Chapter 5: Real-Valued Functions of a Real Variable: Limits and Continuity

Background Let $D \subseteq \mathbb{R}$.

A function f from D into $\mathbb{R}$ is a rule which associates with each $x \in D$ one and only one $y \in \mathbb{R}$. Notation: $f : D \to \mathbb{R}$.

D is called the **domain** of the function.

If $x \in D$, then the element $y \in \mathbb{R}$ which is associated with x is called the **value of** f **at** x or the **image of** x **under** f. y is denoted by $f(x)$.

If $U \subseteq D$, then

$$f(U) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in U\}.$$

If $U = D$, then $f(D)$ is called the **range** of f.

If $y \in \mathbb{R}$, then

$$f^{-1}(y) = \{x \in D \mid f(x) = y\}.$$

Note: 1. $f^{-1}(y)$ might be $\emptyset$. (y is not in the range of f.)

2. $f^{-1}(y)$ might have more than one element.

3. f has an **inverse function** if for each $y \in f(D)$ there is one and only one $x \in f^{-1}(y)$.

Let $V \subseteq \mathbb{R}$. Then

$$f^{-1}(V) = \{x \in D \,|\, f(x) \in V\}.$$

Operations on functions

1. **Arithmetic:** $f, \ g \, : \, D \to \mathbb{R}$

a. $(f \pm g)(x) = f(x) \pm g(x)$

b. $(f \cdot g)(x) = f(x) \cdot g(x)$

c. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \quad g(x) \neq 0$

2. **Composition:**

Let $f : D \to \mathbb{R}$ and let $g : E \to \mathbb{R}$.

If $f(D) \subseteq E$, then g **composed with** f is the function $g \circ f : D \to \mathbb{R}$ defined by

$$(g \circ f)(x) = g\left[f(x)\right].$$

The Elementary Functions

1. Polynomial functions:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o$$

where n is a nonnegative integer,

$a_n, \ldots a_1, a_o \in \mathbb{R}$, $a_n \neq 0$.

2. Rational functions:

$$r(x) = \frac{p(x)}{q(x)}, \quad p(x), q(x) \text{ polynomials.}$$

3. Trigonometric functions and inverse trigonometric functions.

4. Exponential and logarithmic functions.

5. Combinations of the above.

Section 20: Limits of Functions

Def. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. A number L is the **limit of** f **at** c if to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$x \in D \quad \text{and} \quad 0 < |x - c| < \delta.$$

Equivalently:

Def. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. A number L is the **limit of** f **at** c if to each neighborhood V of L there corresponds a deleted neighborhood U of c such that $f(U \cap D) \subset V$.

Notation $\lim\limits_{x \to c} f(x) = L$.

THEOREM 20.1: Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. If $\lim\limits_{x \to c} f(x) = L$ exists, then it is unique. That is, f can have only one limit at c.

Examples:

1. $\lim_{x \to 3} (5x - 3) = 12.$

2. $\lim_{x \to 2} \dfrac{2x^2 + 4x - 16}{x - 2} = 12.$

3. $\lim_{x \to 5} (x^2 - 3x + 1) = 11.$

THEOREM 20.2: Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then

$$\lim_{x \to c} f(x) = L$$

if and only if for every sequence (s_n) in D such that $s_n \to c$, $s_n \neq c$ for all n, $f(s_n) \to L$.

THEOREM 20.3: Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. The following are equivalent:

1. $\lim_{x \to c} f(x)$ does not exist.

2. There exists a sequence (s_n) in D such that $s_n \to c$, but $(f(s_n))$ does not converge.

THEOREM 20.4: If

$$\lim_{x \to c} f(x) = L,$$

then there exists a neighborhood $N(c)$ of c, such that f is bounded on $N(c)$. That is, there is a number M such that

$$|f(x)| \leq M \quad \text{for all} \quad x \in D \cap N(c).$$

THEOREM 20.5: (Arithmetic)

Let $f,\ g : D \to \mathbb{R}$ and let c be an accumulation point of D. If

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M,$$

then

1. $\lim_{x\to c} [f(x) + g(x)] = L + M$,

2. $\lim_{x\to c} [f(x) - g(x)] = L - M$,

3. $\lim_{x\to c} [f(x)g(x)] = LM$,

4 $\lim_{x \to c} [k\, f(x)] = kL,\ \ k$ constant,

5 $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$.

THEOREM 20.6: ("Pinching Theorem") Let $f,\ g,\ h : D \to \mathbb{R}$ and let c be an accumulation point of D. Suppose that

$$f(x) \le g(x) \le h(x)$$

for all $x \in D,\ x \ne c$. If

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then $\lim_{x\to c} g(x) = L$.

Some basic limits:

1. $\lim_{x \to c} k = k$ for any constant k.

2. $\lim_{x \to c} x = c$.

3. $\lim_{x \to c} |x| = |c|$.

4. For any positive number c,

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}.$$

5. $\lim_{x \to c} p(x) = p(c)$ for any polynomial function $p(x)$.

6. $\lim_{x \to c} R(x) = R(c)$ for any rational function $R(x)$, provided $R(c) \neq 0$.

7. $\lim_{x \to 0} \sin x = 0$

8. $\lim_{x \to 0} \cos x = 1$

THEOREM 20.7: The following are equivalent:

$$\lim_{x\to c} f(x) = L, \qquad \lim_{h\to 0} f(c+h) = L,$$

$$\lim_{x\to c} (f(x)-L) = 0, \qquad \lim_{x\to c} |f(x)-L| = 0.$$

9. $\lim_{x \to c} \sin x = \sin c$

10. $\lim_{x \to c} \cos x = \cos c$

THEOREM 20.8: Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. If

$$\lim_{x \to c} f(x) = L > 0,$$

then there exists a deleted neighborhood $N^*(c)$ of c such that $f(x) > 0$ for all $x \in N^*(c) \cap D$.

One-sided limits:

Def. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. A number L is the **right-hand limit of** f **at** c if to each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$x \in D \quad \text{and} \quad c < x < c + \delta.$$

Notation: $\lim\limits_{x \to c^+} f(x) = L.$

A number M is the **left-hand limit of** f **at** c if to each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - M| < \epsilon$$

whenever

$$x \in D \quad \text{and} \quad c - \delta < x < c.$$

Notation: $\lim_{x \to c^-} f(x) = M.$

THEOREM 20.9: $\lim_{x \to c} f(x) = L$ if and only if each of the one-sided limits $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exists, and

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L.$$

Section 21: Continuous Functions

Def. Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f **is continuous at** c if to each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta, \ x \in D.$$

(c.f. the definition of $\lim_{x \to c} f(x)$.)

Let $S \subseteq D$. Then f is **continuous on** S if it is continuous at each point $c \in S$. f is **continuous** if f is continuous on D.

Equivalent definition:

f **is continuous at** c if to each neighborhood V of $f(c)$ there is a neighborhood U of c such that $f(U \cap D) \subseteq V$.

See the definitions on pp. 8, 9.

THEOREM 21.1: (Characterizations of Continuity). Let $f : D \to \mathbb{R}$ and let $c \in D$. The following are equivalent:

1. f is continuous at c.

2. If $\{x_n\}$ is a sequence in D such that $x_n \to c$, then

$$f(x_n) \to f(c).$$

Furthermore, if c is an accumulation point of D, then **1** and **2** are equivalent to:

3. $\lim_{x \to c} f(x) = f(c)$.

See Theorem 20.2.

What's the problem here??

If c is an isolated point of D, then f is continuous at c.

Examples:

1. Let $p(x)$ be a polynomial. Then

$$\lim_{x\to c} p(x) = p(c) \quad \text{for every } c \in \mathbb{R}$$

" polynomials are continuous functions."

2. Let $R(x) = \dfrac{p(x)}{q(x)}$ be a rational function. Then

$$\lim_{x\to c} R(x) = R(c)$$

for every $c \in \mathbb{R}$ such that $q(c) \neq 0$.

3. Since

$$\lim_{x \to c} \sin x = \sin c$$

and

$$\lim_{x \to c} \cos x = \cos c$$

for every $c \in \mathbb{R}$, sine and cosine are continuous functions.

THEOREM 21.2: Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at c if and only if there is a sequence $\{x_n\}$ in D such that $x_n \to c$ but $\{f(x_n)\}$ does not converge to $f(c)$.

See Theorem 20.3.

Combinations of Functions

THEOREM 21.3: (Arithmetic)

Let f, $g : D \to \mathbb{R}$ and $c \in D$. If f and g are continuous at c, then

1. $f \pm g$ is continuous at c.

2. fg is continuous at c; kf is continuous at c for constant k.

3. f/g is continuous at c provided $g(c) \neq 0$.

THEOREM 21.4: (Composition) Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at $c \in D$ and g is continuous at $f(c) \in E$, then the composition of g with f, $g \circ f : D \to \mathbb{R}$, is continuous at c.

THEOREM 21.5: A function $f : D \to \mathbb{R}$ is continuous on D if and only if for each open set G in $\mathbb{R}$ there is an open set H in $\mathbb{R}$ such that $H \cap D = f^{-1}(G)$.

Corollary: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(G)$ is open whenever G is open.

THEOREM 21.6: Let $f : D \to \mathbb{R}$ be continuous. If $f(c) > 0$, then there is a neighborhood $N(c)$ of c such that $f(x) > 0$ for all $x \in N(c)$, $x \in D$.

See Theorem 20.8.

Section 22. Properties of Continuous Functions

Def. A function $f : D \to \mathbb{R}$ is **bounded** if there exists a number M such that $|f(x)| \leq M$ for all $x \in D$. That is, f is bounded if $f(D)$ is a bounded subset of $\mathbb{R}$.

Examples:

1. $f(x) = \sin x$ and $g(x) = \cos x$ are bounded functions on $D = \mathbb{R}$.

$|\sin x| \le 1$, $|\cos x| \le 1$ for all $x \in \mathbb{R}$.

2. $f(x) = \dfrac{1}{1-x}$, $x \in [0,1)$ is not bounded.

3. Polynomial functions of degree $n \ge 1$ on $D = \mathbb{R}$ are not bounded.

The Extreme-Value Theorem.

THEOREM 22.1: Let $f : D \to \mathbb{R}$ be continuous. If D is compact, then $f(D)$ is compact.

Def. Let $f : D \to \mathbb{R}$. $f(x_0)$ is the **minimum value** of f on D if $f(x_0) \leq f(x)$ for all $x \in D$; $f(x_1)$ is the **maximum value** of f on D if $f(x) \leq f(x_1)$ for all $x \in D$.

Corollary 1. If $f : D \to \mathbb{R}$ is continuous and D is compact, then f has a maximum value and a minimum value. That is, there exist points x_0, $x_1 \in D$ such that

$$f(x_0) \leq f(x) \leq f(x_1) \text{ for all } x \in D.$$

Corollary 2. If $f : D \to \mathbb{R}$ is continuous and D is compact, then $f(D)$ is closed and bounded.

THEOREM 22.2: Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a)$ and $f(b)$ have opposite sign, then there is at least one point $c \in (a, b)$ such that $f(c) = 0$.

The Intermediate-Value Theorem.

THEOREM 22.3: Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) \neq f(b)$. If k is a number between $f(a)$ and $f(b)$ then there is at least one point $c \in (a, b)$ such that $f(c) = k$.

Corollary 1. If $f : D \to \mathbb{R}$ is continuous and $I \subseteq D$ is an interval, then $f(I)$ is an interval.

Corollary 2. If $f : D \to \mathbb{R}$ is continuous and $I \subseteq D$ is a compact interval, then $f(I)$ is a compact interval.

Examples:

1. Suppose $f : [a, b] \to [a, b]$ is continuous. Then there is at least one point $x \in [a, b]$ such that

$$f(x) = x.$$

Such a point x is called a **fixed point** of f

2. If $f, g : [a,b] \to [a,b]$ are continuous, then there is at least one point $x \in [a,b]$ such that

$$f(x) = g(x).$$

WHAT????

2. If $f, g : [a, b] \to [a, b]$ are continuous, and if $f(a) \leq g(a)$ and $f(b) \geq g(b)$, then there is at least one point $x \in [a, b]$ such that

$$f(x) = g(x).$$

3. Prove that there is a least one real number r such that $r^2 = 2$.

4. Prove that if p is a polynomial of odd degree, then there is at least one real number c such that $p(c) = 0$.

5. Let $\mathcal{R}$ be the set of all rectangles with perimeter $P = 10$. Prove that there is a member of $\mathcal{R}$ that has maximum area.