6.4.2

AI25BTECH11002 - Ayush Sunil Labhade

Question: Find the shortest distance between the lines:

$$\mathbf{r} = 4\hat{\imath} - \hat{\jmath} + \lambda(1\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$$
$$\mathbf{r} = \hat{\imath} - \hat{\jmath} + 2\hat{k} + \mu(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$$

Solution:

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points on the given lines respectively.

$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$

Let $\mathbf{A} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Let $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -3 & -5 \end{pmatrix}$

$$(\mathbf{M}\ \mathbf{B} - \mathbf{A}) = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -1 \\ -3 & -5 & 2 \end{pmatrix} \tag{1}$$

Row Transformation-1: $R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 5 \\ -3 & -5 & 2 \end{pmatrix} \tag{2}$$

Row Transformation-2: $R_3 \to R_3 + 3R_1$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 5 \\ 0 & 1 & -7 \end{pmatrix} \tag{3}$$

Row Transformation-3: $R_3 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -7 \\ 0 & 0 & 5 \end{pmatrix} \tag{4}$$

Therefore, The Rank is 3 $\Rightarrow$ The Lines are Skew Lines.

$$\text{Let } \mathbf{K} = \begin{pmatrix} \lambda \\ -\mu \end{pmatrix} \tag{5}$$

$$(\mathbf{M}^\top \mathbf{M})\mathbf{K} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \tag{6}$$

$$\begin{pmatrix} 14 & 25 \\ 25 & 45 \end{pmatrix} \mathbf{K} = \begin{pmatrix} -9 \\ -15 \end{pmatrix} \tag{7}$$

The Augmented Matrix from Equation 6,

$$\left(\begin{array}{cc|c} 14 & 25 & -9 \\ 25 & 45 & -15 \end{array}\right) \tag{8}$$

After Row Reductions,

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -\frac{9}{5} \end{array}\right) \tag{9}$$

$$\therefore\ \mathbf{K} = \begin{pmatrix} 3 \\ -\frac{9}{5} \end{pmatrix} \tag{10}$$

$$\therefore\ \lambda = 3 \,\text{and}\, \mu = \frac{9}{5} \tag{11}$$

From Equation 10,

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix} \tag{12}$$

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \frac{9}{5} \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{23}{5} \\ \frac{31}{5} \\ -\frac{23}{5} \end{pmatrix} \tag{13}$$

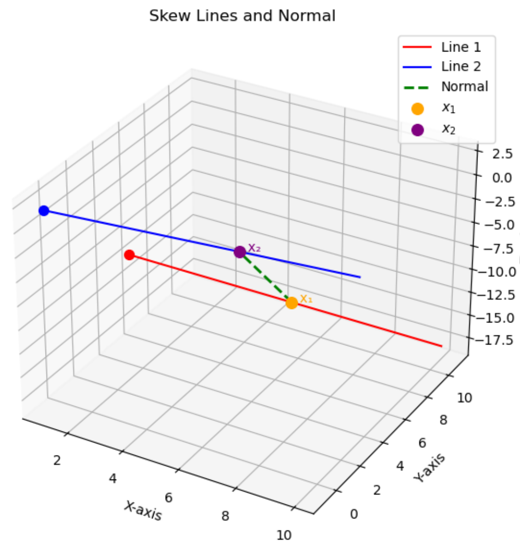
The Minimum Distance between the given skew lines is $\|\mathbf{x}_2 - \mathbf{x}_1\|$

$$\|\mathbf{x}_2 - \mathbf{x}_1\| = \sqrt{(\mathbf{x}_2 - \mathbf{x}_1)^\top (\mathbf{x}_2 - \mathbf{x}_1)} = \frac{13}{\sqrt{5}} \tag{14}$$

The Minimum Distance between the given Lines =

$$\frac{13}{\sqrt{5}}$$

Graph:



Skew Lines and Normal
Line 1
Line 2
Normal
x_1
x_2
x₂
x₁
X-axis
Y-axis
Z-axis

Fig. 1