Quantum Walks and Monte Carlo: A 2-page Summary

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Abstract

The work in [CV22] features a rapid quantum algorithm that simulates the classical Galton Board, an interesting object known for its demonstration of core probability ideas, including Gaussian distributions and the Central Limit Theorem, as well as its applications in modeling, machine learning, and system simulations [CV22]. This paper aims to demonstrate our team's understanding of the concepts behind the algorithm, as well as propose a method of generalizing it to any number of levels.

Introduction

The Galton Board is an interesting statistical device useful for conducting random experiments. It is intriguing because it is a practical demonstration of fundamental ideas in probability (such as Bernoulli trials and the Central Limit Theorem), as well as its multiple uses in applied fields due to its ability to act as a statistical simulator [CV22]. In this paper, we discuss our team's method of implementing a Galton board of arbitrary size using an $O(N^2)$-deep quantum circuit. We explain our understanding of the concepts behind the algorithm described in [CV22], and demonstrate that it can be generalized to any number of levels.

Base Case (1-level QGB)

The basis for the Quantum Galton Board is a 4-qubit circuit that represents a quantum 'peg', with the ball 'dropped' on the peg by initializing the third qubit in the $|1\rangle$ state. The main idea is that when the ball hits a peg, it bounces either left or right, and then it hits another peg. This idea is reflected in the quantum circuit by setting the control qubit in equal superposition, and then applying controlled swaps that represent the two paths the ball can take. The exact mechanics of the circuit are summarized in the following theorem:

Theorem 1. *The circuit below is representative of a ball bouncing off a peg; in particular, exactly one of $|q_1\rangle$ and $|q_3\rangle$ is in the state $|1\rangle$, each with 50% probability.*

Proof. The state $|q_3q_2q_1q_0\rangle$ evolves like so:

$$\begin{aligned}|0000\rangle \to |0100\rangle \to \frac{1}{\sqrt{2}}\left(|0100\rangle + |0101\rangle\right) &\to \frac{1}{\sqrt{2}}\left(|0100\rangle + |0011\rangle\right) \\ &\to \frac{1}{\sqrt{2}}(|0101\rangle + |0011\rangle) \to \frac{1}{\sqrt{2}}\left(|1001\rangle + |0011\rangle\right),\end{aligned}$$

and so $|q_3q_1\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right)$. The result follows. □

2-level QGB: The intermediate CNOT

Recall that a 2-level Galton Board has 3 pegs arranged in a triangular fashion. The circuit for the 2-level QGB mimics this architecture, replicating the quantum 'peg' in Figure 1 three times to achieve the layout shown in Figure 2.

The crucial addition to the circuit is a CNOT gate controlled by the qubit serving as the input channel for the 'ball' acting on the control qubit, placed in the middle of the two 'lower' quantum peg circuits. This CNOT plays a really important role: it is possible (in fact, 25% likely) for the 'ball' to go left after the first peg, and then go right and land on its original channel after the second peg. In this case, the control qubit would be a 1. The CNOT then switches the control back to 0 and thus prevents the qubit from activating the last quantum peg, incorrectly sending the 'ball' to the right.

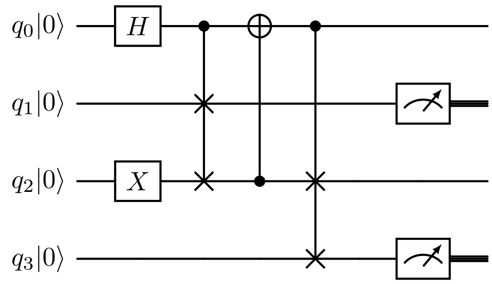

Figure 1: The Quantum Peg circuit

Through a computation similar to that in the proof of Theorem 1 (or by an inductive-style argument), it can be shown that the circuit in Figure 2 really does simulate a 2-level GB.

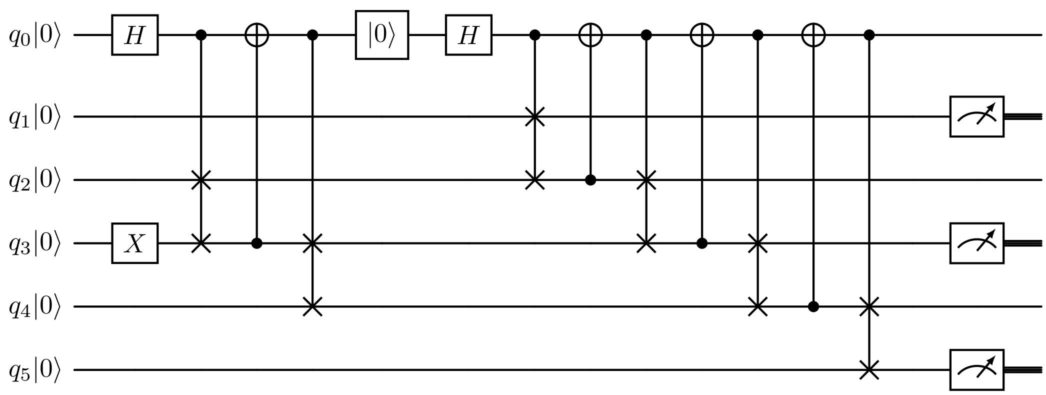

Figure 2: The 2-level QGB (featuring the very important CNOT)

Generalization: n-level QGB

Recall from the 2-level QGB that the 'ball' would incorrectly fall to the rightmost 'peg' in the last layer without the CNOT gate in between the pegs. This scenario occurs every time the ball completes its course on one of the interior qubits (as per what holds in the classical GB). Hence, in the n-level case, a CNOT would need to be placed between any two pegs in the same layer. Besides this, the logic in the 1- and 2- level QGBs can easily be generalized to the n-level case.

Conclusion

We have shown how a quantum peg can be modeled using a quantum circuit featuring only H, CNOT, and SWAP gates, and demonstrated a method of generalizing the construction to simulate a classical Galton Board with an arbitrary number of levels. This circuit computes 2^n trajectories using $O(n^2)$ quantum operations [CV22], providing an example of a quantum algorithm that offers an exponential speedup compared to classical analogues.

The QGB described here is designed assuming that the ball goes left or right with equal probability, and thus produces a binomial distribution. Future work would involve creating a biased QGB that is capable of achieving other distributions, such as exponential, Hadamard, or log-normal distributions.

References

[CV22] Mark Carney and Ben Varcoe. *Universal Statistical Simulator*. arXiv:2202.01735 [quant-ph]. 2022. arXiv: `2202.01735 [quant-ph]`. URL: `https://arxiv.org/abs/2202.01735`.