

Task 2 :

$$M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

$$M = U \Sigma V^T$$

$$M^T = V \Sigma U^T$$

$$M^T M = V \Sigma U^T U \Sigma V^T$$

$$M^T M = V \Sigma^2 V^T \quad \text{--- ①}$$

$$M M^T = U \Sigma^2 U^T \quad \text{--- ②}$$

$$M^T M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 4 \\ 12 & 13 & 5 \\ 4 & 5 & 5 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 10 \\ 10 & 14 \end{bmatrix}$$

$$(17-\lambda)(14-\lambda) = 100$$

$$(\lambda-14)(\lambda-17) - 100 = 0$$

$$\lambda^2 - 31\lambda + 170 + 68 - 100 = 0$$

$$\lambda^2 - 31\lambda + 138 = 0$$

$$\lambda = \frac{31 \pm \sqrt{31^2 - 4 \times 138}}{2}$$

$$\lambda = \frac{31 + \sqrt{409}}{2}, \quad \frac{31 - \sqrt{409}}{2}$$

$$\begin{bmatrix} 17-\lambda & 10 \\ 10 & 14-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(17-\lambda)x_1 + 10x_2 = 0 \quad (\lambda-14-\lambda)$$

$$10x_1 + (14-\lambda)x_2 = 0$$

$$(17-\lambda)(14-\lambda)x_1 + 10(14-\lambda)x_2 = 0$$

$$100x_1 + 10(14-\lambda)x_2 = 0$$

$$\lambda^2 - 31\lambda + 238 = 100$$

$$10x_1 = \frac{(\lambda-14)}{10}x_2$$

$$x_2 = \frac{10}{\lambda-14}x_1$$

$$(\lambda_1 > \lambda_2)$$

e-vectors: $\begin{bmatrix} 1 \\ \frac{16}{\lambda_1 - 14} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{16}{\lambda_2 - 14} \end{bmatrix}$

eigen (value) $\begin{bmatrix} 1 \\ 0.8611 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{\sqrt{409+3}}{-20} \end{bmatrix}$
 $\frac{\sqrt{409-3}}{20}$

norm $\rightarrow = \sqrt{409+9-6\sqrt{409}} + 1$
 $= \sqrt{818-6\sqrt{409}}$
 $= \frac{\sqrt{818+6\sqrt{409}}}{20}$

$U = \begin{bmatrix} \frac{20}{\sqrt{818-6\sqrt{409}}} & \frac{20}{\sqrt{818+6\sqrt{409}}} \\ \frac{\sqrt{409-3}}{\sqrt{818-6\sqrt{409}}} & \frac{-\sqrt{409+3}}{\sqrt{818+6\sqrt{409}}} \end{bmatrix} \quad \text{--- (1)}$

$\Sigma = \begin{bmatrix} \sqrt{\frac{31+\sqrt{409}}{2}} & 0 \\ 0 & \sqrt{\frac{31-\sqrt{409}}{2}} \end{bmatrix} \quad \text{--- (2)}$

Similarly, $M = U \Sigma V^T \Rightarrow U^T M = \Sigma V^T \Rightarrow$
 $\Sigma^{-1} U^T M = V^T \Rightarrow \boxed{V = M^T U \Sigma^{-1}} \quad (\Sigma^T = \Sigma)$
 (not calculated here)