CS 383C
CAM 383C/M 383E Numerical Analysis: Linear Algebra Fall 2008

Solutions to Homework 2

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Keywords: *Matrix Norm, Singular Value Decomposition*

1. $\|A\|_\infty = \sup_{\|\boldsymbol{x}\|_\infty=1} \|A\boldsymbol{x}\|_\infty = \sup_{\|\boldsymbol{x}\|_\infty=1} \max_i |\boldsymbol{a}_i^*\boldsymbol{x}| = \max_i \sup_{\|\boldsymbol{x}\|_\infty=1} |\boldsymbol{a}_i^*\boldsymbol{x}|$. Clearly, $\sup_{\|\boldsymbol{x}\|_\infty=1} |\boldsymbol{a}_i^*\boldsymbol{x}| = \sum_j |a_{ij}| = \|\boldsymbol{a}_i^*\|_1$ when $\|x\|_\infty = 1$. Equality in the above inequality is achieved by the vector $\boldsymbol{x}_j = e^{-i\theta}$ where $a_{ij} = re^{i\theta}$. Hence, $\|A\|_\infty = \max_i \|a_i^*\|_1$.

2. (a) Suppose $I - A$ is singular. Then $\exists y \in \mathbb{C}^n$, $y \neq 0$ s.t. $(I-A)y = 0 \Rightarrow Ay = y$. Since $\|A\| = \sup_{y\neq 0} \frac{\|Ay\|}{\|y\|}$ for any induced norm $\|\cdot\|$, $\|A\| \geq 1$. But this is a contradiction.

   (b) Note that $\|BC\| \leq \|B\|\|C\|$, for all induced matrix norms. If $\|A\| \leq 1$, then $\|A^k\| \leq \|A\|^k$ and $\lim_{k\to\infty} A^k = 0$. Thus the series $\sum_k^\infty A^k$ is convergent, and $(I-A)(\sum_{k=0}^\infty A^k) = \sum_{k=0}^\infty A^k - \sum_{k=1}^\infty A^k = I$, so $(I-A)^{-1} = \sum_{k=0}^\infty A^k$.

   (c) For all induced matrix norms, $\|I\| = \|AA^{-1}\| \leq \|A\|\|A^{-1}\|$. Now $\|I\| = 1$, so $\|A\|\|A^{-1}\| \geq 1$.

   (d) Using part (c), $\|(I-A)\|_p\|(I-A)^{-1}\|_p \geq 1$. Also $\|I-A\|_p \leq \|I\|_p + \|A\|_p$. Thus, $\|(I-A)^{-1}\|_p \geq \frac{1}{1+\|A\|_p}$. Using part (b), $\|(I-A)^{-1}\| = \|\sum_{k=0}^\infty A^k\|_p \leq \sum_{k=0}^\infty \|A\|_p^k = \frac{1}{1-\|A\|_p}$