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AI25BTECH11002 - Ayush Sunil Labhade

Question : Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map satisfying

$$T(e_1) = e_2, \qquad T(e_2) = e_3, \qquad T(e_3) = 0, \qquad T(e_4) = e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. Then determine which of the following statements are true:

 (a) T is idempotent.
 (b) T is invertible.
 (c) rank $T = 3$.
 (d) T is nilpotent.

Solution:
The matrix of T with respect to the standard basis has its jth column equal to the coordinates of $T(e_j)$. Hence $[T]_{\{e_i\}}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

Compute T^2:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

So T^2 is not the zero.
Now compute T^3:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

T^3 is a null matrix
$\therefore$ T is a nilpotent matrix.