First order logic

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0 Sets and relations

Logic is the language of mathematical relations: logic allows you to describe properties of relations. To understand the meaning and value of logic, you need to understand what relations are. This section introduces several elementary concepts, which we assume as known in this course, hence the section number 0. Make sure you understand this section before continuing.

More information can be found online of in basic math books:
`http://en.wikipedia.org/wiki/Set_(mathematics)`
`http://en.wikipedia.org/wiki/Finitary_relation`
`http://en.wikipedia.org/wiki/Theory_of_relations`
`http://en.wikipedia.org/wiki/Binary_relation`
`http://en.wikipedia.org/wiki/Cartesian_product`
`http://en.wikipedia.org/wiki/Function_(mathematics)`

Definition 0.1. *A* set *is a potentially infinite collection of unique objects.*

Examples of sets are:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
- The set of all people in the world.
- $\{\bigstar; \clubsuit; \blacksquare; \heartsuit; \triangle\}$
- The set of all polynomials.
- $\{p | p = 2 * n + 1, n \in \mathbb{N}\}$

Concepts such as *subset*, *intersection*, *superset*, *set difference*, *cardinality* of a set, are considered known.

Definition 0.2. *A* tuple *is a finite list of objects, for which order matters. Objects are allowed to occur multiple times in the same tuple.*

Examples of tuples are:

- $(0, 0)$
- $(\bigstar, \clubsuit, \blacksquare, \blacksquare)$

- Each point in a Cartesian plane can be represented as tuple consisting of two numbers.
- () is the empty tuple

Definition 0.3. *Given a sequence of sets* $\{S_1, \ldots, S_n\}$*, the Cartesian product is written down as* $S_1 \times \ldots \times S_n$*. This product is the set of all possible tuples* $(s_1, \ldots, s_n)$ *for which* $s_i \in S_i$*. Formally:* $S_1 \times \ldots \times S_n = \{(s_1, \ldots, s_n) | s_1 \in S_1, \ldots, s_n \in S_n\}$*.*

A Cartesian product of n *sets is* n-dimensional*. The* k*-dimension Cartesian product of a sequence of equal sets* S *is abbreviated as* S^k*.*

Examples:

- The Euclidean plane can be expressed as $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$
- If $S_1 = \{\clubsuit; \heartsuit\}$, and $S_2 = \{0; 1\}$, then $S_1 \times S_2 = \{(\clubsuit, 1); (\clubsuit, 0); (\heartsuit, 1); (\heartsuit, 0)\}$.
- If A is the set of all letters in the Roman alphabet, then A^k is the set of all possible k-tuples of letters.
- The 0-dimensional Cartesian product is the unique set $\{()\}$ (the set consisting of only the empty tuple).

Definition 0.4. *A* relation R *is a subset of a Cartesian product* C*. We say* R *is a relation* over C*.* R*'s* arity *is* n *of* C *is* n*-dimensional. A relation of arity* 1 *is called a unary relation, a relation of arity* 2 *is called binary and a relation of arity* 3 *is called ternary.*

Remark: For unary relations you can drop the brackets around tuples. $\{(3); (4); (5)\}$ and $\{3; 4; 5\}$ represent the same set.

Examples:

- The relation "is-divisor-of" is a subset of $\mathbb{Z}^2$, and contains, among others, the tuples $(1, 1), (2, 8), (-13, 13)$, but not the tuples $(0, 1), (3, 8), (2345, 13), (-3, 4)$.
- Every Cartesian product is a relation.
- The empty set $\{\}$ is a relation over every Cartesian product.
- The set tuples $(x, y, \sqrt{x + y})$ for which $x, y \in \mathbb{R}$ form a ternary relation over $\mathbb{R} \times \mathbb{R} \times \mathbb{C}$.
- The binary relation $\subseteq$ ("is-subset-of")
- The binary relation $<$ ("is-smaller-than")
- The set of people living in Belgium.

Exercise 1

1. Assume $S = \{0; 1; 2\}$. Give the largest possible ternary relation R over S^3 so that the third element so that the third element of the tuples in R is the difference of the first two elements. In other words, R only contains tuples of the form $(x, y, x - y)$.
2. Give a plausible Cartesian product over which the $<$ ("is-smaller-than") relation can be defined.
3. If S is a set consisting of n elements, how many different relations exist over S^k?

Definition 0.5. *Given a Cartesian product* $S_1 \times \ldots \times S_n$ *and a set* S*, then a* function $F : S_1 \times \ldots \times S_n \to S$ *is a relation over* $S_1 \times \ldots \times S_n \times S$*, such that for each tuple* $(s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$ *there is exactly one* $s \in S$*, for which* $(s_1, \ldots, s_n, s) \in F$*. We call* $S_1 \times \ldots \times S_n$ *the* domain *of* F*, and* S *the* co-domain *of* F*.*

Examples:

- the divisor function $/ : \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$; contains, among others, the tuples $(6, 3, 2)$ and $(0, \sqrt{2}, 0)$, but not $(\pi, 2, \pi)$ and $(0, 0, 0)$.
- The root function $\surd : \mathbb{R} \to \mathbb{C}$
- The function $father : P \to P$ where P is the set of al people, and $x = father(y)$ if $x, y \in P$ and x is the father of y.
- The r-number system is a function of the set of students to the set of natural numbers, preceded with an r.

The previous definition states every function is a relation, and that a function $F : S_1 \times \ldots \times S_n \to S$ is a relation of arity $n + 1$. We agree to ignore the co-domain of a function when deciding its arity:

Definition 0.6. *A function $F : S_1 \times \ldots \times S_n \to S$ has arity n.*

So $F : S_1 \times \ldots \times S_n \to S$ is both a function of arity n and a relation of arity $n + 1$.

Exercise 2

1. Take $S = \{0; 1\}$, write the function $S \times S \to S : (x, y) \mapsto (x * y)$ completely, i.e. write the set of tuples it represents.
2. Take $S = \{0; 1; 2\}$. Assume R is a relation over S^3, and $R = \{(x, y, z) \mid z = x - y\}$. In other words, R is the set of all tuples (x, y, z) where $z = x - y$ and $x, y, z \in S$. Does R represent a binary relation $S \times S \to S$? Hint: write the set of tuples in R completely, and check Definition 0.5.
3. If we take F to be the function consisting of tuples (x, y, z) where $z = x/y$, what is an appropriate domain and co-domain for F so that F is a function according to Definition 0.5?
4. If S is a set which contains n elements, how many functions $F : S^k \to S$ exist?

1 Vocabularies and logical structures

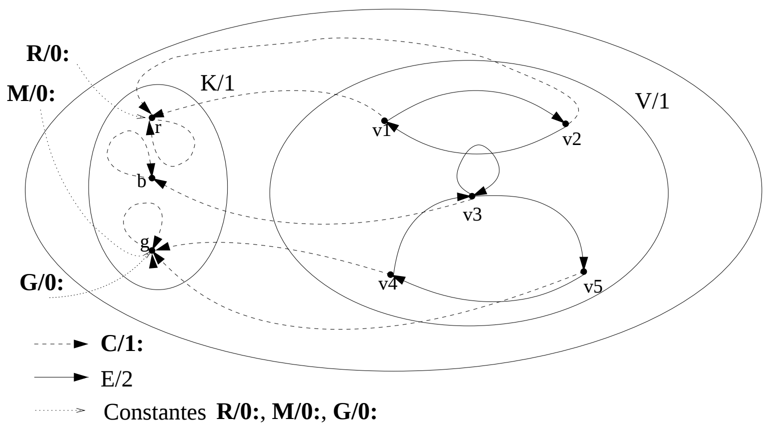
Definition 1.1. *A* vocabulary Σ *is a set of* predicate symbols, function symbols *and* constant symbols*. Each predicate symbol and function symbol is associated with an arity. Constant symbols have arity* 0*.*

We denote a predicate symbol P with arity n as P/n, a function symbol F with arity n as $\mathbf{F/n}$:, and a constant symbol C as $\mathbf{C/0}$:. Instead of the term function symbol we occasionally use the word *functor*.

Definition 1.2. *Given a vocabulary Σ, there exists a* structure $\mathfrak{A}$ *from a* domain $D_{\mathfrak{A}}$*, and for every symbol $\sigma \in \Sigma$ there exists an* interpretation $\sigma^{\mathfrak{A}}$*. A domain is a non empty set of objects called*domain elements*. An interpretation for a predicate symbol P/n is a relation $P^{\mathfrak{A}}$ over $D_{\mathfrak{A}}^n$, an interpretation for a function symbol $\mathbf{F/n}$: is a function $F^{\mathfrak{A}} : D_{\mathfrak{A}}^n \to D_{\mathfrak{A}}$, and an interpretation for a constant symbol $\mathbf{C/0}$: is a domain element $C^{\mathfrak{A}} \in D_{\mathfrak{A}}$.*

Exercise 3 We choose a vocabulary Σ consisting of $IsParentOf/2$, $Man/1$, $Woman/1$, $IsBrotherOf/2$, $IsSisterOf/2$, the function symbol **Age/1** : the constant symbol **OldestDaughter/0** :. The informal meaning of every symbol is clear. Now, construct a structure $\mathfrak{A}$ with as domain $D_{\mathfrak{A}}$ the union of the natural numbers and the set $\{marge; homer; lisa; bart; maggie\}$, famous as the family of the television series "The Simpsons". Ensure a correct mathematical interpretation of the function symbol $Age/1$!

Exercise 4 The next figure is a graphical representation of a structure $\mathfrak{A}$.



Domain elements are represented in lower case, symbols in upper case and function and constant symbols are **bold** faced.

Remark: In the figure r,b,g,v1,...,v5 are **not** constants of the vocabulary. They are domain elements.

1. Over which vocabulary is $\mathfrak{A}$ a structure?
2. Express this structure in the correct mathematical notation. In other words, specify $D_{\mathfrak{A}}$ and the interpretation $\sigma^{\mathfrak{A}}$ for each symbol σ.
3. C is a functor, but does $C^{\mathfrak{A}}$ satisfy all conditions needed to be considered a function? Change the structure so that $C^{\mathfrak{A}}$ is correct.

2 Geo-world: a structure with a fixed vocabulary

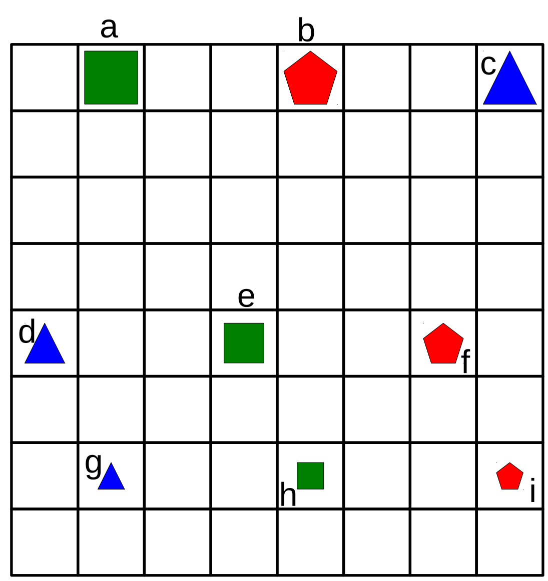


Figure 1: A Geo-world with 9 figures (Geo111).

A Geo-world is a structure with a fixed vocabulary. The domain of a geo-world consists of figures, which are located on a square board (see Figure 1). So, each figure represents a domain element, and together they represent the domain the a geo-world. In this example each figure is attributed an identifier, which make up the domain $\{a, b, c, d, e, f, g, h, i\}$ of the geo-world. Each figure is either a triangle, a square, or a pentagon. A figure is either large (a, b, c), medium (d, e, f), or small (g, h, i). Several relations exist over these figures, for which we use the following vocabulary:

... is a triangle	$Triangle/1$
... is a square	$Square/1$
... is a pentagon	$Pentagon/1$
... is small	$Small/1$
... is medium	$Medium/1$
... is large	$Large/1$
... is (strictly) smaller than ...	$Smaller/2$
... is (strictly) larger than ...	$Larger/2$
... is more to the left than ...	$LeftOf/2$
... is more to the right than ...	$RightOf/2$
... is more to the front than ...	$FrontOf/2$
... is more to the back than ...	$BackOf/2$
... is in between ... and ...	$Between/3$

Most of these predicate symbols speak for themselves. We explain the rest:

We say a figure x is more to the left than a figure y, so $LeftOf(x, y)$, iff the column to which x belongs is to the left of the column to which y langs (so these columns can definitely not be the same). The definition of $RightOf(x, y)$ is analogous.

We say a figure x is more to the front than a figure y, so $FrontOf(x, y)$, iff the row to which x belongs is below the row to which y belong. (so, these rows can definitely not be the same). The definition of $BackOf(x, y)$ is analogous.

We say a figure z is in between figures x and y, so $Between(z, x, y)$, iff z is in between x and y in a single column, row, or 45° diagonal, and x, y and z occupy three different fields.

Exercise 5 Figure 1 represents a geo-world structure $\mathcal{G}$ with domain $D_{\mathcal{G}} = \{a, b, c, d, e, f, g, h, i\}$. Give the interpretation of the following symbols in $\mathcal{G}$.

- $Large/1$
- $Between/3$

What is the size of the interpretation of $FrontOf/2$?

3 Terms and Atoms

Given a vocabulary we can build logical formulas, which describe properties of structures. Two building blocks of logical formulas are *terms* and *atoms*.

Definition 3.1. *A* term *is*

- *a constant symbol, e.g.* C
- *an application of a function symbol* F/n *on a tuple of* n *terms, e.g.* $F(t_1, \ldots, t_n)$ *where the* t_i *are terms.*

Definition 3.2. *An* atom *is*

- *the equating of two terms, e.g.* $F(C) = D$.
- *an application* P/n *on a tuple of* n *terms, e.g.* $P(t_1, \ldots, t_n)$ *where the* t_i *are terms.*

Exercise 6 Given the vocabulary from exercise 4, specify for each of the following strings whether they are a term, an atom, or neither.

1. R
2. $v3$
3. $R = G$
4. $R = K(R)$
5. $V(K)$
6. $K(M)$
7. $K(K(M))$
8. $C(G)$
9. $G(C)$
10. $E(M)$
11. $E(C(R), G)$
12. $C(C(R), G)$
13. $C(r)$
14. $C(C(C(M)))$
15. $C(C(C(C)))$
16. $C(C(C(M))) = C(G)$

If a vocabulary Σ contains all symbols in a term t, we say t *is a term of* Σ. If t is a term of Σ, then t has an interpretation in a structure $\mathfrak{A}$ over Σ:

Definition 3.3. *Given a vocabulary* Σ*, a structure* $\mathfrak{A}$ *over* Σ*, and a term* t *of* Σ*, then* $t^{\mathfrak{A}}$ *is the unique interpretation which maps* t *onto a domain element of* $\mathfrak{A}$*.* $t^{\mathfrak{A}}$ *is defined as:*

- *if* t *is a constant* C*, then* $t^{\mathfrak{A}} = C^{\mathfrak{A}}$
- *if* t *is a function application* $F(t_1, \ldots, t_n)$*, then* $t^{\mathfrak{A}} = F^{\mathfrak{A}}(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$

If a vocabulary Σ contains all symbols in a atom a, we say a *is an atom* Σ. If a is an atom of Σ, then a has an interpretation in a structure $\mathfrak{A}$ over Σ:

Definition 3.4. *Given a vocabulary* Σ*, a structure* $\mathfrak{A}$ *over* Σ*, and an atom* a *of* Σ*, then* $a^{\mathfrak{A}}$ *is the unique interpretation which maps* a *to* truth value *in* $\mathfrak{A}$*.* $a^{\mathfrak{A}}$ *is defined as:*

- *if* a *is of the form* $t_1 = t_2$*, then* $a^{\mathfrak{A}}$ *is true iff* $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$.
- *if* a *is of the form* $P(t_1, \ldots, t_n)$*, then* $a^{\mathfrak{A}}$ *is true iff* $(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}}) \in P^{\mathfrak{A}}$

Exercise 7 Given the vocabulary and structure from exercise 4, what is the truth value of all atoms and the interpretation of all terms of exercise 6?

4 Logical connectives and quantorless formulas

Terms are expressions which can be mapped onto a domain element by structures, while atoms are true or false in a structure. Atoms can form *formulas* by means of *logical connectives*:

Definition 4.1. *Given a vocabulary* Σ*, then a* formula *is either:*

- *an atom*

or a combination of formulas ψ*,* ψ' *with a connective:*

- *a negation:* $(\neg\psi)$
- *a conjunction:* $(\psi \wedge \psi')$
- *a disjunction:* $(\psi \vee \psi')$
- *an implication:* $(\psi \Rightarrow \psi')$
- *an equivalence:* $(\psi \Leftrightarrow \psi')$

or a quantification of a constant symbol x *in a formula* ψ*:*

- *a universal quantification:* $(\forall x)(\psi)$
- *an existential quantification:* $(\exists x)(\psi)$

Remark: the symbols $=, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, \forall, \exists$ are called *logical symbols*. The symbols '(', ')' and ',' are called *helper symbols*. Predicate symbols, function symbols and constant symbols are called *non-logical symbols*. When we talk about *symbols*, we usually mean *non-logical symbols*.

In this part of the exercises we first focus on quantorless formulas, i.e. formulas formed by atoms and combinations of formules, without the quantifiers $\forall$ and $\exists$. Remark: since it is cumbersome to always use brackets to identify a subformula, there is an agreement on order of connectives: $\neg < \wedge < \vee < \Rightarrow < \Leftrightarrow$. This means $\wedge$ *binds stronger* than $\vee$, just as $*$ binds stronger than $+$. The order of the connectives allows us to remove brackets, and yet still keep a formula with a unique meaning.

Examples of (quantorless) formulas:

- $Mother(Alfred) = Bea$
- $Mother(Alfred) = Bea \Rightarrow Woman(Bea)$
- $P(x, y) \wedge Q(y, z)$
- $A \Rightarrow B \Leftrightarrow \neg A \vee B$
- $ToBe \vee \neg ToBe$

Exercise 8 Add all removed brackets back into the formula $A \Rightarrow B \Leftrightarrow \neg A \vee B$, based on the order of connectives.

Just as atoms, quantorless formulas have a truth value in an appropriate structure:

Definition 4.2. *Given a vocabulary* Σ*, a structure* $\mathfrak{A}$ *over* Σ*, and a quantorless formula* ϕ *of which all non-logical symbols are intrerpreted in* $\mathfrak{A}$*, then* $\phi^{\mathfrak{A}}$*, the truth value of* ϕ *in* $\mathfrak{A}$*, is defined as follows:*

- *if ϕ is an atom a, then$\phi^{\mathfrak{A}} = a^{\mathfrak{A}}$*
- *if ϕ consists of $\neg\psi$, then $\phi^{\mathfrak{A}}$ is true iff $\psi^{\mathfrak{A}}$ is **not** true*
- *if ϕ consists of $\psi_1 \wedge \psi_2$, then ϕ is true iff ψ_1 and ψ_2 are both true*
- *if ϕ consists of $\psi_1 \vee \psi_2$, then ϕ is **false** iff ψ_1 and ψ_2 are **false***
- *if ϕ consists of $\psi_1 \Rightarrow \psi_2$, then ϕ is **false** iff ψ_1 is true and ψ_2 is **false**.*
- *if ϕ consists of $\psi_1 \Leftrightarrow \psi_2$, then ϕ is true iff ψ_1 has the same truth value as ψ_2.*

Exercise 9 Assume we have the following vocabulary: $\Sigma = \{P/2, Q/1, \mathbf{R/1:}, \mathbf{S/0:}, \mathbf{T/0:}, \mathbf{x/0:}, \mathbf{y/0:}\}$ with structure $\mathfrak{A}$:

- domain $D_{\mathfrak{A}} = \{a, b, c\}$
- $P^{\mathfrak{A}} = \{(a, a); (a, b); (a, c)\}$
- $Q^{\mathfrak{A}} = \{b, c\}$
- $R^{\mathfrak{A}} = \{(a, b); (b, c); (c, a)\}$
- $S^{\mathfrak{A}} = a$
- $T^{\mathfrak{A}} = c$
- $x^{\mathfrak{A}} = a$
- $y^{\mathfrak{A}} = b$

What is the truth value of the following quantorless formulas:

1. $\neg(R(T) = S)$
2. $R(S) = T \Rightarrow R(T) = S$
3. $P(x, y) \Leftrightarrow \neg Q(x)$
4. $Q(T) \vee P(S, T)$
5. $Q(R(R(T))) \wedge Q(T)$
6. $R(T) = S \Leftarrow \neg Q(y) \wedge Q(S)$

Remark: we use the abbreviation $t_1 \neq t_2$ for formulas $\neg(t_1 = t_2)$, where t_1,t_2 are terms.

5 Quantifiers and sentences

In general formules will contain quantifiers. Some examples:

- $(\forall x)(Man(x) \vee Woman(x))$
- $(\exists y)(\forall z)(y \neq z \Rightarrow SmallerThan(y, z))$
- $(\exists x)(P(x)) \vee (\exists x)(Q(x))$
- $(\forall z)(R(x, y) = z)$

Quantifiers give meaning to constant symbols in a formula, more specifically to the constant symbol over which the quantifier *quantifies*. A quantified constant symbol $x$is called a *variable*, and we say x occurs *variable* or *bound* in the subformula which is quantified over. A symbol which is not bound, is *free*.

Definition 5.1. *A formula ϕ is called a* sentence *of a vocabulary Σ if all free symbols in ϕ belong to Σ.*

Exercise 10 Suppose we have a vocabulary $\Sigma = \{\text{Man}/1; \text{Woman}/1; \text{SmallerThan}/2; \text{P}/1; \text{Q}/1; \mathbf{R/2{:}}\}$. Which of the formulas above are sentences of Σ?

Every sentence of a vocabulary Σ has a unique truth value in a structure over Σ:

Definition 5.2. *Given a vocabulary Σ, a structure $\mathfrak{A}$ over Σ, and a sentence ϕ of Σ, then $\phi^{\mathfrak{A}}$, the truth value of ϕ, is defined as follows:*

- *if ϕ is quantorless,then $\phi^{\mathfrak{A}}$ is defined in Definition 4.2*
- *If ϕ consists of $(\forall x)(\psi)$, then $\phi^{\mathfrak{A}}$ is true iff for each assignment of a value d from the domain $D_{\mathfrak{A}}$ to x, $\psi^{\mathfrak{A}[x:d]}$ is true.*
- *If ϕ consists of $(\exists x)(\psi)$, then $\phi^{\mathfrak{A}}$ is true iff there exists an assignment of a value d from the domain $D_{\mathfrak{A}}$ to x so that $\psi^{\mathfrak{A}[x:d]}$ is true.*

$\mathfrak{A}[x : d]$ *is the modified structure $\mathfrak{A}$ where the constant symbol x is interpreted by the domain element d.*

Exercise 11 Given the vocabulary Σ and structure $\mathfrak{A}$ from Exercise 9, give the truth value of the following sentences:

1. $(\forall x)(Q(x))$
2. $(\forall x)(\exists y)(R(y) \neq x)$
3. $(\forall y)(P(T, y))$
4. $(\exists x)(\forall y)(P(x, y) \wedge (S = x))$
5. $(\forall x)(S = R(T))$
6. $(\forall x)(Q(x) \Rightarrow \neg(\exists y)(P(x, y)))$

Note that in all sentences de constant symbols x and y only occur bound, and that the interpretation for x and y in structure $\mathfrak{A}$ of Exercise 9 was not relevant for Exercise 11. Usually we only use structure which do not interpret variables.

6 Construct simple sentences yourself

Exercise 12 We use the vocabulary of the Geo-world, as explained in Section 2. Give a translation to predicate logic of the following English sentences, where we use the constant **a/0:** to denote a certain figure.

1. a is a small triangle.
2. If a is large, then it is a square.
3. All figures are pentagons.
4. There exists a triangle.
5. All figures are large squares.
6. There exists a figure which is a small triangle.
7. All figures which are large, are also square.
8. There exists a triangle which is behind a square.
9. All pentagons are in front of all triangles.
10. There is a triangle between each two squares.

Exercise 13 In the exercise we look at the logical sentences of the last exercise. Try to give an answer to the following questions:

1. What is the difference between sentences 5 and 7? Give a situation where on sentence is true and the other is not.
2. Why can we not translate sentence 6 as $(\exists x)(Triangle(x) \Rightarrow Small(x))$? Again, show this by constructing a geo-world (a structure) which demonstrates the difference.
3. Is the order of the quantifiers in question 8 important? If so, what is the effect on the meaning when you switch the quantifiers?
4. Is the order of the quantifiers in question 9 important? If so, what is the effect on the meaning when you switch the quantifiers?
5. Is the order of the quantifiers in question 10 important? If so, what is the effect on the meaning when you move the existential quantifier to the front?