

## Solutions to Homework 8

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## 1. Problem 25.1

- (a) Consider the matrix  $B = A - \lambda I$ . Note that  $B$  is also a symmetric tridiagonal matrix such that all the off-diagonal elements are nonzero. Consider a vector  $x \in \text{Null}(B)$ . Fix the first component of  $x$ , i.e. fix  $x_1$ . It is seen that given  $B$  all the other components of  $x_i, \forall i \geq 2$  are functions of  $x_1$  alone. We prove this claim using induction. It is easy to see that  $x_2$  depends on  $x_1$  alone as:  $x_2 = -(B_{11}x_1)/B_{12}$ , as  $B_{12}$  is nonzero. Now consider the  $k$ -th row of  $B$ . Clearly,  $x_{k+1} = -(B_{k,k-1}x_{k-1} + B_{kk}x_k)/B_{k,k+1}$ , as  $B_{k,k+1}$  is nonzero. Using inductive hypothesis, both  $x_{k-1}$  and  $x_k$  are fixed given  $x_1$ . Thus using inductive hypothesis,  $x_{k+1}$  can be computed deterministically using  $x_1$ . Also, it can be seen easily that scaling  $x_1$  amounts to scaling the complete  $x$  by the exact same amount. Hence, the vector  $x$  lies in at most 1-dimensional subspace. Since  $A$  is hermitian, it is diagonalizable, i.e. algebraic multiplicity of each eigenvalue is exactly equal to the geometric multiplicity. Hence, every eigenvalue is distinct.

- (b) Consider the following matrix:  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ . 0 is an eigenvalue of  $A$  with algebraic multiplicity 2.

## 2. Problem 27.2

- (a) We first show that the set  $W(A)$  is a convex set, i.e. if  $\alpha, \beta \in W(A)$  then  $p\alpha + (1-p)\beta \in W(A)$  for all  $p \in [0, 1]$ .

Consider any two points in set  $W(A)$ :  $\alpha$  and  $\beta$  s.t.  $u^*Au = \alpha$  and  $v^*Av = \beta$ ,  $\|u\|_2 = 1$ ,  $\|v\|_2 = 1$ . Assume, that  $u$  and  $v$  are linearly independent as otherwise  $p\alpha + (1-p)\beta \in W(A)$  trivially. Define,

$$B = \frac{-\beta}{\alpha - \beta}I + \frac{1}{\alpha - \beta}A, X = \frac{1}{2}(B + B^*), Y = \frac{1}{2i}(B - B^*).$$

Note that,  $B = X + iY$ ,  $u^*Bu = 1$ ,  $v^*Bv = 0$ ,  $u^*Xu = 1$ ,  $v^*Xv = 0$ ,  $u^*Yu = 0$ ,  $v^*Yv = 0$ . Also WLOG assume that  $u^*Yv$  is purely imaginary, for otherwise  $v$  can be replaced by  $\exp(i\theta)v$  for an appropriate choice of  $\theta$ . Now consider the vector  $z(t) = \frac{tu + (1-t)v}{\|tu + (1-t)v\|_2}$ , where  $t \in [0, 1]$ .  $z(t)^*Bz(t) = z(t)^*Xz(t) = \frac{t^2 + 2t(1-t)\text{Re}(v^*Xu)}{\|tu + (1-t)v\|_2^2}$ , which is a continuous real-valued function mapping 0 to 0 and 1 to 1. Thus there exists a  $t \in [0, 1]$  such that for every  $p \in [0, 1]$ ,  $z(t)^*Bz(t) = p$ . Let  $w = z(t)$  for that particular choice of  $t$ . Then,  $w^*Aw = (\alpha - \beta)(\frac{\beta}{\alpha - \beta} + w^*Bw) = p\alpha + (1-p)\beta$ . Thus for any two points  $\alpha$  and  $\beta$  in  $W(A)$  and for every  $p \in [0, 1]$ ,  $p\alpha + (1-p)\beta$  also lies in  $W(A)$ . Hence  $W(A)$  is a convex set. Also, note that  $v_i^*Av_i = \lambda_i$ , where  $v_i$  is the  $i$ -th eigenvector of  $A$ . Thus, all the eigenvalues of  $A$  lies in  $W(A)$ . Hence, the convex hull of eigenvalues of  $A$  also lies in  $W(A)$  because  $W(A)$  is a convex set.

- (b) If  $A$  is a normal matrix, then  $A$  is unitarily diagonalizable, i.e.  $A = Q\Lambda Q^*$ . Consider a point  $\alpha \in W(A)$ , i.e.  $\alpha = u^*Au$ , where  $\|u\|_2 = 1$ . Let  $u = Qr$ , where  $r \in C^{m \times 1}$ ,  $\|r\|_2 = 1$ . Then,  $\alpha = \sum_i r_i^2 \lambda_i$  where  $\sum_i r_i^2 = 1$ . Hence, every  $\alpha \in W(A)$  lies in the convex hull of eigenvalues. Thus using part (a),  $W(A)$  is equal to the convex hull of the eigenvalues of  $A$ .

3. Note that the solution to Problem 25.1 suggests a constructive method to compute the eigenvector given an eigenvalue. Also, as noted in the solution to Problem 25.1 that scaling the first component (or the last component) amounts to scaling the whole matrix. This can introduce severe round-off error because if  $x_1 = \delta \approx 0$ , then the whole matrix is scaled up by a factor of  $1/\delta$ , thus inducing large round-off errors.
4. Consider the following matrix  $S = \begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}$ . Now,  $S^{-1} = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}$ . Clearly,  $S^{-1} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} S = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ . Hence, both the matrices have the same set of eigenvalues.
5. Since  $A$  is a symmetric matrix, so its eigenvalue and Schur decomposition is same, and is given by  $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*$ .

Both the eigenvalues of  $B$  are given by 0. However,  $B$  is a rank-1 matrix, i.e.  $\text{Null}(B)$  has dimensionality 1. Thus algebraic multiplicity of eigenvalue 0 is greater than its geometric multiplicity. So,  $B$  is not diagonalizable. Schur decomposition of  $B$  is given by  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^*$ .

Eigenvalue decomposition of  $C$  is given by:  $C = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1}$ . Schur decomposition of  $C$  is given by:  $C = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}^*$ .