Counting Process

A counting process is a type of stochastic process that records the occurrence of events over time. It's essentially a function that takes time as input, denoted by 't' and output a non-negative integer representing the number of events that have happened up to that point in time usually denoted by N(t).

Definition
Counting is a non-decreasing, integer-valued process, typically denoted as {N(t), t ≥ 0}, where:

- **Non-negative and Integer-Valued**: N(t) in a counting process represents the total number of events that have occurred up to and including time t. It is a random variable that takes non-negative integer values i.e., N(t) ∈ {0,1, 2, …} for all time t ∈ $[0, \infty)$
- **Starts at Zero (usually):** By convention, the count typically starts at zero and no events have occurred at the initial time or at the beginning. N (0) represents the total number of events that have occurred at time t = 0.
- **Non-Decreasing**: N(t) is non-decreasing meaning that as time progresses, the count can only stay the same or increase. So, if s ≤ t, then N(s) ≤ N(t).
- **Independent increments**: N(t) typically has independent increments, meaning the number of events occurring in disjoint time intervals is independent.
- **Stationary increments**: N(t) often has stationary increments, meaning that the probability distribution of the number of events occurring in any time interval depends only on the length of the interval, not on its position.
- **No** events in a given interval: For $0 \leq s < t, N(t) - N(s)$ is the number of events occurred during the interval $(s, t]$

Example of counting process:

1. **Poisson process**: One of the most well-known counting processes is the Poisson process. The Poisson process is used to model events that occur randomly and independently over time and at constant rate of λ. It's widely applied in various fields, including telecommunications, biology, finance, and more.
2. **Renewal Process**: Counts events with independent and identically distributed inter-arrival times (e.g., lightbulb replacements).
3. **Birth Process:** Models population growth through "birth" events (e.g., bacterial reproduction)

Binomial Process

The binomial distribution models the number of successes in a fixed number of independent trials, where each trial has:

- two possible outcomes: success or failure.
- a constant probability of success *p*.

The binomial distribution is not a stochastic process by itself, but it can be considered part of a stochastic process when used to model cumulative counts over time or sequential trials. In such cases, the sequence of binomial random variables indexed by time forms a stochastic process.

Hence binomial process is a *stochastic process* because it describes how the number of successes *accumulates over time*.

A Binomial process X(n) is a random counting system where there are 'n' independent identical trials, each one of which has the same probability of success 'p', which produces 's' successes from those 'n' trials ($0 \le s \le n$, $n > 0$).

Binomial process is a discrete-time discrete-space counting stochastic process. It is Markov process, and therefore a Markov chain.

Notation:

λ = Arrival rate or average no of successes per one unit of time

$= \frac{p}{\Delta}$

Δ = Frame size or the time interval of each Bernoulli trial

p = Probability of arrival (success) during one frame (trial)

t = No. of frame x frame size = n. Δ

Problem: Jobs are sent to a mainframe computer at a rate of 2 jobs per minute. Arrivals are modelled by a Binomial counting process.

(a) Choose such a frame size that makes the probability of a new job during each frame equal 0.1
(b) Using the chosen frame, compute the probability of more than 3 jobs received during 1 minute
(c) Compute the probability of more than 30 jobs during 10 minutes
(d) What is the average interarrival time, and what is the variance?
(e) Compute the probability that the next job doesn't arrive during the next 30 seconds.

Solution:

Here,

(a) λ = Arrival rate

$= 2 / \text{min} = 2\ \text{min}^{-1}$

p = probability of an arrival during one frame = 0.1

Δ = frame size

$= \frac{p}{\lambda} = \frac{0.1}{2} = 0.05$ min = 3 sec

(b) Time (t) = 1 min

No. of frames in 1 min (n) $= \frac{Time}{Frame\ Size}$

$= \frac{t}{\Delta} = \frac{60}{3} = 20$ frames

During 1 minute, no. of jobs X(n) is Binomial random variable with n = 20 and p = 0.1 i.e.

X (n) ~ B (n = 20, p = 0.1)

P{X(n) > 3} = 1 – P {X (n) ≤ 3}

$= 1 - \sum_{x=0}^{3} n_{c_x}\, p^x\, q^{n-x}$

$= 1 - \sum_{x=0}^{3} 20_{c_x}\, (0.1)^x\, (0.9)^{n-x}$

= 1 – {0.1216 + 0.2702 + 0.2852 + 0.1901}

= 1 – 0.8671

= 0.1329

Hence, there is 13.29 % chance that more than 3 jobs are received during 1 minute time.

(c) New time interval t = 10 mins = 600 secs

No. of frames (n) = $\frac{Time}{Frame\ size}$

$= \frac{600}{3} = 200$

Hence there are 200 frames in 10 minutes time.

Thus, X (n) ~ B (n = 200, p = 0.1)

P {X (n) > 30} =?

We use normal distribution to approximate this probability.

The Z transformation of X(n) is given by,

$$Z = \frac{X(n) - np}{\sqrt{np(1-p)}}$$

$$= \frac{29.5 - 200\ x\ 0.1}{\sqrt{200\ x\ 0.1\ x\ 0.9}} = 2.24$$

Now,

P {X (n) > 30} = P (Z > 2.24) = 0.01255 = 1.26 %

There is chance of 1.26 % that more than 30 jobs are received during 10 minutes time interval.

(d) Average interarrival times is given by,

$$E(T) = \frac{1}{\lambda} = \frac{1}{2}$$

= 0.5 min = 30 secs

Variance of interarrival time is given by,

$$V(T) = \frac{1-p}{\lambda^2} = \frac{1-0.1}{2^2}$$

$= 0.225\ min^2$

Poisson Process

A Poisson process is a simple and widely used stochastic process for modelling the times at which arrivals enter a system. It is continuous time, discrete space stochastic process. It is many ways the continuous-time version of the Binomial process.

A continuous-time process can be viewed as a limit of some discrete-time process whose frame size gradually decreases to zero, therefore allowing more frames during any fixed period of time.

Consider a Binomial counting process that counts arrivals which is occurring at a rate λ. Let X(t) denote the number of arrivals occurring until time ‘t’. If the frame size Δ decreases then the number of frames ‘n’ during time ‘t’ increases. The arrivals occur at the same rate ‘λ’ regardless of the choice of frame size ‘Δ’.

$$n = \frac{t}{\Delta} \uparrow \infty \; as \; \Delta \downarrow 0$$

As the frame size ‘Δ’ decreases to 0, the number of frames ‘n’ during the time ‘t’ increases to infinity. The probability of an arrival ‘p’ during each frame is proportional to frame size ‘Δ’, so it decreases to zero as frame size decreases to zero.

$$p = \lambda \, \Delta \rightarrow 0 \text{ as } \Delta \rightarrow 0$$

Expected number of arrivals during time ‘t’ is a Binomial process with number of frames ‘n’ and probability of a new arrivals during each frame ‘p’, is given by,

$$E\{X(t)\} = np$$

$$= \frac{t}{\Delta} \,.\, \Delta \, \lambda$$

$$= \lambda \, t$$

So in the limiting case, as Δ → 0, n → ∞, and p → 0, X(t) becomes Poisson variable with parameter λt.

Variance of number of arrivals during time ‘t’ is given by,

$$V\{X(t)\} = \lambda \, t$$

The inter-arrival time T becomes a random variable with cummulative distribution function (cdf), as follows,

$$F_T(t) = P\{T < t\}$$

$$= P\ \{Y \le n\}$$

$$= 1 - (1 - p)^n$$

$$= 1 - \left\{1 - \frac{\lambda t}{n}\right\}^n$$

$$= 1 - e^{\lambda t} \qquad \text{(using expansion formula)}$$

Problem: Previous problem

Solution:

Arrival rate (λ) = 2 min^{-1}

Probability of arrival (p) = 0.1

Frame size (Δ) = $\frac{p}{\lambda} = \frac{0.1}{2}$ = 0.05 min = 3 sec

Probability of more than 3 jobs during 1 min time

P{X(t) > 3}

= 1- P{X(t) ≤ 3}

= 1 - $\sum_{x=0}^{3} \frac{e^{-2}\lambda^{x}}{x!}$

= 1 – $\sum_{x=0}^{3} \frac{2.7183^{-2}2^{x}}{x!}$

= 1 – {0.1353+0.2707+0.2707+0.1805}

= 1 – 0.8571

= 0.1428 = 14.28 %

Probability of more than 30 jobs during 10 mins

Time (t) = 10 mins

Arrival rate per 10 min ($\lambda^{/}$) = λ t = 2 x 10 = 20

Thus,

P{X(t) > 30} = 1 – P{X(t) ≤ 30}

The Z transformation of X(t) is,

Z = $\frac{X(t)-\lambda}{\sqrt{\lambda}} = \frac{(30+0.5)-20}{\sqrt{20}}$ = $\frac{29.5 - 20}{\sqrt{20}}$ = 2.12

Now,

P{X(t) > 30} = P{Z > 2.12} = 0.017 = 1.7 %

Hence, using Poisson distribution, there is a chance of 1.7 % that more 30 jobs are received to mainframe computer during 10 minutes time.