4.13.71

AI25BTECH11002 - Ayush Sunil Labhade

October 6, 2025

Question:

If the vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a \neq b \neq c \neq 1)$ are co-planar, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=\cdots$.

Solution:

The vectors are coplanar if they are linearly dependent, i.e., the matrix formed by them has rank less than 3.

Let $\mathbf{u}=a\hat{i}+\hat{j}+\hat{k}$, $\mathbf{v}=\hat{i}+b\hat{j}+\hat{k}$, $\mathbf{w}=\hat{i}+\hat{j}+c\hat{k}$.

Form the matrix

$$\mathbf{M}=\begin{pmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{pmatrix}. \tag{0.1}$$

For rank < 3, perform row reduction to find the condition.

After row reducing the matrix:

$$\begin{pmatrix} 1 & b & 1 \\ 0 & 1-ab & 1-a \\ 0 & 1-b & c-1 \end{pmatrix}. \tag{0.2}$$

Assuming $1 - ab \neq 0$ (consistent with $a \neq b \neq 1$), to eliminate the second column in R_3, compute the multiplier $k = \frac{1-b}{1-ab}$.
Replace R_3 with $R_3 - k\ R_2$:
The second entry becomes $1 - b - k(1 - ab) = 0$ by construction. The third entry becomes $c - 1 - k(1 - a) = 0$ for the row to be zero (ensuring rank 2).
For rank < 3, the third entry must be zero:

$$c - 1 = k(1-a) = \frac{1-b}{1-ab}(1-a). \tag{0.3}$$

Solving for c:

$$c = 1 + \frac{(1-b)(1-a)}{1-ab} = \frac{1-ab+(1-a)(1-b)}{1-ab} = \frac{2-a-b}{1-ab}. \tag{0.4}$$

Now, compute $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$.
First, find $1 - c$:

$$1 - c = 1 - \frac{2 - a - b}{1 - ab} = \frac{-(1-a)(1-b)}{1 - ab}. \tag{0.5}$$

Thus,

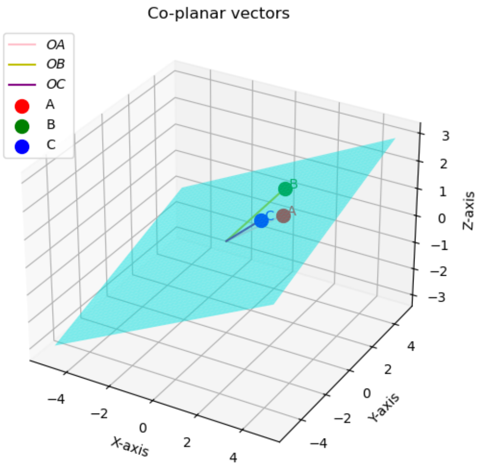
$$\frac{1}{1-c} = \frac{1 - ab}{-(1-a)(1-b)} = -\frac{1 - ab}{(1-a)(1-b)}. \tag{0.6}$$

The sum is:

$$\frac{1}{1-a} + \frac{1}{1-b} - \frac{1 - ab}{(1-a)(1-b)} = \frac{(1-b) + (1-a) - (1-ab)}{(1-a)(1-b)} \tag{0.7}$$

$$= \frac{(1-a)(1-b)}{(1-a)(1-b)} = 1. \tag{0.8}$$

Thus, the value is 1.
Graph:



Figure