UNIT 4: 3D Geometric Transformation

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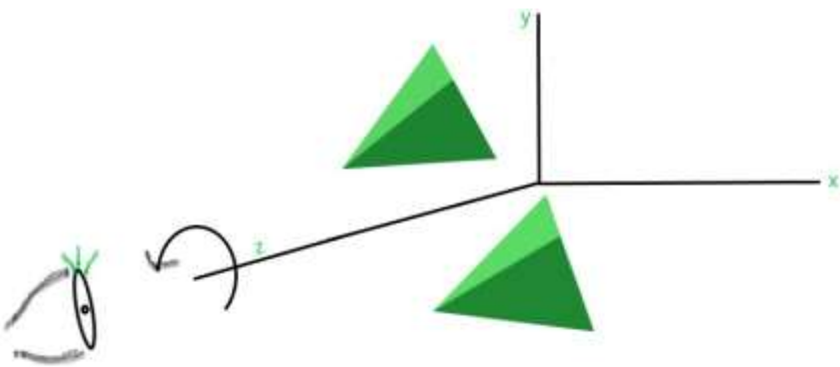
Bsc.CSIT(3rd Semester)

Rotation about an arbitrary point in space

Rotation about Z-axis

- $x' = x\cos\theta - y\sin\theta$
- $y' = x\sin\theta + y\cos\theta$
- $z' = z$

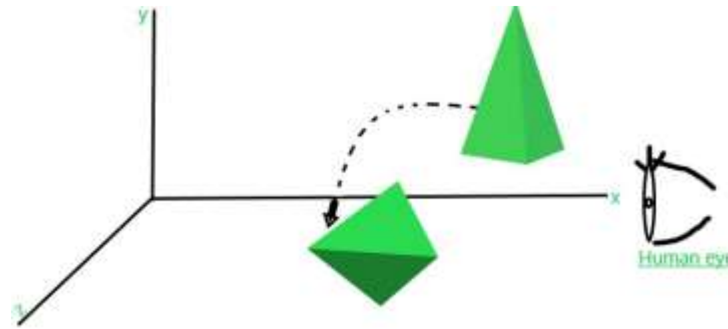


$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about an arbitrary point in space

Rotation about X-axis

- $x' = x$
- $y' = y\cos\theta - z\sin\theta$
- $z' = y\sin\theta + z\cos\theta$

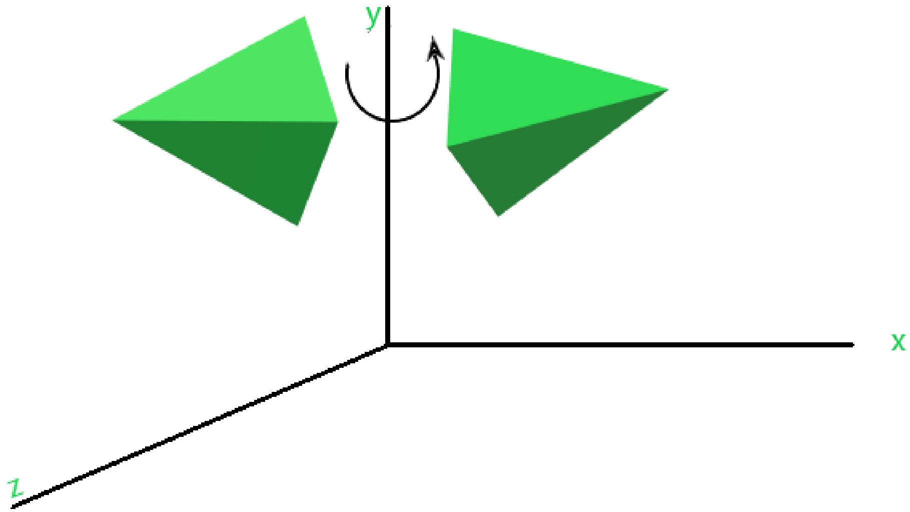


$$\therefore R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

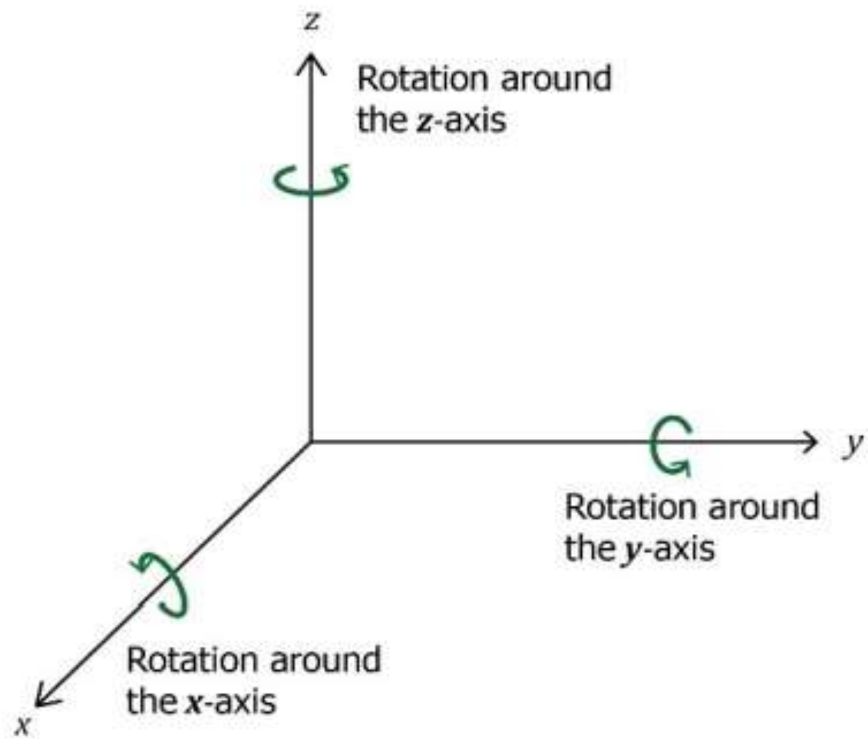
Rotation about an arbitrary point in space

Rotation about Y-axis

- $y' = y$
- $z' = z\cos\theta - x\sin\theta$
- $x' = z\sin\theta + x\cos\theta$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

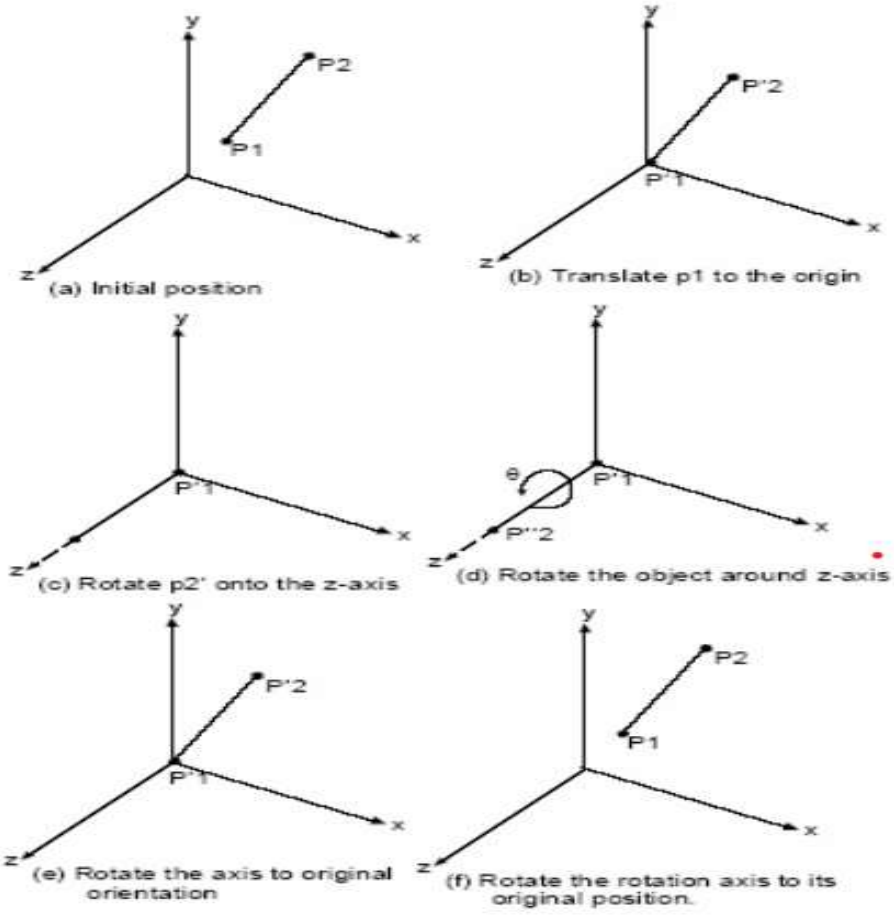

z
Rotation around the z-axis
y
Rotation around the y-axis
x
Rotation around the x-axis

Not Parallel to any of the Co-axis

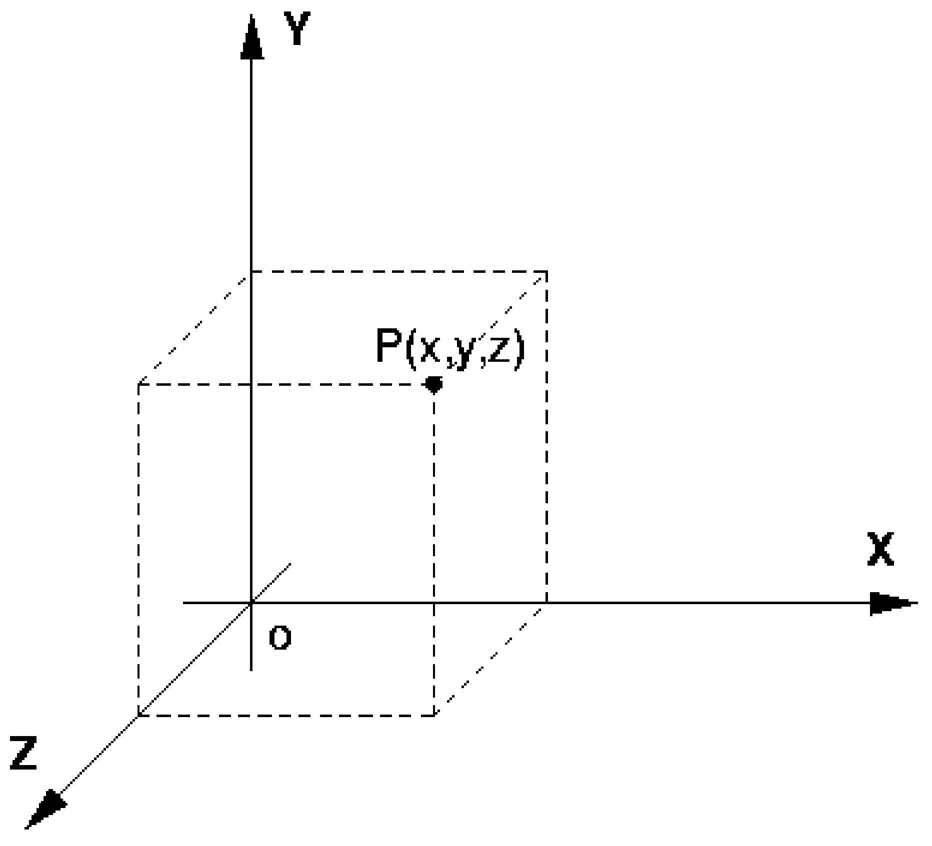
- When an object is to be rotated about an axis that is not parallel to one of the co-ordinate axes, we need to perform some series of transformation.

1. Translate the object such that rotation axis passes through co-ordinate origin.
2. Rotate the object such that axis of rotation coincides with one of the co-ordinate axis.
3. Perform the specific rotation about selected coordinate axis.
4. Apply inverse rotation to bring the rotation axis back to its original orientation.
5. Apply inverse translation to bring the rotation axis back to its original position.

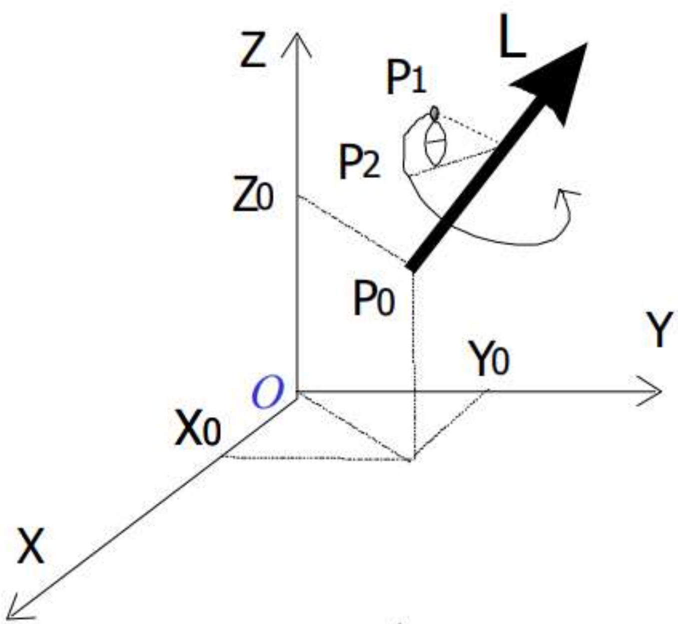
Not Parallel to any of the Co-axis

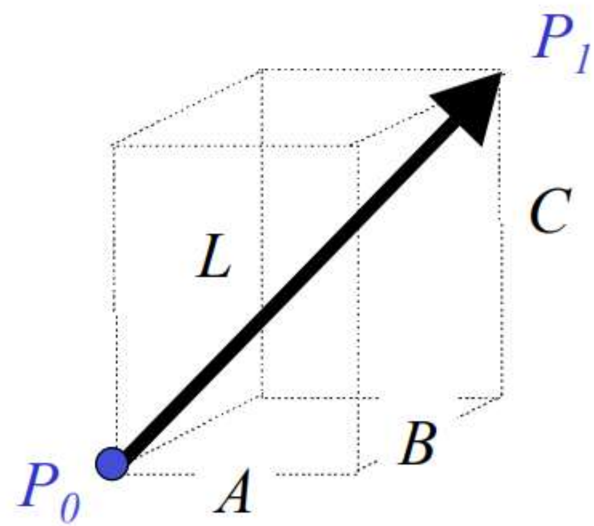


Not Parallel to any of the Co-axis



Rotation about an Arbitrary Axis (Line)





$$x = Au + x_0$$

$$y = Bu + y_0 \qquad 0 <= u <= 1$$

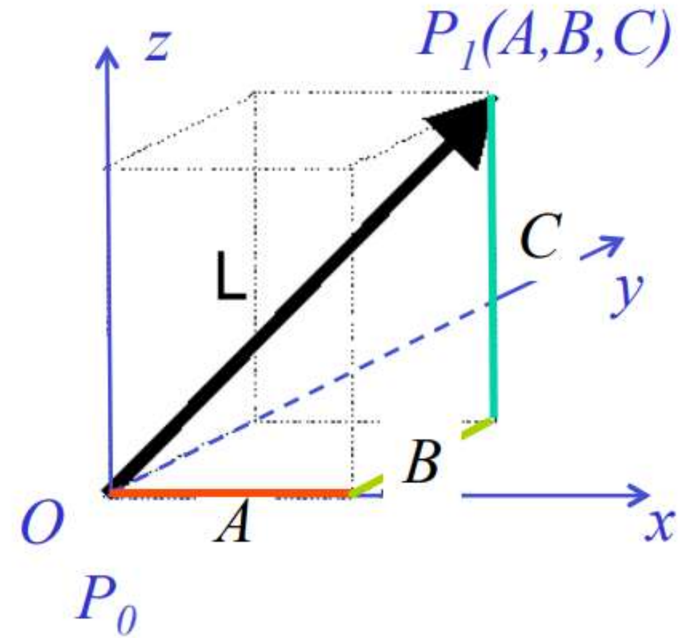
$$z = Cu + z_0$$

$$L = \sqrt{A^2 + B^2 + C^2}\,u$$

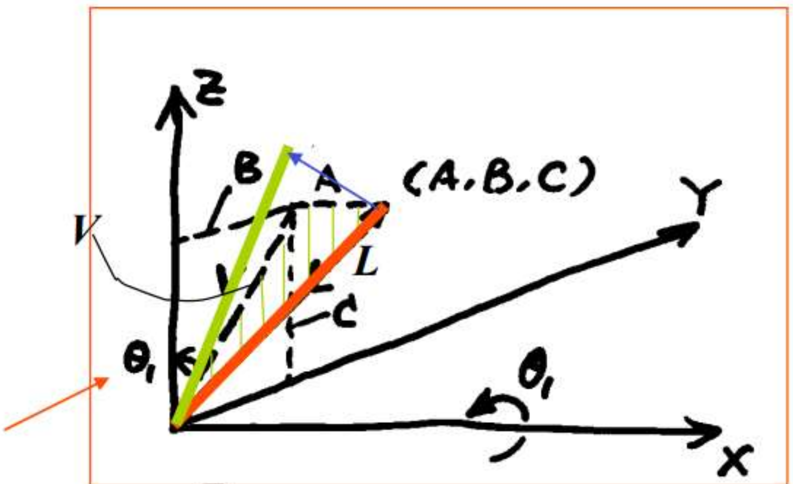
Step 1: Translate Point P_0 to Origin O

$$P_0 = [x_o \; y_o \; z_o]^T$$

$$[D] = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 2: Rotate Vector about X Axis to get into the x - z plane

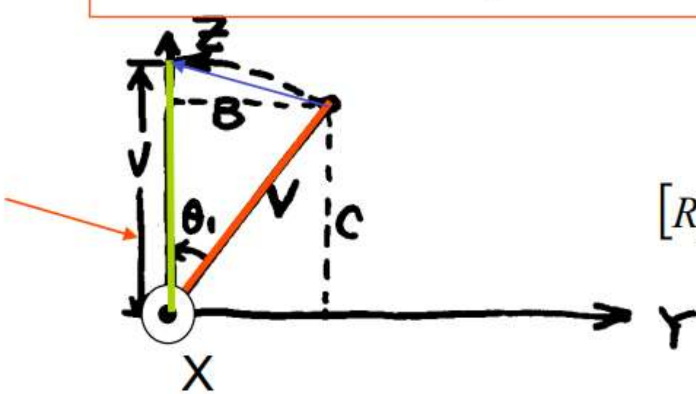


$$L = \sqrt{A^2 + B^2 + C^2}$$

$$V = \sqrt{B^2 + C^2}$$

$$\sin\theta_1 = \frac{B}{V}$$

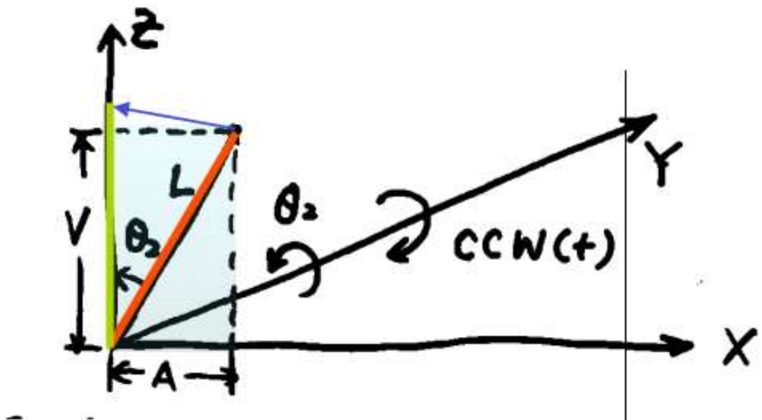
$$\cos\theta_1 = \frac{C}{V}$$



$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & -\frac{B}{V} & 0 \\ 0 & \frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Rotate about the Y axis to get it in the Z direction

Rotate a negative angle (CW)!

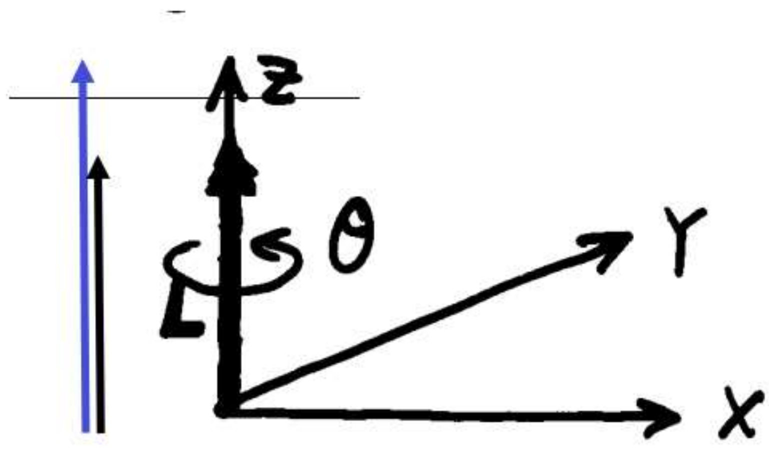


$$\sin\theta_2 = -\frac{A}{L}$$

$$\cos\theta_2 = \frac{V}{L}$$

$$[R_y] = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{V}{L} & 0 & -\frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 4: Rotate angle θ about axis $\vec{L}$



$$[R_z] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Reverse the rotation about the Y axis

$$[R_y]^{-1} = \begin{bmatrix} \frac{V}{L} & 0 & \frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{V}{L} & 0 & -\frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Rotation:

Replace θ by $-\theta$

$\sin\theta$ by $-\sin\theta$

$\cos\theta$ remains $\cos\theta$ (why?)

Step 6: Reverse rotation about the X axis

$$[\mathrm{R_x}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\mathrm{C}}{\mathrm{V}} & \dfrac{\mathrm{B}}{\mathrm{V}} & 0 \\ 0 & -\dfrac{\mathrm{B}}{\mathrm{V}} & \dfrac{\mathrm{C}}{\mathrm{V}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{C}{V} & -\dfrac{B}{V} & 0 \\ 0 & \dfrac{B}{V} & \dfrac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 7: Reverse translation

$$[D]^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Overall Transformation

$$[T] = [D]^{-1}[R_x]^{-1}[R_y]^{-1}[R_z^\theta][R_y][R_x][D]$$

$$P_2 = [T]P_1$$

- **3D Scaling**

Matrix representation for scaling transformation of a position $P = (x, y, z)$ relative to the coordinate origin can be written as;

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

→For scaling of point P(x, y, z) w.r.t to fixed point (x_f, y_f, z_f) can be represented with the following transformation.

1. Translate the fixed point to the origin. $T(-x_f, -y_f, -z_f)$
2. Apply scaling w.r.to origin. $S(s_x, s_y, s_z)$
3. Translate the fixed point back to its original position. $T(x_f, y_f, z_f)$

$$= \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q. Find the new co-ordinates of a unit cube 90 degree rotated about an axis defined by its end points A(2, 1, 0) and B(3, 3, 1).

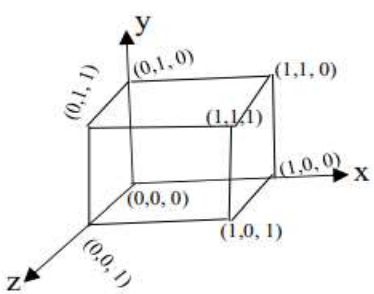


Fig: unit cube

Now,
Translating the point (A) to the origin,

$$T = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, rotate the $A'B'$ about x-axis by angle α until vector $\vec{u}$ lies on xz-plane. Where,
$\vec{v} = \vec{B}$ - $\vec{A}$ =(3, 3, 1)-(2, 1, 0)=(1, 2, 1)

Unit vector along $\vec{v}$, $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{(1,\ 2,\ 1)}{\sqrt{6}} = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}) = (a, b, c)$ (say)

And, $d = \sqrt{b^2 + c^2} = \sqrt{\frac{5}{6}}$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Again, rotating $A'B'$ about y-axis by angle β until it coincides with z-axis.

$$R_y(\beta) = \begin{bmatrix} cos\beta & 0 & sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\beta & 0 & cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{5/6} & 0 & -1/\sqrt{6} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{6} & 0 & \sqrt{5/6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotating the unit cube 90 degree about z-axis.

$$R_z(90^0) = \begin{bmatrix} cos90^0 & -sin90^0 & 0 & 0 \\ sin90^0 & cos90^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The combined transformation rotation matrix about the arbitrary axis becomes,

$$R(\theta) = T^{-1} R_x^{-1}(\alpha).R_y^{-1}(\beta).R_z(90^0).R_y(\beta).R_x(\alpha).T$$

$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5/6} & 0 & 1/\sqrt{6} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{6} & 0 & \sqrt{5/6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{5/6} & 0 & -1/\sqrt{6} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{6} & 0 & \sqrt{5/6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, multiplying $R(\theta)$ by the matrix of original unit cube;

$$P' = R(\theta).P$$

$$P' = \begin{bmatrix} 0.116 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 2.650 & 1.667 & 1.834 & 2.816 & 2.7525 & 1.742 & 1.909 & 2.891 \\ -0.558 & -0.484 & 0.258 & 0.184 & -1.225 & -1.152 & -0.409 & -0.483 \\ 1.467 & 1.301 & 0.650 & 0.817 & 0.726 & 0.566 & -0.091 & 0.076 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$