P.S #4
Solutions

Problem 15.1

a) $\tilde{f}(x) = fl(x) \oplus fl(x) = [x(1+\varepsilon_1) + x(1+\varepsilon_1)](1+\varepsilon_2)$

$= 2x(1+\varepsilon_1)(1+\varepsilon_2) = 2x(1+\varepsilon_3) = 2\tilde{x}$

$= f(\tilde{x})$

Now since $\frac{|x-\tilde{x}|}{|x|} = O(\varepsilon_{mach})$ we have <u>backward stability</u>

b) $\tilde{f}(x) = fl(x) \otimes fl(x) = [x(1+\varepsilon_1) \times (1+\varepsilon_1)](1+\varepsilon_2)$

$= x^2(1+\varepsilon_1)^2(1+\varepsilon_2) = [x(1+\varepsilon_3)]^2 = \tilde{x}^2$

$= f(\tilde{x})$

since $\frac{|\tilde{x}-x|}{|x|} = O(\varepsilon_{mach})$ we have <u>backward stability</u>.

c) $\tilde{f}(x) = fl(x) \oslash fl(x) = \frac{x(1+\varepsilon_1)}{x(1+\varepsilon_1)}(1+\varepsilon_2) = 1+\varepsilon_2$

$f(\tilde{x}) = 1$. for any $\tilde{x}$.

$\frac{|\tilde{x}-x|}{|x|} = O(\varepsilon_{mach})$ and $\frac{|\tilde{f}(x) - f(\tilde{x})|}{|f(\tilde{x})|} =$

$= |1+\varepsilon_2 - 1| = |\varepsilon_2| = O(\varepsilon_{mach})$. So this is <u>stable but not backward stable</u>.

d) $\tilde{f}(x) = fl(x) \ominus fl(x) = [x(1+\varepsilon_1) - x(1+\varepsilon_1)](1+\varepsilon_2)$

$= 0 = f(\tilde{x})$

$\frac{|\tilde{x}-x|}{|x|} = O(\varepsilon_{mach})$ so this is <u>backward stable</u>.

Problem 15.2

a) Backward stability means that the SVD matrices $\tilde{U}, \tilde{\Sigma}, \tilde{V}$ of A are the SVD matrices for ~~the~~ slightly perturbed matrix $A + \delta A$ where $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{mach})$.

b) In general $\tilde{U}, \tilde{V}$ are not unitary, they are just approximates for unitary.

c) A stable SVD factorization means that the relative error between $\tilde{U}, \tilde{V}, \tilde{\Sigma}$ of a matrix A and the exact SVD of a perturbation $A + \delta A$ is of the order of $O(\varepsilon_{mach})$. This is:

$$\frac{\|\tilde{U}\tilde{\Sigma}\tilde{V}^* - (A+\delta A)\|}{\|A + \delta A\|} = O(\varepsilon_{mach}),$$

$$\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{mach}).$$

Problem 16.1

a) Take first $B = QA$, Q unitary.

$$\tilde{f}(A) = (Q+\delta Q)(A+\delta A) = Q\underbrace{[Q^*(Q+\delta Q)(A+\delta A)]}_{\tilde{A}}$$

$$= Q\tilde{A}$$

We show that $\frac{\|\tilde{A} - A\|}{\|A\|} = O(\varepsilon_{mach})$.

Now $\frac{\|\tilde{A}-A\|}{\|A\|} \le \frac{\|\delta A\|}{\|A\|} + \frac{\|Q^*\delta Q A\|}{\|A\|} + \frac{\|Q^*\delta Q \delta A\|}{\|A\|}$

Now $\frac{\|\delta A\|}{\|A\|} \le O(\varepsilon_{mach})$

(*) $\frac{\|Q^*\delta Q A\|}{\|A\|} \le \|Q^*\|\|Q\| \cdot \frac{\|\delta Q\|}{\|Q\|} \cdot \frac{\|A\|}{\|A\|} = O(\varepsilon_{mach})$

$\frac{\|Q^*\delta Q \delta A\|}{\|A\|} \le O(\varepsilon_{mach})$.

So $\frac{\|\tilde{A}-A\|}{\|A\|} \le O(\varepsilon_{mach})$ hence <u>backward stability</u>

Now for a general k we just have to apply and inductive procedure.

b) Q unitary allowed us to estimate (*), namely we used $\|Q^*\|\|Q\| = 1$. If Q is replaced by X then $\|X^{-1}\|\|X\|$ might be huge.

Problem 16.2

```
a) [U,X] = qr(randn(50));
   [V,X] = qr(randn(50));
   S = diag(flipud(sort(rand(50,1))));
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```
= U*S*V';
U2,S2,V2]= svd(A);
norm(U-U2)
norm(V-V2)
norm(S-S2)
norm(A-U2*S2*V2')
```

The errors in U and V are of agnitud 2.0 while the errors in e singular values and A are ~~...~~ f O(Emach). The problem with U&V o that SVD factorization is not unique, since the columns of U & U can be multiplied by -1 without changing the product.

) To be able to compare the atrices U and V, ~~the~~ signs an be changed if necessary. We can do this by:

```
change = sign(diag(U2'*U))
U2= U2* diag(change)
V2= V2* diag(change)
```