PS. #2

Solutions

6.3 Sol 1. Write $A = U\Sigma V^*$ where $\Sigma = \begin{pmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_n\end{pmatrix}$ $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k > 0$ and $\sigma_{k+1} = \dots = \sigma_n = 0$.

Then $\operatorname{rank}(A) = k$. Computing $A^*A = V\Sigma^2 V^*$ we see that $\operatorname{rank}(A^*A) = \operatorname{rank}(\Sigma^2) = k$.

So A^*A is nonsingular $\Leftrightarrow \operatorname{rank}(A^*A) = n$ $\Leftrightarrow \operatorname{rank}(\Sigma^2) = n \Leftrightarrow \operatorname{rank}(A) = n$.

Sol 2. If A has full rank then A^*A is nonsingular since $A^*Ax = 0 \Rightarrow$
$\Rightarrow x^*A^*Ax = 0 \Rightarrow \|Ax\| = 0 \Rightarrow Ax = 0 \Rightarrow$
$\Rightarrow \operatorname{Null}(A) \neq \{0\}$ if $x \neq 0$, so a contradiction.
If A^*A is nonsingular then A has full rank since $Ax = 0 \Rightarrow A^*Ax = 0$ and then $x = 0$.

7.3 Write $A = QR$, Q unitary ($Q^*Q = I$). So $\det(Q) = \pm 1$ and then $|\det A| = |\det R|$. But $R = \begin{pmatrix} r_{11} & \cdots \\ 0 & r_{nn}\end{pmatrix}$ and $|\det R| = \prod_j |r_{jj}|$.

Now $r_{jj} = \|P_\perp a_j\|_2 \leq \|a_j\|_2$ where $P_\perp$ is the projection on the orthogonal space to $\langle q_1, \dots, q_{j-1}\rangle$.

The geometric interpretation comes from the fact that $|\det A|$ is the "volume" of the parallelipiped determined by the columns vectors of A. The volume is maxim (if we fix the lenghts of the vectors) when the vectors are orthogonal to each other, in which case the volume is just the product of the lenghts.

7.5 a) $A = \hat{Q}\hat{R}$. Then A has rank n if and only if $\hat{R}$ has diagonal entries nonzero. Indeed if $Ax = 0$ then $\hat{Q}\hat{R}x = 0$ and then $\hat{R}x = 0$. Now let $\hat{R} = \begin{pmatrix} r_{11} & \cdots \\ 0 & r_{nn}\end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}$. If $r_{nn} \neq 0 \dots r_{k+1,k+1} \neq 0$ but $r_{kk} = 0$ we can show that $\hat{R}x = 0$ and this implies $x_n = x_{n-1} = \dots = x_{k+1} = 0$ and then we can choose $x_k = 1$ and determine the rest of $x_{k-1}, \dots, x_1$. So we get a nonzero vector x such that $\hat{R}x = 0 \Rightarrow Ax = 0$, so a contradiction.

If all r_{ii}'s are nonzero then A has full rank and if A has full rank then the r_{ii}'s are nonzero, this follow by the argument given above or the reverse of it (just take and prove $\operatorname{rank}(\hat{R}) = n$ iff all r_{ii}'s are nonzero).

b) Using the argument above it can be shown that if $\hat{R} = \begin{pmatrix} r_{11} & \cdots & & \\ & r_{kk} & & \\ & & 0 & \\ & & & \ddots \; 0 \end{pmatrix}$ then $\operatorname{rank}(A) \geq k$. So $\operatorname{rank}(A) \in \{k, \dots, n-1\}$ because A can not have full rank.

There is an example when $k = 1$ and $\operatorname{rank}(A)$ can be any value in $\{k, \dots, n-1\}$.

Take $\hat{R} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & \boxed{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{k \times k} & \\ & 0 & & 0 & \end{pmatrix}$

and $\hat{Q} = [q_1 | \cdots | q_n]$

we get $A = \hat{Q}\hat{R} = [q_1 | 0 | 0 \cdots | q_2 | \cdots | q_k |]$,

so A has rank K.

Problem 8.2

```
function modif GS= mgs(A)
% compute the reduced QR decomposition of A using the modified
GS algorithm.
% compute m and n, the number of rows and columns of A
m = size(A,1); n= size(A,2);
for i= 1:n
    v=A(:,i);
    R(i,i) = norm(v);
    Q(:,i)= v/R(i,i); % set the ith orthogonal vector of Q.
% inner loop of the modified GS;
% every time a new orthogonal vector is computed
% project all remaining columns vectors of A onto the space
of orthogonal to that vector.

for j=(i+1):n
    R(i,j)= conj(Q(:,i)'* A(:,j))
    A(:,j)= A(:,j) - R(i,j)*Q(:,i)
end
end.
R=[R zeros(m-n,n)];
modif GS=[QR]
% Recover Q and R
mQR = clgs(A);
m = size(mQR,1); n= size(mQR,2)/2 ;
Qmgs= mQR(:,1:m); Rmgs = mQR(1:n,n+1:2*n).
```