Q10 Indicator func and Fubini's Theorem

To show: $E[X] = \int_0^\infty [1-F(x)]\,dx$

By showing: $\int_\Omega \int_0^\infty 1_{[0,\,X(\omega)}\,dx\,dP(\omega)$

$= EX = \int_0^\infty [1-F(x)]\,dx$

$$X(\omega) = \int_0^\infty 1_{[0,X(\omega)]}(x)\,dx$$

Taking expectation value of both sides

$$E[X(\omega)] = E\left[\int_0^\infty 1_{[0,X(\omega)]}(x)\,dx\right]$$

Applying "Fubini's Theorem"

$$E[X] = \int_\Omega \int_0^\infty 1_{[0,X(\omega)}(x)\,dx\,dP(\omega)$$

$$= \int_0^\infty \left(\int_\Omega 1_{[0,X(\omega)}(x)\,dP(\omega)\right)dx$$

Now,

$$\int_\Omega 1_{[0,X(\omega)}(x)\,dP(x) = P(X>x)$$

$$= 1-F(x)$$

E(X
E(X
Thus
E

$$E(X) = \int_0^\infty (1 - F(x))\, dx$$

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Thus,

$$E(X) = \int_\Omega \int_0^\infty 1_{[0, X(\omega))}(x)\, dx\, dP(x))$$

$$= \int_0^\infty (1 - F(x))\, dx$$

Q7 (a) People = P_i for $i = 0, 1, 2, \ldots n$, starting with "P_0"

Event = "no one ever tells the rumor back to P_0 in first r steps'

At any step 'i', there are $(n+1)$ people one of which is P_0. So probability of "P_0" not being chosen

$$P(A) \quad 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Also,
Every step is independent of past, future or any other step, thus according to law of independent events

Ans: $$P_r(E) = \left[P(A)\right]^r$$
$$= \left[\frac{n}{n+1}\right]^r$$

(b) Probability that rumor is told r times without being repeated to any person

P(E) After 1st step = 1 = n/n
P(E) after 2nd step = $\frac{n-1}{n}$

as '$n-1$' are remaining

Thus after r $P(E) = \frac{n-(r-1)}{n}$

$= \frac{n-r+1}{n}$

$$P(E) = \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) - - \left(\frac{n+1-r}{n}\right)$$

$$= \frac{n(n-1)(n-2) \text{---} (n-r+1)}{n^r}$$

$$= \frac{n!}{(n-r)! \cdot n^r}$$

$$\left[n!/(n-r)! = (n-0)(n-1)(n-2) \text{---} (n-r+1) \right]$$

a) When Each Step tells to Group of 'N' random people

$$P(E) = \frac{\left({}^{n}C_N\right)}{\left({}^{n+1}C_N\right)} = \frac{n(n-1) - - (n-N+1)}{(n+1)(n) - - (n-N+2)}$$

(b) No one is told twice

$$P_r = \prod_{k=1}^{r} \frac{\left({}^{n+1-N(k-1)-1}C_N\right)}{\left({}^{n+1}C_N\right)}$$

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Q4 (a) A Discrete random variable x for which $E(x)$ is finite but $E(x^2) \to \infty$ (b)

Sol Let x be random variable such that we define $x \in N$

$$N = \{1, 2, 3, \ldots\} \quad [\text{finite}]$$

$$P(x = n) = \frac{c}{n^p} \quad \boxed{n \geq 1}$$

~~where~~

Now,

$$E(x) = \sum_{n=1}^{\infty} n \cdot \frac{c}{n^p} = c \sum_{1}^{\infty} \frac{1}{n^{p-1}}$$

$$E(x^2) = \sum_{1}^{\infty} n^2 \frac{c}{n^p} = c \sum_{1}^{\infty} \frac{1}{n^{p-2}}$$ (c)

Let $p > 2$

Therefore $\sum \frac{1}{n^{p-1}}$ will converge

But $\sum \frac{1}{n^{p-2}}$ will diverge

Ex] Say $p = 3$ (say)

$$c \sum_{1}^{\infty} \frac{1}{n} \to \infty \quad \text{but} \quad c \sum_{1}^{\infty} \frac{1}{n^{-2}} \text{ will}$$

be finite.

(b) $f(x) = \frac{c}{x^p}$ $x \geq 1$ and $p > 1$

$E(x) = \int_1^{\infty} x \cdot \frac{c}{x^p} \, dx = c \int_1^{\infty} x^{(1-p)} dx$

$E(x^2) = \int_1^{\infty} x^2 \frac{c}{x^p} \, dx = c \int_1^{\infty} x^{2-p} dx$

Now Let $\boxed{p \cancel{>} 2}$ say $\boxed{p = 2.5}$

Then, $\boxed{\begin{matrix} E(x) < \infty \\ E(x^2) = \infty \end{matrix}}$ Thus, exist

(c) $E(e^{-x}) \geq e^{-E(x)} \geq e^{-1}$
≥ 0.3679

Thus for random variable with $E(x) = 1$ we must have

$\boxed{E(e^{-x}) \geq 1/3}$

Ans: Impossible or does not exist

Q3] (a) $P(A \cap B | C) = P(A | B \cap C) P(B | C)$

$P(A | B \cap C) = P(A \cap B \cap C) / P(B \cap C)$

multiply by $P(B | C)$

$$P(A | B \cap C) P(B | C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \times \frac{P(B \cap C)}{P(C)}$$

$\boxed{\text{Hence, True}}$

(b) $P(A \cap B | C) = P(A | C) P(B | C)$

A & B are independent events

$$P(A \cap B \cap C) = \frac{P(A \cap C) P(B \cap C)}{P(C)}$$

$P(A \cap B | C) = P(A \cap B \cap C) / P(C)$

Not always true, independent events A and B does not guarantee indepen-dence of C

$\boxed{\text{Hence, True}}$

Q3] (c) A is more likely to occur when B is absent

i.e B^C ~~for~~

for D $A|B^C > A|B$

for D^C $A|B^C > A|B$

This probability is just ~~[illegible]~~ weighted sum of regional probabilities

Ans: Hence $P(A|B^C) > P(A|B)$

$\boxed{\text{Hence, True}}$

Q8] Independent events have independant components

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c)$$

so,

$$\prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} \left[1 - P(A_i)\right]$$

for any real no. $1 - x \leq e^{-x}$

This gives

$$\prod_{i=1}^{n} \left[1 - P(A_i)\right] \leq \prod_{i=1}^{n} e^{-P(A_i)}$$

$$\prod_{i=1}^{n} e^{-P(A_i)} = e^{-\sum_{i=1}^{n} P(A_i)}$$

Thus,

$$\boxed{P\left(\bigcap A_i^c\right) \leq e^{-\sum_{i=1}^{n} P(A_i)}}$$