Video 2.1
Sampath Kannan

Optimization

- **Objective Function:** A function that assigns a value to each feasible solution

Optimization

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- **Optimization Problem:** Find the solution with the maximum (or minimum) objective function value

Optimization: Examples

Optimization problems appear everywhere!

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- **Shortest** path from location A to location B
- **Maximum** value of goods you can buy on a budget
- **Smallest** number of changes you need to make to transform one string into another
- Locating k hospitals in a community to **minimize** the maximum time anyone has to travel
- Compute the value of a function in the **fewest** steps

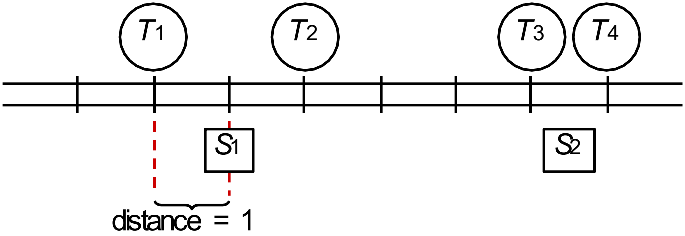
Optimization: Solutions

- Brute Force approach: look at objective function value of each possible solution and take the best
- There can be exponentially many solutions. Brute-force approach can take too long
- **Dynamic Programming** - efficient way to find the optimal solution for some problems
  - When can we use dynamic programming?
  - How can we apply it?

Video 2.2
Sampath Kannan

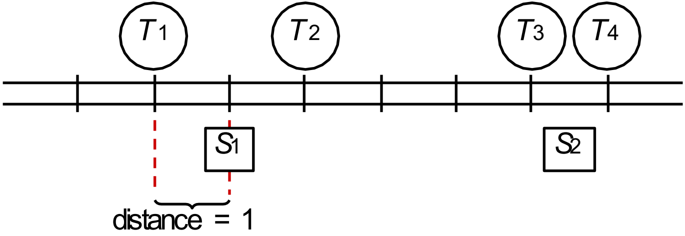
Station Placement

We want to place k stations along a train line so that the maximum distance between a town and its nearest station is minimized.



This diagram shows a cost 1 solution for $k = 2$ where the towns are located at positions 1, 3, 6 and 7.

Notation



- Notation: The towns are a sequence of integers $t_1, t_2, \cdots, t_n$ and the stations are a sequence of rational numbers $s_1, s_2, \cdots, s_k$

Top Level Decisions

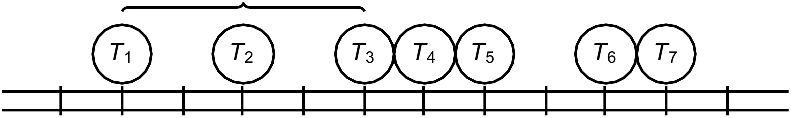
- Once we have the stations, we know which towns will use which.

Top Level Decisions

- Once we have the stations, we know which towns will use which.
- We can make a **top-level decision** about how many towns will use the leftmost station. Don't know the answer so we have to try all possibilities!
- This is the idea of dynamic programming, we explore all choices and take the best one

Top Level Decision

If the first i towns use the left most station…



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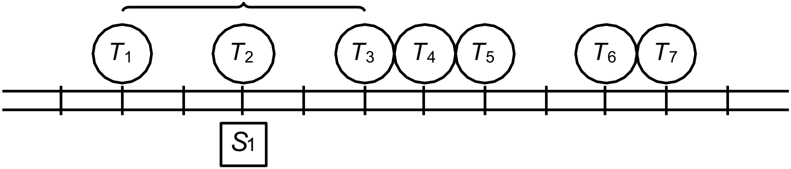

station should be at $(t_i + t_1)/2$

Top Level Decision

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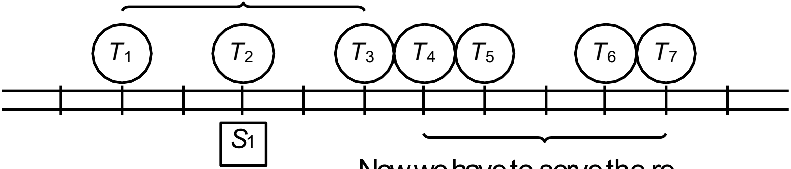


station should be at $(t_i + t_1)/2$

Now we have to serve the remaining $n - i$ towns with $k - 1$ stations

Recurrence

- Locate($[i, j]$, l) finds the best location for l stations to serve towns t_i through t_j and returns the maximum distance from any town to its nearest station.

Recurrence

- Locate([i , j], A) finds the best location for A stations to serve towns t_i through t_j and returns the maximum distance from any town to its nearest station.
- Recursive idea:

$$\text{Locate}([i,j],l) = \min_{x\in[i,j-1]} \max\left\{\frac{t_x - t_i}{2}, \text{Locate}([x+1,j],l-1)\right\}$$

Video 2.3
Sampath Kannan

Computing Locate

- Locate($t[i, j]$, k) finds the best location for k stations to serve towns t_i through t_j and returns the maximum distance from any town to its nearest station.
- Recursive idea: For $k > 1$

$$\text{Locate}(t[i, j], k) = \min_{x \in \{i \ldots j-1\}} \max\left(\frac{t_x - t_i}{2}, \text{Locate}(t[x+1, j], k-1)\right)$$

- What does this mean? Why is it correct?

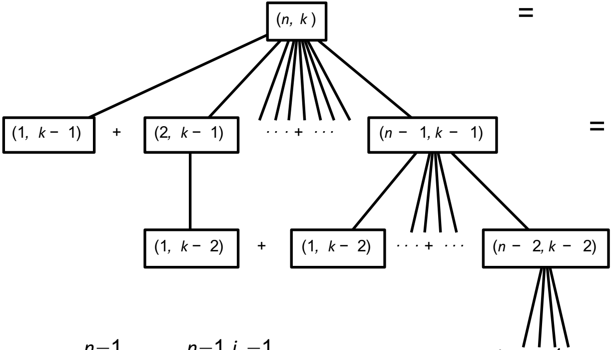
Computing Locate

- We want to compute Locate([1, n], k), the placement of k stations for n towns
- $T(n, k)$ = the time it takes to solve this problem

$$T(n,k) = \sum_{j=1}^{n-1} T(j, k-1) + n$$

$$T(1,k) = 1$$

$$T(n,1) = 1$$



$$= n$$

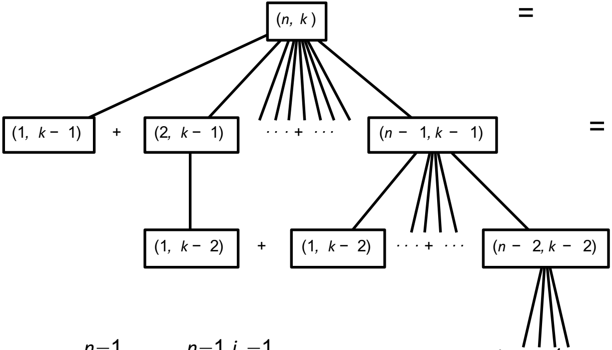
$$= \sum_{i_1=1}^{n-1} i_1$$

$$= \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{i_1-1} i_2$$

$$= n + \sum_{i_1=1}^{n-1} i_1 + \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{i_1-1} i_2 + \cdots + \sum_{i_1=1}^{n-1} \cdots \sum_{i_{k-1}=1}^{i_{k-2}-1} i_{k-1}$$

$$= n + \sum_{i=1}^{k-1} \sum_{m=1}^{n-i} m \binom{n-1-m}{i-1}$$

- This grows exponentially fast!



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- This grows exponentially fast!
- Subproblem $(1, k-2)$ gets called by $(2, k-1), (3, k-1) \ldots$
- If we don't recompute we can save time

Types of Subproblems

- What are the subproblems we see?

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- What are the subproblems we see?
- The new list of towns is always a suffice of the original list ($n - 1$ suffixes)
- The number of stations is always between 1 and k
- The total number of subproblems is at most $k(n - 1)$
- Dynamic programming works when there aren't too many subproblems

Dynamic programming Locate

- Instead of thinking about a recursion tree, think about an array of subproblems C
- $C[i, j]$ = the minimum cost of placing j station to serve towns i through n

Dynamic programming Locate

- Instead of thinking about a recursion tree, think about an array of subproblems *C*
- *C* [*i* , *j*] = the minimum cost of placing *j* station to serve towns *i* through *n*

```
Initialize C[i,j] = null
for all i let C[n,i] = 1
 Locate(t[start,end],k):
   if C[start,k] = null
     c = Inf
     for x in start...end-1 do
       c = min(c, max((t[x]-t[end])/2,
                      Locate(t[x+1,end],k-1)))
     C[start,k] = c
     return c
  else
     return C[start,k]
```

Running Time

- Now each subproblem is only computed once.
- Each sub problem takes at most n operations to solve (remember there are at most $(n-1)k$ subproblems)

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- Each sub problem takes at most n operations to solve (remember there are at most $(n-1)k$ subproblems)
- So the new running time is $O(n^2k)$
- We can also avoid the recursion entirely:

```
Locate(t[start,end],k):
  for all i let C[i,1] =
  (t[end]-t[i])/2   for all i
  let C[end,i] = 0
  for j from 2 to k
    for i from end
      to 1  c = Inf
      for x from i to end-1
        c = min(c, max((t[x]-
        t[end])/2,C[x+1,j-1]))
      C[i,j]
  = c  return
  C[1,k]
```

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Dynamic Programming

What are the general properties of problems where dynamic programming is applicable?

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- **Optimal Substructure:** In order to solve the whole problem optimally, we need to solve certain subproblems optimally

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- **Optimal Substructure:** In order to solve the whole problem optimally, we need to solve certain subproblems optimally
- **Not-too-many subproblems:** The same few subproblems keep recurring, so we do not need to solve too many subproblems.

Dynamic Programming - Optimal Substructure

"Optimal Substructure"- when is it present?

- Another Example: Shortest path between two cities A and B.

Dynamic Programming - Optimal Substructure

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  - Suppose we have a set of cities, connected by roads
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$$A \sim C \sim B$$

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$$A \sim C \sim B$$

  - Shortest path from A to B:
    - Take the shortest path from A to C
    - Then take the shortest path from C to B.
  - This problem has optimal substructure!
    - Why? Optimal solutions is composed of the optimal solution to subproblems
    - Problem: It might be difficult to find this city C

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Video 2.5

Sampath Kannan

Dynamic Programming: LCS

Longest Common Subsequence (LCS)

- Given: two strings $s = s_1 s_2 \ldots s_m$ and $t = t_1 t_2 \ldots t_n$
- Strings are over some alphabet (may be english, may be something else)

Dynamic Programming: LCS

Longest Common Subsequence (LCS)

- Example: s = *TUCSON*,
  t = *HOUSTON*

T	U	C	S	O	N

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Dynamic Programming: LCS

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- One common subsequence: TON (length 3)

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Dynamic Programming: LCS

Longest Common Subsequence (LCS)

- Example: *s* = *TUCSON*,
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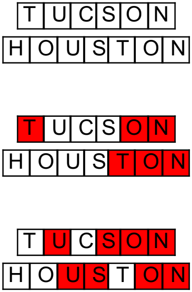
- One common subsequence: TON (length 3)
- Longer one: USON (length 4, best)



LCS: Top-Level Question

Can LCS be solved using dynamic programming?

- Given two subsequences, what might be a question for a top level decision?
- Note: the question must be such that its answers cover all possible solutions
- Idea 1: Does the longest common subsequence include the last letter of both strings?

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Other ideas?

LCS: Top-Level Question

Can we come up with a better top-level decision?

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LCS: Top-Level Question

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- First, a lemma:
- *Lemma:* If the last symbols of two strings match, there is a longest common subsequence that uses these symbols

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- First, a lemma:
- *Lemma:* If the last symbols of two strings match, there is a longest common subsequence that uses these symbols
  - Simple proof by contradiction!
  - Suppose both strings have *x* as their last symbol, but we find a LCS that does not end in *x* .
  - We could add *x* to the end of our current LCS, creating a new, better LCS.
  - If some other occurrence of *x* in either string is used in LCS, it doesn't hurt to replace it with the last occurrence of *x* .

LCS: Top-Level Question

Can we come up with a better top-level decision?

T	U	C	S	O	N	
H	O	U	S	T	O	N

- Good top-level question: If the last symbols don't match, which one do we want to drop from consideration?
  - Note that one of them must be dropped!
  - We are not ignoring any possible solutions
  - How is this question better than our first approach?

LCS: Dynamic Programming

- Let LCS(i,j) represent the length of the longest common subsequence using the first i symbols of s and the first j symbols of t.
- Note that we could store the actual sequences themselves by carefully keeping track of indices

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if s_i = t_j:
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  - We can write a simple recursive algorithm to implement this
  - How slow would it be? Would it be resolving any of the same problems?

LCS: Dynamic Programming

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- Instead, we'll use a bottom-up solution:
  - Solve the smallest subproblems first, build up to bigger solutions
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  - One of our strings is empty, so LCS must be 0
- Which subproblem is our solution to the whole problem?
  - $LCS(m, n)$

LCS: Dynamic Programming

LCS: Dynamic Programming

- overall algorithm:
  - Create an $(m + 1) x (n + 1)$ grid, where slot (i, j) represents the solution to $LCS(i,j)$.
  - Fill in the trivial subproblem solutions ($LCS(i, 0)$ and $LCS(0,j)$)
  - Fill the grid row by row (or column by column) using the previous recurrence.

LCS: Algorithm And Runtime

```
LCS_length(s, t):
    m <- length(s)
    n <- length(t)
    M = an m+1 x n+1 matrix
    for i <- 0 to m
        M[i,0] <- 0
    for j <- 0 to n
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    for i <- 1 to m
        for j <- 1 to n
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- Total runtime = (# subproblems) * (time per subproblem)
- # subproblems = *mn*
- Each subproblem is calculated in constant time (from the recurrence)
- **Total runtime of LCS:** $O(mn)$

LCS: Original Example

T	U	C	S	O	N

H	O	U	S	T	O	N

LCS table

		T	U	C	S	O	N
		0	0	0	0	0	0
H	0						
O	0						
U	0						
S	0						
T	0						
O	0						
N	0						

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O	0	1	1	1			
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LCS: Original Example

T	U	C	S	O	N

H	O	U	S	T	O	N

LCS table

		T	U	C	S	O	N
		0	0	0	0	0	0
H	0	0	0	0	0		
O	0	0	0	0	0		
U	0	0	1	1	1		
S	0	0	1	1	2		
T	0	1	1	1	2		
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U	0	0	1	1	1	1	1
S	0	0	1	1	2	2	2
T	0	1	1	1	2	2	2
O	0	1	1	1	2	3	3
N	0	1	1	1	2	3	4

Video 2.6
Sampath Kannan

Optimal Static Binary Search Trees

- Input: n keys (in sorted order), $k_1 < k_2 < \ldots < k_n$, along with probability of each key being accessed, $p_1, p_2, \ldots, p_n$

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P(access) access time

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key	a	b	c	d	e	f	g	h	i	j
P(access)	0.15	0.05	0.08	0.1	0.2	0.05	0.1	0.07	0.15	0.05

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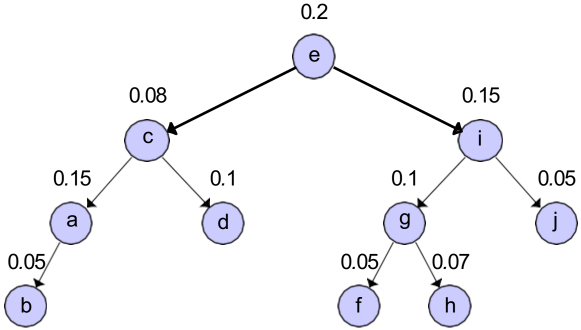

Optimal Static BSTs

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- Which element to make the root?
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  - Search Tree Property:
    - The left subtree must consist of $k_1, \ldots, k_{i-1}$
    - The right subtree must consist of $k_{i+1}, \ldots, k_n$
  - Optimal Substructure Property:
    - The left and right subtrees should also be optimal BSTs for their elements

Optimal Static BSTs

- What subproblems might we have to solve?

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Goal: create optimal BST for keys $k_1 \ldots k_n$

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Choose k_i as root

Optimal Static BSTs

- What subproblems might we have to solve?

Compute right subtree
(Using keys $k_{i+1} \ldots k_n$)

Optimal Static BSTs

- What subproblems might we have to solve?

Choose k_j as root (of the right subtree)

Optimal Static BSTs

- What subproblems might we have to solve?

Compute its left subtree
(Using keys $k_{i+1} \ldots k_{j-1}$)

Optimal Static BSTs

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- What subproblems might we have to solve?
  - Potentially one for each contiguous set of keys $[k_i \ldots k_j]$
- How many are there?
  - Each contiguous set is defined by choosing two keys from our set (smallest and largest key in the interval)
  - $\binom{n}{2} = \frac{n(n-1)}{2} \in O(n^2)$

Optimal Static BSTs: Recurrence

- Can we define a recurrence for this problem?

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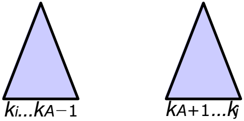

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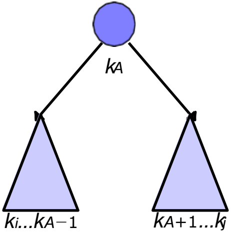

search times for every element in both subtrees increases by 1

P(access element in left subtree) $= \sum_{m=i}^{A-1} p_m$

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$$Min_{i \leq \ell \leq j} \{ \overbrace{T(i, \ell - 1) + 1 * \sum_{m=i}^{\ell-1} p_m}^{\text{left subtree access time}} + \overbrace{T(\ell + 1, j) + 1 * \sum_{m=\ell+1}^{j} p_m}^{\text{right subtree access time}} + \overbrace{1 * p_\ell}^{\text{root access time}} \}$$

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optimal_bst(keys, freq):
  for size = 1 to n: (all possible sizes of contigous sets)
    for i = 1 to n-size-1: (all possible starting points)
      j <- i + size - 1
      T[i,j] <- max_value
      sum_ij = sum(freq, i, j)
        for l = i to j:
          curr <- T[i,l-1] + sum_ij
          if curr < T[i,j]:
            T[i,j] <- curr
  return T[1,n]
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- \# subproblems: $O(n^2)$
- Time required to calculate subproblem (given previous subproblem solutions): $j - i \in O(n)$
- total runtime: $O(n^3)$

Dynamic Programming - Design

- How do we design dynamic programming algorithms?
- Understand if your problem has the two properties
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Dynamic Programming - Design

- How do we design dynamic programming algorithms?
- Understand if your problem has the two properties
- If so, ask what the top-level decision question is
  - There could be many possible questions
  - Picking the right one is an art
- Next, design a recursive algorithm that solves the whole problem by considering each answer to the top-level question, and recursively solving the resulting subproblems

Dynamic Programming - Design

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    - Requires some work to make sure subproblems are being solved in the correct order
- Basic algorithm **only computes the cost of the solution.** Keeping track of the actual solution requires some extra bookkeeping

Video 2.7

Sampath Kannan

Greedy Algorithms

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- For some problems, we can immediately decide what the best answer to the top-level decision question is.
- If so, we can make that choice and solve the resulting smaller problems.
- Algorithms that work like this are called **greedy algorithms:**
  - We greedily choose the best option at the moment because it will also lead to the best solution overall.
- In general, greedy algorithms are easy to design, but hard to prove correct.

Greedy: Example

- **Input:** n jobs, $j_1, j_2, \ldots j_n$, and a time T
- Job j_i takes t_i units of time to complete
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Greedy: Example

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Greedy: Example

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- The greedy choice would be to complete the job that takes the smallest amount of time first
  - Why? Maximizes amount of time remaining after completion of 1 job

Greedy: Example

- Solution:
  - Sort the jobs in increasing order of t_i (time for each job)
  - Complete as many jobs as possible in this order, until T units of time have passed

Greedy: Example

- Solution:
  - Sort the jobs in increasing order of t_i (time for each job)
  - Complete as many jobs as possible in this order, until T units of time have passed
- Correctness of the greedy solution may seem in clear in this case.
- In more complicated problems, this is less obvious!

Video 2.8
Sampath Kannan

Greedy Algorithms

Another (more complicated) scheduling problem:

- **Input:** n events, $e_1, e_2, \ldots, e_n$
- Each event e_i starts at time s_i and finishes at time f_i
- Only one event can be running at any given time
- **Goal:** Find the maximum number of events we can schedule

Scheduling: Some Ideas

Top level decision: Which event to schedule first?
What are some greedy choices that we could make?

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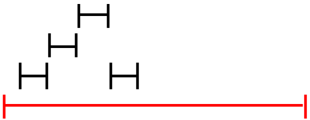

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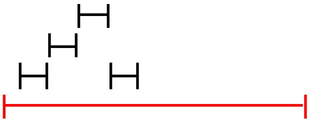

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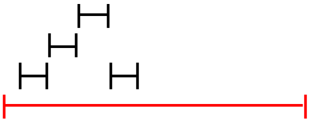

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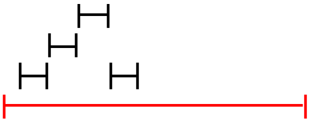

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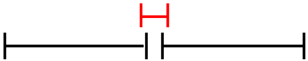

Scheduling: Correct Greedy Algorithm

- Idea: Sort events by finish time!
- Let us renumber the events in order of increasing finish time, $e_1, e_2, \ldots, e_n$
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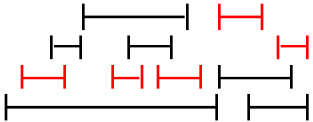

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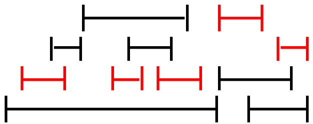

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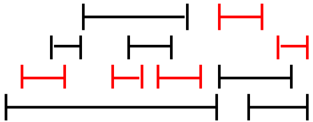

- Intuition: Our greedy first choice leaves the maximum time for remaining events.
- Optimum couldn't do better.

Greedy Algorithm: Correctness

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Greedy Algorithm: Correctness

We must prove that:

- The first choice that the greedy algorithm makes can be continued on to an optimal solution (greedy choice)
- After the greedy choice is made, solving the rest of the problem optimally will solve the entire problem optimally (optimal substructure)

Greedy Algorithm: Correctness

Correctness of the greedy choice:

Greedy Algorithm: Correctness

Correctness of the greedy choice:

- Let $O = e_{o_1} e_{o_2} \ldots e_{o_k}$ be an optimal sequence of events in increasing order of finish time.

Greedy Algorithm: Correctness

Correctness of the greedy choice:

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- **Exchange property:** If $e_{o_1} f = e_1$, we can replace e_{o_1} by e_1 and still get an optimal solution
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Optimal substructure:

- Consider an optimal solution $O = e_{o_1} e_{o_2} \ldots e_{o_k}$
- The solution $e_{o_2} \ldots e_{o_k}$ must be the optimal sequence of events for the time interval $[s_{o_2} \ldots f_{o_k}]$
- Otherwise, take the better sequence and replace $e_{o_2} \ldots e_{o_k}$ with it!

Video 2.9
Sampath Kannan

File Compression

- Suppose we are sending English text over the internet, we need a way to encode the symbols as a sequence of ones and zeros.
- We'd like to minimize the total amount of information sent (the length of the sequence).
- Idea: use shorter encoding for more frequent symbols (e.g., 'e', 'r', 't').

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- Bad Example:
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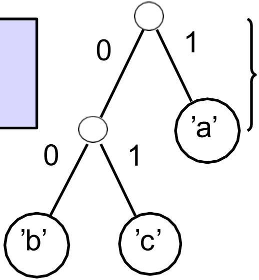
- Problem: 0101110 can be $\boxed{010}\boxed{1110}$ = 'ag', or $\boxed{0101}\boxed{110}$ = 'fs'
- Solution: No code can be a prefix of another

Prefix Codes

Let the alphabet be $\Sigma = \{a, b, c\}$ and for each symbol $x \in \Sigma$ we let $f(x)$ be the frequency of x and $d(x)$ be the length of the encoding of x.

$a \mapsto 1$
$b \mapsto 00$
$c \mapsto 01$



Note: depth(x) = $d(x)$

Goal: minimize $\sum_{x \in Sigma} d(x) f(x)$

Attempt 1 (Greedy):
Most frequent symbol $\mapsto 0$,
recurse on remaining symbols

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$\Sigma = \{a, b, c, d\}$
$f(a) = 0.26, f(b) = 0.255$
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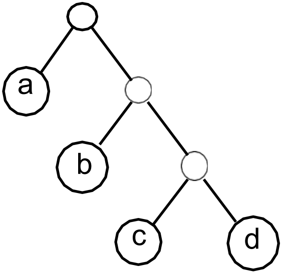

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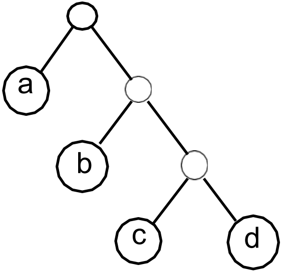

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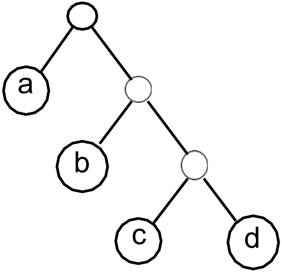

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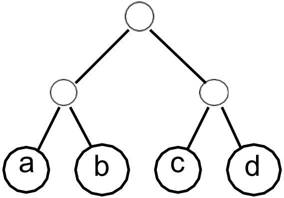
Ignore frequencies, give every symbol code of length $\lceil \log(|\Sigma|) \rceil$



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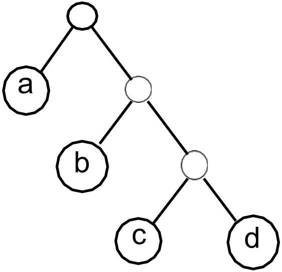

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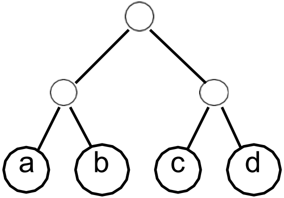
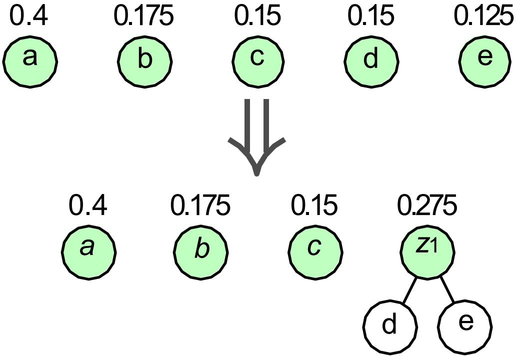


Consider: $f(a) = 1/2$,
$f(b) = 1/4$, $f(c) = 1/8$,
$f(d) = 1/8$.
Cost = 2 but attempt 1 is better (1.75)

Correct Algorithm

Idea: The two lowest frequency symbols x , y will be siblings so replace them with a new node z of frequency $f(x) + f(y)$ and recursively solve the problem



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Huffman Coding

Replace two lowest frequency symbols x, y with a new symbol z of frequency $f(x) + f(y)$ and recurse on $n - 1$ symbols.

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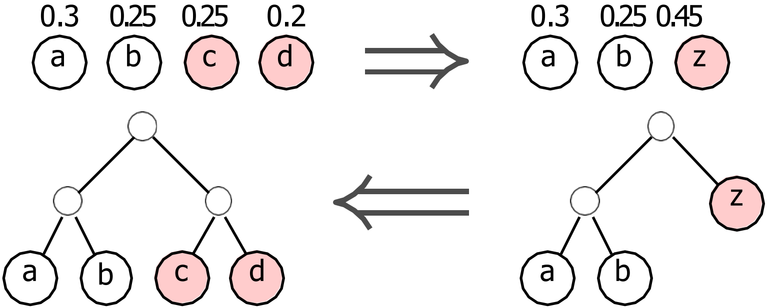
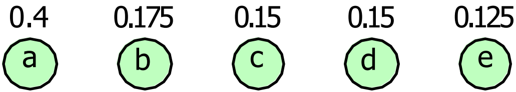
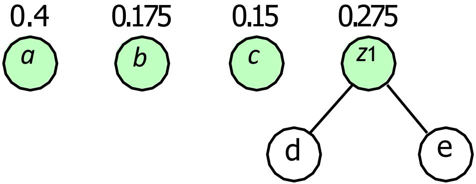


Take the resulting tree on $n - 1$ symbols and replace z with x and y

Example Huffman



Example Huffman



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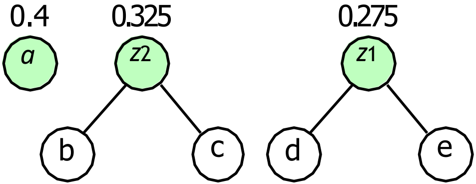

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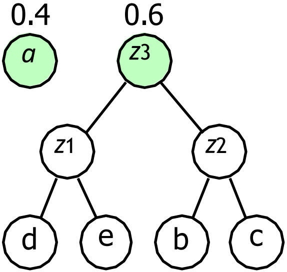

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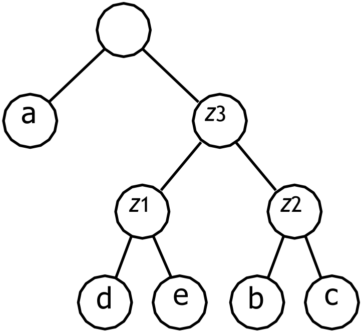

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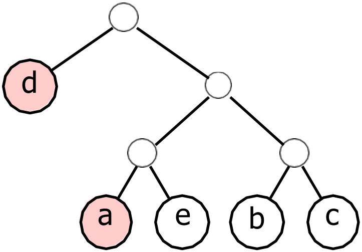
- Some two symbols are siblings at the largest depth
- The algorithm we saw puts symbols with lowest frequencies there
- If an optimal tree existed with higher frequency symbols at the deepest level we could swap the lowest frequency symbol with one of the deepest ones and get a tree with a lower cost. This is contradiction since we started by saying the tree was optimal.

Example Swap

Recall the previous example where $f(a) = 0.4$ and $f(d) = 0.15$. Here is an encoding tree where one of the low- est frequency symbol (d) is swapped with a higher frequency one (a).

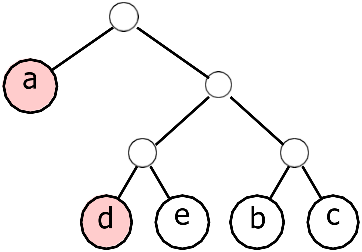


The string "... aadaadaadaa . . . " gets encoded to:

100	100	0	100	100	0	100	100	0	100	100

Example Swap

See how the encoding length shrinks when we swap 'a' and 'd'



"...aadaadaadaa ..." is now encoded to:

0	0	100	0	0	100	0	0	100	0	0