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MTS 102: Introductory Mathematics II
Functions of Real Variables

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1. Definition

A function is a relationship between two sets, in which each member of one set corresponds uniquely to a member of the other set. The following symbols can be used to represent functions: f, $f(x)$, etc. The members of each set are referred to as variables. A set of variables whose values are determined by another set of variables are known as **dependent variables** while the variables which determine their values are known as **independent variables**.

A real-valued function of a real variable is a function which maps a real number $x \in \mathbb{R}$ to another real number $y \in \mathbb{R}$. That is, $f(x) : x \mapsto y$ where $x, y \in \mathbb{R}$.

Many real life phenomena depend on various factors and as such they can be represented mathematically as functions of those factors. For instance, the performance of students in Calculus can be a function of the course lecturers capacities to teach the course, seriousness on the part of the students, the size of the class and a host of other factors. Here, **students' performance** (w) is the dependent variable while **lecturers' capacities** (x), **students' seriousness** (y) and **class size** (z) are the independent variables. We can represent the relationship as $w = f(x, y, z)$.

2. Types of Functions

2.1 Polynomial functions

A polynomial function is a mathematical expression consisting of the sum of a number of terms, each of which is the product of a constant and a variable raised to a positive integral power. Example: $f(x) = 3x^5 - 5x^4 + x^3 + 6x^2 + 2x + 7$.

2.2 Rational functions

A rational function is a quotient or ratio $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Example: $f(x) = \dfrac{3x + 7}{x^2 + 5x - 1}$

2.3 Exponential functions

Functions of the form $f(x) = a^x$ where the base $a > 0$ is a positive constant and $a \neq 1$, are called exponential functions. An exponential function never assumes the value of 0. Examples: $f(x) = 10^x$, $f(x) = e^{2x-1}$, $f(x) = x^x$.

2.4 Logarithmic functions

The logarithmic function with $y = \log_a x$ with base a is the inverse of the base a exponential function $y = a^x$, $(a > 0, a \neq 1)$. There are two main types of logarithm, namely: (a) The Natural/Napierian Logarithm ($\ln x = \log_e x$) ; (b) The Common Logarithm ($\log x = \log_{10} x$). Examples: $f(x) = \ln(4x+1)$, $f(x) = \log 3x$.

2.5 Trigonometric functions

Trigonometric functions involve the six trigonometric ratios, namely sine, cosine, tangent, cosecant, secant and cotangent. Examples: $f(x) = \sin x$, $f(x) = \sec x - \sin x$, $f(x) = 1 + \cos x$.

2.6 Inverse trigonometric functions

Functions that involve the inverse of trigonometric ratios are referred to as inverse trigonometric ratios. It should be noted that inverse trigonometric ratios are not the same as the reciprocals of trigonometric ratios like cosecant, secant and cotangent. Examples: $f(x) = \sin^{-1} x$, $f(x) = \cos^{-1} x$, $f(x) = \tan^{-1} x$.

2.7 Hyperbolic functions

Hyperbolic functions are formed by performing mathematical operations on the two exponential functions e^x and e^{-x}. Hyperbolic functions simplify many mathematical expressions and occur frequently in engineering applications. The hyperbolic sine, hyperbolic cosine and hyperbolic tangent functions are defined by the equations: $\sinh = \dfrac{e^x - e^{-x}}{2}$, $\cosh = \dfrac{e^x + e^{-x}}{2}$, $\tanh = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

2.8 Transcendental functions

These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a **catenary**. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight.

2.9 Explicit functions

An explicit function is a function whose dependent variable can be expressed entirely in terms of the independent variable. All the previous examples of functions given are explicit.

2.10 Implicit functions

An implicit function is a function whose dependent variable cannot be expressed entirely in terms of the independent variable. Examples: $x^2 + y^2 + xy = x + y$, $x^3y^2 - 7x = 3$.

2.11 Periodic functions

A function $f(x)$ is said to be periodic if $f(x) = f(x + L)$, where L is the period of the function. Examples: $f(x) = \sin x^\circ = \sin(x + 180)^\circ$, $f(x) = \cos x^\circ = \cos(x + 360)^\circ$.

2.12 Odd functions

An odd function $f(x)$ is a function such that $f(-x) = -f(x)$. Example: $f(x) = \sin x$, $f(x) = x^3$.

2.13 Even functions

An even function $f(x)$ is a function such that $f(-x) = f(x)$. Example: $f(x) = \cos x$, $f(x) = x^2$.

3. Operations on Functions

3.1 Inverses of Functions

The inverse of a function is a function that exactly reverses it. In order to find the inverse of a function, we make the independent variable dependent by making it a subject of the function. Every function is an inverse of its inverse, i.e., $(f^{-1}(x))^{-1} = f(x)$.

3.1.1 Example 1

Find the inverse of the function $f(x) = \frac{1}{2}(x - 2)$.

Solution

$$\begin{aligned} f(x) &= \frac{1}{2}(x-2) \\ y &= \frac{1}{2}(x-2) \\ 2y &= x - 2 \\ x &= 2y + 2 \\ \therefore f^{-1}(x) &= 2x + 2. \end{aligned}$$

3.1.2 Example 2

Find the inverse of the function $f(x) = \frac{1}{\sqrt{9x-1}}$.

Solution

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{9x-1}} \\
y &= \frac{1}{\sqrt{9x-1}} \\
y^2 &= \frac{1}{9x-1} \\
y^2(9x-1) &= 1 \\
9x-1 &= \frac{1}{y^2} \\
9x &= \frac{1}{y^2}+1 \\
9x &= \frac{1+y^2}{y^2} \\
x &= \frac{1+y^2}{9y^2} \\
\therefore f^{-1}(x) &= \frac{1+x^2}{9x^2}.
\end{aligned}$$

3.1.3 Example 3

Find the inverse of the function $f(x) = \frac{1+x}{1-x}$.

Solution

$$\begin{aligned}
f(x) &= \frac{1+x}{1-x} \\
y &= \frac{1+x}{1-x} \\
y(1-x) &= 1+x \\
y-xy &= 1+x \\
x+xy &= y-1 \\
x(1+y) &= y-1 \\
x &= \frac{y-1}{y+1} \\
\therefore f^{-1}(x) &= \frac{x-1}{x+1}.
\end{aligned}$$

3.2 Composition of Functions

If f and g are functions of real variables, the composite function $f \circ g$ ("f composed with g") is defined by $f \circ g = f(g(x))$. The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

3.2.1 Example 1

If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find (i) $f \circ g(x)$, (ii) $g \circ f(x)$, (iii) $f \circ f(x)$, (iv) $g \circ g(x)$, (v) $f \circ g(0)$, (vi) $g \circ f(0)$, (vii) $f \circ f(-5)$, (viii) $g \circ g(2)$.

Solution

Given that $f(x) = x + 5$ and $g(x) = x^2 - 3$,

(i) To find $f \circ g(x)$,

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x^2 - 3) \\ &= (x^2 - 3) + 5 \\ &= x^2 - 3 + 5 \\ &= x^2 + 2 \end{aligned}$$

$\therefore f \circ g(x) = x^2 + 2.$

(ii) To find $g \circ f(x)$,

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x + 5) \\ &= (x + 5)^2 - 3 \\ &= x^2 + 10x + 25 - 3 \\ &= x^2 + 10x + 22 \end{aligned}$$

$\therefore g \circ f(x) = x^2 + 10x + 22.$

(iii) To find $f \circ f(x)$,

$$\begin{aligned} f \circ f(x) &= f(f(x)) \\ &= f(x + 5) \\ &= (x + 5) + 5 \\ &= x + 5 + 5 \\ &= x + 10 \end{aligned}$$

$\therefore f \circ f(x) = x + 10.$

(iv) To find $g \circ g(x)$.

$$\begin{aligned} g \circ g(x) &= g(g(x)) \\ &= g(x^2 - 3) \\ &= (x^2 - 3)^2 - 3 \\ &= x^4 - 6x^2 + 9 - 3 \\ &= x^4 - 6x^2 + 6 \end{aligned}$$

$\therefore g \circ g(x) = x^4 - 6x^2 + 6.$

(v) To find $f \circ g(0)$.

$$\begin{aligned} f \circ g(x) &= x^2 + 2 \\ f \circ g(0) &= (0)^2 + 2 \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

$\therefore f \circ g(0) = 2.$

(vi) To find $g \circ f(0)$.

$$\begin{aligned} g \circ f(x) &= x^2 + 10x + 22 \\ g \circ f(0) &= (0)^2 + 10(0) + 22 \\ &= 0 + 0 + 22 \\ &= 22 \end{aligned}$$

$\therefore g \circ f(0) = 22.$

(vii) To find $f \circ f(-5)$.

$$\begin{aligned} f \circ f(x) &= x + 10 \\ f \circ f(-5) &= -5 + 10 \\ &= 5 \end{aligned}$$

$\therefore f \circ f(-5) = 5.$

(viii) To find $g \circ g(2)$.

$$\begin{aligned} g \circ g(x) &= x^4 - 6x^2 + 6 \\ g \circ g(2) &= (2)^4 - 6(2)^2 + 6 \\ &= 16 - 24 + 6 \\ &= -2 \end{aligned}$$

$\therefore g \circ g(2) = -2.$

3.2.2 Example 2

If $f(x) = \dfrac{1}{9x-1}$ *and* $g(x) = \dfrac{x-1}{x+1}$, *show that* $(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x)$

Solution

To find $f \circ g(x)$,

$$
\begin{aligned}
f \circ g(x) &= f(g(x)) \\
&= f\left(\frac{x-1}{x+1}\right) \\
&= \frac{1}{9\left(\frac{x-1}{x+1}\right) - 1} \\
&= \frac{1}{\left(\frac{9x-9}{x+1}\right) - 1} \\
&= \frac{1}{\frac{1(9x-9) - 1(x+1)}{x+1}} \\
&= \frac{1}{\frac{9x-9-x-1}{x+1}} \\
&= \frac{1}{\frac{8x-10}{x+1}} \\
&= \frac{x+1}{8x-10} \\
&= \frac{x+1}{2(4x-5)}.
\end{aligned}
$$

$$\therefore f \circ g(x) = \frac{x+1}{2(4x-5)}.$$

To find $(f \circ g)^{-1}(x)$,

$$f \circ g(x) = \frac{x+1}{8x-10)}$$

$$y = \frac{x+1}{8x-10}$$

$$y(8x-10) = x+1$$

$$8xy - 10y = x+1$$

$$8xy - x = 10y+1$$

$$x(8y-1) = 10y+1$$

$$x = \frac{10y+1}{8y-1}.$$

$$\therefore (f \circ g)^{-1}(x) = \frac{10x+1}{8x-1}.$$

To find $g^{-1}(x)$,

$$g(x) = \frac{x-1}{x+1}$$

$$y = \frac{x-1}{x+1}$$

$$y(x+1) = x-1$$

$$xy + y = x-1$$

$$xy - x = -y-1$$

$$x(y-1) = -(y+1)$$

$$x = \frac{-(y+1)}{y-1}$$

$$x = \frac{1+y}{1-y}.$$

$$\therefore g^{-1}(x) = \frac{1+x}{1-x}.$$

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To find $f^{-1}(x)$,

$$\begin{aligned}
f(x) &= \frac{1}{9x-1}\\
y &= \frac{1}{9x-1}\\
y(9x-1) &= 1\\
9xy - y &= 1\\
9xy &= y+1\\
x &= \frac{y+1}{9y}.
\end{aligned}$$

$\therefore f^{-1}(x) = \frac{x+1}{9x}$.

To find $g^{-1} \circ f^{-1}(x)$,

$$\begin{aligned}
g^{-1} \circ f^{-1}(x) &= g^{-1}(f^{-1}(x))\\
&= g^{-1}\left(\frac{x+1}{9x}\right)\\
&= \frac{1+\frac{x+1}{9x}}{1-\frac{x+1}{9x}}\\
&= \frac{\frac{9x(1)+1(x+1)}{9x}}{\frac{9x(1)-1(x+1)}{9x}}\\
&= \frac{\frac{9x+x+1}{9x}}{\frac{9x-x-1}{9x}}\\
&= \frac{\frac{10x+1}{9x}}{\frac{8x-1}{9x}}\\
&= \frac{10x+1}{8x-1}.
\end{aligned}$$

$\therefore g^{-1} \circ f^{-1}(x) = \frac{10x+1}{8x-1}$. So, $(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x)$.

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Exercise

1. Define the following and give two examples of each.
   (i) Onto/ Surjective function
   (ii) One-to-one/ Injective function
   (iii) Bijective Function
   (iv) Step Function
   (v) Periodic Function
2. Find the inverses of the following functions.
   (i) $f(x) = \dfrac{2+x}{1-2x}$
   (ii) $f(x) = \dfrac{2}{3}(4x-7)$
   (iii) $f(x) = \sqrt{2x-11}$
3. Given $f(x) = \dfrac{1}{\sqrt{9x-1}}$ and $g(x) = x^2$,
   (i) find $f \circ g(x)$, $g \circ f(x)$, $f \circ f(x)$, $g \circ g(x)$ and evaluate them at $x = 7.5$
   (ii) show that $(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x)$ and $(f \circ g)^{-1}(x) = f^{-1} \circ g^{-1}(x)$.

4. Graphs of Functions of Real Variables

4.1 Polynomial functions

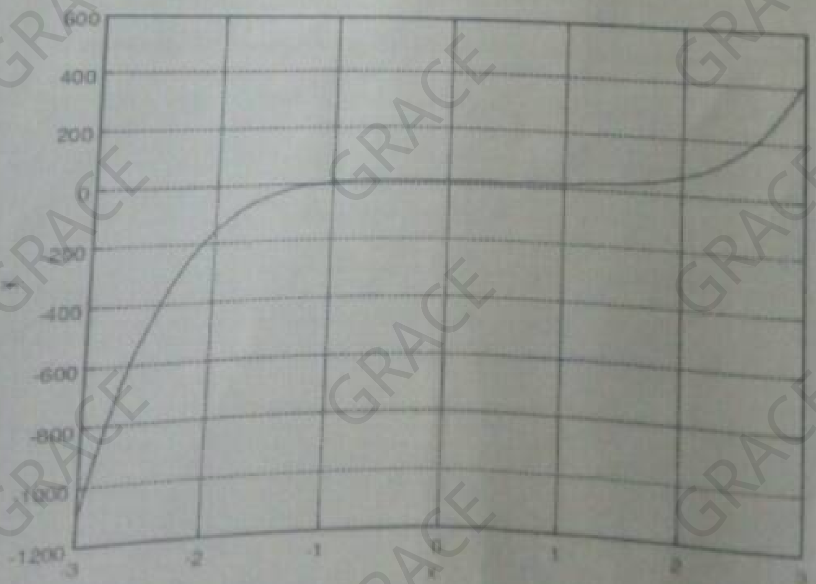

Figure 1: $f(x) = 3x^5 - 5x^4 + x^3 + 6x^2 + 2x + 7$

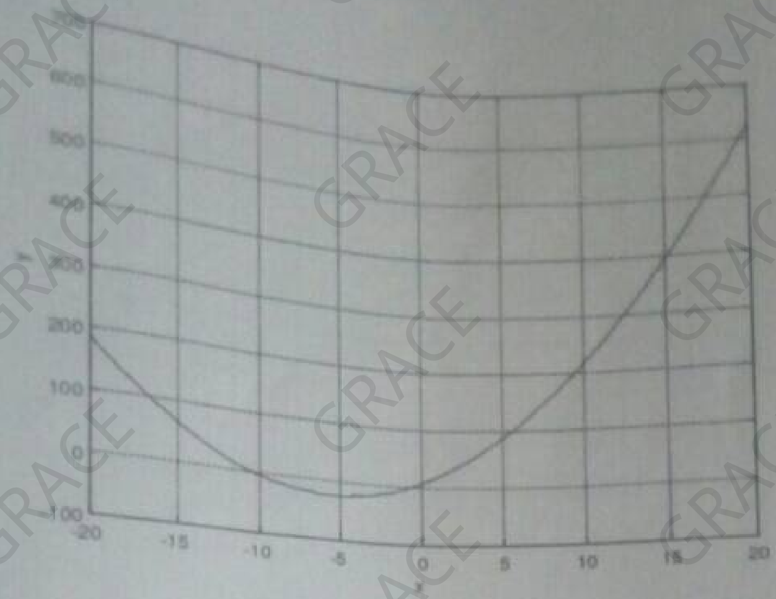

Figure 2: $f(x) = x^2 + 11x + 6$

4.2 Rational functions

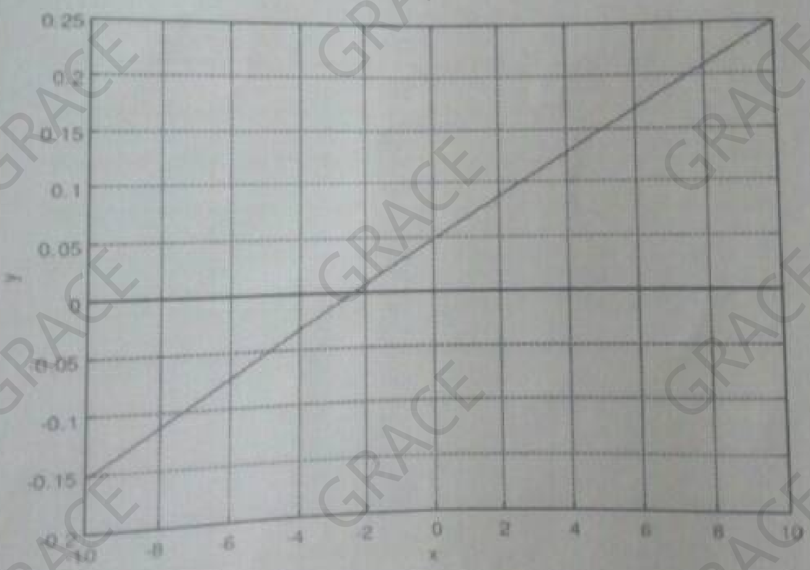

Figure 3: $f(x) = \dfrac{3x+7}{x^2+5x-1}$

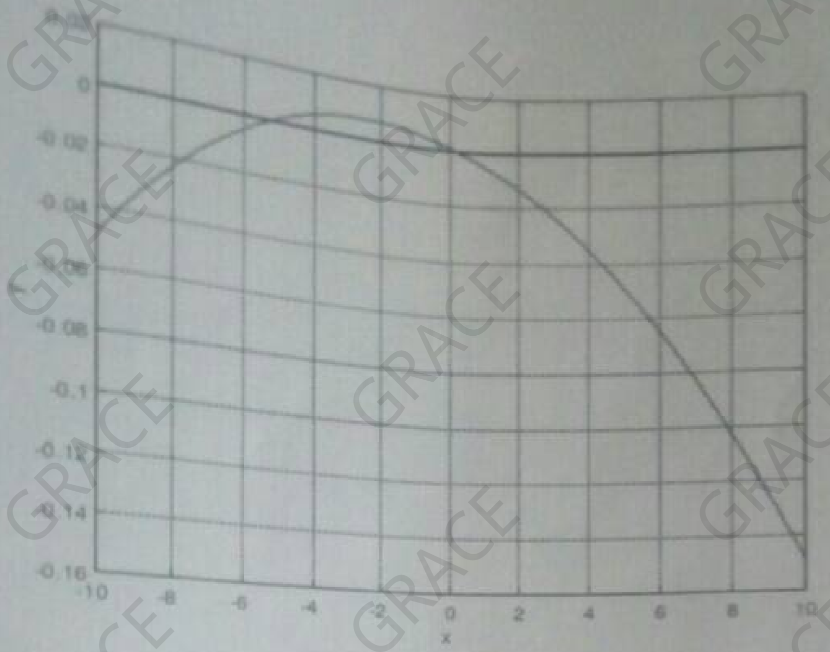

Figure 4: $f(x) = \dfrac{x^2 + 5x - 1}{x^3 - 1}$

4.3 Exponential functions

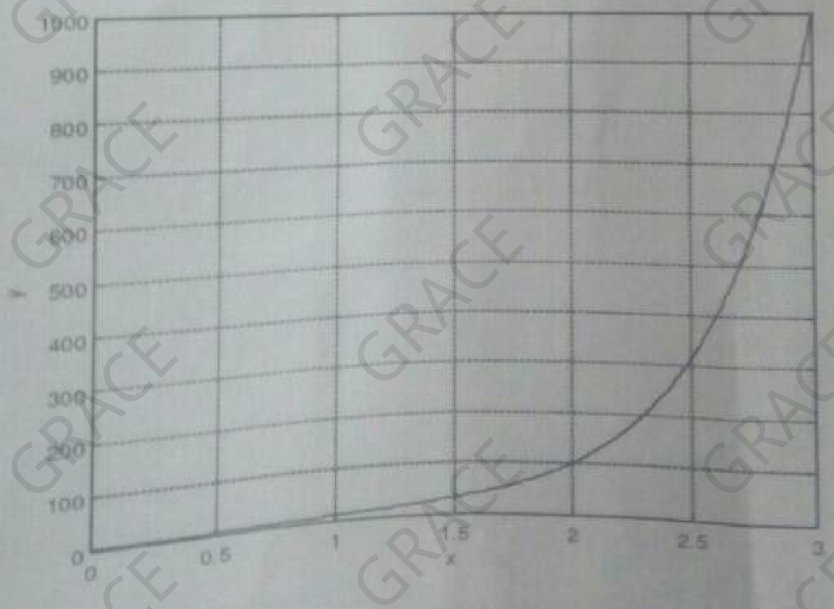

Figure 5: $f(x) = 10^x$

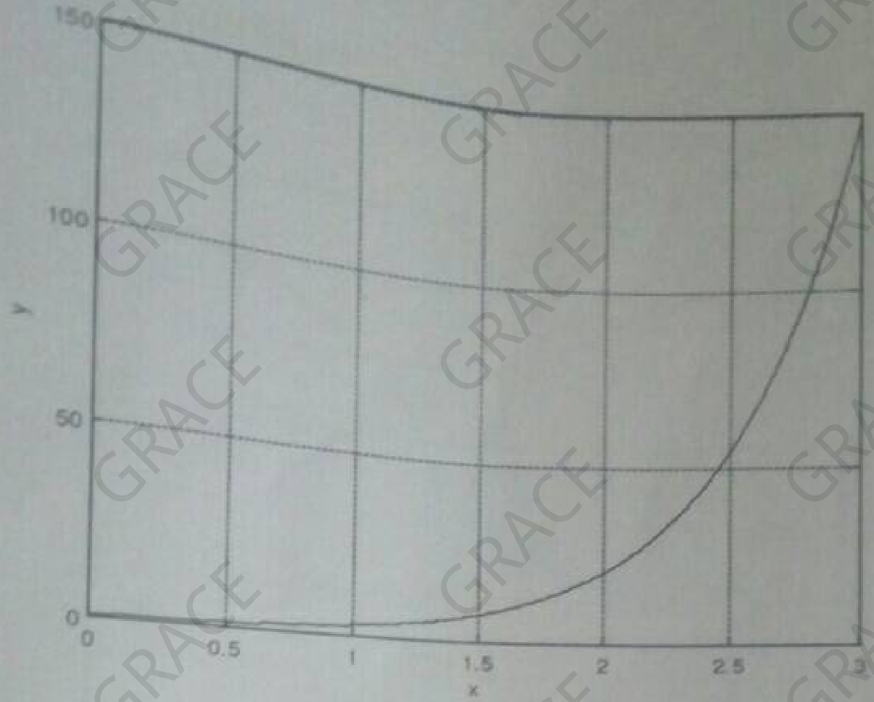

Figure 6: $f(x) = e^{2x-1}$

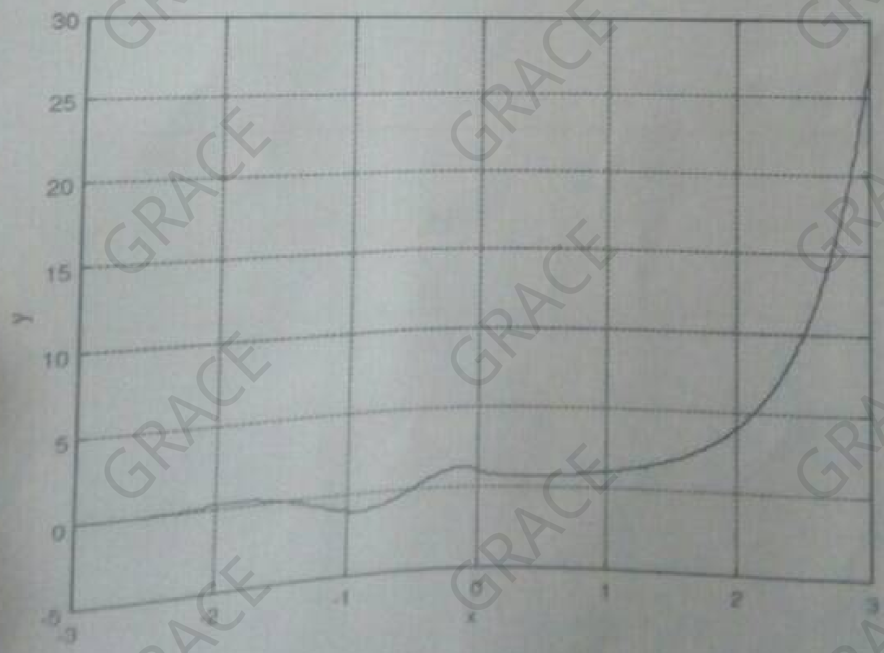

Figure 7: $f(x) = x^x$

4.4 Logarithmic functions

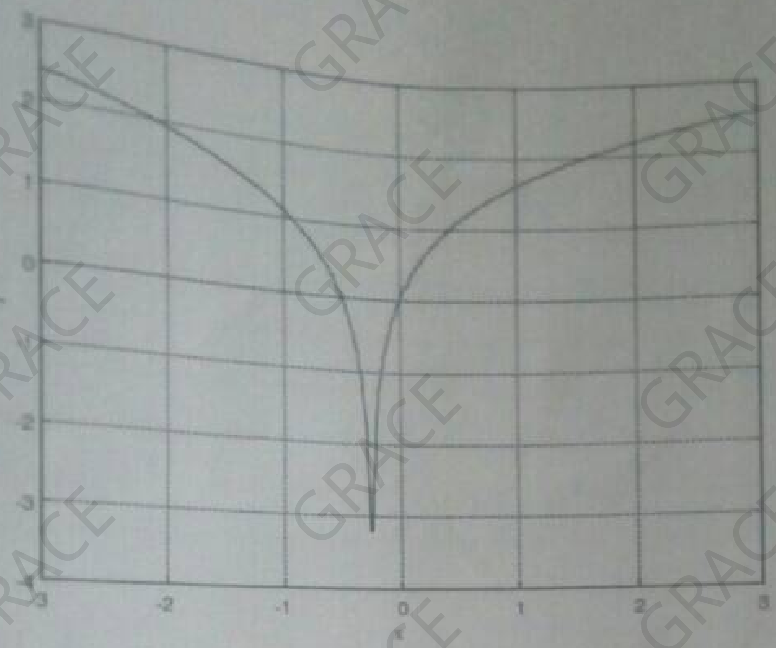

Figure 8: $f(x) = \ln(4x + 1)$

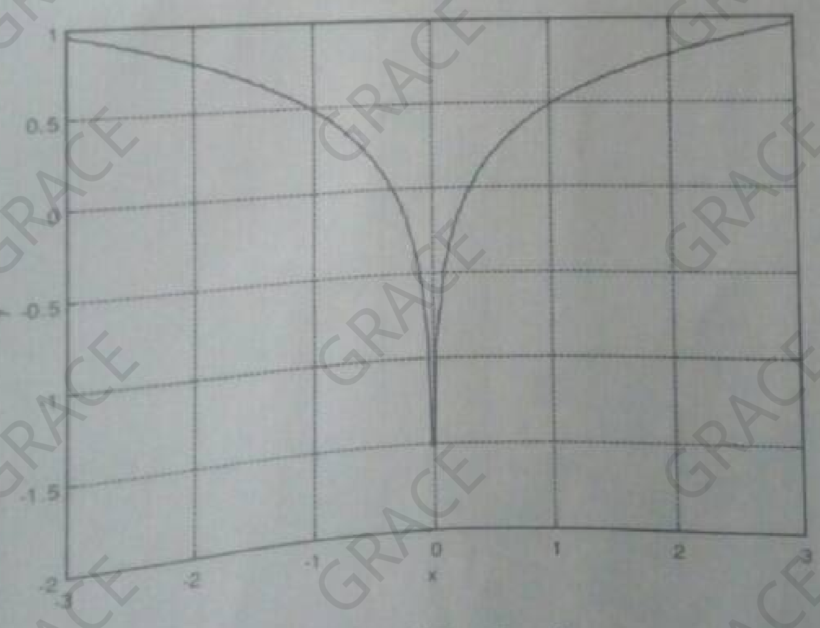

Figure 9: $f(x) = \log 3x$

4.5 Trigonometric functions

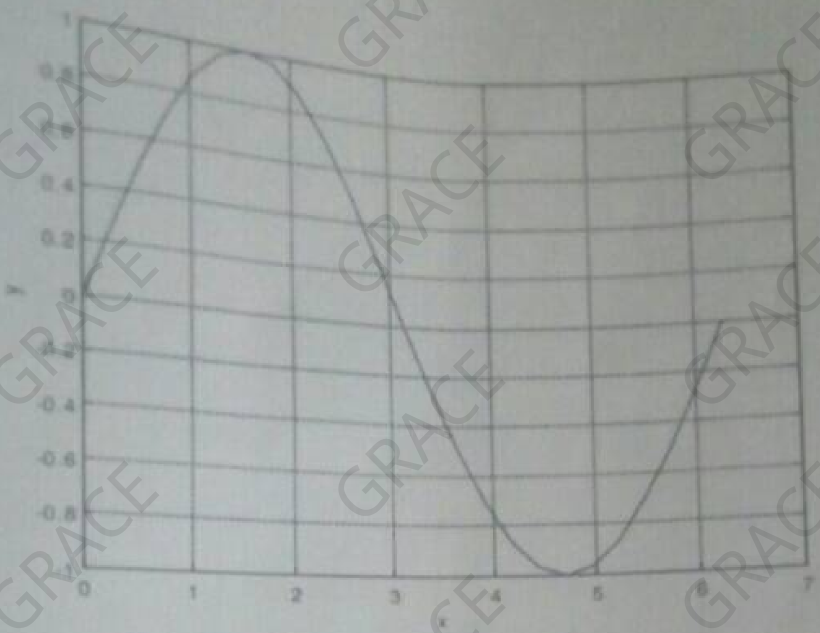

Figure 10: $f(x) = \sin x$

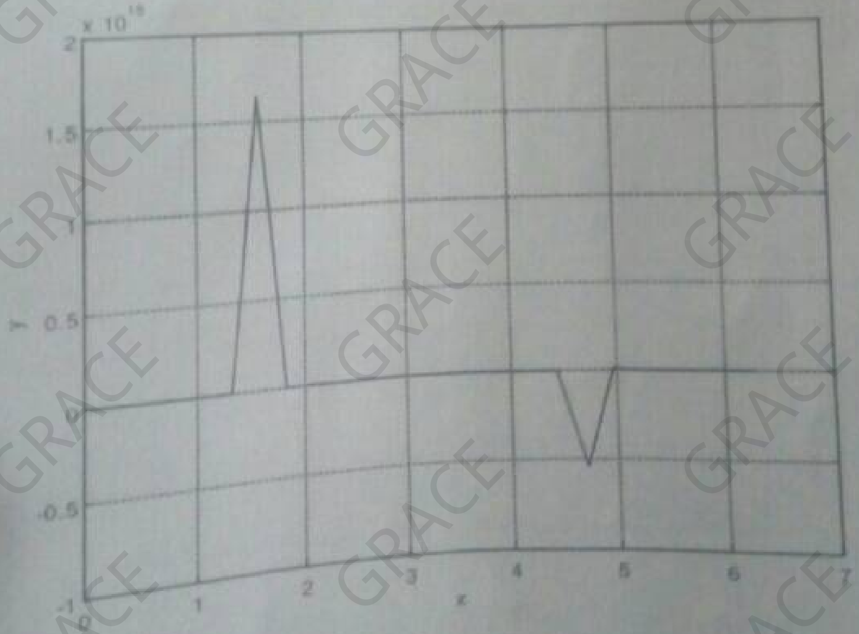

Figure 11: $f(x) = \sec x - \sin x$

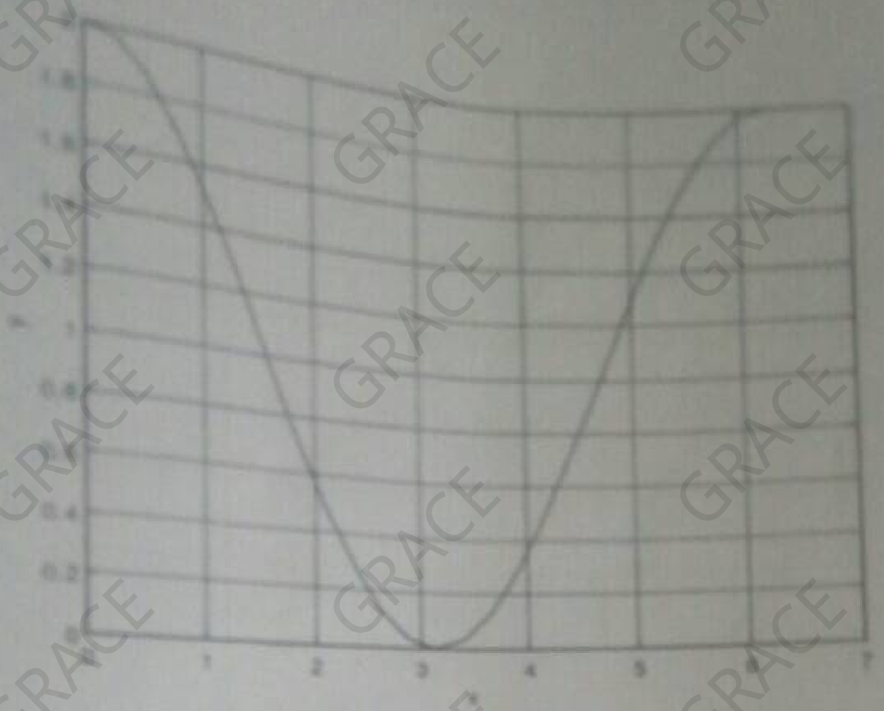

Figure 12: $f(x) = 1 + \cos x$

4.6 Inverse trigonometric functions

Figure 13: $f(x) = \sin^{-1} x$

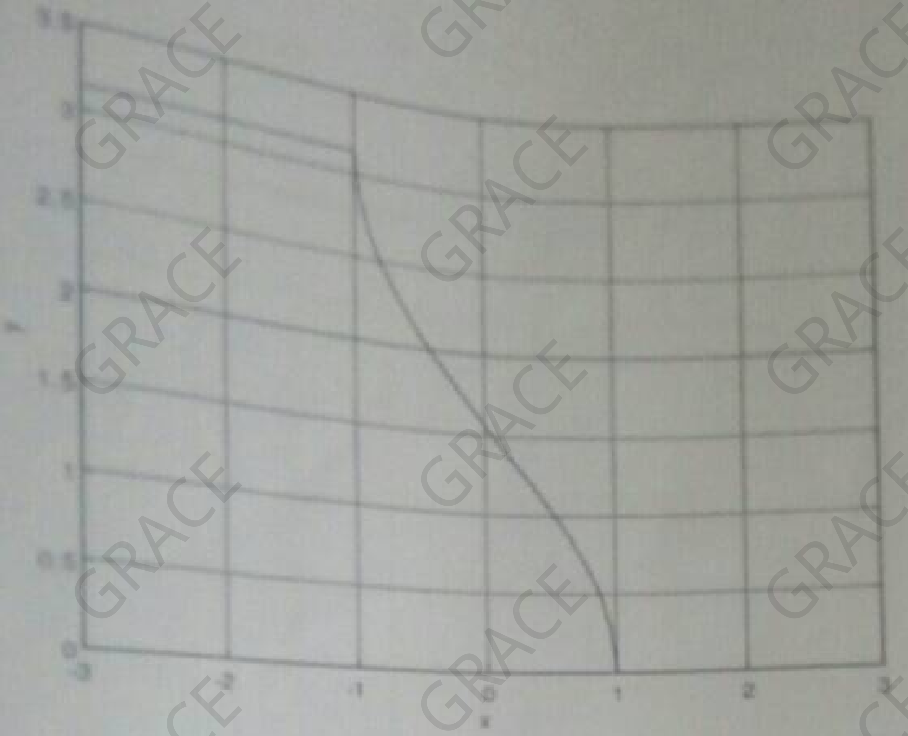

Figure 14: $f(x) = \cos^{-1} x$

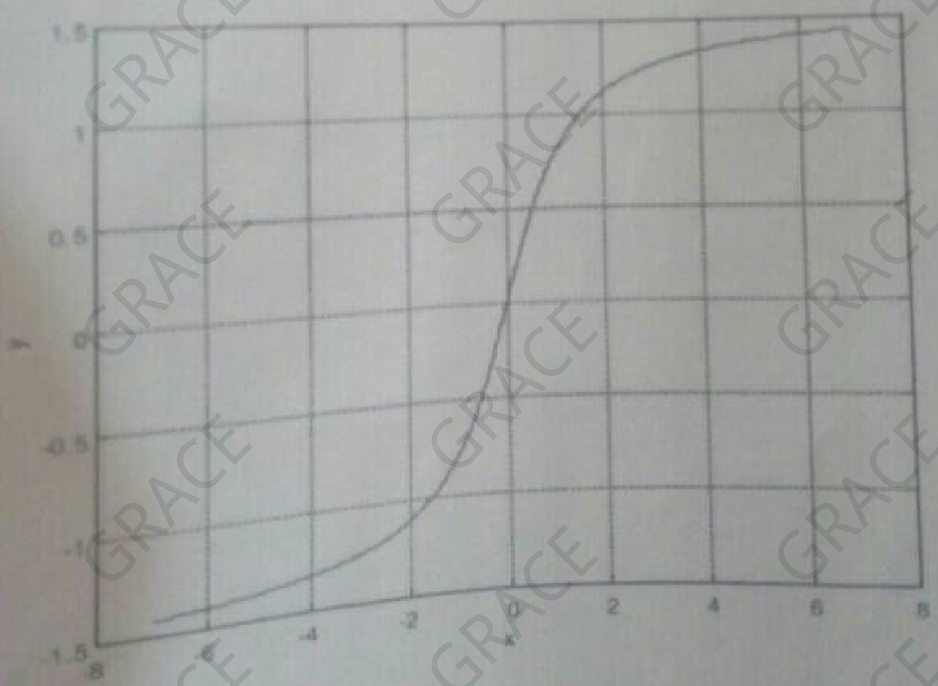

Figure 15: $f(x) = \tan^{-1} x$

1.7 Hyperbolic functions

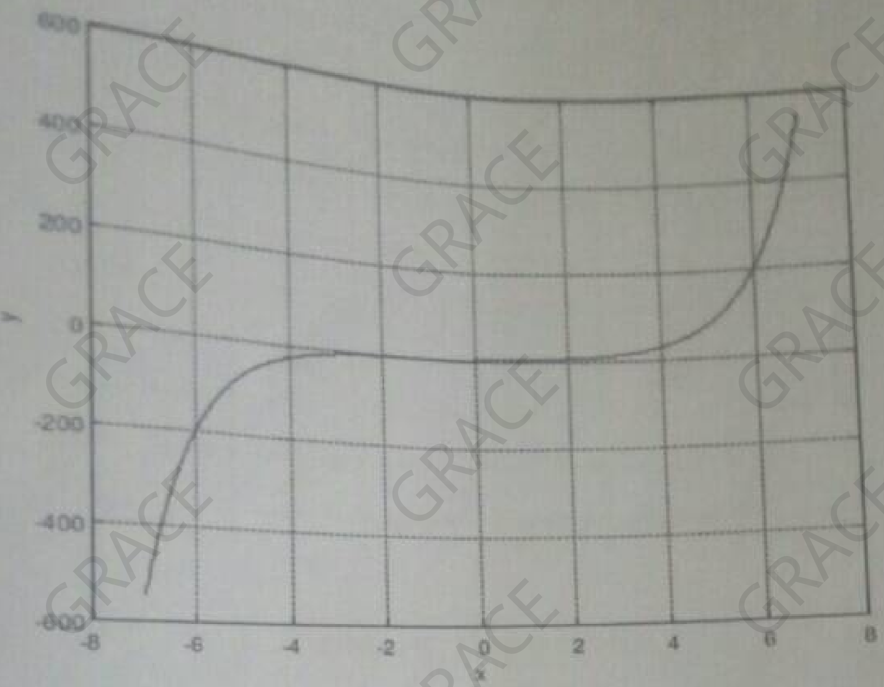

Figure 16: $f(x) = \sinh x$

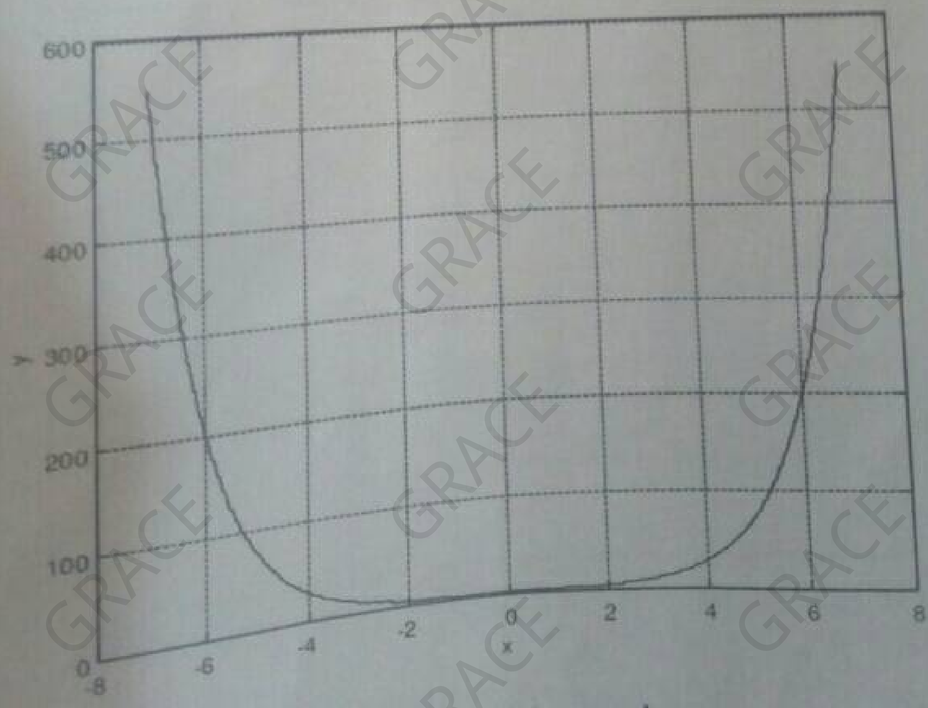

Figure 17: $f(x) = \cosh x$

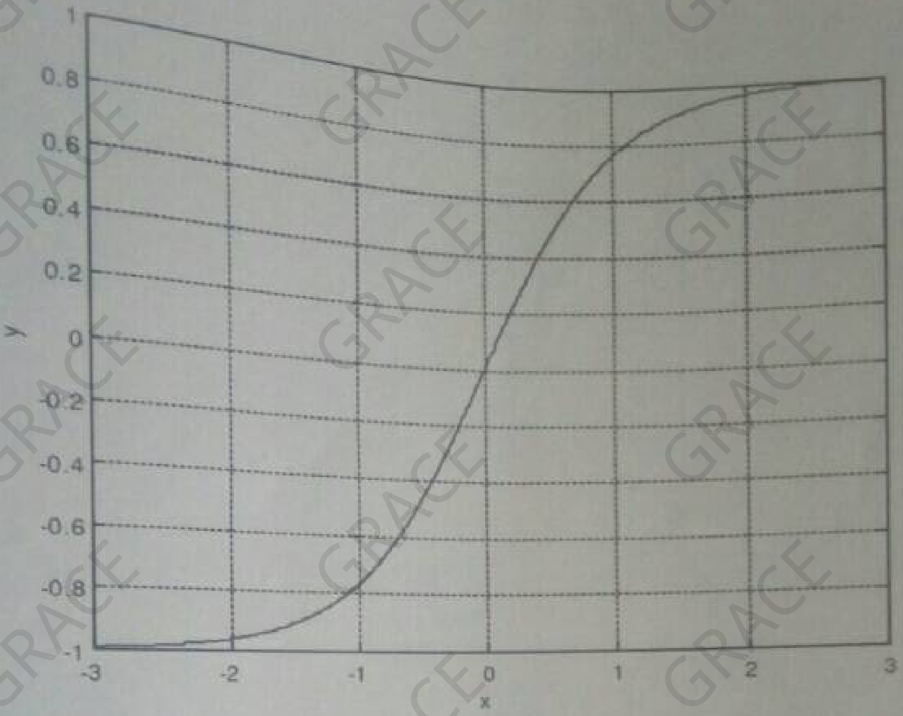

Figure 18: $f(x) = \tanh x$

4.8 Odd functions

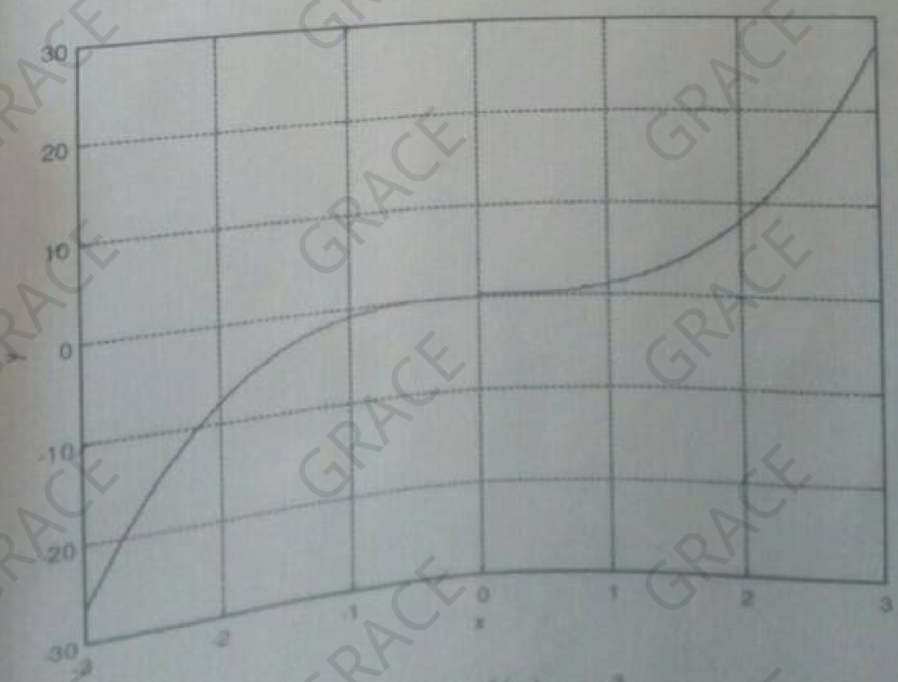

Figure 19: $f(x) = x^3$

4.9 Even functions

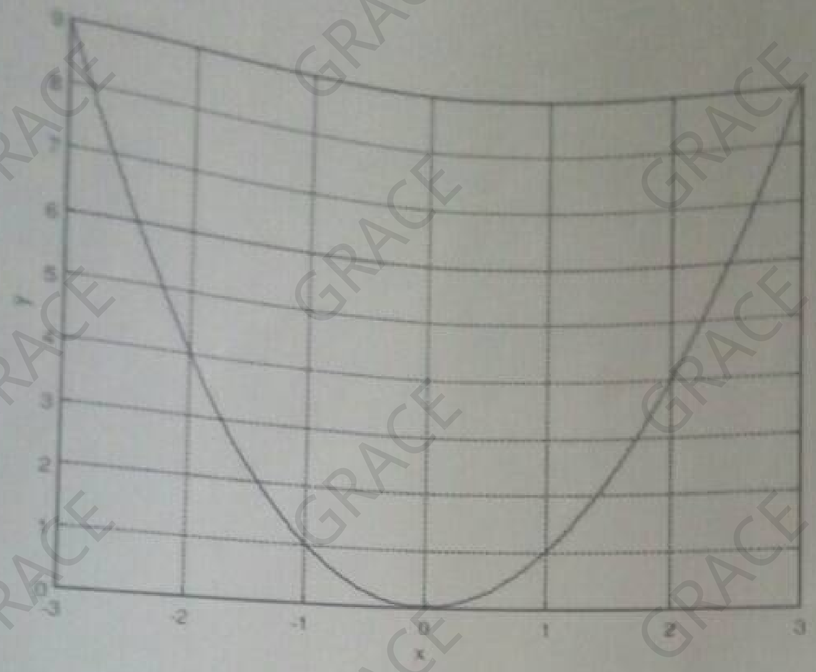

Figure 20: $f(x) = x^2$

4.10 Function and inverse

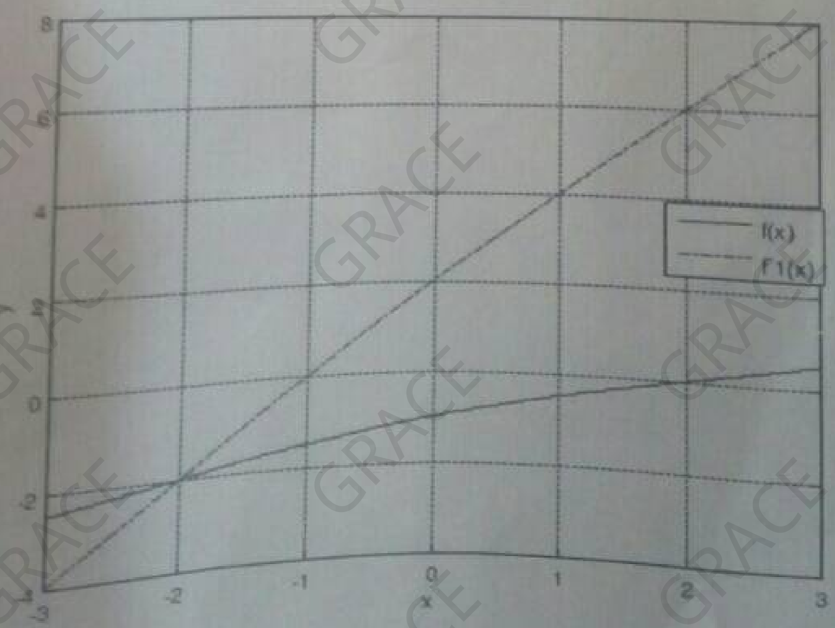

Figure 21: $f(x) = \frac{1}{2}(x - 2); f^{-1}(x) = 2(x + 1)$

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