Volatility Modeling in Quantitative Finance

Ayush Singh

Abstract

Volatility plays a central role in financial modeling, influencing risk management, derivative pricing, and trading strategies. In this document, we provide a rigorous theoretical treatment of volatility as a stochastic process and survey a range of models used to capture its behavior. We begin with an introduction to volatility and its importance in finance, followed by an overview of the statistical characteristics of volatility such as clustering, persistence, leptokurtosis, and mean reversion. We then discuss models where volatility depends on the asset price itself, including the Constant Elasticity of Variance (CEV) model and the path-dependent Hobson-Rogers model for forward volatility. Discrete-time volatility models (ARCH/GARCH) are presented, highlighting their formulation and ability to capture time-varying variance. Next, we explore continuous-time stochastic volatility models, exemplified by the Heston and Hull-White frameworks, and examine their mathematical structure. Important theoretical questions regarding the replicability and hedgeability of volatility are addressed in the context of complete vs. incomplete markets. We further consider the impact of independent vs. correlated volatility shocks, regime-switching dynamics, and the incorporation of jumps (as in the Bates model) on asset prices. Throughout, the emphasis is on mathematical formulations and theoretical insights, laying a foundation for implementing these models (and potential machine-learning surrogates) in an industrial trading context.

1 Introduction

Volatility—broadly defined as the variability or uncertainty in asset returns—is a fundamental concept in financial modeling. It measures the dispersion of returns and thus quantifies risk. In practice, volatility can refer to realized volatility (statistical variance of historical returns) or to implied volatility (the market's expected future volatility inferred from option prices). Accurate modeling of volatility is crucial for **risk management** (e.g. calculating value-at-risk), **derivative pricing** (option values are highly sensitive to volatility assumptions), and **portfolio optimization** (to balance risk and return).

Classical financial models, such as the Black-Scholes model, assumed a *constant* volatility for tractability. However, empirical evidence strongly contradicts this assumption. Market data shows that volatility is dynamic: it changes over time and responds to market conditions. For example, periods of market turmoil exhibit elevated volatility, whereas stable periods show low volatility. Moreover, implied volatilities of options vary across strike and maturity (the *volatility smile/skew* and term structure), indicating that a single constant volatility cannot simultaneously price all options. These observations have motivated a plethora of more sophisticated volatility models.

In what follows, we undertake a deep theoretical examination of volatility viewed as a stochastic process. We outline the well-known statistical properties of asset volatility (and returns), then develop various modeling approaches. These range from models where volatility is directly tied to the asset price, to discrete-time time-series models, to continuous-time stochastic volatility models. We discuss specific examples including the Constant Elasticity of Variance model, the Hobson-Rogers forward volatility model, ARCH/GARCH processes, and popular stochastic volatility models like Heston and Hull-White. We also address key conceptual issues such as whether volatility risk is hedgeable (and the implications for market completeness), the nature of volatility shocks (independent vs. correlated with price movements), and extensions like regime-switching volatility and jump-diffusion volatility models. The emphasis throughout is on mathematical formulations and theoretical properties, providing a rigorous foundation appropriate for an industrial quantitative finance setting.

2 Volatility as a Stochastic Process: Stylized Facts

Volatility in financial markets is itself random and exhibits distinct statistical patterns. Treating volatility (or variance) as a stochastic process allows us to capture the empirical *stylized facts* of asset returns, which include:

- **Volatility Clustering:** Large price moves tend to be followed by large moves (of either sign), and small moves tend to be followed by small moves. This results in **clustered volatility**—periods of high volatility and periods of low volatility persist for some time rather than alternating randomly. Quantitatively, while asset returns themselves may have near-zero autocorrelation, the autocorrelation of absolute or squared returns is significantly positive over many lags, decaying slowly. This indicates that if volatility is high today, it is likely to remain high in the near future (persistence).

- **Persistence and Long Memory:** Volatility exhibits persistence, meaning that shocks to volatility dissipate gradually. In time-series terms, the conditional variance has a high autocorrelation—often decaying hyperbolically, suggesting a long memory in volatility. Practically, this implies that an unexpected surge in volatility can influence the risk level for an extended period (several days, weeks, or more) before reverting toward a long-run average.

- **Leptokurtosis (Heavy Tails):** Asset return distributions are observed to have *fatter tails* than the normal distribution. In other words, extreme returns (both positive and negative) occur more frequently than a Gaussian model would predict. This phenomenon is closely related to volatility dynamics: a mixture of periods of low and high volatility produces a distribution of returns that is a mixture of Gaussians with different variances, leading to **excess kurtosis** (a higher fourth moment). Stochastic volatility models naturally generate leptokurtic return distributions, as the random variance acts as a mixing variable. Empirically, one measures this effect by noting that the sample kurtosis of high-frequency or daily returns is often significantly above 3 (the Gaussian value). Heavy tails have important risk implications, as they increase the probability of extreme losses.

- **Mean Reversion:** Although volatility can undergo prolonged swings, it tends to revert to some typical level over the long run. This means volatility is often modeled as a mean-reverting process: it may wander high or low, but there is a pull towards a normal range. Mean reversion in volatility is consistent with the idea that markets alternate between calm and turbulent periods but do not permanently stay in either extreme.

- **Stationarity:** In many volatility models, it is assumed that volatility (or log-volatility) is stationary in distribution. After sufficient time, the process forgets initial conditions and its probability distribution stabilizes. Stationarity is a desirable property for model tractability and for the existence of well-defined long-run moments. For example, certain stochastic volatility models have a stationary variance distribution (e.g. a Gamma distribution in the Heston model) when their parameters satisfy appropriate conditions. Conversely, if volatility has a unit root or explosive behavior, it would not have a finite long-run variance, which is generally not observed in practice (volatility can spike but not diverge to infinity in a sustained way).

- **Asymmetry (Leverage Effect):** Empirically, volatility tends to rise more after negative asset returns than after positive returns of the same magnitude. This is known as the leverage effect: when stock prices fall sharply, volatility often jumps (stocks become riskier when their value drops). Statistically, one observes a negative correlation between an asset's returns and its subsequent volatility changes. Modeling this asymmetry is important for accurately pricing options (put options often imply higher volatilities due to this effect). Some stochastic volatility models incorporate this via a correlation between the shocks to the asset price and the shocks to volatility.

To mathematically model these properties, one must go beyond simple constant-variance models. The above characteristics have guided the development of both discrete-time and continuous-time volatility models. For instance, the persistence and clustering observed in returns motivated the creation of Autoregressive Conditional Heteroskedasticity (ARCH) models, which directly model a time-varying conditional variance. Likewise, the desire to capture heavy tails and volatility mean-reversion in a continuous-time setting led to stochastic volatility diffusions (like Heston's model) and other extensions.

In summary, volatility is inherently a stochastic process with memory and feedback effects. Any realistic volatility model should reproduce (at least qualitatively) the stylized facts: volatility clustering, persistence, mean reversion, heavy-tailed returns, and correlations between returns and volatility. In the next sections, we explore several major classes of volatility models, each addressing these empirical features to varying degrees and with different mathematical approaches.

3 Price-Dependent Volatility Models

One approach to modeling non-constant volatility is to make the volatility *state-dependent*, i.e. a deterministic function of the asset price (and possibly time). In such models, sometimes called *local volatility models*, the instantaneous volatility $\sigma(S_t, t)$ at time t depends on the current asset price S_t (and time t if needed). These models retain a single source of randomness (the Brownian motion driving the asset price), but allow the amplitude of that randomness (the volatility) to vary with the state of the system.

A key motivation for price-dependent volatility is to be consistent with observed option prices. Dupire's local volatility model (1994) showed that, given the entire surface of option implied volatilities, one can derive a forward-looking volatility function $\sigma(S, t)$ such that, under risk-neutral pricing, the model exactly reproduces those option prices. In a local volatility model, the stock price follows:

$$dS_t = \mu S_t \, dt + \sigma(S_t, t) \, S_t \, dW_t,$$

where W_t is a standard Brownian motion. If $\sigma(S, t)$ is just a constant σ, we recover the Black-Scholes model. If σ varies with S, the distribution of S_t can depart from log-normal, producing implied volatility smiles or skews that align with market data.

Unlike stochastic volatility models, local volatility models are *complete* (since there is no new source of randomness beyond the traded asset). This means any contingent claim on S can in theory be perfectly hedged by dynamic trading in S and a risk-free asset. However, a purely price-dependent volatility cannot capture random spikes of volatility that are uncorrelated with the price's current level (since all variance is explained by S_t). Thus, local volatility models often fit the initial option surface well, but they may not predict the future evolution of the volatility surface as accurately as stochastic volatility models. In practice, traders observe that local volatility models tend to underestimate the variability of future implied volatilities.

In this section, we discuss two important examples of state-dependent volatility modeling: a purely level-dependent diffusion model (the CEV model) and a more complex path-dependent volatility model (the Hobson-Rogers model).

3.1 Constant Elasticity of Variance (CEV) Model

The Constant Elasticity of Variance model, introduced by Cox in the 1970s, is a classic example of a price-dependent volatility model. In the CEV model, the volatility of the asset is a power-law function of the asset price. The SDE for the asset price under the CEV model is typically written as:

$$dS_t = \mu S_t \, dt + \sigma S_t^{\beta} \, dW_t \; , \tag{1}$$

where μ is the drift and $\sigma > 0$ and β are parameters. The parameter β (sometimes denoted α in literature) governs the elasticity of volatility with respect to the price level. We can interpret S_t^{β} as making the instantaneous standard deviation proportional to S_t^{β}. Equivalently, the instantaneous variance rate is $\sigma^2 S_t^{2\beta}$, so the elasticity of variance with respect to price is constant (hence the name of the model).

Different values of β allow the CEV model to produce different volatility behaviors:

- If $\beta = 1$, the diffusion term is σS_t, which reproduces the Black-Scholes model (constant volatility in percentage terms, since $dS_t/S_t = \mu \, dt + \sigma \, dW_t$).

- If $\beta < 1$, volatility in percentage terms increases as the price falls (because $\sigma S_t^{\beta-1}$ grows as S_t decreases). This case is often used for equity markets to incorporate the leverage effect qualitatively: when S_t is low (the stock has fallen), the volatility σS_t^{β} is relatively high compared to the price level. Thus, $\beta < 1$ produces an implicit skew where downward moves are associated with higher volatility.

- If $\beta > 1$, volatility in percentage terms increases as the price rises (volatility elasticity greater than one). This is less common in equity modeling but might be used in commodities or other assets where volatility grows with price.

An important analytical property of the CEV diffusion is the behavior of the process near the boundary $S = 0$. For $\beta < 1$, the origin $S = 0$ is an *absorbing state* that can be reached in finite time with positive probability. In fact, it can be shown that if $\beta < 1$, the stock price will hit zero in finite time almost surely (under the risk-neutral measure, if μ is set to r for pricing). This means the process does not stay strictly positive, modeling a scenario like default or bankruptcy if uncontrolled. For $\beta \geq 1$, the origin is not reachable (for $\beta = 1$, the process is lognormal and never hits zero; for $\beta > 1$, the drift away from zero dominates sufficiently).

The CEV model yields a one-factor diffusion that can generate implied volatility smiles and skews. Intuitively, by choosing $\beta < 1$, one introduces higher volatility for lower prices, which produces a skew reminiscent of equity options (where out-of-the-money puts have higher implied volatilities). For pricing options, the CEV model does not have a closed-form solution as simple as Black-Scholes, but the pricing problem can be handled via numerical methods or (in some parameter regimes) semi-analytical formulas involving confluent hypergeometric functions. The CEV diffusion is closely related to a non-central chi-square distribution, and there exist analytic expressions for transition probabilities for certain β values.

In summary, the CEV model extends the Black-Scholes framework by allowing volatility to vary with the price level through a power-law relationship. This relatively simple modification grants the model flexibility to fit volatility skews. However, one must handle the potential zero boundary issue for $\beta < 1$, and recognize that CEV still assumes the volatility changes are completely driven by the asset price itself (no independent volatility shocks). It remains a one-factor model and thus a complete-market model for contingent claim pricing.

3.2 Hobson-Rogers Model (Path-Dependent Volatility)

While the CEV model ties volatility to the current price, the **Hobson-Rogers model** is an example of a model that ties volatility to the *history* of the price, in a specific way, to achieve a better fit for the volatility surface while maintaining analytical tractability. The Hobson-Rogers (HR) model introduces volatility as a function of a *moving average* or *offset* of the asset's log-price.

In the Hobson-Rogers model, one defines an exponentially-weighted historical average of the log-price. Let $X_t = \ln S_t$ be the log price. Define a *running* *exponential moving average* M_t of X_t:

$$M_t = \lambda \int_0^\infty e^{-\lambda u} X_{t-u}\, du \ ,$$

for some $\lambda > 0$ which determines the weight decay (how far back the moving average effectively reaches). In practice, M_t satisfies an SDE $dM_t = \lambda(X_t - M_t)\, dt$, so it continuously pulls towards the current log-price. The difference $X_t - M_t$ can be called the *offset* of the log-price from its recent average. Intuitively, this offset measures how unusually high or low the current price is relative to its recent trading range or trend.

The Hobson-Rogers model posits that the instantaneous volatility is a deterministic function of this offset. For example, one specification is:

$$\sigma_{\text{HR}}(t) = f\big(X_t - M_t\big) \ ,$$

for some chosen function $f(\cdot)$. When the current price S_t is far above its recent average (large positive offset), the volatility might decrease (as the market is in an uptrend with perhaps less uncertainty), and when S_t is far below its recent average (large negative offset, indicating a sharp drop), the volatility increases (as the market is stressed and uncertain). In effect, this setup can capture a form of mean-reverting volatility that is *endogenously* driven by the asset's trajectory, incorporating a leverage-like effect (volatility rises when the price is unexpectedly low).

Mathematically, the state of the system can be represented by (S_t, M_t) (or equivalently (X_t, M_t)). The model can be written as a 2-dimensional Markov process:

$$\begin{aligned} dS_t &= \mu S_t\, dt + f(X_t - M_t)\, S_t\, dW_t^S \ , \\ dM_t &= \lambda(X_t - M_t)\, dt \ , \end{aligned} \tag{2}$$

with dW_t^S a Brownian motion. There is effectively only one source of randomness (W^S) driving both equations (note M_t changes deterministically given X_t). Despite having an extra state variable, no new random noise is introduced for volatility itself; the volatility is *derived* from the price history. This means the model is **complete**: all randomness comes from the traded asset, so there are no additional unhedgeable risk factors.

The Hobson-Rogers model was designed to better fit the implied volatility surface dynamics. By making volatility depend on a moving average offset, the model can generate realistic **forward volatility surfaces** and patterns of stochastic skew. Empirical tests have shown that volatility indeed tends to vary with such an offset. In fact, some studies have found that this model can fit option prices better than the Heston stochastic volatility model in certain cases. The model can produce **volatility smiles and skews** consistent with market data.

Another notable feature is how the effective correlation between price and volatility arises in HR. Since $f(\cdot)$ is likely a decreasing function (volatility higher when $X_t - M_t$ is negative), the model exhibits an implicit negative correlation between price shocks and subsequent volatility (a leverage effect) even though there is only one Brownian driver. In other words, a downward shock to S_t will make $X_t - M_t$ more negative (price below trend), which immediately boosts volatility; an upward shock does the opposite. Thus, the model has a form of **stochastic correlation** between the spot and future volatility level, driven by the state-dependent rule rather than a fixed parameter.

The Hobson-Rogers framework lies between pure local volatility and full two-factor stochastic volatility models. It introduces path-dependence to enrich dynamics but avoids a second Brownian (maintaining completeness). This makes it a **complete stochastic volatility model** in the sense of Hobson and Rogers (1998): it spans the volatility risk through the underlying asset itself. The cost is a more complex state space and the need to calibrate the function f and parameter λ to market data. Still, it provides a conceptually appealing way to model volatility surfaces in a forward-looking manner and to incorporate the influence of past price trends on current volatility.

4 ARCH and GARCH: Discrete-Time Volatility Models

While the above models operate in continuous time, a major development in volatility modeling came from discrete-time econometric models. In 1982, Robert Engle introduced the Autoregressive Conditional Heteroskedasticity (ARCH) model, which treats volatility as a process that evolves over time based on past errors. The ARCH model and its generalizations have become essential for modeling and forecasting time-varying volatility in financial time series (like daily returns).

4.1 ARCH(q) Processes

In an ARCH model, one directly models the *conditional variance* of returns. Let r_t be the asset return (or log-return) over period t (e.g. daily). An ARCH(q) model specifies:

$$r_t = \mu + \epsilon_t \,, \qquad \epsilon_t = \sigma_t z_t \,,$$

where z_t are i.i.d. standard normal (or another standardized distribution) and σ_t^2 is the conditional variance given past information. The ARCH(q) process defines σ_t^2 as a linear combination of the last q squared residuals:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 \ , \tag{3}$$

with $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$. Here, ϵ_{t-i}^2 are the recent squared returns (or residuals), serving as a proxy for recent volatility. If a large shock occurred in the last q periods, it boosts σ_t^2. This mechanism captures volatility clustering: large ϵ_{t-1}^2 leads to high σ_t^2 today, which in turn makes it more likely that ϵ_t will be large (in absolute value) because its variance is high. Thus, volatility tends to remain high until enough periods of small shocks pass, reproducing persistence.

The parameters α_i control how quickly volatility reacts to new shocks and how long it retains memory of past disturbances. Typically, one finds $\sum_{i=1}^q \alpha_i$ is less than but close to 1 for financial returns, indicating high persistence. The process is covariance stationary (i.e., has a stable long-run variance) if $\sum_{i=1}^q \alpha_i < 1$; in that case the long-run average variance is $\mathbb{E}[\sigma_t^2] = \alpha_0/(1 - \sum_{i=1}^q \alpha_i)$. If the sum equals 1, volatility exhibits unit-root-like behavior (integrated volatility, with no mean reversion, sometimes called IGARCH). If the sum exceeds 1, the model is explosive (variance grows without bound).

4.2 GARCH(p, q) Processes

Bollerslev (1986) generalized ARCH to allow the conditional variance to depend not only on past shocks but also on its own past values. The Generalized ARCH or GARCH(p, q) model is defined as:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \, \epsilon_{t-i}^2 \ + \ \sum_{j=1}^{p} \beta_j \, \sigma_{t-j}^2 \ , \tag{4}$$

with $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$. A common simple specification is GARCH(1,1):

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \ .$$

In GARCH, the last period's variance σ_{t-1}^2 directly feeds into the current variance, allowing for more persistence with fewer parameters. GARCH(1,1) often suffices to capture the slow decay of volatility autocorrelation in many return series.

The GARCH model can be seen as a parsimonious approximation of long-memory volatility. If $\alpha_1 + \beta_1$ is close to 1, shocks to volatility decay slowly. The unconditional variance exists if $\alpha_1 + \beta_1 < 1$. When that sum is near 1, the model produces the high persistence observed empirically.

ARCH/GARCH models capture several stylized facts:

- Volatility clustering (through feedback of past squared returns).

- Heavy-tailed return distributions: Even if z_t are normal, the unconditional distribution of r_t under GARCH is leptokurtic. Essentially, r_t is a scale mixture of normals (variance changes each time), which produces a heavier tail than normal. This helps explain observed excess kurtosis in returns.
- Mean reversion in variance: if $\alpha+\beta < 1$, volatility will revert to $\omega/(1-\alpha-\beta)$ in the long run.

These models are discrete-time and used extensively for volatility forecasting and risk management. They are not directly used for option pricing in their basic form (since they do not specify a risk-neutral process for continuous sample paths), but there are approaches to option pricing under GARCH (e.g. the NGARCH model by Duan, 1995, which creates a link to risk-neutral dynamics).

Interestingly, there is a connection between GARCH and continuous-time stochastic volatility: in a limit of very short time intervals, a GARCH process can converge to a diffusion process for volatility. For example, Nelson (1990) showed that a GARCH(1,1) model, under appropriate scaling, converges to a diffusion akin to a Cox-Ingersoll-Ross (CIR) process for variance. This bridges the gap between discrete and continuous models and justifies using GARCH to approximate certain stochastic volatility models.

In summary, ARCH/GARCH models provide a flexible, econometric approach to modeling volatility directly from return time series. They have become standard tools in financial analysis for capturing time-varying volatility and have inspired or connected to many continuous-time models.

5 Stochastic Volatility Models in Continuous Time

Moving back to continuous-time models, we now consider **stochastic volatility (SV) models** where volatility (or variance) evolves as its own random process, typically driven by a source of randomness independent from (or partially correlated with) the asset price's source. Unlike price-dependent models (where σ is a function of S_t or its history) or GARCH (a discrete recursion), true stochastic volatility models introduce a second source of uncertainty to govern volatility.

The general setup for a stochastic volatility model is a two-factor system:

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^S \ , \\ dV_t &= a(V_t)\, dt + b(V_t)\, dW_t^V \ , \end{aligned} \tag{5}$$

where V_t is the instantaneous variance process (so $\sqrt{V_t}$ is the volatility), and W^S and W^V are Brownian motions (which may be correlated). The functions $a(\cdot)$

and $b(\cdot)$ specify the drift and diffusion of the variance process. Under risk-neutral pricing, μ would be the risk-free rate (less dividends if any), and $a(V_t)$ would include terms ensuring the process is mean-reverting under the risk-neutral measure with appropriate market price of volatility risk.

Stochastic volatility models were pioneered by Hull and White (1987), Scott (1987), Wiggins (1987) and others, with the idea of overcoming the limitations of the Black-Scholes model by introducing a random volatility factor. By having V_t change stochastically, these models can naturally produce implied volatility smiles (as the distribution of S_t is not lognormal) and term structure of volatility (as V_t can wander over time). They also allow a calibration to both the current volatility level and the market prices of volatility-sensitive instruments like variance swaps or options of different maturities.

Two well-known examples of stochastic volatility models are the Heston model and the Hull-White model:

5.1 Heston's Model

Steven Heston (1993) introduced a stochastic volatility model that has become one of the most popular in both academia and industry due to its relative analytical tractability. In the Heston model, the variance follows a mean-reverting square-root process (Cox-Ingersoll-Ross process). The model is:

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sqrt{V_t}\,S_t\,dW_t^S\,, \\ dV_t &= \kappa(\theta - V_t)\,dt + \xi\sqrt{V_t}\,dW_t^V\,, \end{aligned} \tag{6}$$

with parameters $\kappa, \theta, \xi > 0$. Here κ is the rate of mean reversion of the variance, θ is the long-run mean variance, and ξ (sometimes σ_v) is the volatility of volatility (vol-of-vol). The Brownian motions W^S and W^V have a correlation $\rho = \mathrm{Corr}(dW_t^S, dW_t^V)$, which is typically taken to be negative to model the leverage effect (when S drops, V tends to jump up, corresponding to dW_t^S and dW_t^V being negatively correlated).

Several important properties of Heston's model:

- The variance process V_t mean-reverts to θ. This means that if volatility is very high, there is a drift term pulling V_t down toward θ, and if volatility is very low, the drift pushes it up. This reflects the empirical observation of mean reversion in volatility.

- V_t is always non-negative (as long as $2\kappa\theta \geq \xi^2$, the Feller condition, which ensures the origin is not reached). This is a desirable feature since variance cannot be negative. The dV_t equation is a form of the CIR process, which is well-studied in interest rate modeling.

- The correlation ρ allows the model to capture the skew in the implied volatility surface. A negative ρ introduces asymmetry in the distribution of returns (skewness), leading to higher prices for options that pay off in downward moves (puts) relative to those in upward moves, consistent with market data.

- Despite being a two-factor model (and thus an incomplete market), Heston's model yields a semi-analytical solution for European option prices. Heston derived a closed-form expression for the characteristic function of $\ln S_t$ under this model, which can be integrated (via Fourier inversion) to obtain option prices. This allows relatively fast calibration of the model to market option prices.

- Unconditionally, V_t in Heston's model has a stationary distribution (a Gamma distribution, since CIR processes have Gamma stationary law). As a result, returns have a heavy-tailed distribution (a mixture of normals weighted by V_t's distribution), addressing leptokurtosis.

The Heston model is widely used as a benchmark for stochastic volatility. It can fit a variety of implied volatility surfaces by adjusting its five main parameters $(V_0, \kappa, \theta, \xi, \rho)$. However, it has limitations: for instance, it assumes a single volatility factor, which might not capture very complex volatility surface shapes (multi-factor or time-dependent extensions exist). Also, while it can capture the basic skew and term structure, it might struggle with extreme short-term option smiles or very long-dated smile dynamics without additional features (like jumps).

5.2 Hull-White Model

Hull and White (1987) proposed one of the earliest stochastic volatility models. While Heston's model was formulated to allow a closed-form solution, the Hull-White model was formulated a bit differently and focused on examining the impact of stochastic volatility on option prices. One simple version of the Hull-White model assumes the volatility (or variance) follows a lognormal-type process. For example, a variant of the model is:

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^S \;, \\ dV_t &= \alpha\, V_t\, dt + \eta\, V_t\, dW_t^V \;, \end{aligned} \tag{7}$$

where W^S and W^V may be independent or correlated. In the above, dV_t is of the form "geometric Brownian motion" for V_t, meaning $\ln V_t$ performs a Brownian motion with drift $\alpha - \frac{1}{2}\eta^2$. This implies V_t remains positive and can vary widely (not mean-reverting unless α is negative with specific values). Hull and White examined special cases, such as assuming zero correlation (dW^S independent of dW^V), which simplifies analysis.

One key result from Hull and White's analysis is that when the correlation between dW^S and dW^V is zero, the option price in a stochastic volatility setting is just an average of Black-Scholes prices across different volatility scenarios (since effectively the stock's terminal distribution is a mixture of lognormals). They derived that if volatility is stochastic but uncorrelated with the underlying, and if one uses the average level of volatility, Black-Scholes tends to underprice options (because of Jensen's inequality on the convex payoff). They also investigated the case of nonzero correlation and the resultant pricing biases.

While Hull-White is less specific than Heston (it doesn't fix a unique functional form for volatility dynamics beyond something like lognormal variance), it introduced the concept and provided insights:

- The presence of stochastic volatility generally increases option values (relative to a constant volatility with the same mean) because of the convexity of option payoffs in variance.

- Zero correlation implies symmetric effect on calls and puts, whereas negative correlation (leverage effect) skews the distribution and tends to increase the price of out-of-the-money puts (and decrease that of out-of-the-money calls) compared to the Black-Scholes benchmark.

- The model with dV_t proportional to V_t has the advantage that it's relatively easy to simulate or treat analytically (since V_t is lognormal). However, it might not be mean-reverting. Extensions or alternative processes for V_t can incorporate mean reversion.

In practice, the Hull-White model is more of a conceptual framework than a single canonical model—modern usage of the term might refer to any two-factor model they considered. The important aspect is that Hull and White demonstrated how to include a stochastic volatility factor and discuss hedging implications (notably, that volatility risk is not hedgeable with the underlying alone, requiring an incomplete-market treatment or additional instruments to hedge).

Many other stochastic volatility models exist (e.g. Scott's model where $\ln \sigma_t$ is an Ornstein-Uhlenbeck process, the SABR model used in interest rates which has a stochastic volatility with a beta exponent on the price, etc.), but Heston's model remains a go-to due to its balance of complexity and tractability.

6 Volatility Risk and Market Completeness

Introducing stochastic volatility raises an important theoretical question: *Is volatility risk hedgeable?* In other words, can we replicate any derivative payoff (which may depend on the path of volatility) using the primary underlying asset and the money market account alone, or do we need extra instruments?

In a standard Black-Scholes world (constant volatility), the market is **complete**: there is only one source of uncertainty (the stock's Brownian motion), and one can dynamically trade the stock to hedge any contingent claim on that stock. When volatility is stochastic and driven by an independent source of randomness, the market becomes **incomplete**. There are now two sources of uncertainty (the stock price W^S and the volatility W^V), but only one traded asset (ignoring the risk-free asset). Intuitively, we have "one equation (hedging instrument) short" of being able to replicate every payoff. As a result, there is no unique arbitrage-free price for derivative claims that depend on volatility; instead, a continuum of equivalent martingale measures exists, corresponding to different prices of volatility risk.

From a mathematical perspective, in a multi-dimensional diffusion setting, a well-known condition for completeness is that the *volatility matrix* of the traded assets is full rank. With one stock and two Brownian drivers, the 1×2 volatility matrix cannot be full rank (it can have at most rank 1), hence the model is incomplete. The Hobson-Rogers model we discussed earlier is an example where, by making volatility a function of the stock's own path, effectively there is only one Brownian driver in the augmented state, and completeness is preserved.

In incomplete markets, pricing and hedging are more complex:

- **Hedging:** Perfect hedging is impossible with only the underlying and bond. One can form *partial hedges* to minimize variance or take positions in traded options (if available) to hedge some of the volatility risk. For example, one might use a variance swap or an option to hedge part of the exposure to volatility changes.

- **Pricing:** There is no unique risk-neutral measure. Additional assumptions or preferences must be introduced to pick a price. Common approaches include: assuming a particular *market price of volatility risk* (linking the real-world and risk-neutral drift of V_t), or using equilibrium arguments or utility indifference pricing to determine how volatility risk is priced. In practice, model calibration to market option prices is often used to implicitly determine how the market is pricing volatility risk (e.g. calibrating Heston's model to a volatility surface yields an implied κ, θ under risk-neutral measure, which embeds a certain market price of vol risk).

- **Volatility Derivatives:** If additional instruments like VIX futures/options or variance swaps are traded, one can extend the market to include them. These are directly sensitive to volatility. If one includes a full set of such instruments, one can in theory complete the market (e.g. add a traded asset whose payoff is directly linked to V_t). In practice, including these helps but might not fully complete the market due to discrete trading, limited maturities, etc.

- **Implications for Model Selection:** The incomplete nature of stochastic volatility models means that one must be careful: different models (or different risk premia within the same model) can fit the vanilla option surface but have different implications for exotic options. For instance, two stochastic volatility models might price plain options similarly but give different prices for a volatility swap. Traders often choose a model and then calibrate it so that it fits known prices of liquid instruments, hoping it will reasonably price less liquid ones.

An extreme example of completeness vs. incompleteness is the following: In a pure local volatility model (like Dupire's model or CEV), volatility is a deterministic function of S_t and t. All randomness is still from the single Brownian driving S, so the model is complete and one price for each claim exists. However, such a model cannot generate truly random volatility—it pre-determines volatility given the path of S. In contrast, a stochastic volatility model introduces genuine volatility uncertainty but at the cost of incomplete markets.

In summary, whether volatility is replicable/hedgeable depends on the model structure. In one-factor models (including price-dependent or deterministic volatility models), volatility risk is effectively redundant and can be hedged by trading the underlying. In two-factor models (stochastic vol with independent factors, or models with jumps that introduce new risks), volatility risk is non-replicable with the underlying alone, leading to incomplete markets. Practitioners often live with this incompleteness by calibrating models to market prices and managing residual risks through diversification or using available volatility derivatives.

7 Volatility Shocks: Independent vs. Correlated

When modeling stochastic volatility, a crucial consideration is the relationship between the shocks driving the asset price and those driving volatility. We use the term **volatility shocks** to refer to random changes in the volatility level (or variance). These can be:

- **Independent Volatility Shocks:** The simplest assumption is that the source of volatility fluctuations is independent of the source of price fluctuations. In a two-Brownian model, this means $\mathrm{Corr}(dW^S, dW^V) = 0$. Under this assumption, volatility moves on its own, without any contemporaneous correlation with price moves.

- **Correlated Volatility Shocks:** More realistically, one often allows $\mathrm{Corr}(dW^S, dW^V) \neq 0$, usually negative to capture the leverage effect. This means when the price goes down (dW^S negative), the volatility shock dW^V tends to be positive (volatility rises), and vice versa.

7.1 Independent Volatility Shocks

If volatility shocks are independent, the model cannot produce a correlation between returns and volatility changes (no leverage effect). This might be a reasonable approximation for certain assets or over short timescales where the immediate feedback is not pronounced. Models with independent vol shocks are simpler analytically. For example, as mentioned, Hull and White analyzed the case of zero correlation in their stochastic volatility model to obtain semi-analytical pricing formulas. With independent shocks, the distribution of asset returns (integrated over volatility) becomes symmetric in a certain sense, and option pricing involves averaging over volatility states without bias from correlation.

One implication of independent volatility shocks is that large price moves do not, on average, coincide with volatility jumps in that model. If one were to simulate such a model, you might see periods where volatility drifts upward or downward largely unrelated to the immediate price trend. This misses the leverage effect but might capture volatility clustering if the variance process is mean-reverting and persistent.

In terms of pricing, independent volatility still leads to an incomplete market. The independence by itself doesn't restore completeness; it just simplifies some calculations. Risk-neutral valuation would still require specifying the drift of V_t (or equivalently the market price of volatility risk). The independence means that volatility risk might be considered an orthogonal risk factor with its own price of risk parameter.

One might choose independent volatility shocks if the primary goal is to match kurtosis and overall level of implied vols, but not necessarily the skew. If the observed implied volatility smile is mild or symmetric, a zero-correlation model might suffice. However, for equity markets, the skew is important, and thus independent shocks often fall short.

7.2 Correlated Volatility Shocks and the Leverage Effect

Allowing a non-zero correlation between dW^S and dW^V (particularly a negative one) is crucial for capturing the skew in equity options. This correlation is often called the **leverage correlation** since in equity markets, a drop in stock price (increase in leverage) tends to raise volatility. In stochastic volatility models like Heston, ρ is the correlation parameter. Empirically, for many equity indices and stocks, estimates of ρ are significantly below zero (e.g., -0.7 or -0.8 in some calibrations).

The effect of negative correlation:

- The distribution of returns becomes skewed left. Intuitively, if the asset drops, not only do you get a negative return, but volatility likely jumps up, increasing the probability of further large moves (on the downside or upside) in the short term. Conversely, if the asset rises, volatility tends to dip, making extreme upward moves less likely. This asymmetry results in a heavier left tail than right tail for the asset return distribution.

- Option prices: Negative correlation increases the prices of put options (insurance against drops) relative to what they'd be with zero correlation, because when a drop happens, volatility is higher, making the put payoff potentially larger (the underlying likely stays low or goes even lower due to high vol). This mechanism is what produces the implied volatility skew (higher implied vols for lower strike options).

- Hedging: If one were to hedge a short option position by trading the underlying, the presence of correlation means that when the underlying's price falls and your delta-hedge loses money, at the same time volatility (and thus the option's vega) increases, making the option more expensive to buy back. This dynamic is one reason hedging downside options is challenging in practice; as the bad scenario (stock down) is also when volatility (and thus option sensitivity) jumps.

From a modeling perspective, correlated volatility shocks introduce a coupling between the S_t and V_t processes. In the SDE system, $dW^S dW^V = \rho\, dt$. One sometimes uses a single Brownian Z_t and writes $dW^V = \rho\, dW^S + \sqrt{1-\rho^2}\, d\tilde{W}^V$ to correlate them (with $\tilde{W}^V$ independent). This shows the volatility shock has one component perfectly correlated with the price (ρdW^S) and another independent part. If $\rho = -1$, the model becomes degenerate (essentially volatility is perfectly (negatively) correlated with price, which can be shown to reduce to a one-factor model in disguise). If $\rho = 0$, we recover the independent case.

Beyond constant correlation, one might consider **state-dependent or stochastic correlation**. Some research has looked at models where the correlation ρ itself can change over time or depend on the level of volatility, to fit complex dynamics (this adds even more complexity and is not commonly implemented in practice due to identifiability issues). Regime-switching models could also imply different correlations in different regimes.

7.3 Regime-Switching Volatility

A different form of "dependent" volatility shocks arises in **regime-switching models**. These models assume that the market can be in one of several regimes (states)—for example, a low-volatility regime or a high-volatility regime—and that it switches between regimes according to some stochastic rule (often a Markov chain). The volatility of the asset depends on the current regime.

Regime-switching volatility can capture the idea that volatility sometimes changes due to external factors or structural breaks (e.g., entering a financial crisis vs. normal times) rather than as a continuous diffusion. When a regime change occurs, both the volatility level shifts and often the price may jump or behave differently.

A simple regime-switching model might say:

$$dS_t = \mu_{(R_t)} S_t \, dt + \sigma_{(R_t)} S_t \, dW_t \ ,$$

where $R_t \in \{1, 2, \ldots\}$ is a random regime state that evolves as a Markov chain, and $\sigma_{(R_t)}$ is the volatility in regime R_t. For example, Regime 1: $\sigma = 10\%$ (calm), Regime 2: $\sigma = 40\%$ (turbulent). The chain might have transition rates or probabilities that determine how often switching occurs and how persistent each regime is.

This model is not exactly a two-factor diffusion (the volatility changes only when R_t switches), but it introduces *jump-like* shifts in volatility. It's another incomplete-market model because the regime state is not directly traded. The dependence between volatility and price can be incorporated by allowing different drift $\mu_{(R)}$ or even allowing an immediate jump in price at regime switches (to model, say, a sudden crash when entering a crisis regime).

Regime-switching can create a form of volatility clustering (since staying in a high-vol regime will keep volatility high) and abrupt changes (if a sudden switch happens). It can also naturally produce skewed return distributions if, say, the high-vol regime is more likely entered after a big drop (one can model the switching probability as dependent on recent returns, introducing an asymmetry).

One notable aspect: regime-switching models can fit option surfaces with steeper skews or bimodal implied distributions that diffusive models might struggle with. However, calibrating them is complex, and pricing often requires numerical methods (like solving coupled PDEs for each regime or simulation).

In summary, dependent volatility shocks can refer to either continuous correlation between price and volatility innovations (as in Heston's ρ) or more discrete dependencies like regime changes. Both mechanisms acknowledge that volatility does not evolve in isolation: it interacts with price movements or external state variables. Incorporating these dependencies is key to reproducing observed market phenomena like the leverage effect and volatility regime shifts.

8 Stochastic Volatility with Jumps: The Bates Model

Real markets exhibit not only continuous fluctuations but also sudden jumps, due to unexpected news or events. Combining stochastic volatility with jump dynamics provides a rich model that can capture both the smooth evolution of volatility and abrupt price moves. One influential model in this category is the **Bates model**, introduced by Bates (1996), which extends the Heston stochastic volatility model by adding jumps to the asset price process (as in the Merton jump-diffusion model).

In the Bates model, the asset price dynamics under the risk-neutral measure (for instance) can be written as:

$$\frac{dS_t}{S_{t^-}} = r\,dt + \sqrt{V_t}\,dW_t^S + dJ_t\ , \tag{8}$$

where r is the risk-free rate, V_t follows a stochastic volatility process (often the same as Heston's:

$$dV_t = \kappa(\theta - V_t)\,dt + \xi\sqrt{V_t}\,dW_t^V$$

), and dJ_t is a jump process. Typically, J_t is modeled as a compound Poisson process: jumps arrive according to a Poisson process with intensity λ_J, and the jump size (relative change in S) is a random variable (for example, $\Delta S/S = Y$ where Y is distributed as some log-normal or normal distribution). In Merton's model, jump sizes are lognormal: $\ln(1+Y) \sim \mathcal{N}(\mu_J, \delta_J^2)$.

Key features of stochastic volatility with jumps:

- **Jumps in Price:** These account for extreme moves and can create much fatter tails in the return distribution than volatility fluctuations alone. A jump-diffusion can match short-term option prices better, as short-maturity options are very sensitive to jumps (which can occur in the short time frame). Without jumps, a pure diffusion model often underestimates short-dated implied volatilities (it would predict a relatively narrow distribution for a short period, contrary to observations that very short-term options still have a volatility smile indicating crash risk).

- **Stochastic Volatility (SV):** Provides the volatility clustering, persistence, and medium-term skew dynamics. The SV part (like Heston) ensures the model fits longer-dated options and the general term structure of volatility.

- **Combined Effect:** The SV and jumps together allow the model to fit a full volatility surface in both strike and maturity dimensions more flexibly. For instance, the jumps primarily govern the curvature of the implied vol smile at short maturities (and the extreme strike tail behavior), while the stochastic volatility governs the slope of the skew and level for longer maturities.

The Bates model in particular assumes that jumps and diffusive shocks can be correlated with the volatility process as well (for instance, one might assume the jump intensity or size distribution could depend on V_t, although in the simplest version they are independent of V_t but perhaps priced separately). Often, jumps are taken independent of the Brownian motions for simplicity, but one could also introduce correlation between jump occurrences and volatility level (e.g., more jumps when volatility is high, modeling a turmoil state).

Pricing under jump-diffusion SV models like Bates is more complex. Bates derived a characteristic function for the log-price in his model, extending Heston's formula to include jumps, which allows computing option prices via Fourier integrals similar to Heston's method. In general, jump processes break the pure diffusive PDE approach (the pricing equation becomes a partial integro-differential equation, PDE with an integral term due to jumps).

Hedging in models with jumps is even more challenging: not only is there volatility risk (incomplete market from SV), but also jump risk which cannot be hedged by continuous trading. In fact, jumps cause instantaneous discontinuities that no finite amount of trading in the underlying can hedge away (you'd need infinite trading frequency to catch a jump, which is impossible). Thus, jump risk is another unhedgeable component, requiring a risk premium. Usually, the jump intensity and distribution under the risk-neutral measure are treated as calibration parameters (implied by option prices that are sensitive to tail risk).

Despite these complexities, adding jumps provides a much better fit to market realities. For example, the *volatility smile* for equity index options often becomes more pronounced for short expirations—something Heston's model alone may not capture well, but a jump can. Also, during crises, markets experience actual jumps; a model without jumps would require volatility to spike unrealistically to explain one big move, whereas a jump diffusion can attribute it to a rare jump event.

In summary, stochastic volatility models with jumps, exemplified by the Bates model, represent the state-of-the-art in classical derivatives modeling: they incorporate both continuous variance fluctuations and discontinuous price moves. The SDE of such a model combines the Heston-type variance equation with a jump-augmented price equation. The implications include a richer set of phenomena (fat tails, skew that varies by maturity, etc.), and the necessity of dealing with incomplete markets (both vol and jump risks are unhedgeable). These models are powerful but also complex, and calibrating them to market data must balance many parameters (vol-of-vol, correlation, jump intensity, jump size distribution, etc.) to fit observed option surfaces.

9 Conclusion

In this document, we presented a comprehensive overview of volatility modeling approaches in quantitative finance, ranging from statistical properties and simple formulations to advanced stochastic models. We focused on theoretical underpinnings and mathematical formulations, laying the groundwork for practical implementation. In an industrial setting such as a trading firm, these models inform both pricing and risk management. Moreover, the insights from these theoretical models can be leveraged to design machine learning models that capture similar dynamics or to generate simulated data for training. The diversity of models—from ARCH/GARCH time-series to continuous stochastic volatility with jumps—reflects the multifaceted nature of volatility. In practice, model choice depends on the context: high-frequency trading might emphasize different aspects than options market-making or portfolio risk management. By understanding the strengths and assumptions of each model, a practitioner can better adapt and calibrate them to real market behavior or use them as inspiration for data-driven models.