APL Assignment No 4

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EE20B019

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1 Abstract

We will fit two functions, e^x and $\cos(\cos x)$ over the interval [0,2) using the fourier series

2 Introduction

The Fourier Series of a function f(x) with period 2π is computed as follows:

$$f(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos nx + b_n \sin x$$
$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$$

Where,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot \cos nx dx$$
$$b_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot \sin nx dx$$

For the sake of this assignment, Since $\exp x$ doesn't have a period of 2π,We choose to change its definition in a piece-wise manner to satisfy periodicity

3 Problems and Solutions

3.1 Question 1

Define Python functions for the two functions above, that take a vector (or scalar) input, and return a vector (or scalar) value. Plot the functions over the interval [2,4) in Figure 1 and 2 respectively. Determine whether the functions are periodic. What function do you expect to be generated by the fourier series? Compute and plot those functions as well in the respective figures.

```
def exponent(x):
    return exp(x)
def v(f,x,k):
    return f*sin(k*x)
def u(f,x,k):
    return f*cos(k*x)
def cosine(x):
    return cos(x)

x=linspace(-2*pi,4*pi,1001)

figure(1)
semilogy(x,exp(x),label='actual graph')
semilogy(x,exp(x%(2*pi)),label="expected fouriers series")
grid(True)
title("Q1. exp(x) vs. x & the expected fourier series (semilogy)")
xlabel("$x\\rightarrow$")
ylabel("$log(exp(x))\\rightarrow$")
legend()
show()

figure(2)
plot(x,cosine(cosine(x)),label='actual graph')
plot(x,cosine(cosine(x%(2*pi))),label="expected fouriers series")
title("Q2. cos(cos(x)) vs. x & the expected fourier series")
xlabel("$x\\rightarrow$")
ylabel("$cos(cos(x))\\rightarrow$")
grid(True)
legend()
show()
```

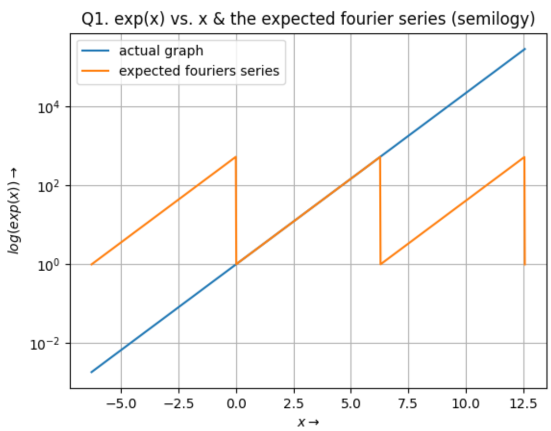

Figure 1: "Question 1(a)"

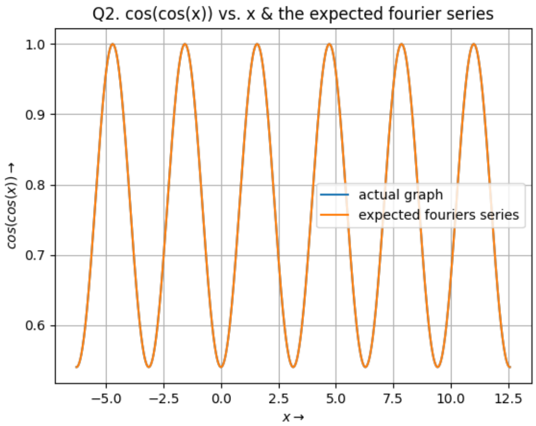

Figure 2: "Question 1(b)"

3.2 Question 2

We use the **scipy quad** function for integration and instead of defining a new 2nd redundant function we used lambda function to get the job done and stored the result in the following format as a numpy column matrix.

$$\begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ \dots \\ a_{25} \\ b_{25} \end{pmatrix}$$

```
def fourier_exp():
    g= lambda k:exp(k)
    a0=float(integrate.quad(g,0,2*pi)[0]/(2*pi))

    coeff_exp_a=array([a0])
    coeff_exp_b=array([0])
    ans_exp=array([a0])
    for i in range(25):
        s=i+1
        a=lambda t,s:u(exp(t),t,s)
        ai=float(integrate.quad(a,0,2*pi,args=(i+1))[0]/(pi))
        coeff_exp_a=append(coeff_exp_a,ai)
        ans_exp=append(ans_exp,ai)
        b=lambda p,s:v(exp(p),p,s)
        bi=float(integrate.quad(b,0,2*pi,args=(s))[0]/(pi))
        coeff_exp_b=append(coeff_exp_b,bi)
        ans_exp=append(ans_exp,bi)

    return coeff_exp_a,coeff_exp_b,ans_exp
def fourier_coscos():
    g= lambda k:cos(cos(k))
    a0=float(integrate.quad(g,0,2*pi)[0])/(2*pi)
    coeff_cos_a=array([a0])
    coeff_cos_b=array([0])
    ans_cos=array([a0])
    for i in range(25):
        s=i+1
        a=lambda t,s:u(cos(cos(t)),t,s)
        ai=float(integrate.quad(a,0,2*pi,args=(s))[0])/(pi)
        coeff_cos_a=append(coeff_cos_a,ai)
```

```
        ans_cos=append(ans_cos,ai)
        b=lambda p,s:v(cos(cos(p)),p,s)
        bi=float(integrate.quad(b,0,2*pi,args=(s))[0])/(pi)
        coeff_cos_b=append(coeff_cos_b,bi)
        ans_cos=append(ans_cos,bi)

    return coeff_cos_a,coeff_cos_b,ans_cos
```

3.3 Question 3

We plot the magnitude of the coefficients we got in question 2 in the same order as the matrix given there, in semilog scale(y-axis) and loglog scale.

```
coeff_exp_a,coeff_exp_b,ans_exp=fourier_exp()
n=linspace(0,25,26)
figure(3)
semilogy(n,abs(coeff_exp_a),linestyle='None',marker='o',color='red')
semilogy(n,abs(coeff_exp_b),linestyle='None',marker='o',color='red')
title("Q3. Magnitude of fourier coefficients of exp(x) (semilog)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
show()
figure(4)
loglog(n,abs(coeff_exp_a),linestyle='None',marker='o',color='red')
loglog(n,abs(coeff_exp_b),linestyle='None',marker='o',color='red')
title("Q4. Magnitude of fourier coefficients of exp(x) (loglog)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
show()
coeff_cos_a,coeff_cos_b,ans_cos=fourier_coscos()

figure(5)
semilogy(n,abs(coeff_cos_a),linestyle='None',marker='o',color='red')
semilogy(n,abs(coeff_cos_b),linestyle='None',marker='o',color='red')
title("Q5. Magnitude of fourier coefficients of cos(cos(x)) (semilogy)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
show()

figure(6)
loglog(n,abs(coeff_cos_a),linestyle='None',marker='o',color='red')
loglog(n,abs(coeff_cos_b),linestyle='None',marker='o',color='red')
title("Q6. Magnitude of fourier coefficients of cos(cos(x)) (loglog)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
show()
```

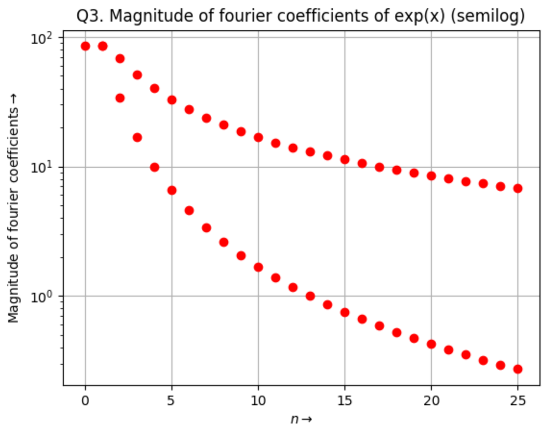

Figure 3: figure 3

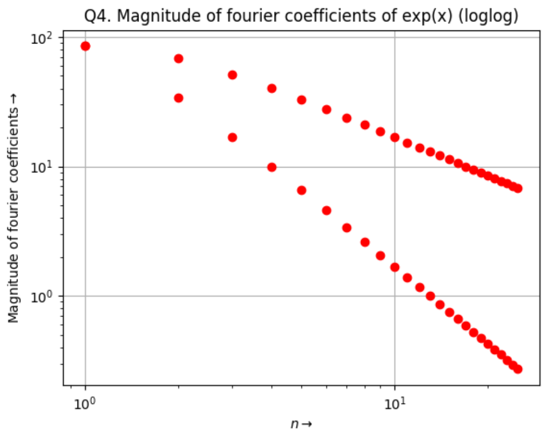

Figure 4: figure 4

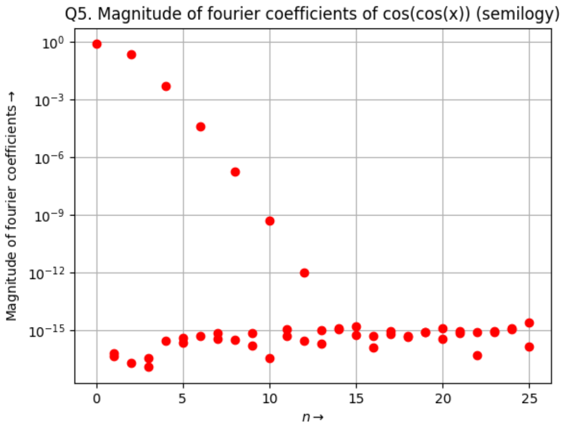

Figure 5: figure 5

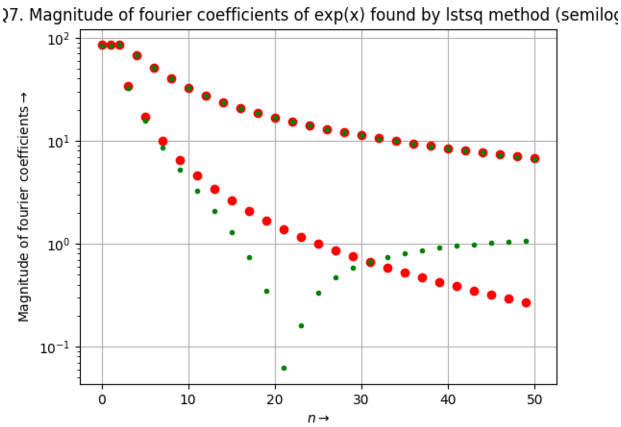

Figure 6: figure 6

3.3.1 If you did Q1 correctly, the b_n coefficients in the second case should be nearly zero. Why does this happen?

This is because $\cos(cosx)$ is a periodic as well as an even function so the odd term should be zero

3.3.2 In the first case, the coefficients do not decay as quickly as the coefficients for the second case. Why not?

In the first case, as an exponential has a number of frequencies in it, it has a wide range of frequencies in its Fourier approximation. On the other hand, the second case has only a low frequency of $\frac{1}{\pi}$, and hence has a low contribution from higher sinusoids.

3.3.3 Why does loglog plot in Figure 4 look linear, whereas the semilog plot in Figure 5 looks linear?

The magnitude of the coefficients of e^x vary as:

$$|a_n|, |b_n| \propto \frac{1}{1+n^2}$$

Thus, with larger values of n, it becomes proportional to $\frac{1}{n^2}$ and hence the log-log plot has a slope of -2$\log n$ and appears to become linear

Similarly for $\cos(\cos x)$, the coefficients vary exponentially with n, and hence, $\log y$ vs x is linear.

3.4 Question 4

We find the Fourier coefficient using **scipy lstsq** function for the following matrix equation:

$$\begin{pmatrix} 1 & \cos x_1 & \sin x_1 & ... & \cos 25x_1 & \sin 25x_1 \\ 1 & \cos x_2 & \sin x_2 & ... & \cos 25x_2 & \sin 25x_2 \\ ... & ... & ... & ... & ... & ... \\ 1 & \cos x_{400} & \sin x_{400} & ... & \cos 25x_{400} & \sin 25x_{400} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ ... \\ a_{25} \\ b_{25} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ ... \\ f(x_{400}) \end{pmatrix}$$

```
t=linspace(0,2*pi,401)
t=t[:-1]
b=exponent(t)
A=zeros((400,51))
A[:,0]=1
for k in range(1,26):
    A[:,2*k-1]=cos(k*t)
    A[:,2*k]=sin(k*t)

c1=lstsq(A,b,rcond=None)[0]
s=linspace(0,50,51)

b2=cos(cos(t))
A2=zeros((400,51))
A2[:,0]=1
for k in range(1,26):
    A2[:,2*k-1]=cos(k*t)
    A2[:,2*k]=sin(k*t)

c2=lstsq(A2,b2,rcond=None)[0]
```

3.5 Question 5

We plot the best fit points and true Fourier coefficients for each function together in both loglog scale and semilog scale.

```
figure(7)
semilogy(s,abs(ans_exp),linestyle='None',marker='o',color='red',label="coeff._by_integration")
semilogy(s,abs(c1),linestyle='None',marker='o',color='green',markersize=3,label='lstq_coeff.')
title("Q7. Magnitude of fourier coefficients of exp(x) found by lstsq method (semilogy)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
show()

figure(8)
loglog(s,abs(ans_exp),linestyle='None',marker='o',color='red',label="coeff._by_integration")
loglog(s,abs(c1),linestyle='None',marker='o',color='green',markersize=3,label='lstq_coeff.')
title("Q8. Magnitude of fourier coefficients of exp(x) found by lstsq method (loglog)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
show()

figure(9)
semilogy(s,abs(ans_cos),linestyle='None',marker='o',color='red',label='coeff. by integration')
semilogy(s,abs(c2),linestyle='None',marker='o',color='green',markersize=3,label='coeff. by listsq ')
title("Q9. Magnitude of fourier coefficients of cos(cos(x)) found by lstsq method (semilogy)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
legend()
show()
figure(10)
loglog(s,abs(ans_cos),linestyle='None',marker='o',color='red',label='coeff. by integration')
loglog(s,abs(c2),linestyle='None',marker='o',color='green',markersize=3,label='coeff. by listsq ')
title("Q10. Magnitude of fourier coefficients of cos(cos(x)) found by lstsq method (loglog)")
xlabel("$n\\rightarrow$")
ylabel("Magnitude of fourier coefficients$\\rightarrow$")
grid(True)
legend()
show()
```

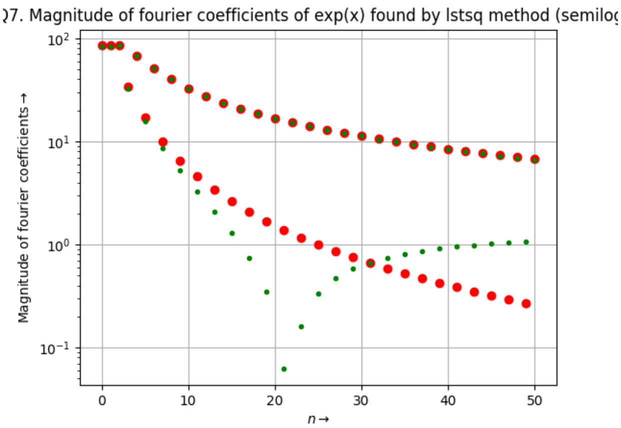

Figure 7: figure 7

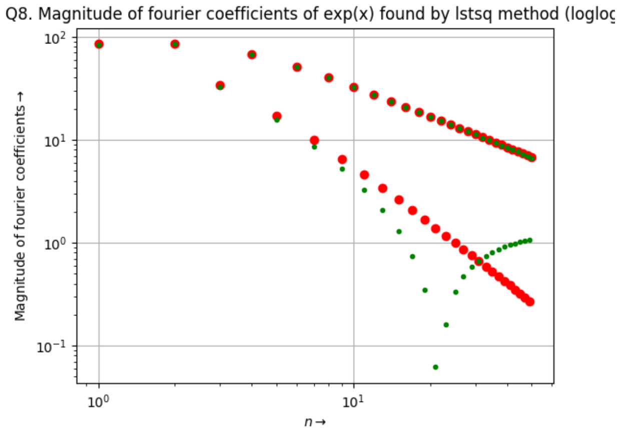

Figure 8: figure 8

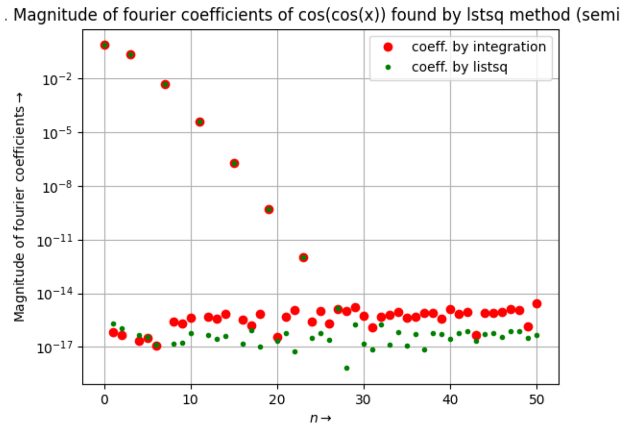

Figure 9: figure 9

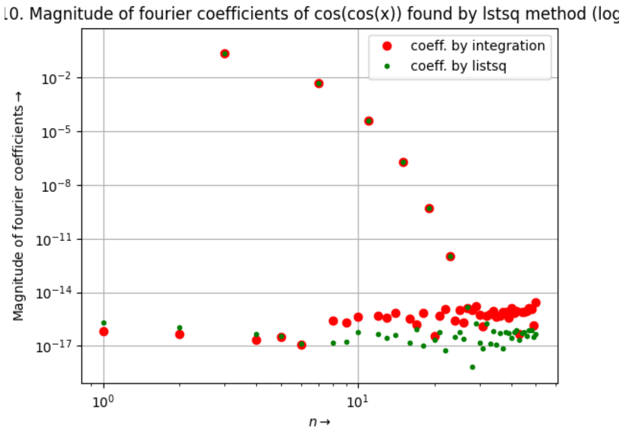

Figure 10: figure 10

3.6 Question 6

We find the absolute error between the best fit values and actual Fourier coefficients and plot them in linear scale.

```
dev_exp=c1-ans_exp
figure(11)
stem(s,abs(dev_exp))
title("Q11. Deviation in coefficients calculated using least squares and direct integration for exp(x) " )
xlabel("$n\\rightarrow$")
ylabel("Deviation in coefficients$\\rightarrow$")
grid(True)
show()

dev_cos=c2-ans_cos
figure(12)
stem(s,abs(dev_cos))
title("Q12. Deviation in coefficients calculated using least squares and direct integration for cos(cos(x)) "
xlabel("$n\\rightarrow$")
ylabel("Deviation in coefficients$\\rightarrow$")
grid(True)
show()
```

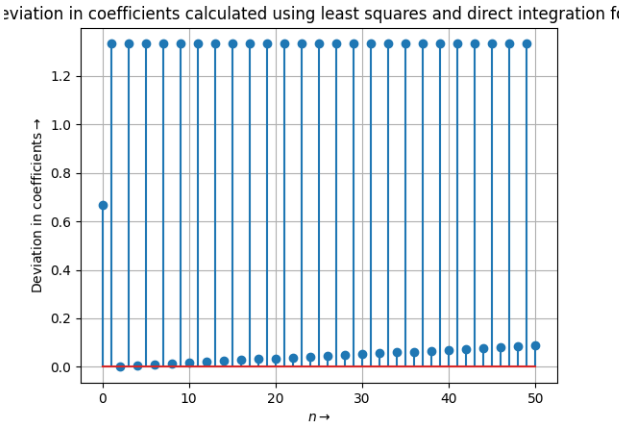

Figure 11: figure 11

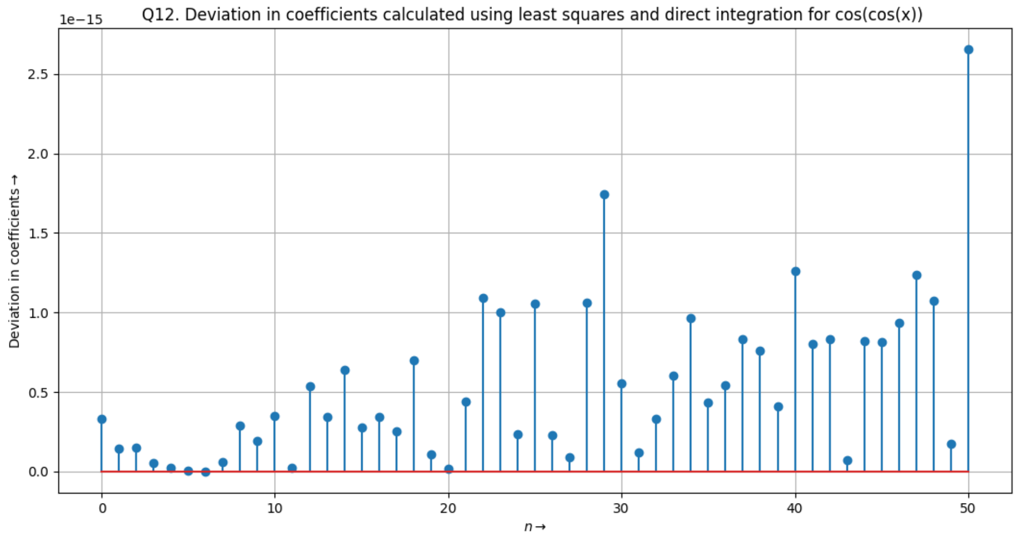

Figure 12: figure 12

3.7 Question 7

e^x is a non periodic function, so we have considered the variation of e^x with period 2π that has the actual value of e^x only in the range $[0,2\pi)$. Hence it is acceptable that there is a large deviation in the predicted value of e^x at the boundaries.

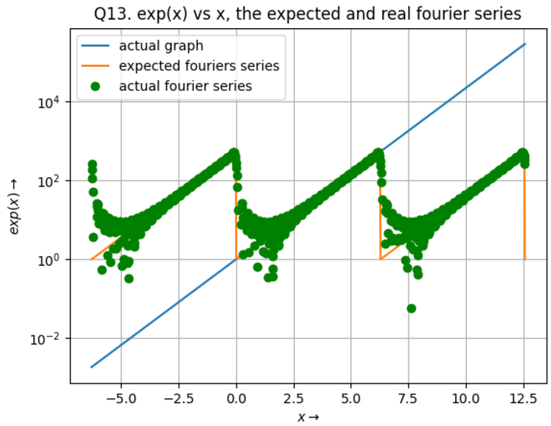

Figure 13: figure 13

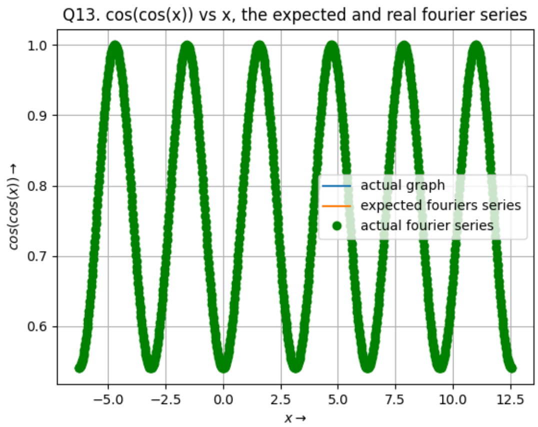

Figure 14: figure 14

4 Conclusion

We found that Fourier series converged for periodic function whereas for a non periodic function it failed to converge outside the region of $[0, 2\pi)$.