4.12.7

AI25BTECH11002 - Ayush Sunil Labhade

Question:
Prove that in any ($\triangle$ ABC),

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \tag{1}$$

where (a,b,c) are the magnitudes of the sides opposite to the vertices (A,B,C) respectively.
Solution:
By definition of the side lengths,

$$c = |\mathbf{B} - \mathbf{A}|, \tag{2}$$
$$b = |\mathbf{C} - \mathbf{A}|, \tag{3}$$
$$a = |\mathbf{C} - \mathbf{B}|. \tag{4}$$

The cosine of angle A is given by

$$\cos A = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})}{|\mathbf{B} - \mathbf{A}| \cdot |\mathbf{C} - \mathbf{A}|} \tag{5}$$
$$= \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})}{bc}. \tag{6}$$

Now, express a^2 in terms of $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{A}$. Observe that

$$\mathbf{C} - \mathbf{B} = (\mathbf{C} - \mathbf{A}) - (\mathbf{B} - \mathbf{A}). \tag{7}$$

Let $\mathbf{u} = \mathbf{C} - \mathbf{A}$ and $\mathbf{v} = \mathbf{B} - \mathbf{A}$ for brevity. Then,

$$\mathbf{C} - \mathbf{B} = \mathbf{u} - \mathbf{v}. \tag{8}$$

Hence,

$$a^2 = |\mathbf{C} - \mathbf{B}|^2 \tag{9}$$
$$= (\mathbf{u} - \mathbf{v})^T(\mathbf{u} - \mathbf{v}) \tag{10}$$
$$= \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v} - 2\mathbf{u}^T\mathbf{v} \tag{11}$$
$$= b^2 + c^2 - 2(\mathbf{C} - \mathbf{A})^T(\mathbf{B} - \mathbf{A}). \tag{12}$$

Rearranging,

$$(\mathbf{C} - \mathbf{A})^T(\mathbf{B} - \mathbf{A}) = \frac{b^2 + c^2 - a^2}{2}. \tag{13}$$

Substitute into the expression for $\cos A$:

$$\cos A = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})}{bc} \tag{14}$$
$$= \frac{1}{bc} \cdot \frac{b^2 + c^2 - a^2}{2} \tag{15}$$
$$= \frac{b^2 + c^2 - a^2}{2bc}. \tag{16}$$

Thus, the required identity is proved.
Graph:

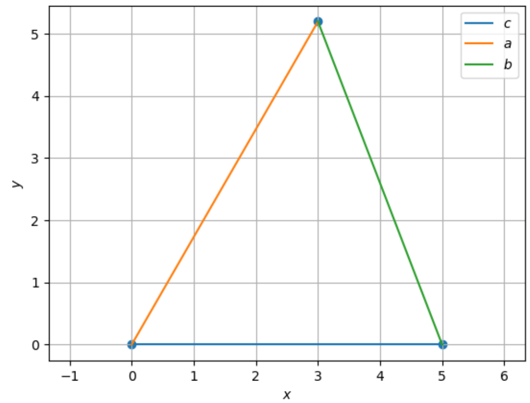

Fig. 1