Fundamentals of Computer Science
Exercises on Regular languages

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Exercise 1. For a given string $w = w_1 w_2 \dots w_n$ ($w_i \in \Sigma$) we denote the reversed string as $w^{\mathcal{R}} = w_n \dots w_2 w_1$. For a given language L we write $L^{\mathcal{R}} = \left\{ w^{\mathcal{R}} \mid w \in L \right\}$. Given L is regular, is $L^{\mathcal{R}}$ regular as well?

Exercise 2. Give for every language below over the alphabet $\Sigma = \{0, 1\}$ a regular expression that determines the language. Construct an NFA that decides the language.

1. $\{w \mid \text{every odd position of } w \text{ is a } 1\}$
2. $\{w \mid w \text{ contains at least two 0s and at most one } 1\}$
3. The complement language of $\{11, 111\}$
4. $\{w \mid \text{the numbers of occurrences of 01 and 10 in } w \text{ are the same}\}$

Exercise 3. Prove that for every $n \geq 1$ there exists an NFA that decides the following languages.

1. $L_n = \left\{ a^k \mid k \text{ is a multiple of } n \right\}$ over the alphabet $\Sigma = \{a\}$.
2. $L_n = \{x \mid x \text{ is the binary representation of a natural number that is a multiple of } n\}$

Exercise 4. Given the alphabet Σ_3:

$$\Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{1}$$

Consider a string over Σ_3 as three rows of 0's and 1's and consider every row to be the binary representation of a binary number (*little-endian*). We define L as the language

$$\{w \mid \text{the last row of } w \text{ is the sum of the two rows above.}\} \tag{2}$$

over Σ_3. Thus $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an element of L, but $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ isn't. Show that L is a regular language.

Exercise 5. An *all-paths*-NFA $\langle Q, \Sigma, \delta, q_0, F \rangle$ differs from an NFA because it accepts a string w only if

- for every possible subdivision $w = y_1 y_2 \dots y_m$, $y_i \in \Sigma_\varepsilon$ and every state sequence $r_0, r_1, \dots, r_m$ such that $r_0 = q_0$ and $r_{i+1} \in \delta(r_i, y_{i+1})$, it holds that $r_m \in F$;
- there is at least one subdivision $w = y_1 y_2 \dots y_m$, $y_i \in \Sigma_\varepsilon$ such that there is a state sequence $r_0, \dots, r_m$ with $r_0 = q_0$ and $r_{i+1} \in \delta(r_i, y_{i+1})$.

Show that L is regular, if and only if there exists an all-paths-NFA that decides L.