UNIT 5: 3D Objects Representation(7Hrs)

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Representing Curves:

Spline Representation

A Spline is a flexible strips used to produce smooth curve through a designated set of points.

A curve drawn with these set of points is spline curve. Spline curves are used to model 3D object surface shape smoothly.

Mathematically, spline are described as piece-wise cubic polynomial functions. In computer graphics, a spline surface can be described with two set of orthogonal spline curves. Spline is used in graphics application to design and digitalize drawings for storage in computer and to specify animation path. Typical CAD application for spline includes the design of automobile bodies, aircraft and spacecraft surface etc.

Representing Curves:

Interpolation and approximation spline

- Given the set of control points, the curve is said to interpolate the control point if it passes through each points.
- If the curve is fitted from the given control points such that it follows the path of control point without necessarily passing through the set of point, then it is said to approximate the set of control point.

Parametric cubic Curves

- Parametric splines are defined by a set of equations that represent a curve or surface. These equations can be used to generate points along the spline, which determines its shape. Parametric splines are commonly used in computer graphics, where they are used to model 3D objects and animations.
- There are many different types of parametric splines, including Bezier curves, B-splines etc. Parametric curves are mathematical representations of curves where the coordinates of points on the curve are defined by one or more parameters.

Parametric continuity condition

- For smooth transition from one curve section on to next curve section we put various continuity conditions at connection points.

Let parametric coordinate functions as

$$x = x(u),\ y = y(u),\ z = z(u) \qquad \because u1 \ll u \ll u2$$

Parametric cubic Curves

1. <u>Zero Order Parametric Continuity</u>

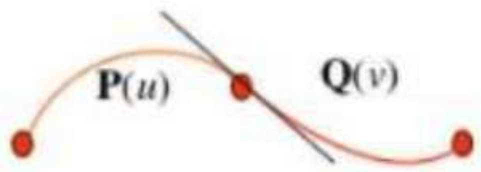
   (C^0)

It meant simply that the curve meets i.e., the values of (x, y, z) evaluated at u_2 for the first curve section is equal to the (x, y, z) values of u_1 for the next curve section.

2. <u>First Order Parametric Continuity</u>

   (C^1)

It means the first parametric derivative (tangent lines) of the coordinate function for two successive crime sections are equal at their joining point.



Parametric cubic Curves

3. <u>Second Order Parametric Continuity</u>
   (C^2)

   It means that both first and second derivative of the coordinate function for two consecutive crime sections are same at their intersection.

Representing Curves:

- **Cubic Spline:**
- Cubic Splines are mostly used for representing path of moving object or existing object shape or drawing.
- It is used for design the object shapes.
- Cubic polynomial offer a reasonable compromise between flexibility and speed of computation. Cubic spline require less calculations compares to higher order polynomials and less memory.
- Given a set of control points, cubic interpolation splines are obtained by fitting the input points with a piecewise cubic polynomial curve that passes through every control points.
- Suppose we have n+1 control points specified with co-ordinates
- pk = (xk , yk ,zk),k = 0,1,2........n A cubic interpolation fit of these points is

We can describe the parametric cubic polynomial that is to be filled between each pair of control points with the following set of equations.

$$x(u) = a_x u^3 + b_x u^2 + c_x u + dx$$
$$y(u) = a_y u^3 + b_y u^2 + c_y u + dy \qquad (0 \le u \le 1)$$
$$z_u(u) = a_z u^3 + b_z u^2 + c_z u + dz$$

Representing Curves:

- For above equation we need to determine for constant a, b and c and d the polynomial representation for each of n curve section.

This is obtained by setting proper boundary condition at the joints.

Common method for setting this conditions are,

1. Natural Cubic splines
2. Hermit interpolation
3. Cardinal Splines
4. Kochanek-Bartels spline

Hermit Interpolation

- Hermite curve named after the French mathematician Charles Hermite is an interpolating piecewise cubic polynomial. It has a specified tangent at each control point.
- The **Hermite curve** in computer graphics is an interpolation spline curve.
- Hermite spline curves can be adjusted locally because each section is only dependent on its endpoint constraints.

Properties of Hermite Curve

- **Interpolation:** Hermit curves interpolate smoothly between their control points.
- **Tangent Control:** Hermit curves allow precise control over the tangent vectors at each control point.
- **Parametric Representation:** Hermite curves are often expressed as parametric equations, where a parameter u varies between 0 and 1.
- **Derivatives:** Hermite curves have continuous first and second derivatives.
- **Local Control:** Changes made to one segment of a Hermite curve affect only that segment, providing local control.
- **Polynomial Form:** The Hermite curve is expressed as a polynomial function, typically a cubic polynomial.
- **Versatility in Applications:** Hermite curves find applications in computer graphics, computer-aided design (CAD), animation, and modeling.

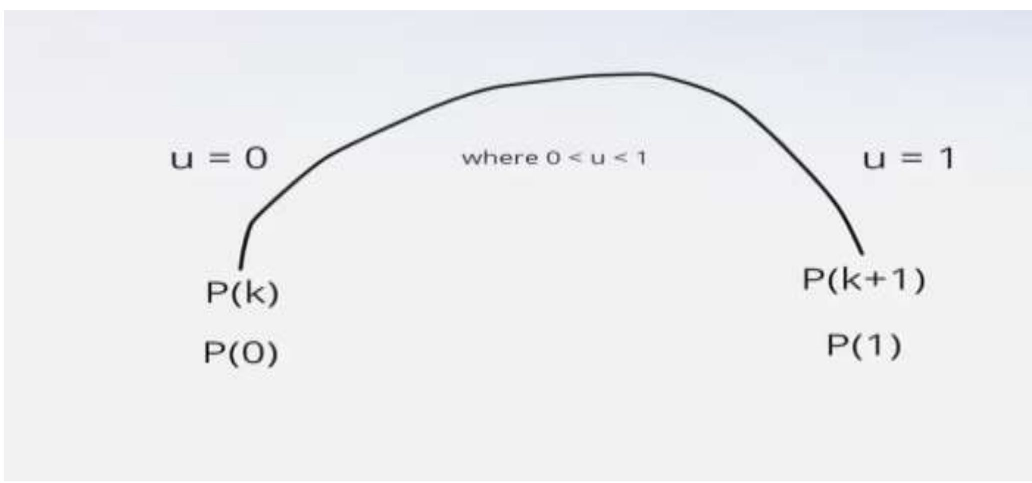
Hermit Interpolation

- Parametric cubic point function for any curve section is,

$$P(u) = au^3 + bu^2 + cu + d$$

$$\text{where } 0 < u < 1$$

The above equation is very important we will use this equation in further steps below. Let's Understand Hermite Curve Derivation,



Hermit Interpolation

- In the above figure, we let two variables i.e.
- **P (k) which is P(0), and P (k+1) which is P(1).**
- Now we have to let the **derivative of P(k) which is D Pk.**
- and **derivative of P(k+1) which is D Pk+1.**

$$p(0) = p_k$$
$$p(1) = p_{k+1}$$
$$p'(0) = dp_k$$
$$p'(1) = dp_{k+1}$$

Now we have to find the **derivative of the Hermite curve Mathematical Expression** as follows:

$$P(u) = au^3 + bu^2 + cu + d$$

$$P'(u) = 3au^2 + 2bu + c + d$$

$$\text{where } 0 < u < 1$$

After finding the derivative of P(u) **we need to put the values 0 and 1 in the u parameter in both the above equations** as follows:

Hermit Interpolation

$$P(u) = au^3 + bu^2 + cu + d$$

$$P(k) = P(0) \quad a.(0)^3 + b.(0)^2 + c.0 + d.0$$

$$= \quad a.0 + b.0 + c.0 + d.0 \longrightarrow 1$$

$$P(k+1) = P(1) \quad a.(1)^3 + b.(1)^2 + c.1 + d.1$$

$$= \quad a.1 + b.1 + c.1 + d.1 \longrightarrow 2$$

$$P'(u) = 3au^2 + 2bu + c + d$$

$$D\,Pk = P(0) \quad 3a.(0)^3 + 2b.(0) + c.1 + d.0$$

$$= \quad a.0 + b.0 + c.1 + d.0 \longrightarrow 3$$

$$DP(k+1) = P(1) \quad 3a.(1)^2 + 2b.(1) + c.1 + d.0$$

$$= \quad a.3 + b.2 + c.1 + d.0 \longrightarrow 4$$

Hermit Interpolation

- So after putting u as 0,1, we have 4 equations as follows:

$$= a.0 + b.0 + c.0 + d.0 \longrightarrow 1$$

$$= a.1 + b.1 + c.1 + d.1 \longrightarrow 2$$

$$= a.0 + b.0 + c.1 + d.0 \longrightarrow 3$$

$$= a.3 + b.2 + c.1 + d.0 \longrightarrow 4$$

Now we have to represent these equations in terms of matrix as shown below so that we can find the Hermite matrix.

Hermit Interpolation

$$\begin{bmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{bmatrix} or \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{K+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Now we can calculate the value of a, b, c by taking the inverse of the equation matrix. The formula for calculating the inverse of the 4x4 matrix is **A inverse= adj(A)/det(A)**

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{K+1} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{K+1} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{K+1} \end{bmatrix}$$

Hermit Interpolation

- So the above figure is a **Hermite matrix** after finding the Hermite matrix we need to multiply these two matrixes on the right side to find the value of a, b, c, d.

$$P(u) = Pk(2u3 - 3u2 + 1) + Pk + 1(-2u3 + 3u2) + DPk(u3 - 2u2 + u) + DPk + 1(u3 - u2)$$

$$P(u) = PkH0(u) + Pk + 1H1(u) + DPkH2(u) + DPk + 1H3(u)$$

Representing Curves:

Bezier Curve and surface:

This spline approximation method was developed by the French Engineer Pierre Bezier for use in the design of automobile body.

Bezier splines have a no of properties that make them highly useful and convenient for curve and surface design. They are easy to implement. For this reason, Bezier spline are widely available in various CAD systems.

In General Bezier curve can be fitted to any number of control points. The no of control points to be approximated and their relative position determine the degree of Bezier polynomial.

The Bezier curve can be specified with boundary conditions, with characterizing matrix or blending functions. But for general blending function specification is most convenient.

Bezier Curve and surface:

suppose we have n+1 control points positions $p_k = (x_k, y_k, z_k), k = 0,1,2,.....n.$ These co-ordinate points can be blended to produce the following position vector p(u) which describes path of and approximating Bezier polynomial function between p_o and p_n.

$$p(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u) \qquad 0 \le u \le 1 \quad \text{———} \quad (1)$$

The Bezier blending functions $BEZ_{k,n}(u)$ the Bernstein polynomial.

$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$ where

$$C(n,k) = \frac{n!}{(n-k)!k!}$$

The vector equation (1) represents a set of three parametric equations for individual curve condition

$$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$

- Bezier curve is a polynomial of degree one less than control points.

Limitation Of Bezier Spline

- Bezier spline limitations include limited shape representation (sharp corners are difficult), global control (moving one point affects the entire curve).
- The no of control points result a cubic polynomial curve, so to reduce the degree of polynomial we need to reduce the no. control points & vice versa.
- **Every Control Point Affects the Entire Curve:** (A change to any control point can affect the entire curve, making it difficult to achieve local control (where changes to one area don't affect others).)
- **Limited Continuity:** (Bezier curves may not always achieve smooth continuity between adjacent curves, potentially resulting in visible discontinuities or sharp transitions.)
- **Requires Careful Arrangement of Control Points:** (To achieve desired levels of continuity, control points need to be carefully arranged, which can reduce local control.)

B-Spline

- B-spline curve is local control. {note: does not depend on control point}
- It depend on the order of polynomial.
- Each vertex affects the shape on the limited range of parameter.
- In B-spline curve the degree of the curve and the no of vertices are independent.
- Thus it is possible to change the degree of curve without changing the no of vertices of the polygon.

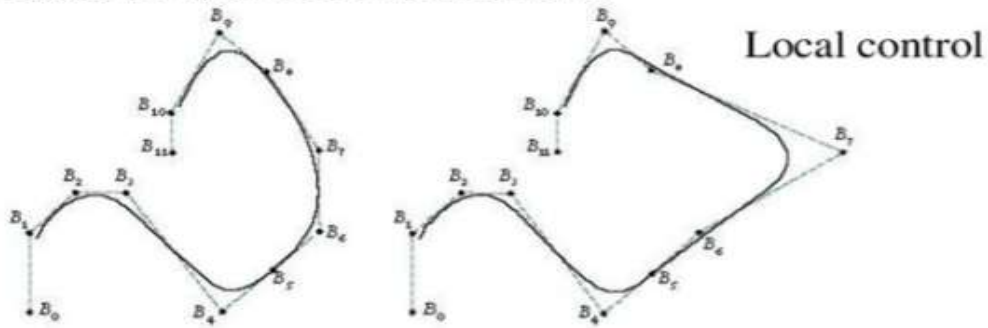
B-Spline

- B-spline blending functions, also known as basis functions, are mathematical functions that combine control points to create smooth curves or surfaces. They are defined recursively and are crucial for generating B-spline curves and surfaces.

The B-spline curve defined by n+1 control points P_i consists of n – 2 curve segments and is given by:

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i \cdot N_{i,k}(u), \qquad 0 \leq u \leq u_{max}$$

where $N_{i,k}(u)$ are the B-spline (blending or basis) functions. The parameter k controls the degree (k-1) of the B-spline curve.



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$$N_{i,k}(u) = (u - u_i)\frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u)\frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}$$

where

$$N_{i,1} = \begin{cases} 1, & u_i \le u \le u_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

B-spline Curves: Important Properties

- The degree of B-spline curve polynomial does not depend on the number of control points which makes it more reliable to use than Bezier curve.
- B-spline curve provides the local control through control points over each segment of the curve.
- We can add/modify any no of control points to change the shape of the curve without affecting the degree of polynomial.
- Approximation spline curve
- Blending function not necessary to be non zero.

Advantages of B-splines:

- **Local Control:** B-splines allow for local control, meaning that modifying a single control point only affects a limited portion of the curve or surface.
- **Flexibility:** B-splines can represent a wide variety of shapes and surfaces.
- **Mathematical Foundation:** B-splines have a solid mathematical foundation, making them a reliable tool for geometric modeling.

Representing Curves:

Quadric Surface

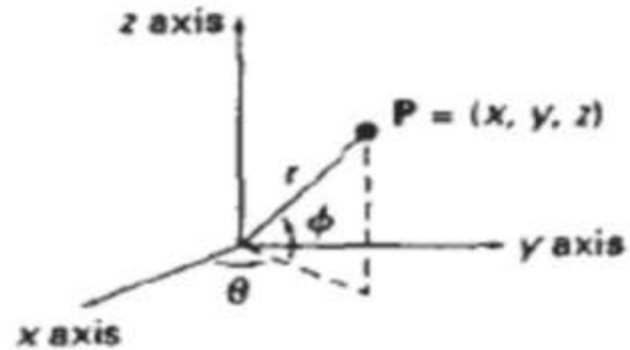
Quadric Surface is one of the frequently used 3D objects surface representation. The quadric surface can be represented by a second degree polynomial. This includes:

1. Sphere: For the set of surface points (x,y,z) the spherical surface is represented as:
   $x^2+y^2+z^2 = r^2$, with radius r and centered at co-ordinate origin.
2. Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where (x,y,z) is the surface points and a,b,c are the radii on X,Y and Z directions respectively.
3. Elliptic parboiled: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$
4. Hyperbolic parboiled : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$
5. Elliptic cone : $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$
6. Hyperboloid of one sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
7. Hyperboloid of two sheet: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

▪ Sphere

- In Cartesian coordinates, ***a spherical surface with radius r centered on the coordinate origin*** is defined ***as the set of points (x, y, z) that*** satisfy the equation

$$x^2 + y^2 + z^2 = r^2$$



- ***The spherical surface in parametric form***

$$x = r\cos\phi\cos\theta, \qquad -\pi/2 \le \phi \le \pi/2$$

$$y = r\cos\phi\sin\theta, \qquad -\pi \le \theta \le \pi$$

$$z = r\sin\phi$$

- **Ellipsoidal**
  - An ellipsoidal surface can be described as an extension of a spherical surface, where the radii in three mutually perpendicular directions can have different values.
  - The Cartesian representation for points over the surface of an ellipsoid centered on the origin is $\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 = 1$
  - And a parametric representation for the ellipsoid in terms of the latitude angle and the longitude angle

$$x = r_x \cos\phi \cos\theta, \qquad -\pi/2 \le \phi \le \pi/2$$

$$y = r_y \cos\phi \sin\theta, \qquad -\pi \le \theta \le \pi$$

$$z = r_z \sin\phi$$