Nullen creëren in een matrix

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Gauss-eliminatie stap

Gauss-eliminatie

- $A \in \mathbb{R}^{m \times n}$
- elementaire transformatiematrix $T_{ij}(m_{ji})$
  $\bar{A} = T_{ij}(m_{ji})A \Rightarrow$ element $\bar{a}_{ji}$ van $\bar{A}$ is nul
- optimale rijpivotering
  $|m_{ji}| \leq 1 \quad \Rightarrow \quad K_1(T_{ij}(m_{ji})) \leq 4$

Givens transformatie

Givens transformatie

- $G = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$ waarbij $\begin{cases} c = \cos\theta \\ s = \sin\theta \end{cases}$
- G is orthogonale matrix want $G^T = G^{-1}$ $(G^T G = I_2)$
- gegeven x en y, zoek c en s zodat

$$\begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}$$

Givens (vervolg)

- Oplossing
  - G is orthogonaal $\Rightarrow z = \pm\sqrt{x^2 + y^2}$
  - $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \begin{cases} c = \frac{x}{\pm\sqrt{x^2+y^2}} \\ s = \frac{y}{\pm\sqrt{x^2+y^2}} \end{cases}$$

- $K(G) = 1$

Householder transformatie

Householder transformatie

- $P = I - \frac{2vv^T}{v^Tv} \quad , \quad v \in \mathbb{R}^{m\times 1}$
- P is orthogonaal want $P^TP = I$

$$\begin{aligned}\left(I - \frac{2vv^T}{v^Tv}\right)\left(I - \frac{2vv^T}{v^Tv}\right) &= \\ I - \frac{4vv^T}{v^Tv} + 4\frac{v^Tv}{(v^Tv)^2}vv^T &= I\end{aligned}$$

Householder (vervolg)

- gegeven $x \in \mathbb{R}^{m \times 1}$, zoek $v \in \mathbb{R}^{m \times 1}$ zodat $Px = \beta \cdot e_1$ oplossing:
  - P is orthogonaal $\Rightarrow \beta = \pm \parallel x \parallel_2$
  - $Px = \left[I - \frac{2vv^T}{v^Tv} \right] x = x - 2\frac{v^Tx}{v^Tv} v = \beta e_1$
    $\Rightarrow \alpha \cdot v = x - \beta e_1 = x \mp \parallel x \parallel_2 e_1$
    $\Rightarrow$ possible choice for v:
    $$v = x + \text{sign}(x_1) \parallel x \parallel_2 e_1$$
- $K(P) = 1$

QR-factorisatie van een matrix

* gegeven: $A \in \mathbb{R}^{m \times n} \quad , \quad m \geq n$
* gevraagd : Q, R zodat
  - $A = Q \cdot R$
  - $Q^T Q = I$
  - $R = \left|\begin{array}{c} \hline \quad * \\ 0 \quad \\ \hline \\ 0 \quad \\ \hline \end{array}\right|$

QR (vervolg)

- oplossing
  - met Givens transformaties

$$
\begin{array}{c} \\ \\ \\ \longrightarrow \\ \longrightarrow \end{array}
\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ \otimes & x & x & x \end{bmatrix}
\Rightarrow
\begin{array}{c} \\ \\ \longrightarrow \\ \longrightarrow \\ \\ \end{array}
\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ \otimes & x & x & x \\ 0 & x & x & x \end{bmatrix}
\Rightarrow
\begin{array}{c} \\ \longrightarrow \\ \longrightarrow \\ \\ \\ \end{array}
\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ \otimes & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}
$$

$$
\begin{array}{c} \longrightarrow \\ \longrightarrow \\ \\ \\ \\ \end{array}
\begin{bmatrix} x & x & x & x \\ \otimes & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}
\Rightarrow
\begin{array}{c} \\ \\ \\ \longrightarrow \\ \longrightarrow \end{array}
\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & \otimes & x & x \end{bmatrix}
\Rightarrow \ldots \Rightarrow
\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

QR (vervolg)

- met Householder transformaties

$$\begin{bmatrix} \boxed{\mathrm{x}} & x & x & x \\ \otimes & x & x & x \\ \otimes & x & x & x \\ \otimes & x & x & x \\ \otimes & x & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x & x \\ 0 & \boxed{\mathrm{x}} & x & x \\ 0 & \otimes & x & x \\ 0 & \otimes & x & x \\ 0 & \otimes & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & \boxed{\mathrm{x}} & x \\ 0 & 0 & \otimes & x \\ 0 & 0 & \otimes & x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & \boxed{\mathrm{x}} \\ 0 & 0 & 0 & \otimes \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $H_4 H_3 H_2 H_1 A = R$

$$\begin{aligned} A &= (H_1^T H_2^T H_3^T H_4^T) \cdot R \\ &= Q \cdot R \end{aligned}$$