Exercise session 7: LTL and CTL

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1 Check equivalence

Which of the following pairs of CTL formulas are equivalent? For those which are not, exhibit a model of one of the pair which is not a model of the other:

1. $\mathrm{EF}\ \phi$ and $\mathrm{EG}\ \phi$

   Solution: Not equivalent.

2. $\mathrm{EF}\ \phi \vee \mathrm{EF}\ \psi$ and $\mathrm{EF}\ (\phi \vee \psi)$

   Solution: Equivalent.

3. $\mathrm{AF}\ \phi \vee \mathrm{AF}\ \psi$ and $\mathrm{AF}\ (\phi \vee \psi)$

   Solution: Not equivalent.

4. $\mathrm{AF}\ \neg\phi$ and $\neg\mathrm{EG}\ \phi$

   Solution: Equivalent.

5. $\mathrm{EF}\ \neg\phi$ and $\neg\mathrm{AF}\ \phi$

   Solution: Not equivalent.

6. $\mathrm{A}\ (\phi_1\ \mathrm{U}\ \mathrm{A}\ (\phi_2\ \mathrm{U}\ \phi_3))$ and $\mathrm{A}\ (\mathrm{A}\ (\phi_1\ \mathrm{U}\ \phi_2)\ \mathrm{U}\ \phi_3)$, hint: it might make it simpler if you think first about models that have just one path **Solution:** Not equivalent.

7. $\top$ and $\mathrm{AG}\ \phi \rightarrow \mathrm{EG}\ \phi$

   Solution: Equivalent.

8. $\top$ and $\mathrm{EG}\ \phi \rightarrow \mathrm{AG}\ \phi$

   Solution: Not equivalent.

9. $\mathrm{A}\ [\phi\ \mathrm{U}\ \psi]$ and $\phi \wedge \mathrm{AF}\ \psi$

   Solution: Not equivalent.

10. $\mathrm{A}\ [\phi\ \mathrm{U}\ \psi] \vee \mathrm{A}\ [\tau\ \mathrm{U}\ \psi]$ and $\mathrm{A}\ [(\tau \vee \phi)\ \mathrm{U}\ \psi]$

    Solution: Not equivalent.

2 Express in CTL and LTL

Express the following properties in CTL and LTL whenever possible. If neither is possible, try to express the property in CTL*:

1. Whenever p is followed by q (after finitely many steps), then the system enters an 'interval' in which no r occurs until t.

   Solution: Possible solution:

   $$\mathrm{AG}\ (p \rightarrow \mathrm{AX}\ \mathrm{AG}\ (\neg q \vee \mathrm{A}\ [\neg r\ \mathrm{U}\ t]))$$

2. Event p precedes s and t on all computation paths. (You may find it easier to code the negation of that specification first)

   Solution: Possible solution:

   $$\neg\mathrm{EF}\ ((s \vee t) \wedge \mathrm{EF}\ (p)) \equiv \mathrm{AG}\ (\neg((s \vee t) \wedge \mathrm{EF}\ (p)))$$

3. After p, q is never true. (Where this constraint is meant to apply on all computation paths.)

   Solution: Possible solution:

   $$\mathrm{AG}\ (p \rightarrow \neg\mathrm{EF}\ q) \text{ or } \mathrm{AG}\ (p \rightarrow \neg\mathrm{EX}\ \mathrm{EF}\ q)$$

4. Between the events q and r, event p is never true.

   Solution: Possible solution:

   $$[\mathrm{AG}\ (q \rightarrow \neg\mathrm{EF}\ (p \wedge \mathrm{EF}\ r))] \wedge [\mathrm{AG}\ (r \rightarrow \neg\mathrm{EF}\ (p \wedge \mathrm{EF}\ q))]$$

5. Transitions to states satisfying p occur at most twice.

   Solution: Possible solution:

   $$\neg(\mathrm{EF}\ (p \wedge \mathrm{EX}\ \mathrm{EF}\ (p \wedge \mathrm{EX}\ \mathrm{EF}\ p)))$$

6. Property p is true for every second state along a path.

   Solution: Possible solution:

   $$\mathrm{X}\ \mathrm{X}\ p$$

   $$\mathrm{AX}\ \mathrm{AX}\ p$$

3 Expressable in ...

1. Give example of an LTL-formula for which equivalent translation in CTL does not exist.

   Solution: $\mathrm{G}\ p \rightarrow \mathrm{G}\ q$

2. Give example of an CTL-formula for which equivalent translation in LTL does not exist.

   Solution: $\mathrm{EX}\ p$

3. Give example of an CTL*-formula for which equivalent translation in LTL either in CTL does not exist.

   Solution: $\mathrm{A}\ (\mathrm{G}\ p \rightarrow \mathrm{G}\ q) \vee \mathrm{EX}\ p$

4 Proof the equivalence

Given the definitions:

- $\pi \models \psi \mathrm{~U~} \phi$ iff there is some $i \geq 1$ such that $\pi^i \models \phi$ and for all $j = 1, \ldots, i-1$ we have $\pi^j \models \psi$
- $\pi \models \psi \mathrm{~R~} \phi$ iff either there is some $i \geq 1$ such that $\pi^i \models \psi$ and for all $j = 1, \ldots, i$ we have $\pi^j \models \phi$, or for all $k \geq 1$ we have $\pi^k \models \phi$

Proof the following theorem:

$$\neg(\psi \mathrm{~U~} \phi) \equiv \neg\psi \mathrm{~R~} \neg\phi \tag{1}$$

Solution: Proof. Let $\mathcal{M}$ be a transition structure, s a state of $\mathcal{M}$. We must show that for every path π of $\mathcal{M}$ starting in s, $\pi \models \neg(\psi \mathrm{~U~} \phi)$ iff $\pi \models \neg\psi \mathrm{~R~} \neg\phi$,
The trick of the proof is in first simplifying the definition of $\mathrm{~R~}$:

$$\pi \models \psi \mathrm{~R~} \phi \text{ iff for all } i \geq 1 \text{ such that } \pi^i \models \neg\phi, \text{ there exists a } j < i \text{ such that } \pi^j \models \psi \tag{2}$$

Assume that $\pi \models \psi \mathrm{~R~} \phi$. Then either all $i \geq 1$ satisfy $\pi^i \models \phi$, or equivalently, there is no $i \geq 1$ such that $\pi^i \models \neg\phi$. In this case, the righthand of the equivalence is trivially satisfied. Or, there is an $i \geq 1$ such that $\pi^i \models \psi$ and $j \leq i$ implies $\pi^i \models \phi$. It follows for any $i' \geq 0$ such that $\pi^{i'} \models \neg\phi$, that $i < i'$ and $\pi^i \models \psi$. Again, the righthand is satisfied.

Assume that $\pi \not\models \psi \mathrm{~R~} \phi$. Hence, there is an $i \geq 1$ such that $\pi^i \models \neg\phi$ and ϕ has not been released, i.e., ψ is false in each of $\pi^1, \ldots, \pi^{j-1}$. Hence, for this i we have $\pi^i \models \neg\phi$ and there is no $j < i$ such that $\pi^j \models \psi$.

Given this equivalence, the theorem can no easily be proven:

$$\pi \models \neg(\psi \mathrm{~U~} \phi) \text{ iff for all } i \geq 1 \text{ such that } \pi^i \models \phi, \text{ there is a } j < i \text{ such that } \pi^j \models \neg\psi \text{ iff } \pi \models \neg\psi \mathrm{~R~} \neg\phi. \tag{3}$$

5 Nim game

If you have time left, and haven't made the Nim game yet last session, complete this exercise. The assignment from last week can still be found on Toledo.