X))/\simeq$, such that a given map from B to the classifying space realizes its corresponding Diff(X)-bundle as the pullback by this map of a so-called "universal Diff(X)-bundle" over the classifying space.

Therefore, notably, $\Rightarrow$ $\{$Characteristic classes of X-bundles over B$\}$ then correspond to characteristic classes of Diff(X)-bundles over B, which correspond via pullback to

$\leftrightarrow$ $H^*(\mathrm{BDiff}(X); \mathbb{Z})$.

For this reason, there's been Lots of interest in $H^*(\mathrm{BDiff}(X); \mathbb{Z})$ & the cohomology $H^*($ of the analogous classifying space rel ∂ $\mathrm{BDiff}(X, \partial); \mathbb{Z})$ among topologists for decades. Some cool results are even known in dimension 4, thanks to work of Konno, Jianfeng Lin, Watanabe, and more. However, as it stands,

Little is known concretely about $H^*(\mathrm{BDiff}(X^4, \partial); \mathbb{Z})$

Now, let's try to think of how to get at such concrete information. Well, we've been thinking about the smooth mapping class group rel ∂, so let's start there. From the definifion of the classifying space, we have an isomorphism $\pi_0(\mathrm{Diff}(X^\circ, \partial)) \cong \pi_1(B\mathrm{Diff}(X^\circ, \partial))$. Then, we can abelianize this first homotopy group to get a map to the first homology of the classifying space $\xrightarrow{ab} H_1(B\mathrm{Diff}(X^\circ, \partial); \mathbb{Z})$. We then denote by $\underset{\cdot^{ab}}{\curvearrowright}$ the composition of these maps, so an element of the abelianized smooth mapping class group is a first homology class in the classifying space, which then corresponds to a characteristic class of smooth X°-bundles.

This notion of abelianized mapping class groups is Well-understood in dim. 2 thanks to work (of Mumford, Birman, Powell); they showed that the abelianized mapping class group of a closed oriented surface is generated by a single abelianized Dehn twist when the surface has genus 1 or 2 and is trivial otherwise. There's been a lot of related work in dimension 2 and Some results are known in higher dim. thanks to work (of Galatius, Randal-Williams, and Krannich, ...), but these are mostly to do with dimension 6 and above.

In some sense, the most concrete result I've seen on abelianized mapping class groups in dimension 4 concerns Abelianized boundary Dehn twists $t_X^{ab} \in H_1(B\mathrm{Diff}(X^\circ, \partial); \mathbb{Z})$;

these are the images of the ∂ Dehn twists we introduced before under the abelianization map. Motivated by how central ∂ Dehn twists are in the study of smooth mapping class groups rel ∂ and how much interest there's been in abelianizations of these groups, the following Question was asked (by Y.ujie Lin in '25)

For which simply-connected closed X^4 is $t_X^{ab} \in H_1(BDiff(X^\circ, \partial); \mathbb{Z})$ trivial? Now, Lin made a certain observation related to this question which provides an approach for how to answer it: generalizing an observation of Kronheimer and Mrowka, Lin noticed that a smooth X-bundle over an oriented surface whose total space has non-vanishing second Stiefel-Whitney class gives rise to an equation realizing the ∂ Dehn twist as a product of commutators in the smooth mapping class group rel ∂, whose abelianizations are then trivial, & vice versa, that such an expression tells how to build such a bundle.

This was formalized as a Prop (Y. Lin '25) saying that t_X^{ab} is trivial iff $\exists$ a SM X-bundle over an ori. surface w/ total space having $w_2 \neq 0$ (so, the bundle has no families spin structure).

Combining this prop. with + some pretty deep machinery in the form of the global Torelli thm. for K3 and an + obstruction of Baraglia-Konno ('22) for spin families, Lin proved the $\Rightarrow$

Thm (Y. Lin '25) $\exists$ a SM K3-bundle over T^2 w/ $w_2 \neq 0$, so t_{K3}^{ab} is trivial.

Now, Lin noted This approach can't directly work to give the corresponding statements for other X known to have t_X nontrivial due to a lack of a global Torelli theorem in these settings (and also, in some cases, because the obstruction of Baraglia and Konno would fail to apply). We got around this issue by just applying an Explicit construction + direct computation to prove the $\Rightarrow$

Thm 1 (L. '26) came here to talk about today:

Let X be any connected sum of complete intersections in products of cplx. proj. spaces. We may build a SM X-bundle over T^2 w/ $w_2 \neq 0$, so when $\dim_{\mathbb{R}} X = 4$ and $\pi_1 X \cong 1$, t_X^{ab} is trivial

We'll define these complete intersections, but first, let's discuss some Examples of mflds. this applies to:

- $S^2 \times S^2$ } on which it's already Known t_X is trivial
- and it also applies to K3
- $K3 \# (S^2 \times S^2)$
- $K3 \# K3$
- Any of the ∞ many complete ints. in a cplx. proj. space w/ $c_1 \equiv 0$, $\sigma \equiv 16 \pmod{32}$

As mentioned before, on all these manifolds, it's Known t_X is nontrivial

- Then, in addition, it also applies to the elliptic surfaces $E(n)$, and their generalizations $X(m,n)$, and
- Any #s of these. All these are simply-connected, so we do,

in fact, get that the abelianized ∂ Dehn twists on the punctures of all of these are trivial.

But what are these so-called complete intersections in products of complex projective spaces which these manifolds are or are connected sums of?

We'll Def (Cplete ints. in one $\mathbb{CP}^n$) for simplicity, these are what people normally mean when they say "complete intersections". Just note everything we'll say is essentially identical but with worse notation for those in products of complex projective spaces.

Ok, now Consider homogeneous polys. $p_1, \dots, p_m : \mathbb{C}^{n+1} \to \mathbb{C}$ w/ degrees $d_1, \dots, d_m$.

By Homogeneity, we have that $\Rightarrow$ $p_i(\lambda z) = \lambda^{d_i} p_i(z)$

$\Rightarrow p_i(\lambda z) = 0$ iff $p_i(z) = 0 \ \forall \lambda \in \mathbb{C} \setminus 0$

$\Rightarrow \{ [z] \in \mathbb{CP}^n \mid p(z) = 0 \}$ is well-defined, let's call it $=: Z_i$.

For a generic polynomial p_i, this will then be a complex hypersurface of $\mathbb{CP}^n$. Then, we Say $X := \bigcap_{i=1}^{m} Z_i \subset \mathbb{CP}^n$ is a cplete. int. if X is a cplx. $(n-m)$-dim. submfld of $\mathbb{CP}^n$; it can be seen from the implicit function theorem that this condition is satisfied for generic choice of polynomials p_i. Moreover, an

Observation (of Thom) states that

Say $p_1, \dots, p_m$ and $p_1', \dots, p_m'$ have degrees $d_1, \dots, d_m$ and cut out cplx. $(n-m)$-dim. submflds. $X, X' \subset \mathbb{CP}^n$. We have $X \cong_{SM} X'$.

The idea behind this is that the set of polynomials of given degree which cut out something singular has codimension at least 2, so you can deform complete intersections smoothly to one another avoiding anything singular. This is key to the proof of our main theorem today, as we use it to pick polynomials with certain symmetries we can make use of. Specifically,

$\Rightarrow$ For $X \subset \mathbb{CP}^n$ cplete. int., we can pick $p_1, \dots, p_m$ s.t. each p_i has:

(1) only even powers of 1^{st} $\mathbb{C}$-coord. z_1

(2) only real coeffs.

(3) a real zero ξ w/ $\xi_1 = 0$

which cut out some $X' \cong_{SM} X$

Note that to apply the observation of Thom to be able to select cutting polynomials of this form, we do need to show it isn't the case that every polynomial of this form cuts out something singular; this is a technical genericity argument we'll ignore for today.

Now, as we're working in the smooth category, let's just say WLOG $X = X'$.

For simplicity, let's also say $\dim_{\mathbb{R}} X = 4$.

Now, For maps $A: z \mapsto (-z_1, z_2, \dots, z_{n+1})$ and

$B: z \mapsto \bar{z}$

we have that assumption (1) $\Rightarrow p_i(A(z)) = p_i(z)$, and

assumption (2) $\Rightarrow p_i(B(z)) = \overline{p_i(z)}$

$\Rightarrow p_i(A(z)) = 0 \Leftrightarrow p_i(z) = 0 \Leftrightarrow p_i(B(z)) = 0$

$\Rightarrow$ the maps $[z] \mapsto [A(z)]$, $[z] \mapsto [B(z)]$ on $\mathbb{CP}^n$
restrict to commuting diffeos. $a, b : X \circlearrowleft$, and
assumption (3) $\Rightarrow$ a, b have a shared fixed pt. $\xi \in X$
at which $da_\xi, db_\xi : T_\xi X \circlearrowleft$ act as $\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$

Let's call this fact $(*)$. Draw box around

That these differentials act this way on the tangent space isn't immediately obvious, but perhaps it's vaguely intuitive from their definitions. Formally, this is just an application of the implicit function theorem. Now, we want to show the boundary Dehn twist on punctured X is trivial. Lin's proposition reduces this to building a smooth X-bundle over an oriented surface with non-vanishing w_2, so let's do exactly that. Now, how can we do this using our commuting diffeomorphisms a and b on X? Well, we can consider the Mapping torus $X_{a,b}$ of $a, b := [0,1]^2 \times X / \sim$, where $\sim$ is defined by the relations that $(0, s, a(x)) \sim (1, s, x)$
and $(s, 0, b(x)) \sim (s, 1, x)$

This is well-def. b/c a, b commute and is a
SM X-bundle over T^2, as desired. So,
Lin's prop. $\Rightarrow$ if $w_2(X_{a,b}) \neq 0$, t_X^{ab} is trivial
Thus, $\Rightarrow$ we Only need to show $w_2(X_{a,b}) := w_2(TX_{a,b}) \neq 0$
to prove the thm. So, let's directly compute this!

To do this, let's consider a section $\sigma : T^2 \to X_{a,b}$ of our mapping torus given by taking a point z on the torus $\mapsto$ the pair (z, ξ), where ξ is our shared fixed point of a and b so this map is continuous. Now, If $w_2(\sigma^* TX_{a,b}) \neq 0$, we have by naturality of Stiefel-Whitney classes that

$\Rightarrow \sigma^* w_2(TX_{a,b}) \neq 0 \Rightarrow w_2(TX_{a,b}) \neq 0$.

Therefore, we Just need to show this, that w_2 of this pullback bundle vanishes.

Now, from our definition of the mapping torus, we can see that $\sigma^* TX_{a,b}$ decomposes $\cong$ into the part coming from the base torus TT^2 and $\oplus$ the part coming from the fiber $(T_\xi X)_{da_\xi, db_\xi}$, where this $\underbrace{\qquad\qquad}_{:=}$ is the tangent space of X twisted by the differentials of a and b at ξ, which we formally write as $[0,1]^2 \times T_\xi X / \sim$, where $\sim$ is defined by the relations that $(0, s, da_\xi(v)) \sim (1, s, v)$

and $(s, 0, db_\xi(v)) \sim (s, 1, v)$

Again, to see this, just compare this with our definition of the mapping torus $X_{a,b}$. Then, our fact $(*)$ characterizing the differentials of a and b at ξ shows that

$\Rightarrow (T_\xi X)$ decomposes into $\cong$ a direct sum

$L_{1,0} \oplus L_{1,1} \oplus L_{0,1} \oplus L_{0,0}$, where $L_{i,j} \to T^2$ is the RLB over T^2 with total space $[0,1]^2 \times \mathbb{R} \Big/ \begin{smallmatrix} (0,s,(-1)^i r) \sim (1,s,r) \\ (s,0,(-1)^j r) \sim (s,1,r) \end{smallmatrix}$

so the line bundle has a twist around the first S^1 when i is 1, and around the second S^1 when j is odd. Writing this formally, let's set $A := S^1 \times *$, $B := * \times S^1 \subset T^2$ alongside

$$\alpha := PD(A), \beta := PD(B).$$

We can see that $L_{ij}|_A$ ori. $\Leftrightarrow$ i even and

$L_{ij}|_B$ ori. $\Leftrightarrow$ j even, so

$\Rightarrow w(L_{ij}) = 1 + i\alpha + j\beta.$

Now, we also have $TT^2 \cong L_{0,0} \oplus L_{0,0}$, so

$\Rightarrow_{\text{the Whitney prod. formula}}$ shows that

$$w(\sigma^* TX_{a,b}) = 1 \cdot 1 \cdot (1+\alpha)(1+\alpha+\beta)(1+\beta)$$
$$= (1+\alpha+\beta+\alpha+\alpha^2+\alpha\beta)(1+\beta)$$

Now, we have $\alpha^2 = 0$ $\cancel{\alpha^2}$, and since we're working mod 2, $\alpha+\alpha = 0$ $\cancel{\alpha}\,\cancel{\alpha}$, so this all equals

$$= 1+\beta+\beta+\beta^2+\alpha\beta+\alpha\beta^2$$

$\Rightarrow w_2(\sigma^* TX_{a,b}) = \alpha\beta \neq 0$

So great! We then discussed above how, by naturality, this means that $\Rightarrow w_2(X_{ab}) := w_2(TX_{ab})$, so

We've built a SM X-bundle over an ori. surface (T^2, specifically) w/ $w_2 \neq 0$.

The existence of such a bundle is then exactly what the proposition of $\Rightarrow_{\text{Lin}}$ says is equivalent to $t_X^{ab} = 0 \in H_1(BDiff(X^\circ, \partial))$, so we're done! $\square$

Thanks very much for listening.