Markov Process

A Markov process is a random process whose future behaviour can't be accurately predicted from its past behaviour except the current or present behaviour and which involves random chance or probability. These processes are named after **Andrei Markov** and are among the most important types of random processes

So, in MP future probabilities are determined by its most recent or current values. Past history of variable and the way that the present has emerged from past are irrelevant. Knowing the present, we get no information from the past that can be used to predict the future.

$$P(\,future\,/\,past,\ present) = P(\,future\,/\,present)$$

For the future development of a MP, only its present state is important and it doesn't matter how the process arrived to this state.

Mathematically,

Stochastic process X (t, w) is Markov if for any $t_1 < \ldots < t_n < t$ and any sets $A;\ A_1,\ \ldots\ , A_n$

$$P\left\{X(t) \in A \,\middle|\, X(t_1) \in A_1,\ \ldots,\ X(t_n) \in A_n\right\}$$

$$= P\left\{X(t) \in A \,|\ X(t_n) \in A_n\right\}$$

The random process which follows MP are:

- Behaviour of share market (Stock price)
- Internet connections
- Flow of traffic
- Chemical reactions

Markov Chain

A Markov Chain is a discrete time, discrete state Markov Stochastic process. Since time is discrete, we can look at Markov chain as a random sequence,

$$\left\{X(0),\ X(1),\ X(2),\ldots\right\}$$

Transitional probability

The transitional probability p_{ij} is the probability that the Markov chain is at the next time point in state j, given that it is at the present time point at state i.

The probability of moving from state i to state j by means of h transitions is an h-step transition probability is given by,

$$p_{ij}(h) = P\{X(t+h) = j | X(t) = i\}$$

Transition Matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{is} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{s1} & p_{s2} & \cdots & p_{sj} & \cdots & p_{ss} \end{bmatrix}$$

Total of transition probability from a state I to all other states must be 1 i.e. $\sum_{j=1}^{S} p_{ij} = 1$. The row sums of p are equal to 1.

Conditions:

1. All states of the Markov chain communicate with each other (i.e. it is possible to go from each state, possibly in more than one step, to every other state)
2. The Markov chain is not periodic (a periodic chain is a chain in which, e.g., you can only return to a state in an even number of steps)
3. The Markov chain does not drift away to infinity.

Example of transition matrix

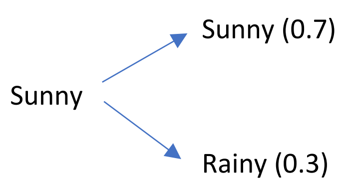
In some town, each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, where as a rainy day is followed by a sunny day with probability 0.4. Prepare one step transition matrix.

Solution:

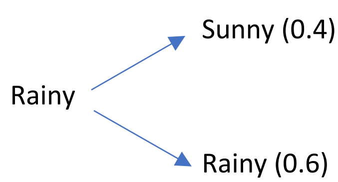
Let State 1 = 'Sunny Day'

and State 2 = 'Rainy Day'

Weather conditions represent homogeneous MC with two states. The transition probabilities are:



p_{11} = 0.7 p_{12} = 0.3



p_{21} = 0.4 p_{22} = 0.6

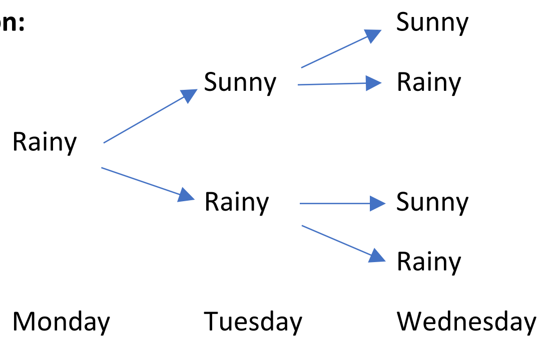
Thus, one-step transition matrix is given by

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Two-step transition probabilities

(a) If it rains on Monday, make forecasts for Wednesday

Solution:



Transition probabilities

Two step transition matrix is given by,

$$P^2 = \begin{bmatrix} p_{11}(2) & p_{12}(2) \\ p_{21}(2) & p_{22}(2) \end{bmatrix}$$

$$p_{21} = P\left\{wednesday\ is\ sunny / monday\ is\ rainy\right\}$$

$$= P\{X(1) = 1/X(0) = 2\}\,.\,P\{X(2) = 1/X(1) = 1\} +$$

$$P\{X(1) = 2/X(0) = 2\}\,.\,P\{X(2) = 1/X(1) = 2\}$$

$$= p_{21} \cdot p_{11} + p_{22} \cdot p_{21}$$

= 0.4 x 0.7 + 0.6 x 0.4

= 0.52

$$p_{22} = P\{wednesday\ is\ rainy/monday\ is\ rainy\}$$

$= 1 - p_{21}$= 1 – 0.5 = 0.48

If it is rainy day on Monday there is 52 % chance of sunny day and 48 % chance of rainy day on Wednesday.

(b) If it is sunny on Monday, make forecast for Wednesday

$$p_{11} = P\{wednesday\ is\ sunny\ /\ monday\ is\ sunny\}$$

$$= P\{X(1) = 1/X(0) = 1\}\,.\,P\{X(2) = 1/X(1) = 1\}$$
$$+P\{X(1) = 2/X(0) = 1\}\,.\,P\{X(2) = 1/X(1) = 2\}$$

$$= p_{11} \cdot p_{11} + p_{12} \cdot p_{21}$$

= 0.7 x 0.7 + 0.3 x 0.4

= 0.61

$$p_{12} = P\{wednesday\ is\ rainy\ /\ monday\ is\ sunny\}$$

$= 1 - p_{11} = 1 - 0.61 = 0.39$

If it is sunny day on Monday, there is 61 % chance of sunny day and 39 % chance on Wednesday.

Two step transition matrix is

$$P = \begin{bmatrix} p_{11}(2) & p_{12}(2) \\ p_{21}(2) & p_{22}(2) \end{bmatrix} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

In general,

P (2) = P (1) x P (1)

P (3) = P (2) x P (1) and so on.

Example

In a computing system, the probability of an error on each cycle depends on whether or not it was preceded by an error. We will define 1 as the error state and 2 as the non-error state. Suppose the probability of an error if preceded by an error is 0.75, the probability of an error if preceded by a non-error is 0.50, the probability of a non-error if preceded by an error is 0.25, and the probability of non-error if preceded by non-error is 0.50. Find two-step and three-step transition matrices

Solution:

The one step transition matrix is given by

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \end{array} = \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix}$$

For two step transition matrix we have,

$$P^2 = P \times P = \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix} = \begin{bmatrix} 0.688 & 0.312 \\ 0.625 & 0.375 \end{bmatrix}$$

The three-step transition matrix is given by,

$$P^3 = P^2 \times P = \begin{bmatrix} 0.688 & 0.312 \\ 0.625 & 0.375 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{bmatrix} = \begin{bmatrix} 0.672 & 0.328 \\ 0.656 & 0.344 \end{bmatrix}$$

Steady-state distribution

The concept of a steady state distribution in the context of Markov chains refers to a probability distribution that remains constant over time.

The steady state distribution also known as **stationary distribution** represents the long-term behaviour of a Markov Chain. This distribution is reached when the process has settled into a stable state where the probabilities of different outcomes no longer change over time i.e. Markov chain will continue to follow the same distribution in subsequent time steps if we have a Markov chain with a transition matrix **P**, the steady state distribution **π** satisfies the following equation.

$$\pi P = \pi$$

Example: Previous example of weather

The one step transition matrix of probability of sunny and rainy days is given by,

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Let π_1 and π_2 be the steady-state probabilities of being in states S_1 = sunny and S_2 = rainy, respectively.

The steady-state equation (Stationary Condition) for this Markov chain is given by,

$$\pi P = \pi$$

Or $$[\pi_1,\ \pi_2] = [\pi_1,\ \pi_2] \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= [0.7\,\pi_1 + 0.4\,\pi_2,\ 0.3\,\pi_1 + 0.6\,\pi_2]$$

Thus, system of equations is:

$$\begin{aligned} 0.7\pi_1 + 0.4\pi_2 &= \pi_1 \\ 0.3\pi_1 + 0.6\pi_2 &= \pi_2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} 0.4\pi_2 &= 0.3\pi_1 \\ 0.3\pi_1 &= 0.4\pi_2 \end{aligned}$$

$$\Rightarrow \quad \pi_2 = \frac{3}{4}\ \pi_1$$

Two systems of equations reduced to one.

Using normalizing equation $\sum \pi_x = 1$, we have

$$\pi_1 + \pi_2 = 1$$

Or $\pi_1 + \frac{3}{4}\pi_1 = \frac{7}{4}\pi_1 = 1$

Or, $\pi_1 = 4/7 = 57.14\% \approx 57\ \%$

and $\pi_2 = 42.86\ \%$

This means that, in the long run, the system will be in state of S_1 = sunny about 57.14% of the time and in state of S_2 = rainy about 42.86% of the time. So, in a long history of the city, there is 57 % of days sunny and 43 % of days rainy.

Example: Suppose in a small town there are three places to eat, two restaurants (one Chinese and another one is Mexican restaurant) and the third place is pizza place. Everyone in town eats dinner in one of these places or has dinner at home. Assume that 20 % of those who eat in Chinese restaurant go to Mexican next time, 20 % eat at home, and 30 % go to pizza place. From those who eat in Mexican restaurant, 10% go to pizza place, 25 % go to Chinese restaurant, and 25 % eats at home next time. From those who eat at pizza place 30 % eat at home, 30 % eat at Chinese and 10 % eat at Mexican restaurant next time. Those who eat at home, 20 % go to Chinese, 25 % go to Mexican place and 30 % to pizza place. What is the probability of going to each of the restaurants when the system is in equilibrium (π_1, π_2, π_3, π_4)

(State 1 = Home, State 2 = Chinese, State 3 = Mexican restaurant, State 4 = pizza place)

Solution:

We define the state first as follows:

State 1 (S1): Home

State 2 (S2): Chinese Restaurant

State 3 (S3): Mexican Restaurant

State 4 (S4): Pizza Place

The one step transition probability matrix is given by,

$$P = \begin{bmatrix} p11 & p12 & p13 & p14 \\ p21 & p22 & p23 & p24 \\ p31 & p32 & p33 & p34 \\ p41 & p42 & p43 & p44 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.20 & 0.25 & 0.30 \\ 0.20 & 0.30 & 0.20 & 0.30 \\ 0.25 & 0.25 & 0.40 & 0.10 \\ 0.30 & 0.30 & 0.10 & 0.30 \end{bmatrix}$$

The steady-state equation for this Markov chain is given by,

$$\pi P = \pi$$

or, $$[\boldsymbol{\pi_1} \quad \boldsymbol{\pi_2} \quad \boldsymbol{\pi_3} \quad \boldsymbol{\pi_4}] \begin{bmatrix} 0.25 & 0.20 & 0.25 & 0.30 \\ 0.20 & 0.30 & 0.20 & 0.30 \\ 0.25 & 0.25 & 0.40 & 0.10 \\ 0.30 & 0.30 & 0.10 & 0.30 \end{bmatrix} = [\boldsymbol{\pi_1} \quad \boldsymbol{\pi_2} \quad \boldsymbol{\pi_3} \quad \boldsymbol{\pi_4}]$$

Thus, the system of equations are:

$0.25\,\pi_1 + 0.20\,\pi_2 + 0.25\,\pi_3 + 0.30\,\pi_4 = \pi_1$ ... (i)

$0.20\,\pi_1 + 0.30\,\pi_2 + 0.20\,\pi_3 + 0.30\,\pi_4 = \pi_2$ ... (ii)

$0.25\,\pi_1 + 0.25\,\pi_2 + 0.40\,\pi_3 + 0.10\,\pi_4 = \pi_3$ ... (iii)

$0.30\,\pi_1 + 0.30\,\pi_2 + 0.10\,\pi_3 + 0.30\,\pi_4 = \pi_4$ ... (iv)

$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ ... (v)

Rearranging the terms we have,

- $0.75\,\pi_1 + 0.20\,\pi_2 + 0.25\,\pi_3 + 0.30\,\pi_4 = 0$ … (vi)

$0.20\,\pi_1 - 0.70\,\pi_2 + 0.20\,\pi_3 + 0.30\,\pi_4 = 0$ … (vii)

$0.25\,\pi_1 + 0.25\,\pi_2 - 0.60\,\pi_3 + 0.10\,\pi_4 = 0$ … (viii)

$0.30\,\pi_1 + 0.30\,\pi_2 + 0.10\,\pi_3 - 0.70\,\pi_4 = 0$ … (ix)

We take any three equations and equation (v) (one equation is left because of redundancy), Let's take equation (vii), (viii), (ix) and important one (v), then we have following system of equations.

$0.20\,\pi_1 - 0.70\,\pi_2 + 0.20\,\pi_3 + 0.30\,\pi_4 = 0$

$0.25\,\pi_1 + 0.25\,\pi_2 - 0.60\,\pi_3 + 0.10\,\pi_4 = 0$

$0.30\,\pi_1 + 0.30\,\pi_2 + 0.10\,\pi_3 - 0.70\,\pi_4 = 0$

$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

On solving these equations, we following estimates of π_1, π_2, π_3 and π_4

- $\pi1 \approx 0.2667$ (Home)
- $\pi2 \approx 0.2667$ (Chinese)
- $\pi3 \approx 0.20$ (Mexican)
- $\pi4 \approx 0.2667$ (Pizza)

So, in the long run, about 26.67% of people will be at home, 26.67% at the Chinese restaurant, 20% at the Mexican restaurant, and 26.67% at the pizza place.

Why one equation is redundant:

Because the rows of the transition matrix P sum to 1, one of the equations derived from $\pi P = \pi$ will be linearly dependent on the others. This means it doesn't provide any *new* information. That's why we use the sum equation instead of one of the equations obtained directly from $\pi P = \pi$.