Question 1: Markov Chain

(a) Transition matrix Q
states - $\{1, 2, 3, 4\}$

Transition probabilities:

From state 1: $P(1 \to 1) = 0.5$
$P(1 \to 2) = 0.5$

From state 2: $P(2 \to 1) = 0.25$
$P(2 \to 3) = 0.75$

From state 3: $P(3 \to 3) = 0.75$
$P(3 \to 4) = 0.25$

From state 4: $P(4 \to 3) = 0.25$
$P(4 \to 4) = 0.75$

Transition Matrix Q:

$$Q = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0 & 0.75 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.75 & 0.75 \end{pmatrix}$$

(b) Recurrent and transient states:

1. State 1 and 2
   - $1 \leftrightarrow 2$ (they communicate with each other)
   - From state 1, you can return to state 1
   - From state 2, you can return to state 2 via state 1
   - Thus state 1 and 2 form a recurrent class.

2. State 3 and 4
   - $3 \leftrightarrow 4$ (they can communicate)
   - From state 3, you can return to state 3
   - From state 4, you can return to state 4

   Thus state 3 and 4 are recurrent.

(c) Stationary Distributions

stationary distribution satisfies: $\pi Q = \pi$ and $\sum \pi_i = 1$

1. For recurrent class $\{1, 2\}$

- Let π_1, π_2 be probabilities for state 1 and 2.
- equations:

$$\pi_1 = 0.5\pi_1 + 0.25\pi_2$$
$$\pi_2 = 0.5\pi_1 + 0\pi_2$$

- second eqn: $\pi_2 = 0.5\pi_1$
- substitute into the first eqn;

$$\pi_1 = 0.5\pi_1 + 0.25 \times 0.5\pi_1 \Rightarrow \pi_1 = 0.625\pi_1$$

This shows $\pi_1 = 0$, which is not valid.

$$\pi_1 = 0.5\pi_1 + 0.25\pi_2$$
$$\pi_2 = 0.5\pi_1$$

Substituting in first eqn:

$$\pi_1 = 0.5\pi_1 + 0.25 \times 0.5\pi_1 = 0.5\pi_1 + 0.125\pi_1 = 0.625\pi_1$$
$$\pi_2 = 0.5\pi_1$$
$$\Rightarrow \pi_1 = 2/3, \ \pi_2 = 1/3$$

- Thus one stationary distribution is:

$$\pi^{1} = \left(\tfrac{2}{3}, \tfrac{1}{3}, 0, 0\right)$$

- Similarly for recurrent class $\{3, 4\}$

Eqns: $\pi_3 = 0.75\pi_3 + 0.25\pi_4$

$\pi_4 = 0.25\pi_3 + 0.75\pi_4$

$\Rightarrow$ as $\pi_3 + \pi_4 = 1$
$\Rightarrow \pi_3 = \pi_4 = 0.5$

$\pi^{(2)} = (0, 0, 0.5, 0.5)$

final answer: $\pi^{(1)} = (2/3, 1/3, 0, 0)$
$\pi^{(2)} = (0, 0, 1/2, 1/2)$

Question 2:

(a) Let $W \to$ Win, $L \to$ Loss

the transitions,

$P(W \to W) = 0.8$
$P(W \to L) = 0.2$
$P(L \to W) = 0.3$
$P(L \to L) = 0.7$

So, $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$

we find stationary distribution $\pi = (\pi_W, \pi_L)$ satisfying

$\pi = \pi P, \quad \pi_W + \pi_L = 1$

$\pi_W = 0.8\pi_W + 0.3\pi_L$
$\pi_L = 0.2\pi_W + 0.7\pi_L$

now substitute $\Rightarrow \pi_L = 1 - \pi_W$

$\Rightarrow \pi_W = 0.6, \ \pi_L = 0.4$ (long run proportion of wins π_W)

(b) the dinner probabilities,

after a win $\to 0.7$, after a loss $\to 0.2$

the expected proportion of dinners,

$\pi_W \cdot 0.7 + \pi_L \cdot 0.2$

$= 0.5$ (long-run proportion of games with dinner)

(c) The expected waiting time until a dinner is the inverse of the probability of dinner in a single game.

expected no. of games $= 1/0.5 = 2$

Question 3 :

(a) Cat Chain

- cat moves 2 rooms (Room 1 and Room 2)
- It moves to other room with probab 0.8, so it stays in the same room with probab 0.2

$\pi_c(1)$ = long-run probab: Cat is in Room 1
$\pi_c(2) = 1 - \pi_c(1)$

Transition matrix $P_c = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$

Stationary distribution $\pi_c = [\pi_1, \pi_2]$

$\pi_c \cdot P_c = \pi_c$, $\pi_1 + \pi_2 = 1$

From eqn: $\pi_1 = 0.2\pi_1 + 0.8\pi_2$
$\Rightarrow 0.8\pi_1 = 0.8\pi_2 \Rightarrow \pi_1 = \pi_2 \Rightarrow 0.5$

$\pi_c = [0.5, 0.5]$

Mouse distribution:

$\pi_m = [\pi_1, \pi_2]$

$P_M = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$

$\pi_1 = 0.7\pi_1 + 0.6\pi_2$
$\pi_1 + \pi_2 = 1$ $\Rightarrow \pi_1 = 2/3$, $\pi_2 = 1/3$

$\pi_M = [2/3, 1/3]$

(b) 4 possible joint states,
$(1,1), (1,2), (2,1), (2,2)$

Let Z_n be the joint state of (cat room, mouse room)

$\Rightarrow$ Z_n is a Markov Chain with 4 states, as transition to next state depends on previous one.