

8.2.36

AI25BTECH11002 - Ayush Sunil Labhade

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Question: Find the equation of the conic that satisfies the given conditions.

Vertex $(0, 4)$, Focus $(0, 2)$.

Solution:

Since only one focus is given, the conic is a **parabola**. Let

$$\mathbf{V}_0 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad (1)$$

As both lie on the Y-axis, the axis is vertical. Hence, let

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \|\mathbf{n}\| = 1. \quad (2)$$

For the vertex point, by definition of a conic,

$$\|\mathbf{V}_0 - \mathbf{F}\| = e \frac{|\mathbf{n}^\top \mathbf{V}_0 - c|}{\|\mathbf{n}\|}. \quad (3)$$

Since $\|\mathbf{V}_0 - \mathbf{F}\| = 2$ and $e = 1$,

$$2 = |4 - c| \Rightarrow c = 6. \quad (4)$$

Thus the directrix is $\mathbf{n}^T \mathbf{x} = 6$.

(As $\mathbf{n}^T \mathbf{x} = 2$ passes through the focus which is not possible)

The general matrix form of a conic is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0, \quad (5)$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2(\mathbf{n}\mathbf{n}^T), \quad \mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2. \quad (6)$$

Substituting $e = 1$, $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $c = 6$, $\mathbf{F} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (7)$$

$$\mathbf{u} = 6 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad (8)$$

$$f = 4 - 36 = -32. \quad (9)$$

Hence,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} - 32 = 0, \quad (10)$$

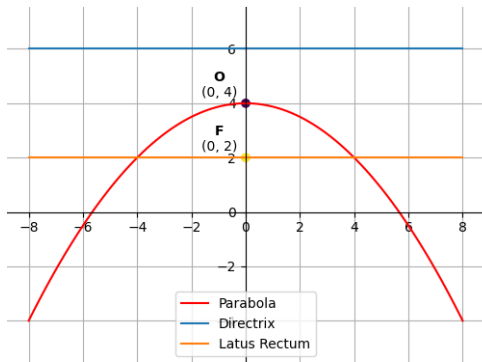
$$x^2 + 8y - 32 = 0 \Rightarrow y = 4 - \frac{x^2}{8}. \quad (11)$$

Eigen decomposition of \mathbf{V} gives

$$\lambda_1 = 1, \lambda_2 = 0, \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (12)$$

Since one eigenvalue is zero, the conic is confirmed as a **parabola**.

Graph:



Figure