

12.424

AI25BTECH11002 - Ayush Sunil Labhade

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Question:

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear map defined by

$$T(x, y, z, w) = (x + z, 2x + y + 3z, 2y + 2z, w).$$

Solution:

Write the input vector in coordinates:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad (0.1)$$

Each component of $T(x, y, z, w)$ is a linear combination of x, y, z, w . So , we can match the coefficients:

$$T_1(x, y, z, w) = 1 \cdot x + 0 \cdot y + 1 \cdot z + 0 \cdot w, \quad (0.2)$$

$$T_2(x, y, z, w) = 2 \cdot x + 1 \cdot y + 3 \cdot z + 0 \cdot w, \quad (0.3)$$

$$T_3(x, y, z, w) = 0 \cdot x + 2 \cdot y + 2 \cdot z + 0 \cdot w, \quad (0.4)$$

$$T_4(x, y, z, w) = 0 \cdot x + 0 \cdot y + 0 \cdot z + 1 \cdot w. \quad (0.5)$$

Therefore the matrix T is the matrix whose rows are the coefficient vectors:

$$[T] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (0.6)$$

On row reducing we get,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{rank}([T]) = 3, \quad (0.7)$$

(0.8)