CS 383C
CAM 383C/M 383E — Numerical Analysis: Linear Algebra — Fall 2008

Solutions to Homework 9

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Keywords: *Iterative Methods, Arnoldi Iteration, Lanczos Method*

1. **Problem 33.2**

   (a) Since $h_{n+1,n} = 0$, $H = \begin{bmatrix} H_n & B \\ 0 & H_{m-n} \end{bmatrix}$, where B is a $n \times (m-n)$ matrix. Also, $AQ_n = Q_n H_n$because $h_{n+1,n} = 0$.

   (b) Let $\boldsymbol{v} \in \mathcal{K}_n$. Since Q_n forms an orthogonal basis for $\mathcal{K}_n$, $\boldsymbol{v} = Q_n \boldsymbol{x}$ for some $\boldsymbol{x} \in \mathbb{C}^n$. Now, $A\boldsymbol{v} = AQ_n\boldsymbol{x} = Q_n(H_n\boldsymbol{x}) = Q_n\boldsymbol{y}$, where $\boldsymbol{y} = H_n\boldsymbol{x}$. Therefore, $A\boldsymbol{v} \in \mathcal{K}_n$. Hence proved.

   (c) $n+1$-th basis vector of $\mathcal{K}_{n+1}$ is given by $A^n b = A(A^{n-1}b)$. Using part(b), $A^n b \in \mathcal{K}_n$. Hence, $\mathcal{K}_{n+1} \subseteq \mathcal{K}_n$. Also, $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$ trivially. Hence, $\mathcal{K}_n = \mathcal{K}_{n+1}$ and using induction, $\mathcal{K}_n = \mathcal{K}_{n+i}$, $1 \leq i \leq (m-n)$.

   (d) Let $\boldsymbol{v}$ be an eigenvector of H_n with eigenvalue λ. Using part (b), $AQ_n = Q_n H_n$. Hence, $A(Q_n\boldsymbol{v}) = Q_n H_n \boldsymbol{v} = \lambda(Q_n\boldsymbol{v})$. Therefore, every eigenvalue of H_n is an eigenvalue of A.

   (e) $\boldsymbol{x} = A^{-1}b = A^{-1}Q_n\boldsymbol{e_1}\|b\| = A^{-1}Q_nH_nH_n^{-1}\boldsymbol{e}_1\|b\| = A^{-1}AQ_nH_n^{-1}\boldsymbol{e}_1\|b\| = Q_n(H_n^{-1}\boldsymbol{e}_1\|b\|) \in \mathcal{K}_n$.

2. **Problem 36.1** Any $\boldsymbol{x} \in \mathcal{K}_n$ can be represented as $\boldsymbol{x} = Q_n\boldsymbol{y}$. Also, $T_n = Q_n^T A Q_n$. Thus, the Rayleigh quotient when restricted to $\mathcal{K}_n$ is given by: $r(\boldsymbol{x}) = \frac{\boldsymbol{y}^T Q_n^T A Q_n \boldsymbol{y}}{\boldsymbol{y}^T\boldsymbol{y}} = \frac{\boldsymbol{y}^T T_n \boldsymbol{y}}{\boldsymbol{y}^T\boldsymbol{y}}$, where $\boldsymbol{x} = Q_n\boldsymbol{y}$. Now, eigenvalues of a matrix are stationary points of the Rayleigh quotient, thus the Ritz values, i.e. eigenvalues of T_n are all stationary points of Rayleigh quotients of A when restricted to $\mathcal{K}_n$.

3. **Problem 38.5**

   (a) $\phi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}$, $\bigtriangledown\phi(\boldsymbol{x}) = A\boldsymbol{x} - \boldsymbol{b}$.

   (b) Now, $\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \alpha_n r_{n-1}$. Optimal α_n is the one that minimizes $\phi(\boldsymbol{x}_n)$. Thus, to obtain the optimal α_n we set gradient of $\phi(\boldsymbol{x}_n)$ w.r.t. α_n to be 0:

   $$r_{n-1}^T A r_{n-1}\alpha_n + r_{n-1}^T(A\boldsymbol{x}_{n-1} - \boldsymbol{b}) = 0.$$

   Simplifying, we get: $\alpha_n = \frac{r_{n-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}}$.

   (c)
      i. $\boldsymbol{x}_0 = 0$, $r_0 = \boldsymbol{b}$
      ii. **for** $n = 1, 2, 3....$
      iii. $\quad\alpha_n = \frac{r_{n-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}}$
      iv. $\quad\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \alpha_n r_{n-1}$
      v. $\quad r_n = b - A\boldsymbol{x}_n$