Integration

Integration is the reverse of differentiation. Given the derivative of a function, the process of finding the original function is the reverse of differentiation, f is said to be an antiderivative of f'. Given $f'(x)=4$, then $f(x)=4x+c$.

Example 1

Find the antiderivative of any constant of $f'(x)=0$.

Solution:

The derivative of any constant function is zero. Therefore, the antiderivative is $f(x)=c$.

Example 2

Find the antiderivative of $f'(x)=2x-7$.

Solution:

The antiderivative is $f(x)=x^2-7x+c$

Definition

If $F(x)$ is a function of x such that $\frac{d}{dx}F(x)=f(x)$, then we define integral of $f(x)$ with respect to (w.r.t) x to be the function $F(x)$ and we write

$$\int f(x)dx=F(x)$$

Thus $\frac{d}{dx}[F(x)]=f(x)\Rightarrow\int f(x)dx=F(x)$

For example $\frac{d}{dx}[x^5]=5x^4$, we have $\int 5x^4dx=x^5$.

Moreover, if c is any constant, then $\frac{d}{dx}[x^5+c]=5x^4$.

So, in general, $\int 5x^4dx=x^5+c$.

NOTE: Different values of c will give different integrals and thus integral of a function is unique.

Thus in general if f is a continuous function, then

$$\int f(x)dx=F(x)+c$$

If $F'(x)=f(x)$, where c is the constant of integration.

Rules of Integration

Rule 1. Constant functions

$\int k dx = kx + c$, where k is any constant.

Example

Evaluate the following integrals

(i) $\int -7 dx$

(ii) $\int \frac{1}{3} dx$

(iii) $\int \sqrt{3}\, dx$

(iv) $\int -\frac{10}{7} dx$

(v) $\int 0 dx$

(vi) $\int \frac{\sqrt{3}}{2} dx$

Solution:

(i) $\int -7 dx = -7x + c$

(ii) $\int \frac{1}{3} dx = \frac{1}{3} x + c$

(iii) $\int \sqrt{3}\, dx = \sqrt{3} x + c$

(iv) $\int -\frac{10}{7} dx = -\frac{10}{7} x + c$

(v) $\int 0 dx = 0x + c = c$

(vi) $\int \frac{\sqrt{3}}{2} dx = \frac{\sqrt{3}}{2} x + c$

Rule 2: Power Rule

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$, provided that $n \neq -1$.

Example

Evaluate the following integrals

(i) $\int x dx$

(ii) $\int x^2 dx$

(iii) $\int \sqrt{x}dx$

(iv) $\int \frac{1}{x^4}dx$

(v) $\int \sqrt[3]{x}dx$

(vi) $\int \sqrt[7]{x}dx$

(vii) $\int \frac{1}{\sqrt{x}}dx$

Solution:

(i) $\int xdx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$

(ii) $\int x^2dx = \frac{x^{2+1}}{2+1} + c = \frac{x^3}{3} + c$

(iii) $\int \sqrt{x}dx = \int x^{1/2}dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$

(iv) $\int \frac{1}{x^4}dx = \int x^{-4}dx = \frac{x^{-4+1}}{-4+1} + c = \frac{-1}{3}x^{-3} + c$

(v) $\int \sqrt[3]{x}dx = \int x^{1/3}dx = \frac{x^{1/3+1}}{1/3+1} + c = \frac{x^{4/3}}{4/3} + c = \frac{3}{4}x^{4/3}+c$

(vi) $\int \sqrt[7]{x}dx = \int x^{1/7}dx = \frac{x^{1/7+1}}{1/7+1} + c = \frac{x^{8/7}}{8/7} + c = \frac{7}{8}x^{8/7} + c$

(vii) $\int \frac{1}{\sqrt{x}}dx = \int \frac{1}{x^{1/2}}dx = \int x^{-1/2}dx = \frac{x^{-1/2+1}}{-1/2+1} + c = \frac{x^{1/2}}{1/2} + c = 2x^{1/2} + c$

Rule 3: Constant Times a Function

$\int kf(x)dx = k\int f(x)dx$, where k is any constant.

Example

Evaluate the following integrals

(i) $\int 4xdx$

(ii) $\int \frac{x^2}{5}dx$

(iii) $\int 3\sqrt{x}dx$