12.424

AI25BTECH11002 - Ayush Sunil Labhade

Question:
Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined by

$$T(x, y, z, w) = (\, x + z, \; 2x + y + 3z, \; 2y + 2z, \; w \,).$$

Solution:

Write the input vector in coordinates:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \tag{1}$$

Each component of $T(x, y, z, w)$ is a linear combination of x, y, z, w. So , we can match the coefficients:

$$T_1(x, y, z, w) = 1 \cdot x + 0 \cdot y + 1 \cdot z + 0 \cdot w, \tag{2}$$

$$T_2(x, y, z, w) = 2 \cdot x + 1 \cdot y + 3 \cdot z + 0 \cdot w, \tag{3}$$

$$T_3(x, y, z, w) = 0 \cdot x + 2 \cdot y + 2 \cdot z + 0 \cdot w, \tag{4}$$

$$T_4(x, y, z, w) = 0 \cdot x + 0 \cdot y + 0 \cdot z + 1 \cdot w. \tag{5}$$

Therefore the matrix T is the matrix whose rows are the coefficient vectors:

$$[T] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

On row reducing we get,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{rank}([T]) = 3, \tag{7}$$

$$\tag{8}$$