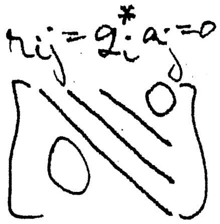
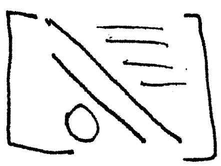
PS# 7
Solutions

28.2 a) looking at Gram-Schmidt algorithm we see that each column j of Q is a linear combination of the columns 1 to j of A, so Q is upper Hessenberg. On the other hand since A is tridiagonal we have $r_{ij} = q_i^* a_j = 0$ for all $j > i+2$. So $R =$



and $Q =$



b) The matrix RQ must be upper Hessenberg and also $RQ = Q^T A Q$ is symmetric when A is symmetric so RQ is upper Hessenberg and symmetric which means tridiagonal.

c) At step i only the element $a_{i+1,i}$ needs to be made 0. This can be done by using a 2×2 Householder reflection based on a_{ii} and $a_{i+1,i}$. This will affect only 6 elements of the matrix, namely $a_{ii}, a_{i+1,i}, a_{i,i+1}, a_{i+1,i+1}, a_{i,i+2}, a_{i+1,i+2}$. So we need 6 flops ~~and~~ for multiplications and 24 flops at each step. A total of $n-1$ steps have to be done, ending with $24n$ operations, compared with $O(n^3)$ for a full matrix.

30.3 Let A be the matrix and A' the updated matrix. Let $|a_{jk}| = \max_{i\neq l} |a_{il}|$.

If a_{jk} is zeroed out then:

$$a_{jj}^2 + a_{kk}^2 + 2a_{jk}^2 = (a'_{jj})^2 + (a'_{kk})^2 + 2(a'_{jk})^2 = (a'_{jj})^2 + (a'_{kk})^2.$$

Now
$$\text{off}^2(A') = \sum_{i\neq l} (a'_{il})^2 = \sum_{\substack{l\neq i\\ l,i\neq j,k}} (a'_{il})^2 + (a'_{jj})^2 + (a'_{kk})^2 = \text{off}^2(A) - 2a_{jk}^2$$

Since $\text{off}^2(A) = \sum_{i\neq l} a_{il}^2 \le m(m-1)a_{jk}^2$ we get $\text{off}^2(A') \le \text{off}^2(A)\left(1 - \frac{2}{m(m-1)}\right)$.

or $\text{off}^2(A') \le \left(1 - \frac{2}{m(m-1)}\right)\text{off}^2(A)$.

30.6 Simple computations show that $p^{(-1)}(2) = 0$, $p^{(0)}(2) = 1$, $p^{(1)}(2) = -1$, $p^{(2)}(2) = 0$, $p^{(3)}(2) = 1$, $p^{(4)}(2) = 1$ so Sturm sequence is $1, -1, 0, 1, 1$ and there are two sign changes which means that 2 eigenvalues of A are smaller than 2.

Now $p^{(-1)}(1) = 0$, $p^{(0)}(1) = 1$, $p^{(1)}(1) = 1$, $p^{(2)}(1) = -1$, $p^{(3)}(1) = -1$, $p^{(4)}(1) = -1$, so Sturm sequence is $+1, 1, -1, -1, -1$ and there is one sign change, which means that 1 eigenvalue of A is smaller than 1.

So there is only one eigenvalue of A in $[1, 2]$.