Question 1:

N letters are to be put in N separate envelopes $\underline{N=50}$

$$P\left(\begin{matrix}\text{at least one letter is in} \\ \text{the correct envelop}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{No letter is in} \\ \text{correct} \\ \text{envelop}\end{matrix}\right)$$

$$\Downarrow$$

$$1 - \frac{D_n}{n!} \qquad D_n = \left[\frac{n!}{e} + \frac{1}{2}\right]$$

$$1 - \frac{D_{50}}{50!} = 1 - \frac{1}{50!}\left[\frac{50!}{e} + \frac{1}{2}\right] \approx 1 - \frac{1}{e} \approx 0.6321$$

Question 5

1, 2, 3, ... N ; $n \le N$ tickets are drawn with replacement, only max prize ticket is to be kept,

M = prize money obtained, $E(M) = ?$

$$P(M \le k) = P(\text{all } n \text{ draws} \le k) = \left(\frac{k}{N}\right)^k$$

$$P(m = k) = P(m \le k) - P(M \le k-1) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

$$E[M] = \sum_{k=0}^{N} k \cdot P(M=k) = \sum_{k=1}^{N} k\left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]$$

envelopes N=50

(No letter is in correct envelope)

$\approx 1 - \frac{1}{e} \approx 0.6321$

with

to be kept,

Question 11

$u \in \mathbb{R}$, $\phi(x) = e^{ux}$ $\forall$ $x \in \mathbb{R}$

x is a normal random variable, $\mu = E(x)$, $\sigma = [E(x-\mu)^2]^{1/2}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[e^{ux}] = e^{u\mu + \frac{1}{2}\sigma^2 a^2}$$

$$\text{LHS} = E[e^{ux}] = E[e^{ux - u\mu + u\mu}] = e^{u\mu} E[e^{u(x-\mu)}]$$

$$E[e^{ux}] = e^{u\mu} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{u(x-\mu)} \cdot e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$= e^{u\mu} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 u(x-\mu)\right]}$$

$$(x-\mu)^2 - 2\sigma^2 u(x-\mu) = (x-\mu)^2 - 2\sigma^2 u(x-\mu) + \sigma^4 u^2 - \sigma^4 u^2$$

$$\Rightarrow (x-\mu-\sigma^2 u)^2 - \sigma^4 u^2$$

$$E(e^{ux}) = e^{u\mu} e^{\frac{\sigma^2 u^2}{2}} \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{[(x-\mu-\sigma^2 u)]}{2\sigma^2}} dx$$

$$\Rightarrow E(e^{ux}) = e^{u\mu + \frac{1}{2}\sigma^2 u^2} \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 u))^2}{2\sigma^2}} dx = 1 \right)$$

$$E[e^{ux}] = e^{u\mu + \frac{1}{2}\sigma^2 u^2}$$

$\Rightarrow E(\phi(x)) \geq \phi[E(x)]$

Verification: $E[\phi(x)] = e^{u\mu + \frac{1}{2}\sigma^2 u^2} = e^{u\mu} \cdot e^{\frac{1}{2}\sigma^2 u^2} = \phi[E(x)] \cdot e^{\frac{1}{2}\sigma^2 u^2}$

Now, $e^x \geq 1$, if $x \geq 0$ $\Rightarrow$ Jensen's inequality holds.

$\Rightarrow E[\phi(x)] \geq \phi[E(x)]$

Question 9: Convolution of 2 distribution function is also a distribution function

Let F and G be distribution functions of two random variables X and Y

let $Z = X + Y$

Convolution of F and G can be written

$$H(x) = (F * G)(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y)$$

Properties of distribution function:

→ non-decreasing, be right-continuous, $\lim_{x \to -\infty} H(x) = 0$ and $\lim_{x \to \infty} H(x) = 1$

① Let $x_1 < x_2$, for fixed y

$$F(x_1 - y) \leq F(x_2 - y)$$

as F is non-decreasing, integration both side w.r.t. G,

$$H(x_1) = \int F(x_1 - y)\, dG(y) \leq \int F(x_2 - y)\, dG(y) = H(x_2)$$

⇒ $H(x)$ is non-decreasing

② as $x \to -\infty$, for any y, $x - y \to -\infty$, ⇒ $F(x-y) \to 0$

Hence $\lim_{x \to -\infty} H(x) = \int 0\, dG(y) = 0$.

as $x \to \infty$, $F(x-y) \to 1$ $\forall y$, $\lim_{x \to \infty} H(x) = \int 1\, dG(y) = 1$

③ To show: $H(x) \to H(x_0)$ as $x \to x_0^+$

∵ F is right continuous, the integrand $F(x-y)$ is also right-continuous in x for each y.

Thus, H(x) inherits right-continuity from F.

Question 6:

Let X and Y be chosen uniformly on $[0, d]$

$P(|X-Y| < d/3)$

as X and Y are chosen independently and uniformly their joint distribution is uniform over square $[0,d] \times [0,d]$

$0 \leq X, Y \leq d$

$|X-Y| < d/3$

This is a square with area d^2 $\quad X-Y = \pm d/3$

This region is a band of width $2d/3$ along the diagonal $X=Y$ centered within the square, so desired probability is area of region where $|X-Y| < d/3 \,/\, d^2$

$A_{Total} = d^2$

Area outside band $|X-Y| \geq d/3$ → This consists of two right-angled triangles one below $Y = X - d/3$ and one above $Y = X + d/3$

Each of the triangles has legs of length $= d - d/3 = 2d/3$

Area of each triangle $= \frac{1}{2} \cdot \frac{2d}{3} \cdot \frac{2d}{3} = \frac{2d^2}{9}$

Area outside band $= 4d^2/9$

Area inside band $= 5d^2/9$

$P(|X-Y| < d/3) \; \frac{5d^2/9}{d^2} = 5/9$

Question 2

~~Case 1: Gift in Present 1~~

~~probability of host opening present 2 = 1/2~~

~~prior probability = 1/3~~

~~Total probability = 1/6~~

Let G_i be gift in present i, $i = 1,2,3$

Let H_2 be host opens present 2 and shows it is empty.

I chose present 1, initially.

I need, $E[\text{Win} / H_2, \text{ I switch to present 3}]$

$$= 1000 \cdot P(G_3/H_2) + 0 \cdot P(G_1/H_2)$$

$$= 1000 \cdot P(G_3/H_2)$$

All gifts are equally likely $= P(G_1) = P(G_2) = P(G_3) = 1/3$

Case 1: if gift in Present 1

$P(H_2/G_1) = 1/2$

Case 2: if gift in Present 2

$P(H_2/G_2) = 0$

Case 3: if gift in Present 3

$P(H_2/G_3) = 1$

$$P(H_2) = \sum_{i=1}^{3} P(H_2/G_i) \cdot P(G_i)$$

$$= \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \frac{1}{3} = \frac{1}{2}$$

$$P(G_3/H_2) = \frac{P(H_2/G_3) \cdot P(G_3)}{P(H_2)}$$

$$= \frac{1 \cdot 1/3}{1/2} = 2/3$$

Hence, $E[\text{Win}/H_2, \text{switch}]$

$$= 1000 \cdot 2/3 = ₹ 666.6$$