4.13.71

AI25BTECH11002 - Ayush Sunil Labhade

Question:
If the vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a \neq b \neq c \neq 1)$ are co-planar, then the value of $\dfrac{1}{1-a}+\dfrac{1}{1-b}+\dfrac{1}{1-c}=\cdots$.

Solution:
The vectors are coplanar if they are linearly dependent, i.e., the matrix formed by them has rank less than 3.
Let $\mathbf{u}=a\hat{i}+\hat{j}+\hat{k}$, $\mathbf{v}=\hat{i}+b\hat{j}+\hat{k}$, $\mathbf{w}=\hat{i}+\hat{j}+c\hat{k}$.
Form the matrix

$$\mathbf{M}=\begin{pmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{pmatrix}. \tag{1}$$

For rank <3, perform row reduction to find the condition.
After row reducing the matrix:

$$\begin{pmatrix} 1 & b & 1 \\ 0 & 1-ab & 1-a \\ 0 & 1-b & c-1 \end{pmatrix}. \tag{2}$$

Assuming $1-ab \neq 0$ (consistent with $a \neq b \neq 1$), to eliminate the second column in R_3, compute the multiplier $k=\frac{1-b}{1-ab}$.
Replace R_3 with $R_3 - k\ R_2$:
The second entry becomes $1-b-k(1-ab)=0$ by construction. The third entry becomes $c-1-k(1-a)=0$ for the row to be zero (ensuring rank 2).
For rank <3, the third entry must be zero:

$$c-1=k(1-a)=\frac{1-b}{1-ab}(1-a). \tag{3}$$

Solving for c:

$$c=1+\frac{(1-b)(1-a)}{1-ab}=\frac{1-ab+(1-a)(1-b)}{1-ab}=\frac{2-a-b}{1-ab}. \tag{4}$$

Now, compute $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$.
First, find $1-c$:

$$1-c=1-\frac{2-a-b}{1-ab}=\frac{-(1-a)(1-b)}{1-ab}. \tag{5}$$

Thus,

$$\frac{1}{1-c}=\frac{1-ab}{-(1-a)(1-b)}=-\frac{1-ab}{(1-a)(1-b)}. \tag{6}$$

The sum is:

$$\frac{1}{1-a}+\frac{1}{1-b}-\frac{1-ab}{(1-a)(1-b)}=\frac{(1-b)+(1-a)-(1-ab)}{(1-a)(1-b)} \tag{7}$$

$$=\frac{(1-a)(1-b)}{(1-a)(1-b)}=1. \tag{8}$$

Thus, the value is 1.
Graph:

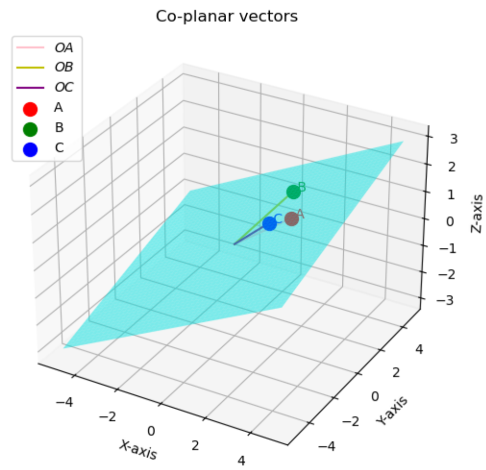

Fig. 1