PS #1

solutions

2.6 Start looking for $\alpha \in \mathbb{R}$ such that $(I + uv^*)(I + \alpha uv^*) = I$. This is the same as $I + uv^* + \alpha uv^* + \alpha uv^* uv^* = I \Leftrightarrow uv^* + \alpha uv^* + \alpha uv^* uv^* = 0$.

Because v^*u is a scalar we get

$$uv^* + \alpha uv^* + \alpha (v^*u) uv^* = 0 \Leftrightarrow$$

$$[1 + \alpha + \alpha(v^*u)] uv^* = 0.$$

So a good guess is $\alpha = -\dfrac{1}{1+v^*u}$ as long as $v^*u \neq -1$.

In the case $v^*u = -1$ then A is singular since $Au = u + uv^*u = u - u = 0$

So $A^{-1} = I - \dfrac{1}{1+v^*u} uv^*$ if $A = I + uv^*$ is nonsingular ($\Leftrightarrow v^*u \neq -1$).

If $Ax = 0$ then $x + uv^*x = 0$. Since $-v^*x = k$ is a scalar we get $x = ku$

Thus $\text{null}(A) = \{u\}$.

3.2. We know that $\|A\| = \sup_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$.

Choose an eigenvalue λ of A and let $x_\lambda \neq 0$ such that $Ax_\lambda = \lambda x_\lambda$. Then

$$\frac{\|Ax_\lambda\|}{\|x_\lambda\|} = \frac{\|\lambda x_\lambda\|}{\|x_\lambda\|} = \frac{|\lambda|\,\|x_\lambda\|}{\|x_\lambda\|} = |\lambda|.$$

Thus $\|A\| = \sup_{x \neq 0} \dfrac{\|Ax\|}{\|x\|} \geq \dfrac{\|Ax_\lambda\|}{\|x_\lambda\|} = |\lambda|$.

So $\|A\| \geq |\lambda|$ and since this is true for any eigenvalue of A we get

$\|A\| \geq \sup\{|\lambda|,\ \lambda \text{ eigenvalue of } A\} = \rho(A)$.

4.4 False in general. Here is a counterexample. Take $A = I_2 \cdot I_2 \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = I_2 \cdot I_2 \cdot I_2$. Clearly A and B have the same singular values.

If $A = QBQ^*$ for some unitary Q then $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = QI_2Q^* = QQ^* = I_2$ since Q is unitary, hence a contradiction.

5.2 Let $A = U\Sigma V^*$ where $\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_k & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{pmatrix}$

Define for $\varepsilon > 0$ $\Sigma_\varepsilon = \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_k & & \\ & & & \varepsilon & \\ & & & & \ddots \; \varepsilon \end{pmatrix}$ and

$A_\varepsilon = U\Sigma_\varepsilon V^*$. Then we have

$$A - A_\varepsilon = U \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \varepsilon & \\ & & & & \ddots \; \varepsilon \end{pmatrix} V^*.$$

Thus $\|A - A_\varepsilon\|_2 = \varepsilon \xrightarrow[\varepsilon \to 0]{} 0$.

So, since A_ε is full-rank matrix we have just proved that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m \times n}$.