CS 383C
CAM 383C/M 383E **Numerical Analysis: Linear Algebra** **Fall 2008**

Solutions to Homework 1

Lecturer: Inderjit Dhillon
Date Due: Sept 10, 2008
Keywords: *Linear Algebra Basics*

1. Problem 2.1
   Claim: If a matrix A is upper triangular and unitary then it is a diagonal matrix with nonzero diagonal elements.
   Proof: We prove the above claim using induction on size of the matrix (n). Base case is trivial. For induction step, assume that the claim holds for all matrices of size $(n-1) \times (n-1)$. Let A be an $n \times n$ upper triangular unitary matrix:
$$A = \begin{bmatrix} A_{n-1} & \boldsymbol{a}_n \\ 0 & a_{nn} \end{bmatrix},$$
   where A_{n-1} is an $(n-1) \times (n-1)$ upper triangular matrix and $\boldsymbol{a}_n$ is a column vector of dimensional $(n-1)$. Now,
$$A^*A = \begin{bmatrix} A_{n-1}^* A_{n-1} & A_{n-1}^* \boldsymbol{a}_n \\ \boldsymbol{a}_n^* A_{n-1} & \boldsymbol{a}_n^* \boldsymbol{a}_n + a_{nn}^* a_{nn} \end{bmatrix}.$$
   Since $A^*A = I_n$, then $A_{n-1}^* A_{n-1} = I_{n-1}$ and thus A_{n-1} is unitary and upper triangular. By the induction hypothesis, A_{n-1} is diagonal with nonzero entries on the diagonal. Further, $A_{n-1}\boldsymbol{a}_n = 0$, which is possible only if $\boldsymbol{a}_n = 0$, which implies $\|a_{nn}\| = 1$, i.e. $a_{nn} \neq 0$. Hence proved.

2. Problem 2.2
   (a) $||\boldsymbol{x}_1 + \boldsymbol{x}_2||_2^2 = ||\boldsymbol{x}_1||_2^2 + ||\boldsymbol{x}_2||_2^2 + 2\boldsymbol{x}_1^*\boldsymbol{x}_2$. Now $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are orthogonal, so $\boldsymbol{x}_1^*\boldsymbol{x}_2 = 0$ and $||\boldsymbol{x}_1 + \boldsymbol{x}_2||_2^2 = ||\boldsymbol{x}_1||_2^2 + ||\boldsymbol{x}_2||_2^2$.
   (b) We prove using induction on the number of vectors. Base case follows from part (a). For induction step, assume that the claim holds for an orthogonal set of $(n-1)$ vectors. Now, $|| \sum_{i=1}^n \boldsymbol{x}_i||_2^2 = ||\boldsymbol{y} + \boldsymbol{x}_n||_2^2$, where $\boldsymbol{y} = \sum_{i=1}^{n-1} \boldsymbol{x}_i$. Now since $\boldsymbol{x}_n$ is orthgonal to $\boldsymbol{x}_i, \forall 1 \leq i \leq n-1$, therefore $\boldsymbol{x}_n^*\boldsymbol{y} = 0$. Hence using part (a), the claim follows.

3. Problem 2.6
   Let $A = I + \boldsymbol{u}\boldsymbol{v}^*$ and $A^{-1} = I + \alpha\boldsymbol{u}\boldsymbol{v}^*$. Then $AA^{-1} = I + (1 + \alpha + \alpha\boldsymbol{v}^*\boldsymbol{u})uv^* = I \Longrightarrow \alpha = \frac{-1}{1+\boldsymbol{v}^*\boldsymbol{u}}$. So if $\boldsymbol{v}^*\boldsymbol{u} \neq -1$, A is non-singular and A^{-1} is given by $I - \frac{\boldsymbol{u}\boldsymbol{v}^*}{1+\boldsymbol{v}^*\boldsymbol{u}}$ (recall that if inverse of a matrix exists then it is unique). If $\boldsymbol{v}^*\boldsymbol{u} = -1$, then $A\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{v}^*\boldsymbol{u}\boldsymbol{u} = 0$ implying A is singular. To compute Null(A), consider any vector x s.t. $A\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x} = -(\boldsymbol{v}^*\boldsymbol{x})\boldsymbol{u}$. Thus $\boldsymbol{x}$ is parallel to $\boldsymbol{u}$, i.e., Null(A)=span($\boldsymbol{u}$) if $\boldsymbol{v}^*\boldsymbol{u} = -1$ else Null(A)= ϕ.