Modeling and Simulation

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Lecture 6:

Stochastic Processes and Markov-Chain

Stochastic Processes

Many random systems of interest display some form of time-dependent change, evolving from one state to another as time passes, for instance:

- Stock market prices
- The number of incoming calls in a given period of time at a center call
- The number of packets queued in a router's buffer,
- ...

To model such systems, the notion of a **stochastic process** (or **random process**) is useful.

So, What is a stochastic process?

Stochastic Processes

A **stochastic process** represents a situation where uncertainty is present.

In other words, it's a process that has some kind of randomness.

Formal Definition of a Stochastic Process:

A family of random variables, $\{X(t) : t \in T\}$, where t usually denotes time. That is, at every time t in the set T, a random number $X(t)$ is observed.

Stochastic Processes: Types

- $\{X(t) : t \in T\}$ is a **discrete-time process** if the set T is finite or countable.
  - In practice, this generally means $T = \{0, 1, 2, 3, ...\}$
  - Thus a discrete-time process is $\{X(0), X(1), X(2), X(3), ...\}$: a random number associated with every time $0, 1, 2, 3, ...$
- $\{X(t) : t \in T\}$ is a **continuous-time process** if T is not finite or countable.
  - In practice, this generally means $T = [0, \infty)$, or $T = [0, K]$ for some K.
- The **state space**, S, is the set of real values that $X(t)$ can take.
  - Every $X(t)$ takes a value in $\mathbb{R}$, For example, if $X(t)$ is the outcome of a coin tossed at time t, then the state space is $S = \{0, 1\}$.

Recall:

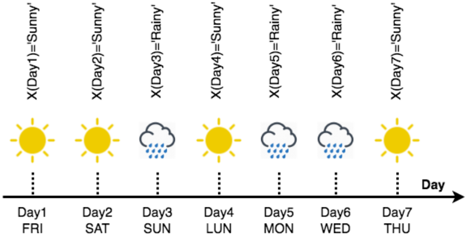
The **state space** S is the set of states that the stochastic process can be in.

Stochastic Processes: The weather Status Example

- $X(Day_i)$: Status of the weather observed each DAY

$X(Day_i) = \{$**'Sunny'** or **'Rainy'**$\}$: Random Variables that varies with the day.

State Space
$S = \{'Sunny', 'Rainy'\}$



Stochastic Processes: Markov Process

- **Markov Process** is a stochastic process that satisfies the Markov property
- **Markov property:** the future of a process does not depend on its past, only on its present

$$X_{t+1} \text{ depends only on } X_t.$$

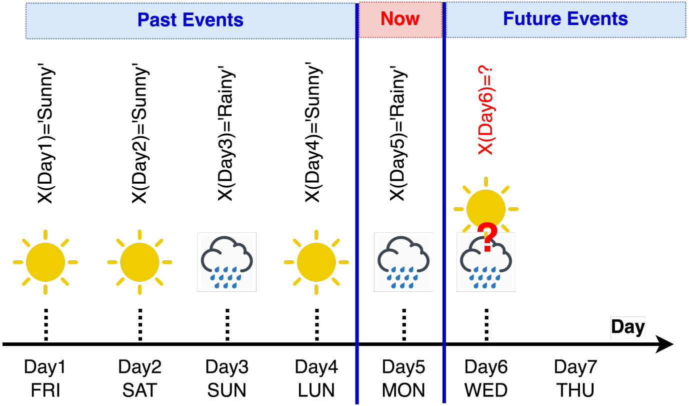
Figure 1: Andrey Markov (1856-1922)

Formal Definition of Markov Property

$$P(X_{t+1} = s | X_t = s_t, X_{t-1} = s_{t-1}, ..., X_0 = s_0) = P(X_{t+1} = s | X_t = s_t),$$

for all $t = 1, 2, 3, ...$ and for all states $s_0, s_1, ..., s_t, s$

Markov Process: Markov Property - Weather Example



Markov Property: The probability that it will be (FUTURE) SUNNY in DAY 6 given that it is RAINNY in DAY 5 (NOW) is <u>independent</u> from PAST EVENTS

Stochastic Processes and Markov-Chain

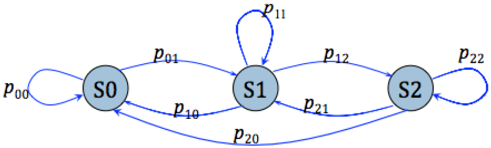
Markov Process: Markov-Chain

Definition: Let $\{X_0, X_1, X_2, ...\}$ be a sequence of discrete random variables. Then $\{X_0, X_1, X_2, ...\}$ is a **Markov chain** if it satisfies the Markov property:

$$P(X_{t+1} = s | X_t = s_t, X_{t-1} = s_{t-1}, ..., X_0 = s_0) = P(X_{t+1} = s | X_t = s_t),$$

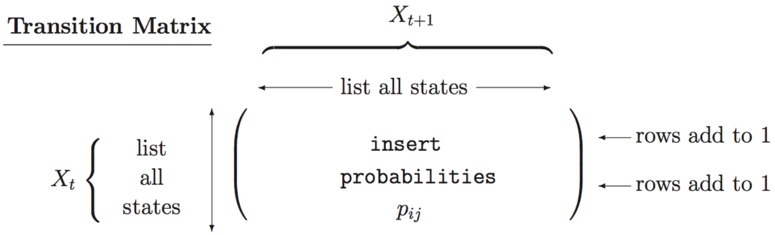
Graph Representation of a Markov Chain:

State Space: $S = \{S0, S1, S2\}$
Discrete Time: $T = \{0, 1, .., k\}$
Transition Probability: P_{ij}



Markov Process: Markov-chain – The Transition Matrix

We can also summarize the Graph probabilities using the **Transition Matrix**:



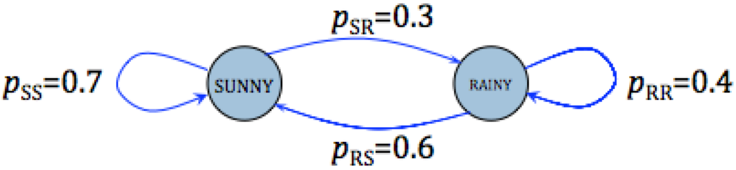
In the transition matrix:

- The ROWS represents **NEXT**, or **TO(**X_{t+1}**)**
- The COLUMNS represents **NOW**, or **FROM(**X_t**)**
- Entry (i, j) is the CONDITIONAL probability :

$$p_{ij} = P(X_{t+1} = j | X_t = i),$$

Markov Process: Markov-chain

The weather Example:



State Space: $S = \{SUNNY, RAINY\}$

The Transition Matrix:

Let $\mathcal{P}_{ij}$ denote the transition Matrix of the weather example. Then:

$$\mathcal{P}_{ij} = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

The t-step Transition probabilities

Let $X_0, X_1, X_2, ...$ be a Markov chain with state space $S = 1, 2, ..., N$.

Question: The 2-step transition

what is the probability of making a transition from state i to state j over 2 steps? I.e. what is $P(X_2 = j|X_0 = i)$?

Answer: The 2-step transition

The probabilities of the 2-step transition are given by the matrix P^2:

$$P(X_2 = j|X_0 = i) = P(X_{n+2} = j|X_n = i) = \mathcal{P}_{ij}^2 \text{ for any } n.$$

Caution: P^2 is just a notation of the $t = 2$ steps, which means that we want to predict for the next $n + 2$ steps the probability that $X_2 = j$ knowing $X_0 = i$.

The t-step Transition probabilities

Question: 3-step transitions:

We can find $P(X_3 = j|X_0 = i)$ similarly, but conditioning on the state at time $t = 2$:

Answer:3-step transitions:

The probabilities of the 3-step transitions are given by the matrix P^3:

$$P(X_3 = j|X_0 = i) = P(X_{n+3} = j|X_n = i) = \mathcal{P}^3_{ij} \text{ for any } n.$$

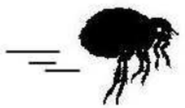
General case: t-step transitions

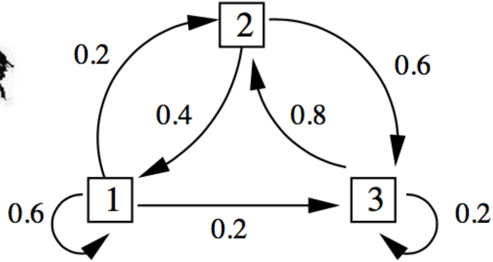
The above working extends to show that the t-step transition probabilities are given by the matrix P^t for any t:

$$P(X_t = j|X_0 = i) = P(X_n + t = j|X_n = i) = \mathcal{P}^t_{ij} \text{ for any } n.$$

Stochastic Processes and Markov-Chain

The t-step Transition probabilities: Working Example





Transition Matrix:

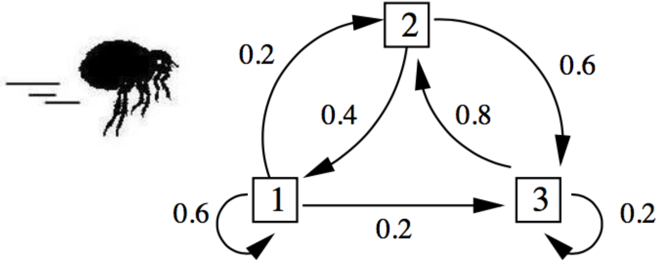
$$\mathcal{P}_{ij} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

- Find $P(X_2 = 3 | X_0 = 1)$ (The Probability that $X_2 = 3$ knowing $X_0 = 1$).

$$P(X_2 = 3|X_0 = 1) = \mathcal{P}^2_{13} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ . & . & . \\ . & . & . \end{pmatrix} * \begin{pmatrix} . & . & 0.2 \\ . & . & 0.6 \\ . & . & 0.2 \end{pmatrix} = 0.28$$

Stochastic Processes and Markov-Chain

The t-step Transition probabilities: Working Example



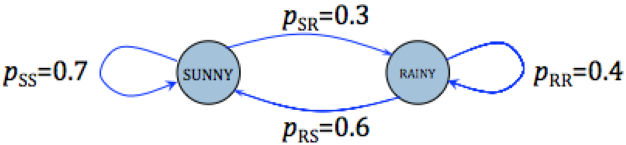
Transition Matrix:

$$\mathcal{P}_{ij} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

- Similarly to find the transition matrix $\mathcal{P}_{ij}^2$ for all i, j

$$\mathcal{P}_{ij}^2 = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix} * \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.28 & 0.28 \\ 0.24 & 0.56 & 0.2 \\ 0.32 & 0.16 & 0.52 \end{pmatrix}$$

The t-step Transition probabilities: The Weather Working Example



Transition Matrix:

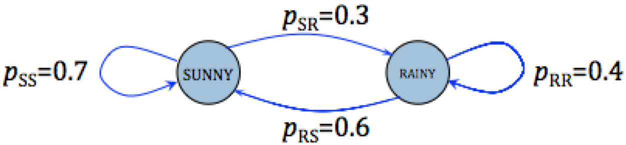
$$\mathcal{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

- What is the probability that the weather is RAINY on day 3 knowing that it is SUNNY on day 1?

$$P(X_{day_3} = RAINY | X_{day_1} = SUNNY) = \mathcal{P}^2_{SUNNY,RAINY}$$

$$P(X_{day_3} = RAINY | X_{day_1} = SUNNY) = \begin{pmatrix} 0.7 & 0.3 \\ . & . \end{pmatrix} * \begin{pmatrix} . & 0.3 \\ . & 0.4 \end{pmatrix} = 0.33$$

The t-step Transition probabilities: The Weather Working Example



Transition Matrix:

$$\mathcal{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix}$$

- Find all the probabilities of the weather status on day 3 given status on day 1

$$\mathcal{P}^2_{i,j} = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} * \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{pmatrix}$$

- Find all the probabilities of the weather status on day 4 given status on day 1

$$\mathcal{P}^3_{i,j} = \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} * \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} * \begin{pmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{pmatrix}$$

Markov-Chain: The Weather Working Example

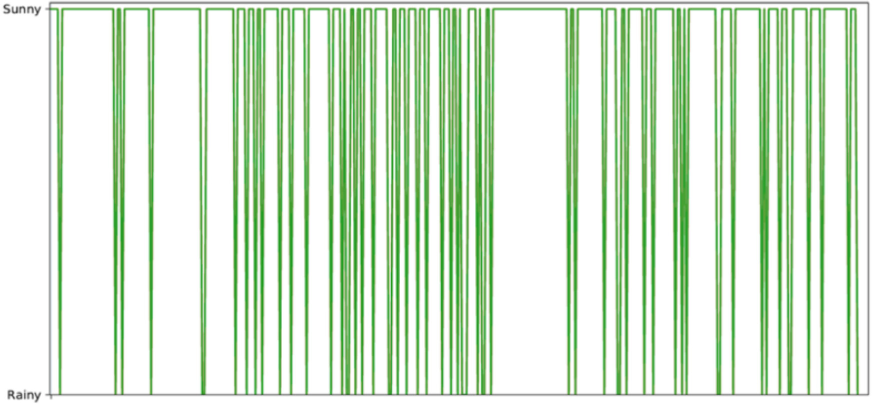


Figure 2: Plot of the weather forecast

Stochastic Processes and Markov-Chain

Markov-Chain: The Weather Working Example

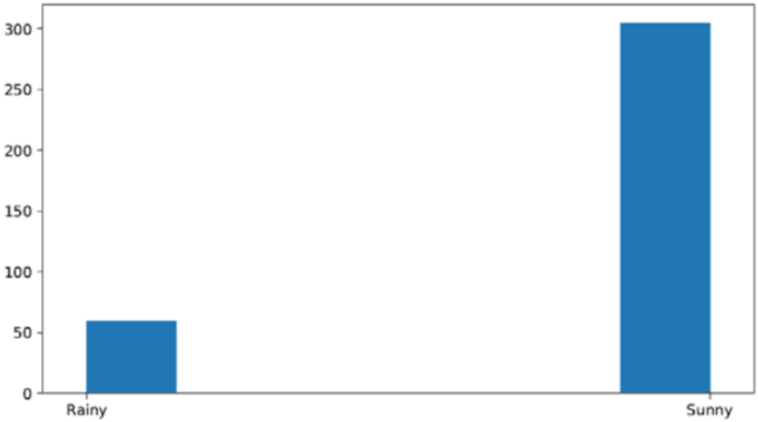


Figure 3: Histogram of the weather forecast

End of Lecture 6:

Stochastic Processes and Markov-Chain