FUNDAMENTALS OF COMPLEX NUMBERS

1. Introduction

Undoubtedly you have heard about numbers, its different classes and categorisations. A particular classification of numbers is their division into real and imaginary numbers, which is the focus of this work. Most of the numbers you might have encountered are real. They are real in the technical- conventional sense and have nothing to do with any linguistic meaning. Examples include integers, rational and irrational numbers, inter alia. Surds are also considered real numbers.

Imaginary numbers on the other hand are not very common except in science and engineering, and together with real numbers form the basis of complex numbers. Do not panic. They are called complex numbers not because they are difficult to understand or work with – this is merely conventional. Although as a precaution, we should be aware that other numbers are not referred to as 'simple' numbers. The core difference between complex numbers and other numbers are few and it only takes understanding few tips about what constitutes complex numbers in order to spot these dissimilarities. Additionally, you will need to apply more principles from other topics in mathematics, particularly surds, to solve complex number problems than using new rules or theorems from this topic.

Without further ado, let us delve into the explanation now. If you are however familiar with this concept, feel free to proceed to the worked examples. Look out for footnotes strategically placed to provide further information regarding the solutions.

2. Complex Number

2.1. Quadratic equations

In case you have not studied this for a while, let me briefly remind you that a quadratic equation is a polynomial of degree two with two unknown variables. The solutions to a quadratic equation can be obtained by using: (i) graphical, (ii) factorisation, (iii) completing the square, or (iv) quadratic formula method.

The solutions to $ax^2 + bx + c = 0$ can be derived using the quadratic formula given by

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\boldsymbol{= \frac{-b \pm \sqrt{D}}{2a}}$$

where

$$\boldsymbol{D = b^2 - 4ac}$$

The formula above indicates that $\boldsymbol{D}$ is a determinant [1] or discriminant [2]. We say that when $D \geq 0$ there are two solutions of x and when $D < 0$, i.e. when D is negative, there are no solution for x. The reason for the 'no-solution' conclusion is primarily because there is no 'solution' to the square root of a negative number. Why? Well, it is generally known that $(-) \times (-) = +$ and $(+) \times (+) = +$. In other words, the square of a number, positive or negative, will only produce a positive number. Conversely, to find the square root of a number, the number must be a product of two positive or negative numbers. Therefore, a negative number cannot be a product of two positive or negative numbers. Let's quickly illustrate this

$$9 = (3) \times (3) \qquad \text{and} \qquad 9 = (-3) \times (-3)$$

Therefore,

$$\sqrt{9} = 3 \qquad \text{and} \qquad \sqrt{9} = -3$$

This is because

$$\sqrt{9} = \sqrt{(3) \times (3)} = 3 \qquad \text{and} \qquad \sqrt{(-3) \times (-3)} = -3$$

Or probably, you can apply one of the laws of indices to have

$$\sqrt{9} = \sqrt{3^2} = (3^2)^{\frac{1}{2}} = 3 \qquad \text{and} \qquad \sqrt{9} = \sqrt{(-3)^2} = [(-3)^2]^{\frac{1}{2}} = -3$$

With this illustration it becomes obvious that using any previously known principle, it is impossible to solve $\sqrt{-9}$ or any negative square root. However, this booklet will shortly show you how to unravel this problem.

Before I move on I would like to draw your attention to an analogy. The no-solution conclusion illustrated above can be compared to asking a child to solve: $\boxed{2 - 3}$ (that is two take away three). The most probable answer is that such arithmetic problem is either incorrect or impossible. Subsequently, the child would realise that this problem is solvable and its answer is $\boxed{-1}$ (negative one). The reason for the initial impossible answer is due to the child's limited knowledge about numbers. Similarly, due to our

[1] This is because it determinates the possibility of having a real or otherwise solution.
2 This is because it 'discriminates' between the two possibilities.

acquaintance with only real numbers, we are unable to find the square root of negative numbers.

2.2. The j operator

j is simply an operator that works just like $\times, \sqrt{}, \cup,$ % or other mathematical operators. Letter j is used in engineering while i is employed in physics and mathematics for vector analysis, although j and i denote the same numerical process.

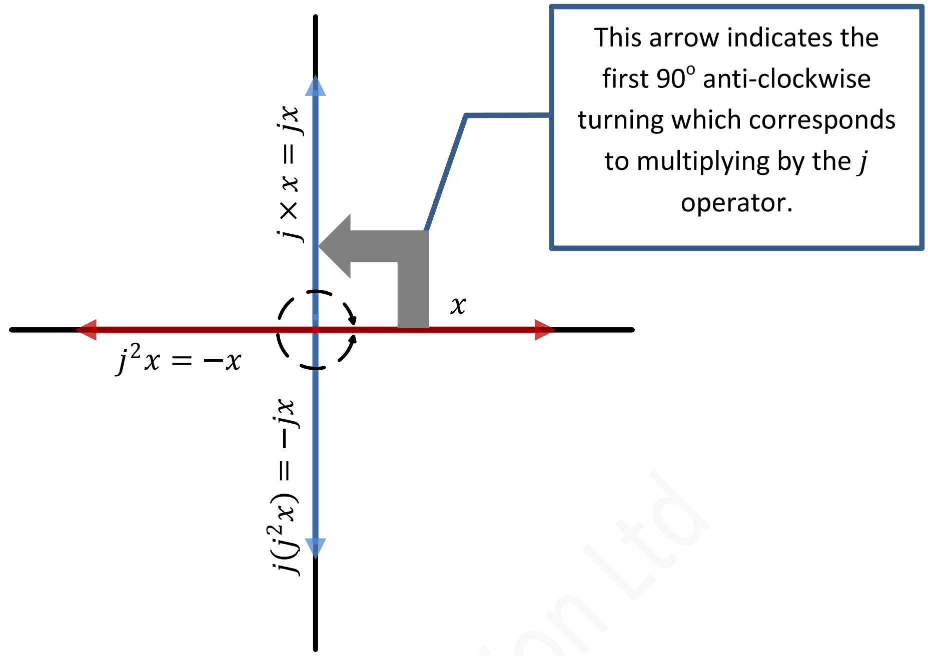
As we know i symbolises electric current in engineering and usually complex number is applied in AC circuit analysis where electric current is also used. Consequently, it will be difficult and confusing to discern when i is used as electric current or complex number operator. For this reason, j is adopted in engineering and this notation will be used throughout this booklet even when solving questions related to vector analysis.

So what is j? The answer is simply

$$\boldsymbol{j = \sqrt{-1}} \quad \text{or} \quad \boldsymbol{j^2 = -1}$$

In other words, j is 90° counter-clockwise rotation of a vector usually taken from the positive x-axis. It should be noted that when such a rotation is performed twice, the vector will be on the same line but pointing towards the opposite direction. It is therefore not surprising that $\boxed{j^2 = -1}$. In addition, when 360° rotation is carried out on a vector, the vector will return to its original position and again $\boxed{j^4 = 1}$. We use j^4 because there are four 90° in 360°.

For example, when a vector of x units originally pointing East (that is on the positive x-axis) is rotated counter-clockwise by 90°, the new vector can be regarded as $\boxed{j * x}$ units pointing northward or on the positive y-axis. If the same vector is rotated again in the same direction and with the same magnitude, i.e. 90° counter-clockwise, the new vector has a value of $\boxed{j * j * x}$ units, which is – x units. It therefore follows that $\boxed{j * j = j^2 = -1}$.



The third and fourth rotations will produce vectors of values $\boxed{j * j * j * x}$ units and $\boxed{j * j * j * j * x}$ units respectively. Obviously after the third rotation, the vector will be pointing southward. Because it is on the negative y-axis, its value is the same in magnitude but opposite to the value of the vector pointing northward. Therefore, $\boxed{j * j * j * x \ = \ -j * x}$ or $\boxed{j^3 = -j}$. Similarly, the fourth rotation will be identical in magnitude and direction to the initial vector, thus $\boxed{j * j * j * j * x \ = \ x}$ or $\boxed{j^4 = 1}$. In summary, we have

$$\boldsymbol{j = \sqrt{-1}}$$

$$\boldsymbol{j^2 = -1}$$

$$\boldsymbol{j^3 = -j}$$

$$\boldsymbol{j^4 = 1}$$

$$\boldsymbol{\frac{1}{j} = -j}$$

$$\boxed{\textbf{Note: } \frac{1}{j} = \frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j}$$

It is important to note that j^n has only four possible values, namely: $1,\ j, -1, -j$. They respectively correspond to values of n which when divided by 4 leave the remainders

0, 1, 2, 3. For example, $j^{137} = j$ since dividing 137 by 4 leaves 1 as a remainder and this corresponds to j.

So the problem of the square roots of negative numbers is finally solved. We can now say that

$$\sqrt{-9} = \sqrt{-1 \times 9}$$

$$= \sqrt{-1} \times \sqrt{9}$$

$$= j \times \pm 3$$

$$= \pm j3$$

To reiterate, whenever we take the square root of a number, we always have two answers. The two solutions are equal in magnitude but with opposite signs. Generally, $\pm x$ is used to denote this, where x is the magnitude (or number).

Numbers such as $\sqrt{-9} = \pm j3$, are called imaginary numbers. A complex number is a number that is made up of two parts, the real and the imaginary, which is expressed as

$$\boldsymbol{z = x + jy}$$

In the above equation, x is the real part (or component) of the complex number, z abbreviated as $Re\ (z)$ and the imaginary part is y, shortened as $Im\ (z)$. Note that the imaginary does not include the j operator itself. Sometimes the j operator is written first before the number, or vice versa i.e. a number before the j operator. Therefore, $j2$ can be written as $2j$. Is there any number that is both real and imaginary? Yes, the answer is 0. This is simply because $+0 = -0$ so we can regard $\sqrt{0}$ as real and $\sqrt{-0}$ as imaginary.

One other point that should be remembered is that in engineering, the real part refers to the active (or in-phase) component and the imaginary is the reactive (or quadrature) part.

2.3. Operation with complex numbers

Addition, subtraction, multiplication and division can be performed on complex numbers. The rules governing these operations are similar to other numbers with a slight manipulation. Let us summarise the tips here:

- **Addition and Subtraction**: add/subtract the real parts and add/subtract the imaginary parts. That is all you need to do.

- **Multiplication**: open the brackets as usual or carry out the multiplication as you would normally do but remember that $j \times j = -1$. In other words, if numbers with j terms or values are multiplied, j will disappear leaving the new term (now a real number) with a negative sign. Again, it is as simple as that.
- **Division**: division of complex numbers – either a complex number by another complex number or a number by a complex number - is not straight forward. It can only be done using the concept of rationalising the denominator. Remember that multiplying a surd with its conjugate results in a rational number [3]. Similarly, when a complex number is multiplied by its conjugate complex number, the result is a real number.

Given a complex number $z = x + jy$, its complex conjugate is denoted as $\bar{z} = x - jy$. Note that we only change the sign between real and imaginary parts of the complex number to obtain a complex conjugate pair. $z*$ is also used to indicate a complex conjugate. The following are true of a complex number and its conjugate.

If

$$z = x + jy$$

then

$$z(\bar{z}) = x^2 + y^2$$

$$\overline{(z_1 z_2)} = \overline{z_1 z_2}$$

$$z + \bar{z} = 2Re(z)$$

$$z - \bar{z} = j2\,Im\,(z)$$

2.4. The Argand diagram

An Argand diagram, named after the French mathematician Jean-Robert Argand (1768 - 1822), is a geometrical plot of complex numbers on x-y Cartesian plane, also known as complex plane. The x-axis (horizontal axis) represents the real parts and the y-axis (vertical axis) represents the imaginary parts of complex numbers. They are called real axis and imaginary axis respectively.

2.5. Polar and exponential forms

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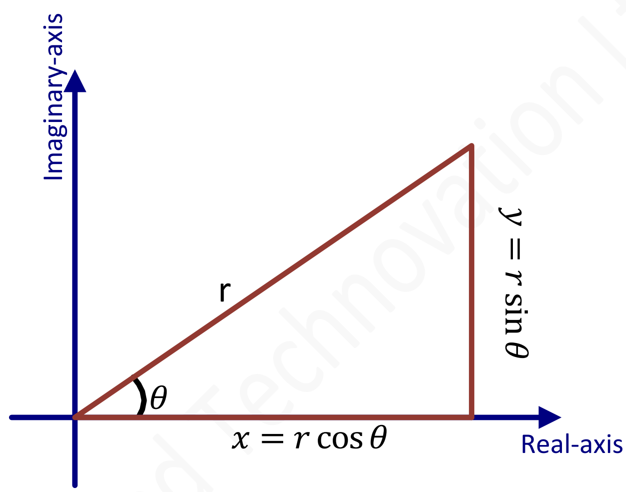
[3] For instance, if we multiply $\{4 + 2\sqrt{3}\}$ by its conjugate $\{4 - 2\sqrt{3}\}$, we obtain 2, i.e. $\boxed{4^2 - 2^2(3)}$.

The format of the complex number used so far is known as the Cartesian (or the rectangular) form and is written as $\boxed{x + jy}$; again x is the real part and y is the imaginary part. There are other two forms, namely polar and exponential. The polar form is denoted by $\boxed{z = r < \theta}$ where

$$r = \sqrt{x^2 + y^2}$$

r is called the modulus (or magnitude) of the complex number, which is the length of the line joining the origin to the point representing the complex number. It is abbreviated as $mod\ z$ or $|z|$. On the other hand, θ is the angle between the positive real-axis and the line joining the complex with the origin as shown in the figure below.



It follows that

$$\theta = \ tan^{-1}\left\{\frac{imaginary\ part}{real\ part}\right\}$$

$$= tan^{-1}\left\{\frac{y}{x}\right\}$$

The angle θ is generally called the argument of the complex number and is written as $\arg(z)$. Note that the angle must be measured from the 1st quadrant i.e. positive x-axis and it must be in the interval $-\pi < \theta \leq \theta$.

The other form is known as the exponential form represented thus:

$$z = re^{\pm j\theta} \quad --- \ exponential\ form$$

where r is the modulus of z and θ the argument of z measured in radians unlike degrees in polar form.

It is important to add that when carrying out addition and subtraction of complex numbers, Cartesian form is useful. However, multiplication and division of these numbers can be easily evaluated in polar or exponential form. For example, consider two complex numbers $z_1 = x_1 + jy_1 = r_1 < \theta_1$ and $z_2 = x_2 + jy_2 = r_2 < \theta_2$, the four operations are

$$\boldsymbol{z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)}$$

$$\boldsymbol{z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)}$$

$$\boldsymbol{z_1 * z_2 = r_1 * r_2 < (\theta_1 + \theta_2)}$$

$$\boldsymbol{\frac{z_1}{z_2} = \frac{r_1}{r_2}(\theta_1 - \theta_2)}$$

The following relationship is also valid and can be proven.

$$\boldsymbol{|z_1 * z_2| = |z_1||z_2|}$$

$$\boldsymbol{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$$

and

$$\boldsymbol{\arg(z_1 * z_2) = \arg(z_1) + \arg(z_2)}$$

$$\boldsymbol{\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)}$$

Conversion between various forms, particular a polar to rectangular and vice-versa, is inevitable. A complex number $z = r < \theta$ in polar form has its rectangular form as $z = x + jy$ such that

$$\boldsymbol{x = r\cos\theta}$$

$$\boldsymbol{y = r\sin\theta}$$

It is important to sketch an Argand diagram to determine the quadrant of a complex number when converting between the various forms since there are two possible values for angle θ between $0°$ and $360°$ $(0 - 2\pi\ rad)$.

The relationship between the different forms of complex numbers is summarised below.

Complex Number Forms

$$z = x + jy \qquad --- \; rectangular\ form$$

$$z = r(\cos\theta \ \pm j\sin\theta\,) \qquad ----polar\ form$$ [4]

$$z = r < \pm\theta \qquad ---\ shortened\ polar\ form$$ [5]

$$z = re^{\pm j\theta} \qquad ---\ exponential\ form$$

2.6. Equation involving complex numbers

When dealing with complex number equations, there are two rules to be applied

i) If a complex number is equal to zero, both the real part and the imaginary part must also be equal to zero. In other words, if $x + jy = 0$ then $x = 0$ and $y = 0$.

ii) If two complex numbers are equal, then their real parts and imaginary parts must be equal. For instance, given that $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ such that $z_1 = z_2$ then $x_1 = x_2$ and $y_1 = y_2$.

The second rule is derived from the first as follows. From rule (ii), $z_1 = z_2$ implies that

$$x_1 + jy_1 = x_2 + jy_2$$

Collect the like terms

$$(x_1 - x_2) + j(y_1 - y_2) = 0$$

Applying the first rule, we have

$$x_1 - x_2 = 0$$

this implies that

$$x_1 = x_2$$

and

$$y_1 - y_2 = 0$$

this gives

$$y_1 = y_2$$

[4] This is also called a trigonometrical form in some textbooks, while some authors consider it as a rectangular form.
[5] It is simply a representation, which can be considered as a derivative of the exponential form except that its angle is measured in degrees while the former is in radians.

as before.

2.7. Phasor and j operator

In AC (alternating current) circuits, the current and voltage are either in phase, current leading the voltage or current lagging the voltage depending on the elements in the circuit. In a purely resistive circuit (in which only resistors are connected) the voltage and the current are said to be in phase. In a RL circuit (having resistance R and inductance L), the voltage leads the current while current leads in a RC (having resistance R and capacitance C) circuit. The lead and lag angle of the current is 90° in a purely-capacitive and purely-inductive circuit respectively.

It is known that the angle between the x-axis and y-axis is 90°, so it is possible that if the positive x-axis is taken to represent the voltage in a RC series circuit, the circuit current can be represented by the positive y-axis. Similarly, in a RL circuit, the current can be represented by the negative y-axis while the voltage is kept on the positive x-axis. In fact, 90° leading is akin to multiplying a quantity by $\boxed{j}$ while 90° lagging is similar to multiplying a number by $\boxed{-j}$. In RLC circuits, the lagging / leading factors vary; it is however possible to identify which element is leading or lagging by expressing both quantities in polar form.

This graphical representation of voltages and currents in an AC circuit is known as phasor diagram, which is similar to an Argand diagram.

The following equations are handy when dealing with an AC circuit analysis.

$$Z_{RC} = R - jX_c \quad --- for\ RC\ circuits$$

for which

$$X_c = \frac{1}{2\pi f C}$$

and

$$Z_{RL} = R + jX_L \quad --- for\ RL\ circuits$$

for which

$$X_L = 2\pi f L$$

where Z = impedance of the circuit, X_C = capacitive reactance, X_L = capacitive inductance, R = resistance, C = capacitance and L = inductance. In general, the impedance in a RLC circuit is given by

$$Z = R + jXZ = R + j\,(X_L - X_c)$$

2.8. De Moivre's theorem

De Moivre's theorem is used to find the roots and powers of complex numbers. It states that if

$$z = r < \theta$$

then

$$z^n = [r < \theta]^n$$

$$= r^n < (n\theta)$$

$$= r^n(\cos n\theta + j\sin n\theta)$$

This is because if

$$z = re^{j\theta}$$

then

$$z^n = [re^{j\theta}]^n$$

$$= r^n e^{j(n\theta)}$$

$$= r^n < (n\theta)$$

This theorem is valid for any real value of n, i.e. positive, negative, whole and fractional numbers.

2.9. nth root of complex number

As previously mentioned, there are two values for the square root of a number. Similarly, when the square root of a complex number is taken, two complex numbers are produced having the same magnitude $|z| = r$ such that their arguments only differ in sign. If one of the angles is θ, the other will be $[\theta + (360°)/2]$ or $(\theta + 180°)$. Hence, the square roots of a complex number z, are $|z| < \theta$ and $|z| < (\theta + 180°)$. You may wonder why angles θ and $(\theta + 180°)$ are said to be equal but only opposite in sign. Yes,