8.2.36

AI25BTECH11002 - Ayush Sunil Labhade

Question: Find the equation of the conic that satisfies the given conditions.
Vertex $(0, 4)$, Focus $(0, 2)$.

Solution:
Since only one focus is given, the conic is a **parabola**. Let

$$\mathbf{V_0} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{1}$$

As both lie on the Y-axis, the axis is vertical. Hence, let

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \|\mathbf{n}\| = 1. \tag{2}$$

For the vertex point, by definition of a conic,

$$\|\mathbf{V_0} - \mathbf{F}\| = e\frac{|\mathbf{n}^\top \mathbf{V_0} - c|}{\|\mathbf{n}\|}. \tag{3}$$

Since $\|\mathbf{V_0} - \mathbf{F}\| = 2$ and $e = 1$,

$$2 = |4 - c| \Rightarrow c = 6. \tag{4}$$

Thus the directrix is $\mathbf{n}^T\mathbf{x} = 6$.
(As $\mathbf{n}^T\mathbf{x} = 2$ passes through the focus which is not possible)
The general matrix form of a conic is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \tag{5}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2(\mathbf{n}\mathbf{n}^\top), \qquad \mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \qquad f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2. \tag{6}$$

Substituting $e = 1$, $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $c = 6$, $\mathbf{F} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

$$\mathbf{u} = 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{8}$$

$$f = 4 - 36 = -32. \tag{9}$$

Hence,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 0 & 4 \end{pmatrix}\mathbf{x} - 32 = 0, \tag{10}$$

$$x^2 + 8y - 32 = 0 \Rightarrow y = 4 - \frac{x^2}{8}. \tag{11}$$

Eigen decomposition of $\mathbf{V}$ gives

$$\lambda_1 = 1,\ \lambda_2 = 0,\ \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12}$$

Since one eigenvalue is zero, the conic is confirmed as a **parabola**.
Graph:

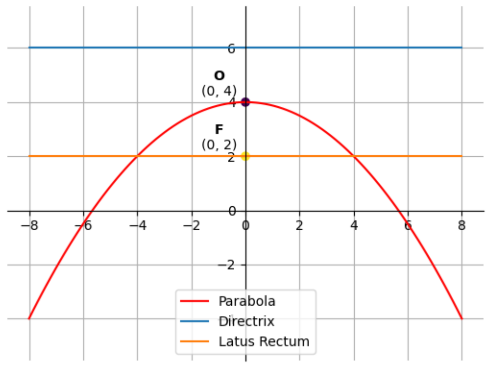

Fig. 1