CS 383C
CAM 383C/M 383E Numerical Analysis: Linear Algebra Fall 2008

Solutions to Homework 6

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Keywords: *Gaussian Elimination, Partial Pivoting, Complete Pivoting*

1. (a) **Problem 21.5**
We find a decomposition of the form: $LBAB^*L^* = D$, where the matrix B need not be a permutation matrix. In fact, a decomposition of the type $LPAP^*L^* = D$ (where P is a permutation matrix) need not even exist in general, e.g., consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Let A_k be the matrix obtained after $k-1$ rounds of GE with symmetric pivoting. Let A_k be of the form $A_k = \begin{bmatrix} D_{k-1} & 0 \\ 0 & G_k \end{bmatrix}$, where D_{k-1} is a $k-1 \times k-1$ diagonal matrix and G_k is a $(n-k+1)\times(n-k+1)$ dimensional matrix. Now, we describe a pivoting procedure that makes the pivot element $A_k(k,k)$ the largest element in the sub-matrix G_k. Let $A_k(s,r) = \max_{i\geq k, j\geq k|} |A_k(i,j)|$. Then exchange the s-th row with the k-th row and s-th column with the k-th column. This operation can be seen as $A_k = PA_kP^T$, where P is a permutation matrix. If $s = r$, then we are done with pivoting as we bring the largest element onto the pivot position. Otherwise, add α times the r-th row to the k-th row and $\bar{\alpha}$ times the r-th column to the k-th column, where $\alpha = e^{i\theta}$ for some θ to be decided later. Now the new A_{kk} element is given by: $A_{kk} = A_{kk} + |\alpha|^2 A_{rr} + 2\alpha A_{sr}$. θ is selected so that αA_{sr} is real and the sign of αA_{sr} is the same as that of $A_{kk} + |\alpha|^2 A_{rr}$. It can be easily seen that the addition operation described above makes the pivot element $A_k(k,k)$ the largest in the sub-matrix G_k. This addition operation can be seen as $A_k = (I + \alpha \boldsymbol{e}_k \boldsymbol{e}_r^*) A_k (I + \bar{\alpha} \boldsymbol{e}_k \boldsymbol{e}_r^*)$. Thus, pivoting operation $B_k = (I + \alpha \boldsymbol{e}_k \boldsymbol{e}_r^*)P$ and $B_k^{-1} = P^*(I - \alpha \boldsymbol{e}_k \boldsymbol{e}_r^*)$. Now, we can use standard Gaussian Elimination to remove the k-th column of A_k to obtain new matrix A_{k+1}. Since, the matrix A_k is symmetric and our pivoting operation is also symmetric, so we can copy lower triangular half of A_{k+1} to the upper triangular half and save 50% of the computational cost. So the overall operation in round k is: $L_k B_k A_k B_k^* L_k^* = A_{k+1}$, and after $n-1$ rounds we get $L_{n-1}B_{n-1}L_{n-2}B_{n-2}\ldots L_1 B_1 A B_1^* L_1^* \ldots B_{n-2}^* L_{n-2}^* B_{n-1}^* L_{n-1}^* = D$. Using an argument similar to GEPP, we get the decomposition $LBAB^*L^* = D$.
Pivoting operation B_k costs $n-k$ floating point operations, i.e. total of $\sum_{k=1}^{n}(n-k) = O(n^2)$ floating point operations. Also we save 50% of the floating point operations during Gaussian elimination phase. So the total computational cost is $\frac{1}{3}n^3 + O(n^2)$ flops.

(b) **Problem 21.6**
Claim: After every step of Gaussian elmination, the resulting sub-matrix is again diagonally dominant.
Proof: We prove the claim for the first round of GE and the general claim follows using induction. Before the GE step (i.e. pre-multiplying by L_1), $\forall k, |A(k,k)| > \sum_{j\neq k}|A(j,k)| > \sum_{j\neq\{k,1\}}|A(j,k)| + |A(1,k)|\frac{\sum_{j\neq 1}|A(j,1)|}{|A(1,1)|}$. Or, $|A(k,k)| - |A(1,k)|\frac{|A(k,1)|}{|A(1,1)|} > \sum_{j\neq\{k,1\}}|A(j,k)| + |A(1,k)|\frac{|A(j,1)|}{|A(1,1)|}$. Using negative form of triangular inequality for the LHS, $|A(k,k) - A(1,k)\frac{A(k,1)}{A(1,1)}| > \sum_{j\neq\{k,1\}}|A(j,k)| + |A(1,k)|\frac{|A(j,1)|}{|A(1,1)|}$. Using triangular inequality for the RHS, $|A(k,k) - A(1,k)\frac{A(k,1)}{A(1,1)}| > \sum_{j\neq\{k,1\}}|A(j,k) - A(1,k)\frac{A(j,1)}{A(1,1)}|$. Hence, the claim holds for the first round of GE. By induction, the claim holds in general.
Using the claim above, the matrix is always diagonally dominant, so no row exchange is required.

2. (a) We tabulate the "rcond" values below:

x\ n	25	50	100
0	0.04	0.02564	0.01852
1	0.02	0.01307	0.00961
2	0.01	0.00990	0.00980

(b) $W_n(x) = LU$, where $L = \begin{bmatrix} 1 & 0 & . & . & 0 & 0 \\ -1 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ -1 & -1 & . & . & 1 & 0 \\ -1 & -1 & . & . & -1 & 1 \end{bmatrix}$, and $U = \begin{bmatrix} 1 & 0 & . & . & 0 & 1 \\ 0 & 1 & . & . & 0 & 2 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 2^{n-2} \\ 0 & 0 & . & . & 0 & x + 2^{n-1} \end{bmatrix}$.

Partial pivoting does not make any difference because at each step, magnitude of the pivot element is the largest in its column.

(c) If $fl(x + 2^{n-1}) = 2^{n-1}$, then $U(x) = U(0)$. For single precision, $fl(x + 2^{n-1}) = 2^{n-1}$ for $n \geq 25$ and for double precision it holds for $n \geq 54$.

(d) For $n \geq 54$ in double precision arithmetic, Matlab's "\" operator gives incorrect results similar to partial pivoting and are completely different from the ones given by complete pivoting. This indicates that "\" operator might use partial pivoting but does not use complete pivoting. In fact, "\" operator is indeed Gaussian Elimination with Partial Pivoting.