**Unit 6: Solution of Partial Differential Equations**

Many physical phenomenon in applied science and engineering when formulated into mathematical models fall into a category of systems known as partial differential equations. A partial differential equation is a differential equation involving more than one independent variable. These variables determine the behavior of the dependent variable as described by their partial derivatives contained in the equation. Some of the problems which lend themselves to partial differential equations include

1. Study of displacement of a vibrating string
2. Heat flow problems
3. Fluid flow analysis
4. Electrical potential distribution
5. Analysis of torsion in a bar subject to twisting
6. Study of diffusion of matter and so on

**Partial Differential Equation (PDE)**

A differential equation with one independent variable is called an ordinary differential equation. An example of such an equation would be

3 +5y2 =3e-x, y(0) = 5

Where, y is dependent variable and x is called independent variable

If there is more than one independent variable, then differential equation is called a partial differential equation. An example of such an equation would be

3 += x2+y2

Where u is the dependent variable and x and y are the independent variables.

**Classification of Partial Differential Equations (PDE’s)**

As an introduction to solve PDE’s, most textbooks concentrate on linear second order PDEs with two independent variables and one dependent variable. The general form of such an equation is:

A + B + C +D = 0

Where A, B and C are functions of x and y and D is a function of x, y, u and , .

Depending on the value of B2-4AC, a second order linear PDE can be classified into three categories.

1. If B2-4AC<0, it is called elliptic
2. If B2-4AC=0 then it is called parabolic
3. If B2-4AC>0., it called hyperbolic

**Deriving Differential Equation**

Consider a two dimensional solution domain as shown in the figure below. The domain is divided into rectangular grids of width h and height k. the value of at the intersection of grid points are functions of two variables x and y and that are represented by f(x, y).

h

0,n

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  | k |
|  |  | i,j |  |  |
|  |  |  |  |  |

m,0

0,0

We know that if the function f(x) has continuous first derivative then its first and second derivative is given by

f’(xi) =

and

f’’(xi) =

When f is a function two variables x and y, we can use the function in x-direction and y-direction to determine partial derivatives with respect to x and y as below

= fx(xi,yi) =

= fy(xi,yi) =

= fxx(xi,yi) =

= fx(xi,yi) =

= fx(xi,yi) =

These finite equivalents of the partial derivatives can be used to construct various types of differential equations.

**Solving Laplace’s Equation**

We know that Laplace’s equation is given by

+ = 0 or ∇2T = 0 …(1)

The operator ∇ 2 = +

Is called Laplace’s operator and equation **(1)** is called Laplace’s equation

To solve the Laplace’s equation in the xy plane, we subdivide the region as shown below where each side of the plate is maintained at a specific temperature. We are interested in finding the temperature within the plate at steady state. No head sinks or source exists in the problem. To find the temperature within the plate, we divide the plate area by a grid as shown in Figure below. The length L along the x-axis is divided into m equal segments, while the width W along the y-axis is divided into n equal segments, hence giving

H = and k =

(i,j)

(i,j)

(i-1,j)

(i,j-1)

(i+1,j)

Now, we apply the finite difference approximation of the partial derivatives at a general interior node (i,j).

= ………….(2)

= ……………(3)

Equations (2) and (3) are central divided difference approximations of the second derivatives. Substituting equations (2) and (3) in equation (1), we get

+ =0 ………………..(4)

For a grid h=k, equation (4) can be simplified as:

+ =0

Or Ti+1,j + Ti-1,j+ Ti,j+1,+ Ti,j-1 -4 i,j = 0

Now, we can write this equation at all the interior nodes of the plate. There are (m-1)\*(n-1) nodes. This will result in an equal number of equations and unknowns. The unknowns are the temperatures at the interior (m-1)\*(n-1) nodes. Solving these equations will give us the two-dimensional profile of the temperature inside the plate.

**Algorithm**

1. Start
2. Read dimension of plate ,say n
3. Read the temperature in left, right, upper and bottom part of the plate say tl, tr, tu, lb
4. Construct a coefficient matrix as:

Set a[i][j] = 0, i=0,1…n

Set a[i][n-i]=1, i=0,1,…..n, i≠j & j≠n-1

1. Construct RHS vector as

K=0

For i=1 to n-1

For j=1 to n-1

1. Use Gauss-Jacobi method iteration method to solve the equation
2. Display the solution vector
3. Stop

**Example 1:** A plate of dimension 2.4×2.4 is subjected to temperatures as follows. Left side at 75 C, right side at 100C, and lower part at 50 C. If square grid length of 0.8m ×0.8 assumed, what will be temperature at interior nodes.

Solution:

Since height and width of grid is same]

H = k = 0.8 cm

Thus, we can divide rectangular plate into m\*n grids where

m = L/h = 3, and n = W/k = 3

The nodes are as shown in the figure below

|  |  |  |
| --- | --- | --- |
| u3 |  | u4 |
| u1 |  | u2 |
|  |  |  |

Figure: Plate with nodes

Now to get the temperature at the interior nodes we have to write equation for all the combinations of i and j, i,….m-1, j=1,…n-1.

For note u1:

u3+u2+75+50-4u1 = 0

🡪-4u1+u2+u3 = -125 ….(1)

For node u2:

U4+u1+100+50-4u2 =0

🡪u1-4u2+u4 = -150…….(2)

For u3:

u4+u1+300+75-4u3 = 0

🡪u1+u4-4u3 = -375………..(3)

For u4:

u2+u3+300+100-4u4 = 0

u2+u3-4u4=-400……(4)

Equations (1) to (4) represent a set of four simultaneous linear equations, which is given below:

-4u1+u2+u3 = -125

u1-4u2+u4 = -150

u1+u4-4u3 =-375

u2+u3-4u4 = -400

We solve above simultaneous equations using Gauss seidel method, for that we can write above equations as:

u1 =

u2 =

u3 =

u4 =

Solving above system of equations by Gauss Seidel method with initial guess u2=0, u3=0, u4=0, we get

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Iterations | u1 | u2 | u3 | u4 |
| 1 | 31.250 | 45.313 | 101.563 | 136.719 |
| 2 | 67.969 | 88.672 | 144.922 | 158.398 |
| 3 | 89.648 | 99.512 | 155.762 | 163.818 |
| 4 | 95.068 | 102.222 | 158.472 | 165.173 |
| 5 | 96.432 | 102.899 | 159.149 | 165.512 |
| 6 | 96.762 | 103.069 | 159.319 | 165.597 |
| 7 | 96.847 | 103.111 | 159.361 | 165.618 |
| 8 | 96.868 | 103.121 | 159.371 | 165.623 |
| 9 | 96.873 | 103.124 | 159.374 | 165.625 |

Thus,

u1 = 96.817, u2 = 103.124, u3 = 159.374, u4 = 165.625

**Example 2: Consider a steel plate of size 15 cm \* 15 cm. If two of the sides are held at 100C and other two sides are held at 0C. What are the steady state temperatures at interior points assuming a grid size of 5 cm \* 5 cm?**

**Solution:**

**C program**

**Solving Poison’s Equation**

We know that general partial differential equation is given by

A + B + C = D

Where A, B and C are functions of x and y and D is a function of x, y, u and , .

When A = 1, B = 0, C =1 and D = g(x,y), the above equation becomes

+ = g(x,y)

Or ∇2f = g(x,y) ………….(1)

This equation is called Poisson’s equation. To solve the equation take a rectangular plate as shown below where each side of the plate is maintained at a specific temperature. We are interested in finding the temperature with the plate at steady state. No heat sinks or sources exist in the problem. To find the temperature within the plate, we divide the plate area by a grid as shown below. The length L along the x-axis is divided into m equal segments, while the width W along the y axis is divided into n equal segments, hence giving

H = and k =

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+ =gi,j ………………..(4)

For a grid h=k, equation (4) can be simplified as:

+ = gi,j

Or fi+1,j + fi-1,j+ fi,j+1,+ fi,j-1 -4 i,j = h2gi,j

Now, we can write this equation at all the interior nodes of the plate. There are (m-1)\*(n-1) nodes. This will result in an equal number of equations and unknowns. The unknowns are the temperatures at the interior (m-1)\*(n-1) nodes. Solving these equations will give us the two-dimensional profile of the temperature inside the plate.

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For j=1 to n-1

1. Use Gauss-Jacobi method iteration method to solve the equation
2. Display the solution vector
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**Example 1:** Solve the Poisson’s equation ∇2f = 2x2y2 over the square domain 0≤x≤3 and 0≤y≤3 with f=0 on the boundary and h=1

Let us divide the domain into grids of 3\*3 as shown below

0

0

0

|  |  |  |
| --- | --- | --- |
| 0  f2 | f4 | 0 |
| 0  f1 | f3 | 0 |
|  |  |  |

0

0

0

0

For node f1:

f2+f3-4f1 = 2\*12\*12 = 2

For node f2:

F1+f4-4f2 = 2\*22\*12 = 8

For node f3:

F1+f4-4f3 = 2\*12\*22 = 8

For node f4:

f2+f3-4f4 = 2\*22\*22 = 32

Now, we have following system of equations

f2+f3-4f1 = 2

F1+f4-4f2 = 8

F1+f4-4f3 = 8

f2+f3-4f4 = 32

now, we rearrange above equations as

f1 =

f2 =

f3 =

f4 =

Solving above system of equations by using Gauss Seidel method, we get

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Iterations | f1 | f2 | f3 | f4 |
| 1 | -0.500 | -2.125 | -2.125 | -9.063 |
| 2 | -1.563 | -4.656 | -4.656 | -10.328 |
| 3 | -2.826 | -5.289 | -5.289 | -10.645 |
| 4 | -3.145 | -5.447 | -5.447 | -10.724 |
| 5 | -3.224 | -5.487 | -5.487 | -10.743 |
| 6 | -3.234 | -5.497 | -5.497 | -10.748 |
| 7 | -3.248 | -5.499 | -5.499 | -10.750 |
| 8 | -3.250 | -5.500 | -5.500 | -10.750 |
| 9 | -3.250 | -5.500 | -5.500 | -10.750 |

Thus, f1 = -3.25, f2 = -5.4, f3 = -5.5, f4 = -10.75

**Example 2:** Solve the Poisson’s equation ∇2f = xy over the square domain 0≤x≤3 and 0≤x≤3 with f=0 the boundary and h=1.

Solution:

Let us divide the domain into grids of 3\*3 as below:

2

2

|  |  |  |
| --- | --- | --- |
| 2  f2 | f4 | 2 |
| 2  f1 | f3 | 2 |
|  |  |  |

2

2

2

2

For node f1:

f2+f3-4f1 +4= 1\*1 = 1

For node f2:

F1+f4-4f2 +4= 1\*2 = 2

For node f3:

F1+f4-4f3 +4= 2\*1 = 2

For node f4:

f2+f3-4f4 +4= 2\*2 =4

Now, we have following system of equations

f2+f3-4f1 = 1

F1+f4-4f2 = 2

F1+f4-4f3 = 2

f2+f3-4f4 = 4

Now, we rearrange above equations as

f1 =

f2 = 031

f3 =

f4 =

Solving above system of equations by using Gauss Seidel method, we get

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Iterations | f1 | f2 | f3 | f4 |
| 1 | 0.750 | 0.688 | 0.688 | 0.344 |
| 2 | 1.094 | 0.859 | 0.859 | 0.430 |
| 3 | 1.180 | 0.902 | 0.902 | 0.430 |
| 4 | 1.201 | 0.913 | 0.913 | 0.451 |
| 5 | 1.207 | 0.916 | 0.913 | 0.457 |
| 6 | 1.208 | 0.916 | 0.916 | 0.458 |
| 7 | 1.208 | 0.917 | 0.917 | 0.458 |

Thus, f1 = 1.208, f2 = 0.917, f3 = 0.917, f4 = 0.458