(5) $E[W_t \mid F_s] = W_s$ for $0 \le s \le t$

$W_t = W_s + (W_t - W_s)$

W_s is known at time s

$E[W_s \mid F_s] = W_s$

$W_t - W_s$ is independent of F_s and has mean $0 \rightarrow E[W_t - W_s \mid F_s] = 0$

$E[W_t \mid F_s] = W_s$

Since, $E[W_t \mid F_s] = W_s$, Brownian Motion is a martingale

To show:

Q3) $E[W_s, W_t] = \min(s,t)$ for $s,t \geq 0$

Assume $s \leq t$

$$W_t = W_s + (W_t - W_s)$$

Then,

$$E[W_s W_t] = E[W_s(W_s + (W_t - W_s))] = E[W_s^2] + E[W_s(W_t - W_s)]$$

Now,

$E[W_s^2] = \text{var}(W_s) = s$

W_s & $W_t - W_s$ are independent

$E[W_t - W_s] = 0$

$$E[W_s(W_t - W_s)] = E[W_s] \cdot E[W_t - W_s] = 0$$

Thus, $E[W_s W_t] = s = \min(s,t)$

Q4) To show: $W_t - W_s \sim N(0, t-s)$

1. Distribution of Increments

$W_t \sim N(0,t)$
$W_s \sim N(0,s)$

Increment $W_t - W_s$

Normal distributed: $W_t - W_s \sim N(0, t-s)$

2. Independence of Non-Overlapping Increments

let interval $[a,b]$ and $[c,d]$ be overlapping

Then, $W_b - W_a$ and $W_d - W_c$ are

independent

This follows from independent increments property of Brownian motion

Thus, $W_t - W_s \sim N(0, t-s)$