Unit-I
Logic and AI

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Sources:

1. Edward A. Bender, Mathematical Methods in Artificial Intelligence, IEEE Computer Society Press Los Alamitos, California, SBN: 9780818672002, 9780818672002
2. Nelson, H , Essential Math for AI, ISBN 9781098107581, O'Reilly Media, January 2023
3. https://www.cs.utexas.edu/~mooney/cs343/

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Syllabus

Unit-I Logic and AI (04 Hours)

What Is Mathematical Logic? Logic and AI, Propositional Logic, Syntax and Semantics of Propositional Logic, Predicate Logic, Syntax and Informal Semantic of Predicate Logic, The Theory of Resolution. Truth versus Proof, Truth, Proof, Resolution and Propositional Calculus, The Resolution Method, Resolution of Horn Clauses, First-Order Predicate Calculus, Skolemization, Unification, Resolution, Soundness and Completeness, Decidability

A story

- You roommate comes home; he/she is completely wet
- You know the following things:
  - Your roommate is wet
  - If your roommate is wet, it is because of rain, sprinklers, or both
  - If your roommate is wet because of sprinklers, the sprinklers must be on
  - If your roommate is wet because of rain, your roommate must not be carrying the umbrella
  - The umbrella is not in the umbrella holder
  - If the umbrella is not in the umbrella holder, either you must be carrying the umbrella, or your roommate must be carrying the umbrella
  - You are not carrying the umbrella
- Can you conclude that the sprinklers are on?
- Can AI conclude that the sprinklers are on?

Knowledge base for the story

- RoommateWet
- RoommateWet => (RoommateWetBecauseOfRain OR RoommateWetBecauseOfSprinklers)
- RoommateWetBecauseOfSprinklers => SprinklersOn
- RoommateWetBecauseOfRain => NOT(RoommateCarryingUmbrella)
- UmbrellaGone
- UmbrellaGone => (YouCarryingUmbrella OR RoommateCarryingUmbrella)
- NOT(YouCarryingUmbrella)

- **This is nothing but the LOGIC!**

Mathematical Logic

- In simple words, logic means to reason. This reasoning can be a legal opinion or even a Mathematical confirmation. The reasoning to solve mathematical problems is mathematical logic.
- In mathematical logic we formalize (formulate in a precise mathematical way) notions used informally by mathematicians such as:
  - Property
  - statement (in a given language)
  - Structure
  - truth (what it means for a given statement to be true in a given structure)
  - proof (from a given set of axioms)
  - algorithm
- Classification of mathematical logic:
  - Set Theory
  - Recursion Theory
  - Model Theory
  - Proof Theory
- Mathematical logical operators:
  - Conjunction or (AND : ^)
  - Disjunction or (OR: v)
  - Negation or (NOT: ~)

Propositional Logic

Propositional Logic (1)

- **Proposition (Statement)** is a declarative statement that is either True or False. It is one of the building blocks of Logic. E.g., Delhi is capital of India. Pune is a city in India. VIT is an autonomous institute.
- Exclamatory sentences, questions, commands, opinions are not statements. E.g., What a beautiful morning!! What is your name ? Mute yourselves. China is responsible for Corona Pandemic.

- **Propositional logic** deals with truth values and the logical connectives ¬ (not), ∧ (and), ∨ (or), etc. Most of the concepts in propositional logic have counterparts in first-order logic.

- Propositional Logic is a formal language. Here are the most fundamental concepts.
  - **Syntax** refers to the formal notation for writing assertions. It also refers to the data structures that represent assertions in a computer. It uses a set of propositional symbols P, Q, R, …and logical connectives, → (implies), ↔ (if and only if) to form atomic formulae. E.g. At the level of syntax, 1 + 2 is a string of three symbols, or a tree with a node labelled + and having two children labelled 1 and 2.

  - **Semantics** expresses the meaning of a formula in terms of mathematical or real-world entities. The semantics of a logical statement will typically be true or false. E.g. While 1 + 2 and 2 + 1 are syntactically distinct, they have the same semantics, namely 3.

  - **Proof theory** concerns ways of proving statements, at least the true ones. Typically, we begin with axioms and arrive at other true statements using inference rules. Formal proofs are typically finite and mechanical: their correctness can be checked without understanding anything about the subject matter.

Propositional Logic (2)

Connective	Symbol	**Example** (p and q are simple statements)
p and q	$\wedge$	$p \wedge q$
p or q	$\vee$	$p \vee q$
not p	$\neg$ or ~	$\neg p$ or ~p
Implies (if then)	$\rightarrow$	$p \rightarrow q$
if and only if (iff)	$\leftrightarrow$	$p \leftrightarrow q$

- The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.
- It uses truth table to get semantics of the logical formulae.

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

  – $A \rightarrow B$ is known as *modus ponens.*

  – By inspecting the table, we can see that $A \rightarrow B$ is equivalent to $\neg A \vee B$ and that $A \leftrightarrow B$ is equivalent to $(A \rightarrow B) \wedge (B \rightarrow A)$. (The latter is also equivalent to $\neg(A \oplus B)$, where $\oplus$ is exclusive-or.)

Propositional Logic (3)

- Syntax of Propositional Logic:
- Let *S* be a set whose elements will be called propositional letters. We'll denote the propositional letters by *p*, *q,* and so on. A propositional logic language *L* with propositional letters *S* is the collection of formulas determined by the following four conditions:
  a) All propositional letters are formulas.
  b) If α is a formula, so is $(\neg \alpha)$.
  c) If α and β are formulas, so are $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \equiv \beta)$.
  d) All formulas are obtained in this manner.
- Some people call formulas well formed formulas, or simply *WFFs*.

- An interpretation, or truth assignment, for a set of formulas is a function from its set of propositional symbols to {1, 0}. An interpretation satisfies a formula if the formula evaluates to 1 under the interpretation.
- In propositional logic, a valid formula is also called a tautology.

Propositional Logic (4)

- Some of the logically equivalent propositions are listed here. They are also called **identities**.
- The symbol $\Leftrightarrow$ shows the logical equivalence.

Fundamental Logical Equivalences

Idempotence[a]
$(P \vee P) \Leftrightarrow P$
$(P \wedge P) \Leftrightarrow P$

Associativity
$[(P \vee Q) \vee R] \Leftrightarrow [P \vee (Q \vee R)]$
$[(P \wedge Q) \wedge R] \Leftrightarrow [P \wedge (Q \wedge R)]$

Commutativity
$(P \vee Q) \Leftrightarrow (Q \vee P)$
$(P \wedge Q) \Leftrightarrow (Q \wedge P)$

Distributivity ($\wedge$ over $\vee$)
$[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$
$[(P \vee Q) \wedge R] \Leftrightarrow [(P \wedge R) \vee (Q \wedge R)]$

Law of the Excluded Middle
$[P \vee (\neg P)] \Leftrightarrow \mathbf{T}$

Law of Double Negation (Involution)
$\neg(\neg P) \Leftrightarrow P$

Law of Simplification
$[(P \wedge Q) \rightarrow P] \Leftrightarrow \mathbf{T}$
$[(P \wedge Q) \rightarrow Q] \Leftrightarrow \mathbf{T}$

Domination
$(P \vee \mathbf{T}) \Leftrightarrow \mathbf{T}$
$(P \wedge \mathbf{F}) \Leftrightarrow \mathbf{F}$

Identity
$(P \vee \mathbf{F}) \Leftrightarrow P$
$(P \wedge \mathbf{T}) \Leftrightarrow P$

De Morgan's Laws
$[\neg(P \vee Q)] \Leftrightarrow [(\neg P) \wedge (\neg Q)]$
$[\neg(P \wedge Q)] \Leftrightarrow [(\neg P) \vee (\neg Q)]$

Distributivity ($\vee$ over $\wedge$)
$[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$
$[(P \wedge Q) \vee R] \Leftrightarrow [(P \vee R) \wedge (Q \vee R)]$

Law of Contradiction
$[P \wedge (\neg P)] \Leftrightarrow \mathbf{F}$

Law of Addition
$[P \rightarrow (P \vee Q)] \Leftrightarrow \mathbf{T}$

[a] An idempotent is an algebraic element for which $x^2 = x$.

Propositional Logic (5)

Q. 1 Prove that $(p \vee p) \wedge (p \rightarrow (q \vee q))$ is equivalent to $\mathbf{p \wedge q}$

<u>Answer:</u> $(p \vee p) \wedge (p \rightarrow (q \vee q)) \Leftrightarrow p \wedge (p \rightarrow q)$

$\Leftrightarrow p \wedge (\neg p \vee q)$

$\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q)$

$\Leftrightarrow F \vee (p \wedge q)$

$\Leftrightarrow (p \wedge q)$

Q.2 $p \rightarrow (q \rightarrow r) \Leftrightarrow \mathbf{(p \wedge q) \rightarrow r}$

<u>Answer</u>: Let us explore the L.H.S first

$p \rightarrow (q \rightarrow r) \Leftrightarrow \neg p \vee (\neg q \vee r)$

$\Leftrightarrow (\neg p \vee \neg q) \vee r$

$\Leftrightarrow \neg(p \wedge q) \vee r$

$\Leftrightarrow (p \wedge q) \rightarrow r$

Propositional Logic (6)

Q. 3 Show that p⇒(p∨q) is a tautology.

Answer:

p	**q**	**p∨q**	**p⇒(p∨q)**
T	T	T	**T**
T	F	T	**T**
F	T	T	**T**
F	F	F	**T**

Q. 4 Show that the statement p ∧∼p is a contradiction.

Answer:

p	**∼p**	**p ∧∼p**
T	F	**F**
F	T	**F**

Propositional Logic (7)

Q. 5 Which of the following is true about the proposition
$p \wedge (\neg p \vee q)$?
a)Tautology
b)Contradiction
c)Logically equivalent to $p \wedge q$
d)None of these

Answer: The proposition can be written as
$(p \wedge \neg p) \vee (p \wedge q) \Leftrightarrow F \vee (p \wedge q)$
$\Leftrightarrow (p \wedge q)$. So the **answer is (c)**

Q. 6 What is the dual value of $(p \wedge q) \vee T$?

Answer: The dual value of any expression is created by replacing $\vee$ with $\wedge$, $\wedge$ with $\vee$, T with F and F with T.
Thus, the dual value of the given expression is **$(p \vee q) \wedge F$**.

Propositional Logic (8)

Q.7 Which of the following proposition is a tautology?

a) $(p \vee q) \rightarrow p$
b) $p \vee (q \rightarrow p)$
c) $p \vee (p \rightarrow q)$
d) $p \rightarrow (p \rightarrow q)$

Answer: There are two ways to solve the problem. The **first** method is by drawing the **Truth-table** for each option given and thereby finding the tautology. This approach sometimes becomes a time and space consuming approach.

The **second** approach is by making the given expression of the form **T $\vee$ (an expression)**. Where T is a known tautology (like $p \vee \neg p$, $q \vee \neg q$ etc..)

Option (a) can be written as $\neg(p \vee q) \vee p$.(There is no tautology).
Option (b) can be written as: $p \vee (\neg q \vee p) \Leftrightarrow (p \vee \neg q) \vee (p \vee p)$
$\Leftrightarrow (p \vee \neg q) \vee p$ (there is no tautology present in this also)
Option (c) can be written as: $p \vee (p \rightarrow q) \Leftrightarrow p \vee (\neg p \vee q)$
$\Leftrightarrow (p \vee \neg p) \vee (p \vee q)$
$\Leftrightarrow T \vee$ (expression)

So, the truth value of the statement is always true. So it is a tautology. **Ans. is (c)**

Propositional Logic (9)

Q. 8 Let X denotes $(p \vee q) \rightarrow r$ and Y denotes $(p \rightarrow r) \vee (q \rightarrow r)$. Which of the following is a tautology?

a) $X \leftrightarrow Y$ b) $X \rightarrow Y$ c) $Y \rightarrow X$ d) $\neg Y \rightarrow X$

Answer: We need to draw truth tables for all the options given.

p	Q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	X	Y	$\neg$ Y	$X \rightarrow Y$	$Y \rightarrow X$	$\neg Y \rightarrow X$
F	F	F	F	T	T	T	T	F	T	T	T
F	F	T	F	T	T	T	T	F	T	T	T
F	T	T	T	T	T	T	T	F	T	T	T
T	T	T	T	T	T	T	T	F	T	T	T
T	F	F	T	F	F	F	T	F	T	F	T
T	T	F	T	F	F	F	F	T	T	T	F
F	T	F	T	T	T	F	T	F	T	F	T
T	F	T	T	T	T	T	T	F	T	T	T

Predicate Logic

Predicate Logic (1)

- Propositional Logic can't say:
  - If X is married to Y, then Y is married to X.
  - If X is west of Y, and Y is west of Z, then X is west of Z.
  - And a million other simple things.
  - Fix it: Extend representation: add predicates. Extend operator(resolution): add unification

- Predicate logic or First-order logic (FOL) extends propositional logic to allow reasoning about the members (such as numbers) of some non-empty universe.

- A predicate is a generalization of a propositional variable. It uses the quantifiers ∀ (for all) and ∃ (there exists). Quantifiers are symbols used with propositional functions. There are two types of quantifiers as shown in the table below.

Name	Symbol	Meaning
Universal Quantifier	$\forall$	" for all"
Existential Quantifier	$\exists$	" there exists at least one"

- Eg: If N is a set of all positive numbers, then the following statements are true.
  - $\forall x \in N, (x + 3 > 2)$.
  - $\exists\, x \in N, (x + 2 < 7)$.

Predicate Logic (2)

- Variables are symbols capable of taking on any constant as value. In fact, a propositional variable is equivalent to a predicate with no arguments, and we shall write p for an atomic formula with predicate name p and zero arguments.
- An atomic formula is a predicate with zero or more arguments. For example, u(X) is an atomic formula with predicate u and one argument, here occupied by the variable X.
- An atomic formula all of whose arguments are constants is called a ground atomic formula.
- A nonground atomic formula can have constants or variables as formula arguments, but at least one argument must be a variable.
- A proposition is a predicate with no arguments, and therefore is a ground atomic formula.
- A literal is either an atomic formula or its negation.
- If there are no variables among the arguments of the atomic formula, then the literal is a ground literal.

- Naming conventions:
  - A variable name will always begin with an upper-case letter. E.g. X, A.
  - Constants are represented either by 1) Character strings beginning with a lower-case letter, 2) Numbers, like 12 or 14.3, or 3) Quoted character strings. E.g. 34, “AIML”, sy.
  - Predicates, like constants, will be represented by character strings beginning with a lower-case letter.
  - There is no possibility that we can confuse a predicate with a constant, since constants can only appear within argument lists in an atomic formula, while predicates cannot appear there.

Predicate Logic (3)

- The Elements of Predicate Logic Language:
- A predicate logic language L consists of the following symbols:
  - an infinite set of variables, denoted by uppercase letters;
  - a set of constants, denoted by lowercase letters, usually *a, b*, etc.;
  - a set of predicates, denoted by lowercase letters, usually *p, q,* etc.;
  - a set of functions, denoted by lowercase letters, usually *f, g*, etc.;
  - the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, and $\equiv$;
  - the quantifiers $\exists$ and $\forall$; and
  - the parentheses) and (.

- The Syntax of Predicate Logic Terms:
- The terms in L are defined recursively as follows:
  a) Every variable and every constant is a term.
  b) If $t_1, \ldots, t_n$ are terms and f is a function that takes n arguments, then $f(t_1, \ldots, t_n)$ is a term.
  c) Every term is obtained in this manner.
- Terms with no variables are called variable-free terms.

Predicate Logic (4)

- The Syntax of Predicate Logic Formulas:
- The formulas in *L* are defined recursively as follows.
  a) If $t_1, \ldots, t_n$ are terms and p is a predicate that takes n arguments, then $p(t_1, \ldots, t_n)$ is a formula, called an atomic formula.
  b) If α is a formula, so is $(\neg \alpha)$.
  c) If α and β are formulas, so are $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \equiv \beta)$.
  d) If V is a variable and α is a formula, then $(\forall V\ \alpha)$ and $(\exists V\ \alpha)$ are formulas.
  e) Every formula is obtained in this manner.

- Semantics for Predicate Logic:
- The syntactic definitions tell us how to construct everything in the language of first-order predicate calculus.
- As in propositional logic, the syntax tells us nothing about what our formulas “mean.”
- To associate meaning with predicate logic formulas, we must know how to interpret them. Unfortunately, this is more complicated than the simple true/false of propositional logic.
- The approach logicians use to define semantics involves the discussion of models.
- Semantics will specify when a formula is true in a recursive manner that parallels the syntactic definition of the formula.

Predicate Logic (5)

- Informal Semantics for Predicate Logic:
  a) If p is a predicate and none of the terms $t_1, \ldots, t_n$ contains variables, then $p(t_1, \ldots, t_n)$ is either true or not according to the interpretation.

  b) If the truths of α and β are known, then the truth of connectives is determined by semantics (using truth tables) of logic formulae $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \equiv \beta)$.

  c) Let V be a variable and α be a formula. If there is some constant c such that replacing every free occurrence of V in α with c gives a true formula, then $(\exists V\, \alpha)$ is true. (The restriction to free V is needed because a quantifier in α might bind some occurrences of a variable that is also called V.) $\exists$ is called an existential quantifier.

  d) Let V be a variable and α be a formula. If, for every constant c, replacing every free occurrence of V in α with c gives a true formula, then $(\forall V\, \alpha)$ is true. $\forall$ is called a universal quantifier.

- A formula is called valid or a tautology if and only if it is true for all possible interpretations.

Predicate Logic (6)

- Informal Semantics for Predicate Logic: (cont..)
- The definition of an informal semantics for predicate logic defines the truth and falsity of formulas only when there are no free occurrences of variables; so, don't try to apply it to a formula where a variable occurs freely. Also, as in propositional logic, the definition is often applied in the reverse direction of the definition of syntax—while syntax builds up, semantics tears down.

- E.g., consider $((\forall X\ p(X)) \rightarrow (\exists X\ p(X)))$ -------------(1)
- To determine the truth of a given predicate logic (1) we must first determine the truth of $(\forall X\ p(X))$ and $(\exists X\ p(X))$.
- There are three relevant possibilities for the truth of *p( )*. Here they are, along with their consequences.
  - *p(c)* is true for all *c*. In this case, both $(\forall X\ p(X))$ and $(\exists X\ p(X))$ are true. Thus (1) is true.
  - *p(c)* is false for all *e*. In this case, both $(\forall X\ p(X))$ and $(\exists X\ p(X))$ are true. Thus (1) is true.
  - *p(c)* is true for some *c* and false for some *c*. In this case, $(\forall X\ p(X))$ is false and $(\exists X\ p(X))$ is true. Thus (1) is true.

Predicate Logic (7)

- Quantifiers Examples: Quantifiers can apply to arbitrary expressions, not just to atomic formulas.
  1. “There exists an individual X such that X gets wet.” The expression: $(\exists X)w(X)$.
  2. “If it rains, then for all individuals X, either X takes an umbrella or X gets wet.”
     The expression: $r \rightarrow (\forall X)$ (u(X) OR w(X))
  3. “Either all individuals X stay dry or, at least one individual Y gets wet.”
     The expression: $((\forall X)$ NOT w(X)) OR $((\exists Y)$w(Y))
- Precedence of Operators in Logical Expressions:
  - NOT (highest), AND, OR, $\rightarrow$, and $\equiv$ (lowest).
  - However, quantifiers have highest precedence of all.
- Order of Quantifiers:
  - A common logical mistake is to confuse the order of quantifiers.
  - E.g., to think that $(\forall X)(\exists Y)$ means the same as $(\exists Y)(\forall X)$, which it does not.
    - E.g., if we informally interpret loves(X, Y) as “X loves Y ,” then $(\forall X)(\exists Y)$loves(X, Y) means “Everybody loves somebody,” that is, for every individual X there is at least one individual Y that X loves.
    - On the other hand, $(\exists Y)(\forall X)$loves(X, Y) means that there is some individual Y who is loved by everyone — a very fortunate Y , if such a person exists.

Predicate Logic (8)

Q.9 Identify the following as constants, variables, ground atomic formulas, or nonground atomic formulas, using the conventions of predicate logic.

a) CS205 b) cs205 c) 205 d) “cs205” e) p(X, x) f) p(3, 4, 5)

<u>Answer</u>: a) CS205 : variable

b) cs205 : constant

c) 205 : constant

d) “cs205” : constant

e) p(X, x) : a nonground atomic formula

f) p(3, 4, 5) : a ground atomic formula

Predicate Logic (9)

Q.10 State TRUE or FALSE for the following statements.

1. p(X, a) is an atomic formula and a literal.
2. p(X, a) is a ground atomic formula.
3. NOT p(X, a) is an atomic formula and a literal.
4. NOT p(X, a) is a ground literal.
5. The expressions p(a, b) is a ground atomic formula and ground literal.
6. NOT p(a, b) is a ground atomic formula and ground literal.

Answer:

1. p(X, a) is an atomic formula and a literal. ------TRUE
2. p(X, a) is a ground atomic formula.------FALSE (It is not ground because of the argument X, which is a variable by our convention.)
3. NOT p(X, a) is an atomic formula and a literal. ------- FALSE (It is a literal, but not an atomic formula.)
4. NOT p(X, a) is a ground literal. -------- FALSE (It is not a ground literal.)
5. The expressions p(a, b) is a ground atomic formula and ground literal.------TRUE
6. NOT p(a, b) is a ground atomic formula and ground literal. -------- FALSE (It is not an atomic formula but is a ground literal.)

Predicate Logic (10)

Q. 11 What is the predicate calculus statement equivalent to the following?
"Every teacher is liked by some student"
a) $\forall x$ [teacher(x) → $\exists y$[student(y) → likes(y,x)]]
b) $\forall x$ [teacher(x) → $\exists y$[student(y) $\wedge$ likes(y,x)]]
c) $\exists y \forall x$[teacher(x) → [student(y) $\wedge$ likes(y,x)]]
d) $\forall x$ [teacher(x) $\wedge$ $\exists y$[student(y) → likes(y,x)]]

Answer: The statement given can also be written as "For all x, if x is a teacher, then there exists a student y who likes x", which can also be represented using the quantifiers as follows.

$\forall x$ [teacher(x) → $\exists y$[student(y) $\wedge$ likes(y,x)]].

It is important to note that implication (→) is almost used in conjunction with the quantifier $\forall$. Mostly the quantifier $\exists$ is associated with $\vee$.

Predicate Logic (11)

Q. 12 p(x): x is a human being.
f(x, y): x is father of y.
m(x, y): x is mother of y.
Write the predicate corresponding to
"x is the father of the mother of y"

Answer: If we try to interpret the predicate using three variables x, y, z we can write "z is a human being and x is the father of z and z is the mother of y". This can be represented as **$(\exists z)(p(z) \wedge f(x,z) \wedge m(z,y))$**.

Predicate Logic (12)

- **Normal Forms:**
- Some important points to remember here are,
  1. An atomic proposition is a proposition containing no logical connectives. E.g.: **p, q, r** etc.
  2. A literal is either an atomic proposition or a negation of an atomic proposition. E.g.: $\neg$**p, q,** $\neg$**r** etc.
  3. A conjunctive clause is a proposition that contains only literals and the connective $\wedge$. E.g.: ($\neg$**p** $\wedge$ **q** $\wedge$ $\neg$**r).**
  4. A disjunctive clause is a proposition that contains only literals and the connective $\vee$. E.g.: ($\neg$**p** $\vee$ **q** $\vee$ $\neg$**r).**

- The problem of finding whether a given statement is a tautology or contradiction in a finite number of steps is called a decision problem. Constructing truth tables is not a practical way.
- We can therefore consider alternate procedure known as reduction to normal forms. Two such normal forms are:
  1. Disjunctive Normal form(DNF)
  2. Conjunctive Normal form(CNF)

Predicate Logic (13)

- **Disjunctive Normal form (DNF):**
  - A proposition is said to be in disjunctive normal form (DNF) if it is a disjunction of conjunctive clauses and literals. E.g.: **$(\neg p \wedge q \wedge \neg r) \vee q \vee (q \wedge r)$.**
  - A proposition is said to be in principal disjunctive normal form if it is a disjunction of conjunctive clauses only. E.g.: **$(\neg p \wedge q \wedge \neg r) \vee (q \wedge r)$.**
  - **Fundamental conjunction:** A conjunction of statement variables and (or) their negations is called as a fundamental conjunction(min term).

    E.g.: p, ~p, $p \wedge q$, $p \wedge$ ~q $\wedge$ ~p are fundamental conjunctions.
  - A statement form which consist of disjunction of fundamental conjunctions is called disjunctive normal form

    E.g.: 1) (p $\wedge$q)V ~q

    2) (~p $\wedge$q)V (p $\wedge$q)V q

    3) (p $\wedge$~q) V (p$\wedge$r)

Predicate Logic (14)

- **Conjunctive Normal form(CNF):**
  - A proposition is said to be in conjunctive normal form (CNF) if it is a conjunction of disjunctive clauses and literals.
    E.g.: $(\neg p \vee q \vee \neg r) \wedge r \wedge (q \vee r)$.
  - A proposition is said to be in principal conjunctive normal form if it is a conjunction of disjunctive clauses only.
    E.g.: $(\neg p \vee q \vee \neg r) \wedge (q \vee r)$.
  - **Fundamental disjunction:-** A disjunction of statement variables and (or) their negations is called as a fundamental disjunction (max term).
    E.g.: p, ~p, p V q , p V ~q V ~p are fundamental disjunction.
  - A statement form which consist of conjunction of fundamental disjunctions is called conjunctive normal form
    E.g.: 1) $p \wedge q$
    2) $\sim p \wedge (p \vee q)$
    3) $(p \vee q \vee r) \wedge (\sim p \vee r)$

Predicate Logic (15)

Q. 13 What is the disjunctive normal form of $p \wedge (p \rightarrow q)$?

<u>Answer</u>: $(p \wedge \neg p) \vee (p \wedge q)$

Q. 14 Obtain the DNF of the form $(p \rightarrow q) \wedge (\sim p \wedge q)$.

<u>Answer:</u> $(p \rightarrow q) \wedge (\sim p \wedge q) = (\sim p \vee q) \wedge (\sim p \wedge q)$ ....(as $p \rightarrow q = \sim p \vee q$)

$= (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$

$= (\sim p \wedge q) \vee (q \wedge \sim p)$------------DNF

Q. 15 Obtain the DNF of $(p \wedge (p \rightarrow q)) \rightarrow q$.

<u>Answer</u>: $(p \wedge (p \rightarrow q)) \rightarrow q = \sim(p \wedge (p \rightarrow q)) \vee q$

$= (\sim p \vee \sim(p \rightarrow q)) \vee q$

$= \sim p \vee \sim(\sim p \vee q) \vee q$

$= \sim p \vee (p \wedge \sim q) \vee q$------------DNF

Predicate Logic (16)

Q. 16 What is the conjunctive normal form of $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$?

<u>Answer</u>: $(\sim p \rightarrow r) \wedge (p \leftrightarrow q) = (\sim p \rightarrow r) \wedge ((p \rightarrow q) \wedge (q \rightarrow p))$

$= (\sim(\sim p) \vee r) \wedge ((\sim p \vee q) \wedge (\sim q \vee p))$

$= (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$ ------------CNF

Q. 17 Obtain the CNF of the form $(p \wedge q) \vee (\sim p \wedge q \wedge r)$.

<u>Answer:</u> $(p \wedge q) \vee (\sim p \wedge q \wedge r) = (p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r))$

$= ((p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)) \wedge ((q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r))$

$= (T \wedge (p \vee q) \wedge (p \vee r)) \wedge ((q \vee \sim p) \wedge (q) \wedge (q \vee r)$

$= (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \wedge (q \vee r)$ -----------CNF

Predicate Logic (17)

- Translating from English to predicate logic:
- E.g., A Lewis Carroll Example:
- The Oxford geometer Charles Dodgson is famous for writing *Alice in Wonderland* and *Through the Looking Glass* under the pen name Lewis Carroll.
- Two years before his death, he published a symbolic logic text containing delightful problems. We'll look at one of his problems in this example.
  1) Colored flowers are always scented;
  2) I dislike flowers that are not grown in the open air;
  3) No flowers grown in the open air are colorless.

  How can we recast these in terms of predicate logic?

Predicate Logic (18)

- A Lewis Carroll Example (cont..)

Answer: Our constants will be flowers and our predicates will be as follows:

- c(X): indicates X is colored,
- d(X): indicates I dislike X,
- g(X): indicates X is grown in the open air,
- s(X): indicates X is scented.

- You should be able to verify that the following are translations of the statements:

1) $\forall X\ (c(X) \rightarrow s(X))$
2) $\forall X\ ((\neg\ g(X)) \rightarrow d(X))$
3) $\neg(\exists X\ (g(X) \wedge (\neg\ c(X))))$

- Other translations are possible, but these are the most direct and also reflect the meaning of the English.
- In this case, one possible conclusion is $\forall X\ ((\neg\ s(X)) \rightarrow d(X))$; that is, I dislike all flowers that are not scented.

Predicate Logic (19)

Q. 18 Translate the following English sentences into predicate logic or first-order-logic.

1. Every kid eats every chocolate.
2. Anyone who eats some chocolate is not a nutrition fanatic.
3. Anyone who eats a watermelon is a nutrition fanatic.
4. Anyone who buys any watermelon either craves it or eats it.
5. Seema is a kid.
6. Seema buys a watermelon.

Answer:

1. $\forall$X, $\forall$Y: ((kid(X) $\wedge$ chocolate(Y)) $\rightarrow$ eats(X,Y))
2. $\forall$X, $\exists$Y: ((chocolate(Y) $\wedge$ eats(X,Y)) $\rightarrow$ $\neg$ nutrition-fanatic(X)) …{Conclusion from 1 & 2:- $\forall$X:(kid(X) $\rightarrow$ $\neg$ nutrition-fanatic(X))}
3. $\forall$X, $\exists$Y : ((watermelon(Y) $\wedge$ eats(X,Y)) $\rightarrow$ nutrition-fanatic(X))
4. $\forall$X, $\forall$Y : ((watermelon(Y) $\wedge$ buys(X,Y)) $\rightarrow$ (carves(X,Y) $\vee$ eats(X, Y)))
5. kid(Seema) ……{Conclusion from 1, 2 & 5:- $\neg$ nutrition-fanatic(Seema)}
6. $\exists$Y : (watermelon(Y) $\rightarrow$ buys(Seema, Y)) ……{Conclusion from 4 & 5:- (craves(Seema, Y) $\vee$ eats(Seema, Y)}

Predicate Logic (20)

Q.19 Translate the following English sentences into predicate logic or first-order-logic.

1. There is a barber in town who shaves all men in town who do not shave themselves.
2. There is a barber in town who shaves only and all men in town who do not shave themselves.

Answer:

1. $\exists x\,(\text{Barber}(x) \wedge \text{InTown}(x) \wedge \forall y\,(\text{Man}(y) \wedge \text{InTown}(y) \wedge \neg\text{Shave}(y,y) \Rightarrow \text{Shave}(x,y)))$
2. $\exists x\,(\text{Barber}(x) \wedge \text{InTown}(x) \wedge \forall y\,(\text{Man}(y) \wedge \text{InTown}(y) \wedge \neg\text{Shave}(y,y) \Leftrightarrow \text{Shave}(x,y)))$

Theory of Resolution

Basic Terminology (1)

- Mathematical Statement/ Proposition/ Statement:
  - It is a statement which is either TRUE or FALSE everywhere and for everyone. A mathematical statement has a specific unambiguous meaning.
  - E.g. (1) **The Earth rotates around the moon.** It is a mathematical statement and has FALSE value.

    (2) **“Math for AI” is an interesting subject.** It is not a mathematical statement as it has different opinions from different persons. Its meaning is ambiguous and subjective. It can be TRUE for few persons and FALSE for others.

    (3) **Read the book.** It is not a mathematical statement as it does not have a TRUE/FALSE value.
- Axiom:
  - It is a mathematical statement that is assumed to be TRUE without any proof.
  - It is also known as a postulate or mathematical result. It can be used as a base for logical reasoning.
  - E.g., 1 + 1 = 2, 10 – 5 = 5, Euclid’s five postulates.

Basic Terminology (2)

- Conjecture:
  - It is an unproved mathematical statement assumed to be either TRUE or FALSE by logical-mathematical reasoning. It does not have sufficient evidence for proof.
  - E.g., The missing number in the series 2, 4, 6, ---, 10, 12, 14 can be predicted as 8 by logical-mathematical reasoning that the given series is of multiple of 2.
- Definition:
  - It precisely and unambiguously describes new concepts with the help of existing ones. It describes the meaning of a mathematical term by presenting all the properties and only those properties that show the truth.
  - E.g., A ray is part of a line that has a starting point but no ending point.
- Undefined Terms:
  - Some terms are implicitly defined by the axioms and are not explicitly specified.
  - E.g., number, variable, point.
- Mathematical System:
  - It is formed by axioms, definitions and undefined terms.
  - E.g., Set theory.

Basic Terminology (3)

- Theorem:
  - It is a significant mathematical result or a proposition that has proof of its truth. A theorem is derived within a mathematical system and is proven by rigorous logical reasoning based on axioms.
  - E.g., The perpendicular segment from a point to a line is the shortest segment from the point to the line.
- Lemma:
  - It is a special type of theorem that is normally helpful in proving another theorem. It is a minor mathematical result generally not too interesting in its way.
  - E.g., Euclid's lemma - If a prime number X divides the product PQ of two integers P and Q, then X must divide at least one of those integers P and Q.
- Corollary:
  - It is a special kind of theorem that directly follows from another theorem. Corollaries are quickly derived from other theorems.
  - E.g., The perpendicular segment from a point to a plane is the shortest segment from the point to the plane.

Truth Vs Proof (1)

- Concepts of proof and truth play an important role in metamathematics, especially in the methodology and the foundations of mathematics.
- Truth is the state or quality of being true to someone or something while proof is an effort, process, or operation designed to establish or discover a fact or truth; an act of testing; a test; a trial.
- We have completely separate definitions of "truth" ($\vDash$) and "provability" ($\vdash$). We would like them to be the same; that is, we should only be able to prove things that are true, and if they are true, we should be able to prove them.
- These two properties are called soundness and completeness.
  - A proof system is sound if everything that is provable is in fact true. In other words, if $\phi_1, \ldots, \phi_n \vdash \psi$ then $\phi_1, \ldots, \phi_n \vDash \psi$.
  - A proof system is complete if everything that is true has a proof. In other words, if $\phi_1, \ldots, \phi_n \vDash \psi$ then $\phi_1, \ldots, \phi_n \vdash \psi$.

- Truth:
  - The classical definition (attributed to Aristotle) says that a statement is true if and only if it agrees with the reality.

Truth Vs Proof (2)

- Proof:
  - It forms the main method of justifying mathematical statements. Only statements that have been proved are treated as belonging to the corpus of mathematical knowledge.
  - A proof of a theorem is a finite sequence of logically valid steps that demonstrate that the premises of a theorem imply its conclusion. Proofs are used to convince the readers of the truth of presented theorems.
  - The proof can be presented by applying either deductive reasoning or inductive reasoning. There are two broad categories of proofs based on applied mathematical reasoning :
    - Deductive Proofs: It establishes a particular inference by referring to general observations. It gives the reasoning from general to particular.
    - Inductive Proofs: It makes a general conclusion by inferring from specific observations. It gives the reasoning from particulars to general.
  - Types of proofs: There are different ways to prove a mathematical statement. Some of them are listed below:
    1) Proof by contradiction
    2) Proof by mathematical induction
    3) Direct proof
    4) Proof by counter example
    5) Proof by contraposition

Truth (1)

- Since truth is defined in the semantics of a language, there is only one notion of truth in a given language. In contrast, there may be many methods of proof because a method of proof is simply a method for manipulating the syntax to obtain valid results.
- The notion of truth for propositional logic and predicate logic are based on truth assignments to the propositional letters and predicates, respectively.

- There are three interrelated truth notions for a formula:
  - Valid: All possible interpretations make the formula true.
  - Satisfiable: Some possible interpretation makes the formula true.
  - Unsatisfiable: No possible interpretations make the formula true.
  - In particular, it follows that α is valid if and only if $(\neg \alpha)$ is unsatisfiable.

- The last two notions are extended to a set S of formulas, but the terminology differs:
  - Consistent: Some interpretation makes all the formulas in S simultaneously true.
  - Inconsistent: No interpretation makes all the formulas in S simultaneously true.

Truth (2)

- In propositional logic, “all possible interpretations” are easy to list:
  - Simply consider all possible assignments of T and F to the propositional letters in the formula. This shows that validity, satisfiability, and unsatisfiability of a formula can all be determined in a finite amount of time.

- In predicate logic, the situation is more complicated:
  - The number of interpretations is infinite. As a result, the semantics of predicate logic does not automatically provide an algorithm for validity the way the semantics of propositional logic does.
  - As per the semantics of predicate logic, it is unclear how many constants, functions, and predicates our interpretation must allow.
  - Our interpretation must allow a number of constants at least equal to the total number of constants and variables in the formula we’re looking at.
  - As far as functions are concerned, we should have all possible functions. Even this is not quite enough because, when we see *f(c),* we may want its value to be different from other constants we’re considering.

Truth (3)

- In proof theory, the turnstile (⊢) is used to denote "provability" or "derivability".
  - It is also referred to as **tee** and is often read as "yields", "proves", "satisfies" or "entails".
  - E.g., if ***P*** is a formal theory and ***Q*** is a particular sentence in the language of the theory then ***P* ⊢ *Q*** *means that **Q** is provable from **P**.*
  - It gives a syntactic consequence.
  - A *modus ponens rule* can be written as a sequent notation as ***P* → *Q***, ***P* ⊢ *Q*.**

- The syntactic consequence of provability should be contrasted to semantic consequence, denoted by the double turnstile (⊨).
  - It is often read as "entails“, "models", "is a semantic/logical consequence of" or "is stronger than".
  - It gives a semantic consequence.
  - One says that *"**P** logically implies **Q**"*, or *"**Q** is a semantic/logical consequence of **P**"*, or ***P* ⊨ *Q***, when all possible valuations in which ***P*** is true, ***Q*** is also true.
- For propositional logic, it may be shown that semantic consequence ⊨ and derivability ⊢ are equivalent to one-another. That is, propositional logic is sound (⊢ implies ⊨) and complete (⊨ implies ⊢).

Truth (4)

- Consequences:
  - Let S be a set, possibly empty, of formulas in the syntax of some logic and let α be another formula in that logic. We say that α is a consequence of S if, whenever all the formulas in S are true, α is also true. This is written as $S \vDash \alpha$.
  - If $S = \varphi$, we usually write $\vDash \alpha$, call α a tautology, and say that α is valid. Note that if S is inconsistent, then $S \vDash \alpha$ for all formulas α in the language because there is no interpretation in which S is true and α is not.
  - If S is a finite set consisting of the formulas $\sigma_1, \sigma_2, \ldots, \sigma_n$ then $S \vDash \alpha$ if and only if $\vDash ((\neg \sigma) \vee \alpha)$ …where σ is the formula $\sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_n$.
    - This is easily shown: By definition, $S \vDash \alpha$ if and only if $\sigma \vDash \alpha$. By definition, the latter is true if and only if, whenever σ is true, α is also true. In other words, either α must be true or σ must be false for every interpretation. This is simply the definition of $\vDash (\alpha \vee (\neg \sigma))$.

Proof (1)

- Proof Method, Soundness, Completeness:
- A proof method is a procedure for manipulating the syntax to deduce conclusions from assumptions. If the set of assumptions is the formulas in S and if the formula α is a conclusion that the proof method deduces for S, we write $S \vdash \alpha$. If $S = \varphi$, we usually write $\vdash \alpha$. Note that $\vdash$ depends on the proof method.

- For a proof method to be useful, it should not prove false formulas, that is, whenever $S \vdash \alpha$, we also have $S \vDash \alpha$. Such a proof method is called sound. The soundness property says that if proof method proves a theorem, then it is a valid formula in the logic.

- It is also helpful if all consequences are provable; that is, whenever $S \vDash \alpha$, we also have $S \vdash \alpha$. Such a proof method is called complete. The completeness property says that if there is a valid formula in the logic then there exists a proof in the proof system.

- When a method of proof is sound and complete, precisely those formulas that are consequences of S are provable. Except for the simplest logics, such proof methods do not exist.

Proof (2)

- In addition to soundness and completeness, certain aspects of proof methods are important from a practical point of view. Given a method of proof, there are at least three questions we should address:
  - Is the method sound? (If not, it is probably useless.)
  - Is the method complete? (If not, it might still be useful.)
  - Can the method be used as the basis for a reasonable computer algorithm?

- A proof method is an algorithm for proving the truth of a given conclusion.
  - It does not address the issue of how to discover interesting things to prove. Discovery is an aspect of learning and is in a more primitive state of development than proof methods.
  - The syntax and semantics of a logic seldom give an operational method for establishing $S \models \alpha$, whereas proof methods give techniques for establishing $S \vdash \alpha$.
  - Thus, it seems reasonable to reserve the term logic for the syntax and semantics and the term calculus for the proof methods. This is seldom done—predicate logic and predicate calculus are used interchangeably.

Proof (3)

- Truth Tables:
  - The manipulation of syntax via truth tables is a method of proof in propositional calculus. This method is the same as the definition of semantics. Hence, the truth table method of proof is complete and sound when the formulas in ***S*** contain only a finite number of propositional letters.
  - Unfortunately, the amount of work required to construct a truth table is exponential in the number of propositional letters since a truth table for ***n*** propositional letters has 2^n rows. As a result, the truth table method is impractical when we have a large number of propositional letters.
  - Drawback: The method cannot be extended to predicate logic, and we need predicate logic for AI.
- Axiomatics:
  - The subject being considered is described by a series of axioms and the methods for manipulating the axioms are described by rules of inference.
  - Let ***S*** be the set of axioms together with hypotheses of a theorem we wish to prove and let ***α*** be the theorem's conclusion.
  - The rules of inference comprise a proof method for establishing $S \vdash \alpha$.
  - Drawback: In order to implement an axiomatic system on computer, guidance concerning the next step is needed.
  - It's not necessary to determine the step completely, because we can always use a search strategy with heuristics.

Proof (4)

- Resolution:
  - It is theorem proving technique that constructs a proof by contradiction; that is, it proves that the formulas are inconsistent.
  - Unification is a key concept in proofs by resolutions. Resolution is a single inference rule which can efficiently operate on the **conjunctive normal form** or **clausal form**.
  - The resolution method is sound and complete for first-order logic (FOL).
  - Informally, in first-order logic, resolution condenses the traditional syllogisms of logical inference down to a single rule. E.g., consider the following example syllogism of term logic:
    - All Greeks are Europeans.
    - Homer is a Greek.
    - Therefore, Homer is a European.
  - More generally:
    - $\forall X\ (p(X) \rightarrow q(X))$
    - $p(A)$
    - Therefore, $q(A)$.

Proof (4)

- Resolution (cont..)
  - There are two features of the resolution method:
    1. It is a method of proof by contradiction.
    2. It requires that formulas be in clausal form.
  - Drawbacks:
    - It can be time-consuming to rewrite formulas in clausal form.
    - There is still the problem of “What should the next step be?” To some extent, we have to expect such problems because establishing validity in propositional logic is NP-complete (here NP stands for **N**ondeterministic **P**olynomial-time problem).

Resolution Rule (1)

- The resolution rule in propositional logic is a single valid inference rule that produces a new clause implied by two clauses containing complementary literals.
- A literal is a propositional variable or the negation of a propositional variable. Two literals are said to be complements if one is the negation of the other (in the following, ¬c is taken to be the complement to c). The resulting clause contains all the literals that do not have complements. Formally:

$$\frac{a_1 \vee a_2 \vee \cdots \vee c, \quad b_1 \vee b_2 \vee \cdots \vee \neg c}{a_1 \vee a_2 \vee \cdots \vee b_1 \vee b_2 \vee \cdots}$$

where

all a_i, b_i, and c are literals,

the dividing line stands for "entails".

The above may also be written as:

$$\frac{(\neg a_1 \wedge \neg a_2 \wedge \cdots) \rightarrow c, \quad c \rightarrow (b_1 \vee b_2 \vee \cdots)}{(\neg a_1 \wedge \neg a_2 \wedge \cdots) \rightarrow (b_1 \vee b_2 \vee \cdots)}$$

Resolution Rule (2)

- A resolution rule can be schematically written as: $$\frac{\Gamma_1 \cup \{\ell\} \quad \Gamma_2 \cup \left\{\bar{\ell}\right\}}{\Gamma_1 \cup \Gamma_2} |\ell|$$

We have the following terminology:

- The clauses $\Gamma_1 \cup \{\ell\}$ and $\Gamma_2 \cup \left\{\bar{\ell}\right\}$ are the inference's premises
- $\Gamma_1 \cup \Gamma_2$ (the resolvent of the premises) is its conclusion.
- The literal ℓ is the left resolved literal,
- The literal $\bar{\ell}$ is the right resolved literal,
- $|\ell|$ is the resolved atom or pivot.

- The clause produced by the resolution rule is called the resolvent of the two input clauses. It is the principle of consensus applied to clauses rather than terms.
- When the two clauses contain more than one pair of complementary literals, the resolution rule can be applied (independently) for each such pair; however, the result is always a tautology.
- Modus ponens can be seen as a special case of resolution (of a one-literal clause and a two-literal clause).

$$\frac{p \to q, \quad p}{q}$$

is equivalent to

$$\frac{\neg p \vee q, \quad p}{q}$$

Resolution of Horn Clauses (1)

- Horn clause is clause (a disjunction of literals) with at most one positive (unnegated) literal and zero or more negative literals.
- E.g., $[\neg p, \neg q, \ldots, \neg t, u]$. It is representation for $(\neg p \lor \neg q \lor \ldots \lor \neg t \lor u)$ or $(p \land q \land \ldots \land t) \supset u$. Its implication is $(p \land q \land \ldots \land t) \rightarrow u$.

- The negative literals correspond to the propositional symbols on the left side of the implication, and the positive literal corresponds to the propositional symbol on the right side of the implication.

Written with implication	**Written with disjunction (Clausal form)**
$A \rightarrow C$	$\neg A \lor C$
$A \land B \rightarrow C$	$\neg A \lor \neg B \lor C$

- Conversely, a disjunction of literals with at most one negated literal is called a dual-Horn clause.

Resolution of Horn Clauses (2)

- Types of Horn Clauses :
  - Definite clause / Strict Horn clause – It has exactly one positive literal.
  - Unit clause – Definite clause with no negative literals. A unit clause without variables is a fact.
  - Goal clause – Horn clause without a positive literal. Also referred to as negative clause or integrity constraint.

Type of Horn clause	Disjunction form	Implication form	Read intuitively as
Definite clause or strict Horn clause	$\neg p \vee \neg q \vee \ldots \vee \neg t \vee u$	$u \leftarrow p \wedge q \wedge \ldots \wedge t$	assume that, if p and q and ... and t all hold, then also u holds
Fact	u	$u \leftarrow true$	assume that u holds
Goal clause	$\neg p \vee \neg q \vee \ldots \vee \neg t$	$false \leftarrow p \wedge q \wedge \ldots \wedge t$	show that p and q and ... and t all hold

- All variables in a clause are implicitly universally quantified with the scope being the entire clause.
  - E.g., $\neg$ human(X) $\vee$ mortal(X)

    stands for: $\forall$X($\neg$ human(X) $\vee$ mortal(X))

    which is logically equivalent to: $\forall$X (human(X) $\rightarrow$ mortal(X))

Resolution of Horn Clauses (3)

- Resolution or Generalized inference rule for clauses have any number of literals:
  - Let's try to generalize modus ponens by allowing it to work on general clauses. This generalized inference rule is called resolution, which was invented in 1965 by John Alan Robinson.
  - The idea behind resolution is that it takes two general clauses, where one of them has some propositional symbol ***p*** and the other clause has its negation **¬p**, and simply takes the disjunction of the two clauses with ***p*** and ***¬p*** removed. Here, $f_1, \ldots, f_n, g_1, \ldots, g_m$ are arbitrary literals.
  - We can verify the soundness of resolution by checking its semantic interpretation. Indeed, the intersection of the models of f and g is a subset of models of $f \vee g$.

$$\neg A \vee B \vee \neg C \vee D \vee \neg E \vee F$$

Example: resolution inference rule

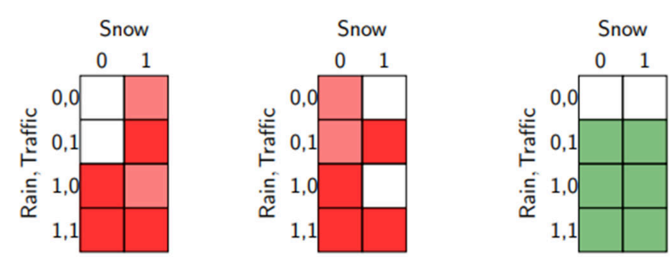
$$\frac{\text{Rain} \vee \text{Snow}, \quad \neg\text{Snow} \vee \text{Traffic}}{\text{Rain} \vee \text{Traffic}}$$

Definition: resolution inference rule

$$\frac{f_1 \vee \cdots \vee f_n \vee p, \quad \neg p \vee g_1 \vee \cdots \vee g_m}{f_1 \vee \cdots \vee f_n \vee g_1 \vee \cdots \vee g_m}$$

$$\frac{\text{Rain} \vee \text{Snow}, \quad \neg\text{Snow} \vee \text{Traffic}}{\text{Rain} \vee \text{Traffic}} \quad \text{(resolution rule)}$$

$$\mathcal{M}(\text{Rain} \vee \text{Snow}) \cap \mathcal{M}(\neg\text{Snow} \vee \text{Traffic}) \subseteq ? \mathcal{M}(\text{Rain} \vee \text{Traffic})$$



Sound!

Resolution of Horn Clauses (4)

- Conversion of proposition to CNF:
- A conjunction of clauses is called a CNF formula, and every formula in propositional logic can be converted into an equivalent CNF.
- Given a CNF formula, we can toss each of its clauses into the knowledge base (KB) using following general rules.
  - First, we try to reduce everything to negation, conjunction, and disjunction.
  - Next, we try to push negation inwards so that they sit on the propositional symbols (forming literals). Note that when negation gets pushed inside, it flips conjunction to disjunction, and vice-versa.
  - Finally, we distribute so that the conjunctions are on the outside, and the disjunctions are on the inside.
- Note that each of these operations preserves the semantics of the logical form.
- Also, when we apply a CNF rewrite rule, we replace the old formula with the new one, so there is no blow-up in the number of formulas.

Resolution of Horn Clauses (4)

- Conversion of proposition to CNF (cont..)

Conversion rules:

- Eliminate $\leftrightarrow$: $\frac{f \leftrightarrow g}{(f \rightarrow g) \wedge (g \rightarrow f)}$
- Eliminate $\rightarrow$: $\frac{f \rightarrow g}{\neg f \vee g}$
- Move $\neg$ inwards: $\frac{\neg(f \wedge g)}{\neg f \vee \neg g}$
- Move $\neg$ inwards: $\frac{\neg(f \vee g)}{\neg f \wedge \neg g}$
- Eliminate double negation: $\frac{\neg\neg f}{f}$
- Distribute $\vee$ over $\wedge$: $\frac{f \vee (g \wedge h)}{(f \vee g) \wedge (f \vee h)}$

Initial formula:

$$(\text{Summer} \rightarrow \text{Snow}) \rightarrow \text{Bizzare}$$

Remove implication ($\rightarrow$):

$$\neg(\neg\text{Summer} \vee \text{Snow}) \vee \text{Bizzare}$$

Push negation ($\neg$) inwards (de Morgan):

$$(\neg\neg\text{Summer} \wedge \neg\text{Snow}) \vee \text{Bizzare}$$

Remove double negation:

$$(\text{Summer} \wedge \neg\text{Snow}) \vee \text{Bizzare}$$

Distribute $\vee$ over $\wedge$:

$$(\text{Summer} \vee \text{Bizzare}) \wedge (\neg\text{Snow} \vee \text{Bizzare})$$

Resolution Algorithm (1)

- After we have converted all the formulas to CNF, we can repeatedly apply the resolution rule, to check $\boldsymbol{KB} \vDash \boldsymbol{f} \leftrightarrow \boldsymbol{KB} \cup \{\neg \boldsymbol{f}\}$ is unsatisfiable.
  - Both testing for entailment and contradiction boil down to checking satisfiability. Resolution can be used to do this very thing. If we ever apply a resolution rule (e.g., to premises A and ¬A) and we derive false (which represents a contradiction), then the set of formulas in the knowledge base is unsatisfiable.
  - If we are unable to derive false, that means the knowledge base is satisfiable because resolution is complete. However, unlike in model checking, we don't actually produce a concrete model that satisfies the KB.

- Resolution-based Inference Algorithm:
  - Add $\neg \boldsymbol{f}$ into ***KB***.
  - Convert all formulas into CNF.
  - Repeatedly apply resolution rule.
  - Return entailment iff derive false.

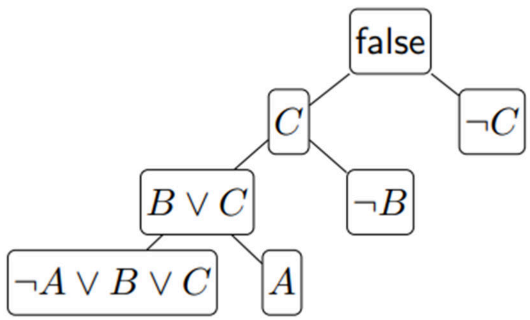
Resolution Algorithm (2)

- Resolution Algorithm Example:
- Here's an example of taking a negation of a formula to a knowledge base, converting it into CNF, and applying resolution. In this case, we derive false, which means that the original knowledge base was unsatisfiable.

Given: After adding ¬*f* into *KB the resulting* KB' = {A → (B ∨ C), A, ¬B, ¬C}

Convert to CNF: KB' = {¬A ∨ B ∨ C, A, ¬B, ¬C}

Repeatedly apply resolution rule:



Conclusion: ***KB*** ⊨ ***f*** (*KB* entails *f*) .

Resolution Algorithm (3)

- Time Complexity:
- We have a sound and complete inference procedure for all of propositional logic (although we didn't prove completeness). But what do we have to pay computationally for this increase?
- If we only have to apply modus ponens, each propositional symbol can only get added once, so with the appropriate algorithm (forward chaining), we can apply all necessary modus ponens rules in linear time.
- But with resolution, we can end up adding clauses with many propositional symbols, and possibly any subset of them! Therefore, this can take exponential time.

Definition: modus ponens inference rule

$$\frac{p_1, \quad \cdots \quad , p_k, \quad (p_1 \wedge \cdots \wedge p_k) \rightarrow q}{q}$$

- Each rule application adds clause with **one** propositional symbol $\Rightarrow$ linear time

Definition: resolution inference rule

$$\frac{f_1 \vee \cdots \vee f_n \vee p, \quad \neg p \vee g_1 \vee \cdots \vee g_m}{f_1 \vee \cdots \vee f_n \vee g_1 \vee \cdots \vee g_m}$$

- Each rule application adds clause with **many** propositional symbols $\Rightarrow$ exponential time

Resolution Algorithm (4)

- To summarize, we can either content ourselves with the limited expressivity of Horn clauses and obtain an efficient inference procedure (via modus ponens).
- If we wanted the expressivity of full propositional logic, then we need to use resolution and thus pay more.

Horn clauses	Any clauses
modus ponens	resolution
linear time	exponential time
less expressive	more expressive

- Two inference procedures based on modus ponens for Horn KBs:
- **Forward chaining:**
  - Idea: Whenever the premises of a rule are satisfied, infer the conclusion. Continue with rules that became satisfied.
- **Backward chaining (goal reduction):**
  - Idea: To prove the fact that appears in the conclusion of a rule prove the premises of the rule. Continue recursively.
- Both procedures are complete for KBs in the Horn form !!!

Forward Chaining (1)

- Forward chaining is also known as a forward deduction or forward reasoning method when using an inference engine. It is a form of reasoning which start with atomic sentences in the knowledge base and applies inference rules (Modus Ponens) in the forward direction to derive all consequences until a goal is reached.
- The Forward-chaining algorithm starts from known facts, triggers all rules whose premises are satisfied, and add their conclusion to the known facts. This process repeats until the problem is solved.
- Inferences cascade to draw deeper and deeper conclusions.
- To avoid looping and duplicated effort, must prevent addition of a sentence to the KB which is the same as one already present.
- Must determine all ways in which a rule (Horn clause) can match existing facts to draw new conclusions.

- Properties of Forward-Chaining:
  - It is a bottom-up approach, as it moves from bottom to top.
  - It is a process of making a conclusion based on known facts or data, by starting from the initial state and reaches the goal state.
  - It is also called as data-driven as we reach to the goal using available data.
  - It is commonly used in the expert system, such as CLIPS, business, and production rule systems.

Forward Chaining (2)

- Example 1: Assume in KB:
  1) parent(X,Y) ∧ male(X) ⇒ father(X,Y)
  2) father(X,Y) ∧ father(X,Z) ⇒ sibling(Y,Z)
- Add to KB: 3) parent(Tom, John)
- Rule 1) tried but can't "fire"
- Add to KB: 4) male(Tom)
- Rule 1: now satisfied and triggered and adds:
  5) father(Tom, John)
- Rule 2: now triggered and adds:
  6) sibling(John, John) {X/Tom, Y/John, Z/John}
- Add to KB: 7) parent(Tom, Fred)
- Rule 1: triggered again and adds:
  8) father(Tom, Fred)
- Rule 2: triggered again and adds:
  9) sibling(Fred, Fred) {X/Tom, Y/Fred, Z/Fred}
- Rule 2: triggered again and adds:
  10) sibling(John, Fred) {X/Tom, Y/John, Z/Fred}
- Rule 2: triggered again and adds:
  11) sibling(Fred, John) {X/Tom, Y/Fred, Z/John}

procedure FORWARD-CHAIN(KB, p)

  if there is a sentence in KB that is a renaming of p **then return**
  Add p to KB
  for each $(p_1 \wedge \ldots \wedge p_n \Rightarrow q)$ **in** KB such that for some i, UNIFY$(p_i, p) = \theta$ succeeds **do**
    FIND-AND-INFER($KB, [p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n], q, \theta$)
  end

---

procedure FIND-AND-INFER($KB, premises, conclusion, \theta$)

  if $premises = [\,]$ **then**
    FORWARD-CHAIN(KB, SUBST$(\theta, conclusion)$)
  else for each p' **in** KB such that UNIFY$(p'$, SUBST$(\theta$, FIRST$(premises))) = \theta_2$ **do**
    FIND-AND-INFER(KB, REST$(premises), conclusion$, COMPOSE(θ, θ_2))
  end

Forward Chaining (3)

- Example2:**"As per the law, it is a crime for an American to sell weapons to hostile nations. Country A, an enemy of America, has some missiles, and all the missiles were sold to it by Robert, who is an American citizen."** Prove that **"Robert is criminal."**
- **Solution:**
- To solve the above problem, first, we will convert all the above facts into first-order definite clauses, and then we will use a forward-chaining algorithm to reach the goal.

A) Facts Conversion into FOL:

- It is a crime for an American to sell weapons to hostile nations. (Let's say P, Q, and R are variables): American (P) ∧ weapon(Q) ∧ sells (P, Q, R) ∧ hostile(R) → criminal(P) ...(1)
- Country A has some missiles: ∃P missile(P) ∧ owns(A, P) . It can be written in two definite clauses by using Existential Instantiation, introducing new constant t1: owns(A, t1) ......(2), missile(t1) ....(3)
- All of the missiles were sold to country A by Robert. ∃P missile(P) ∧ owns(A, P) → sells (Robert, P, A) ......(4)
- Missiles are weapons: missile(P) → weapon(P) ....(5)
- Enemy of America is known as hostile: enemy(A, America) →hostile(A) ....(6)
- Country A is an enemy of America: enemy (A, America) ....(7)
- Robert is American: American(Robert) ....(8)

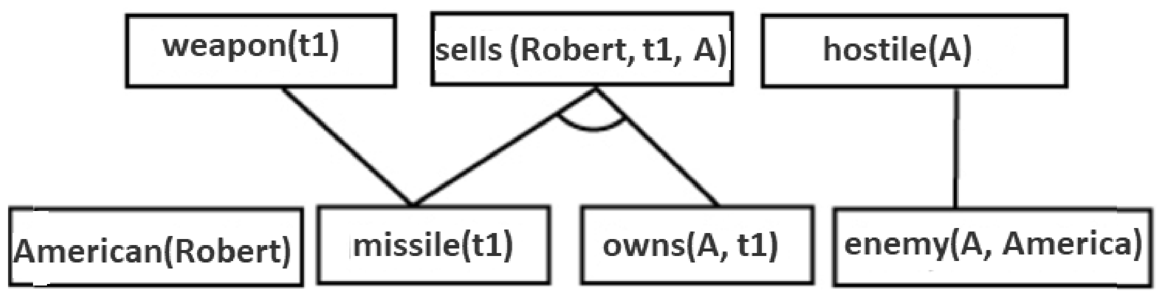
Forward Chaining (4)

- Example2 Solution (cont..):

B) Forward Chaining Proof:

- **Step-1:** In the first step we will start with the known facts and will choose the sentences which do not have implications, such as: **American(Robert), enemy(A, America), owns(A, t1), and missile(t1)**. All these facts will be represented as below.

- **Step-2:** At the second step, we will see those facts which infer from available facts and with satisfied premises.
  - Rule-(1) does not satisfy premises, so it will not be added in the first iteration.
  - Rule-(2) and (3) are already added.
  - Rule-(4) satisfy with the substitution {p/t1}, so sells (Robert, t1, A) is added, which infers from the conjunction of Rule (2) and (3).

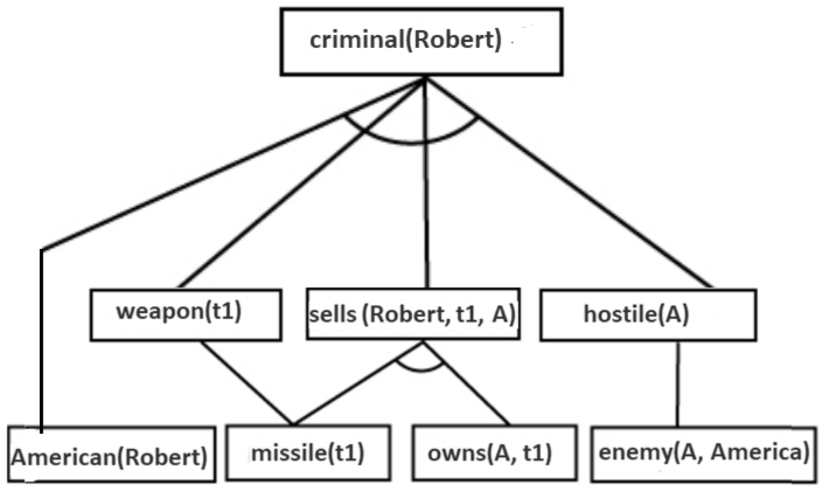


Forward Chaining (4)

- Example2 Solution (cont..):

B) Forward Chaining Proof:

- **Step-3:** At step-3, as we can check Rule-(1) is satisfied with the substitution {P/Robert, Q/t1, R/A}, so we can add criminal(Robert) which infers all the available facts. And hence we reached our goal statement.



- **Hence it is proved that Robert is Criminal using forward chaining approach.**

Backward Chaining (1)

- Backward chaining is also known as a backward deduction or backward reasoning method when using an inference engine. A backward chaining algorithm is a form of reasoning, which starts with the goal and works backward, chaining through rules to find known facts that support the goal.
- Given a conjunction of queries, first get all possible answers to the first conjunct and then for each resulting substitution try to prove all of the remaining conjuncts.
- Assume variables in rules are renamed (standardized apart) before each use of a rule.

- Properties of Forward-Chaining:
  - It is known as a top-down approach.
  - It is based on modus ponens inference rule.
  - In backward chaining, the goal is broken into sub-goal or sub-goals to prove the facts true.
  - It is called a goal-driven approach, as a list of goals decides which rules are selected and used.
  - It is used in game theory, automated theorem proving tools, inference engines, proof assistants, and various AI applications.
  - The backward-chaining method mostly used a depth-first search strategy for proof.

Backward Chaining (2)

- Example 1:
- KB: 1) parent(X,Y) ∧ male(X) ⇒ father(X,Y)

  2) father(X,Y) ∧ father(X,Z) ⇒ sibling(Y,Z)

  3) parent(Tom, John)

  4) male(Tom)

  7) parent(Tom, Fred)

- Query: parent(Tom, X)

  Answers: ({X/John}, {X/Fred})

- Query: father(Tom, S)

  Subgoal: Parent(Tom, S) ∧ male(Tom)

  {S/John}

  Subgoal: Male(Tom)

  Answer: {S/John}

  {S/Fred}

  Subgoal: Male(Tom)

  Answer: {S/Fred}

  Answers: ({S/John}, {S/Fred})

function BACK-CHAIN(*KB*, *q*) **returns** a set of substitutions

  BACK-CHAIN-LIST(*KB*, [*q*], { })

---

function BACK-CHAIN-LIST(*KB*, *qlist*, θ) **returns** a set of substitutions
  inputs: *KB*, a knowledge base
    qlist, a list of conjuncts forming a query (θ already applied)
    θ, the current substitution
  static: *answers*, a set of substitutions, initially empty

  if *qlist* is empty **then return** $\{\theta\}$
  $q \leftarrow$ FIRST(*qlist*)
    for each q'_i **in** *KB* such that $\theta_i \leftarrow$ UNIFY(q, q'_i) succeeds **do**
      Add COMPOSE(θ, θ_i) to *answers*
    end
    for each sentence ($p_1 \wedge \ldots \wedge p_n \Rightarrow q'_i$) **in** *KB* such that $\theta_i \leftarrow$ UNIFY(q, q'_i) succeeds **do**
      answers $\leftarrow$ BACK-CHAIN-LIST(*KB*, SUBST(θ_i, [$p_1 \ldots p_n$]), COMPOSE(θ, θ_i)) $\cup$ *answers*
    end
  return the union of BACK-CHAIN-LIST(*KB*, REST(*qlist*), θ) for each $\theta \in$ *answers*

Backward Chaining (3)

(Example1 Cont..)

- Query: father(F, S)

  Subgoal: parent(F, S) ∧ male(F)

  {F/Tom, S/John}

  Subgoal: male(Tom)

  Answer: {F/Tom, S/John}

  {F/Tom, S/Fred}

  Subgoal: male(Tom)

  Answer: {F/Tom, S/Fred}

  Answers: ({F/Tom, S/John}, {F/Tom, S/Fred})

(Example1 Cont..)

- Query: sibling(A, B)

  Subgoal: father(F, A) ∧ father(F, B)

  {F/Tom, A/John}

  Subgoal: father(Tom, B)

  {B/John}

  Answer: {F/Tom, A/John, B/John}

  {B/Fred}

  Answer: {F/Tom, A/John, B/Fred}

  {F/Tom, A/Fred}

  Subgoal: father(Tom, B)

  {B/John}

  Answer: {F/Tom, A/Fred, B/John}

  {B/Fred}

  Answer: {F/Tom, A/Fred, B/Fred}

  Answers: ({F/Tom, A/John, B/John},{F/Tom, A/John, B/Fred}

  {F/Tom, A/Fred, B/John}, {F/Tom, A/Fred, B/Fred})

Backward Chaining (4)

- Example2:**"As per the law, it is a crime for an American to sell weapons to hostile nations. Country A, an enemy of America, has some missiles, and all the missiles were sold to it by Robert, who is an American citizen."** Prove that **"Robert is criminal."**
- **Solution:**
- To solve the above problem, by backward-chaining algorithm we will write all the rules in FOL.

A) Facts Conversion into FOL:

- It is a crime for an American to sell weapons to hostile nations. (Let's say P, Q, and R are variables): American (P) ∧ weapon(Q) ∧ sells (P, Q, R) ∧ hostile(R) → criminal(P) ...(1)
- Country A has some missiles: ∃P missile(P) ∧ owns(A, P) . It can be written in two definite clauses by using Existential Instantiation, introducing new constant t1: owns(A, t1) ......(2), missile(t1) ....(3)
- All of the missiles were sold to country A by Robert. ∃P missile(P) ∧ owns(A, P) → sells (Robert, P, A) ......(4)
- Missiles are weapons: missile(P) → weapon(P) ....(5)
- Enemy of America is known as hostile: enemy(A, America) →hostile(A) ....(6)
- Country A is an enemy of America: enemy (A, America) ....(7)
- Robert is American: American(Robert) ....(8)

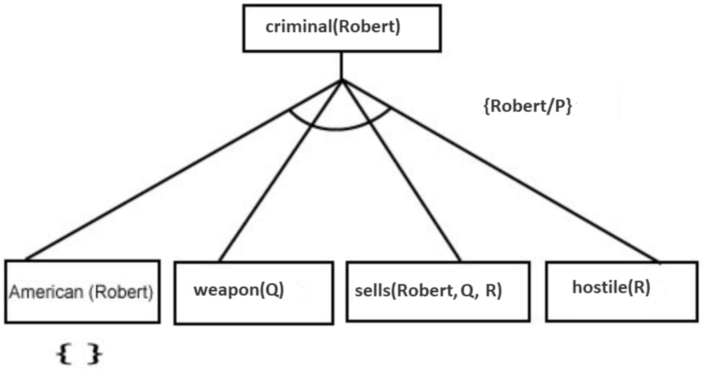
Backward Chaining (5)

- Example2 Solution (cont..):

B) Backward Chaining Proof:

- In Backward chaining, we will start with our goal predicate, which is criminal(Robert), and then infer further rules.
- **Step-1:** At the first step, we will take the goal fact. And from the goal fact, we will infer other facts, and at last, we will prove those facts true. So, our goal fact is "Robert is Criminal," so following is the predicate of it.

- **Step-2:** At the second step, we will infer other facts from goal fact which satisfies the rules. So, as we can see in Rule-1, the goal predicate criminal (Robert) is present with substitution {Robert/P}. So, we will add all the conjunctive facts below the first level and will replace P with Robert. Here we can see American(Robert) is a fact, so it is proved here.

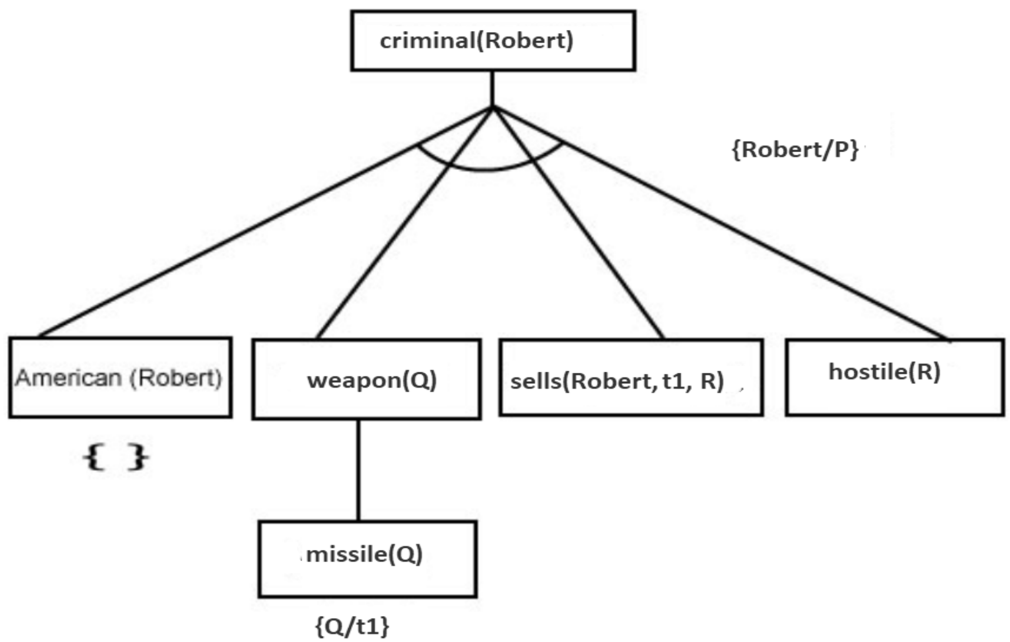


Backward Chaining (6)

- Example2 Solution (cont..):

B) Backward Chaining Proof:

- **Step-3:** At step-3, we will extract further fact missile(Q) which infer from weapon(Q), as it satisfies Rule-(5). weapon(Q) is also true with the substitution of a constant t1 at Q.

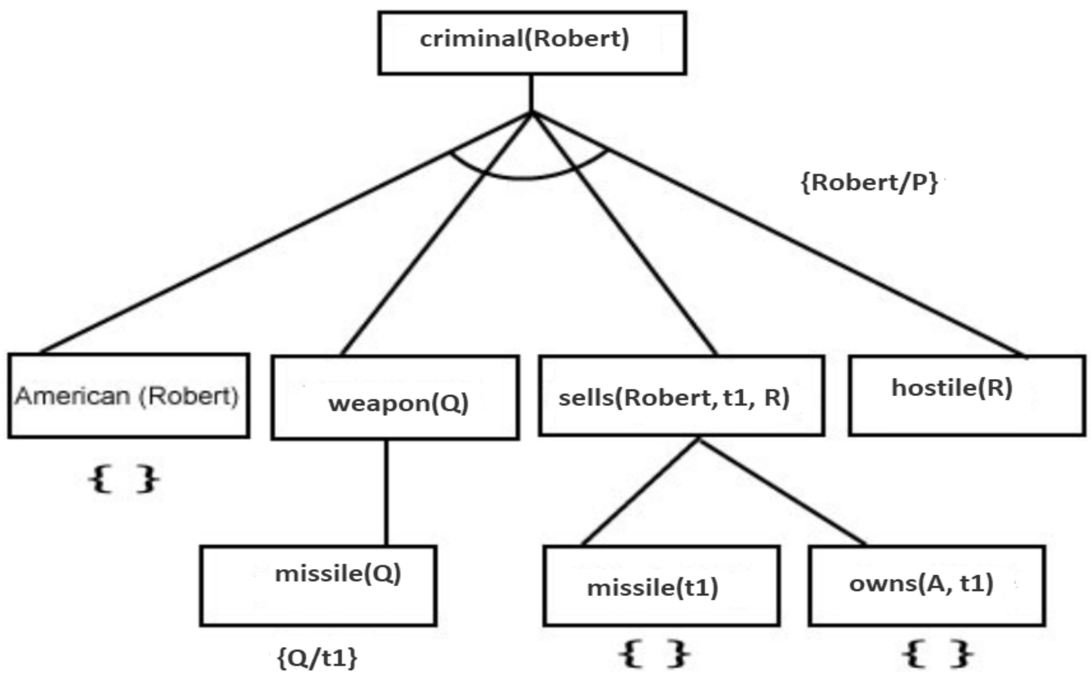


Backward Chaining (7)

- Example2 Solution (cont..):

B) Backward Chaining Proof:

- **Step-4:** At step-4, we can infer facts missile(t1) and owns(A, t1) form sells(Robert, t1, R) which satisfies the Rule- 4, with the substitution of A in place of R. So, these two statements are proved here.

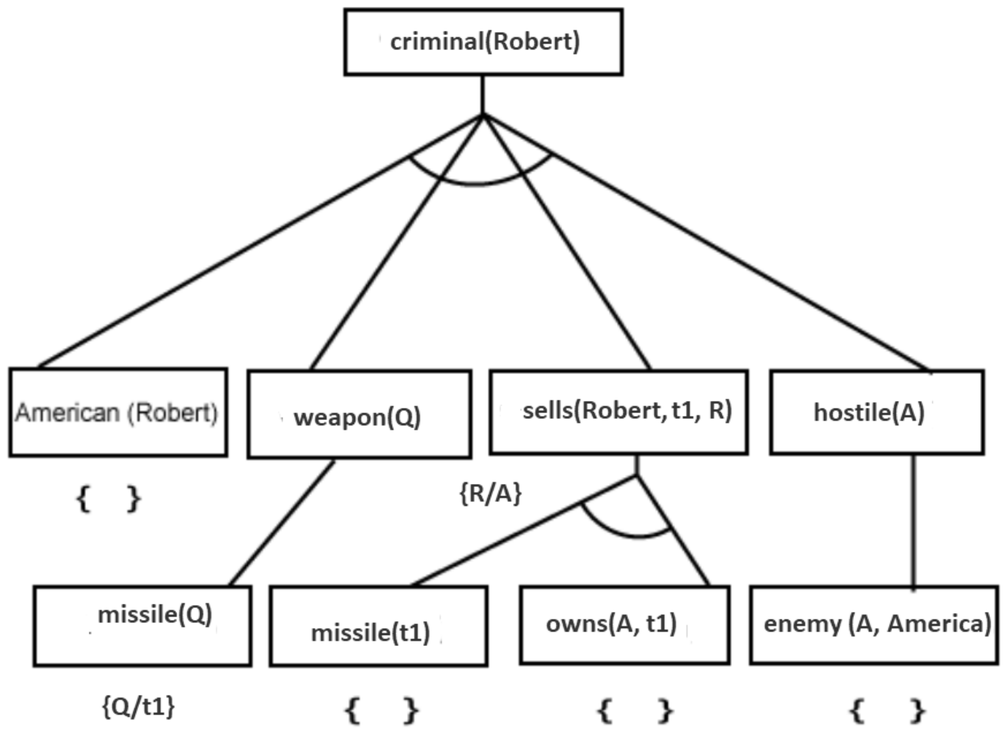


Backward Chaining (8)

- Example2 Solution (cont..):

B) Backward Chaining Proof:

- **Step-5:** At step-5, we can infer the fact enemy(A, America) from hostile(A) which satisfies Rule- 6. And hence all the statements are proved true using backward chaining.



First-Order Predicate Calculus

First-Order Predicate Calculus (1)

- Quantifiers:
- Let α be a formula in which any occurrences of X and Y are free and let β be a formula with no free X. Let * be either v or $\wedge$. Then the following formulas are tautologies:

  $(\forall X(\forall Y\ \alpha)) \equiv (\forall Y(\forall X\ \alpha))$ and $(\exists X(\exists Y\ \alpha)) \equiv (\exists Y/(\exists X\ \alpha))$............(1)

  $(\neg(\forall X\ \alpha)) \equiv (\exists X\ \neg\alpha)$ and $(\neg(\exists X\ \alpha)) \equiv (\forall X\ \neg\alpha)$.......................(2)

  $((\forall X\ \alpha) * \beta) \equiv (\forall X(\alpha * \beta)$ and $((\exists X\ \alpha) * \beta) \equiv (\exists X\ (\alpha * \beta))$.........(3)

- **Proof:** Abuse notation and write α(X) to indicate the free occurrences of X in α. Use $\leftrightarrow$ to stand for the phrase "if and only if." Let's prove (2) here and leave the rest as an exercise.
- By the definition of the semantics,

  $\neg(\forall X\ \alpha(X))$ is true $\leftrightarrow$ $\forall X\ \alpha(X)$ is false

  $\leftrightarrow$ there is some constant c such that $\alpha(c)$ is false

  $\leftrightarrow$ $(\exists X \neg\ \alpha(X))$ is true.

- The right side of (2) follows similarly.

First-Order Predicate Calculus (2)

- To carry out a resolution proof for FOL we must rewrite a formula so that all the quantifiers are universal and on the outside while the formula inside the quantifiers is in clausal form. This can be done as follows:
  - Adapt the conversion rules to produce a clausal form containing quantifiers.

    Conversion rules:

    - Eliminate $\leftrightarrow$: $\frac{f \leftrightarrow g}{(f \rightarrow g) \wedge (g \rightarrow f)}$
    - Eliminate $\rightarrow$: $\frac{f \rightarrow g}{\neg f \vee g}$
    - Move $\neg$ inwards: $\frac{\neg (f \wedge g)}{\neg f \vee \neg g}$
    - Move $\neg$ inwards: $\frac{\neg (f \vee g)}{\neg f \wedge \neg g}$
    - Eliminate double negation: $\frac{\neg\neg f}{f}$
    - Distribute $\vee$ over $\wedge$: $\frac{f \vee (g \wedge h)}{(f \vee g) \wedge (f \vee h)}$
  - Use eq.(2) to move negation inward through quantifiers. …{$(\neg(\forall X\ \alpha)) \equiv (\exists X\ \neg\alpha)$ and $(\neg(\exists X\ \alpha)) \equiv (\forall X\ \neg\alpha)$…(2)}
  - Assign unique names to all quantified (=bound) variables and then use eq.(3) to move the quantifiers to the left side of the formula. The result is said to be in *prenex form*. …{$((\forall X\ \alpha) * \beta) \equiv (\forall X(\alpha * \beta)$ and $((\exists X\ \alpha) * \beta) \equiv (\exists X\ (\alpha * \beta))$…(3)}
  - Use "Skolemization" to eliminate existential quantifiers.
- Once this normal form has been achieved, resolution can begin; however, it's complicated by the need for "unification." The unification must be done so as not to impose any equalities that are not absolutely required—the "most general unification."

Unification

Unification (1)

- To apply inference rules inference system must be able to determine when two expressions are the same or match.
- In propositional calculus, this is trivial: two expressions match if and only if they are syntactically identical.
- In predicate calculus, the process of matching two sentences is complicated by the existence of variables in the expressions.
- Universal instantiation allows universally quantified variables to be replaced by terms from the domain.
- Unification is a process of making two different logical atomic expressions identical by finding a substitution. Unification depends on the substitution process.
- It takes two literals as input and makes them identical using substitution.
- Let p and q be two atomic sentences and σ be a unifier such that, $p\sigma = q\sigma$, then it can be expressed as UNIFY(p, q).
- In order to match antecedents to existing literals in the KB, need a pattern matching routine.

Unification (2)

- The UNIFY algorithm is used for unification, which takes two atomic sentences and returns a unifier for those sentences (If any exist).
- Unification is a key component of all first-order inference algorithms. It returns fail if the expressions do not match with each other.
- The substitution variables are called Most General Unifier or MGU.
- Examples:
  1. UNIFY(Parent(x,y), Parent(Tom, John)) = {x/Tom, y/John}
  2. UNIFY(Parent(Tom,x), Parent(Tom, John)) = {x/John})
  3. UNIFY(Likes(x,y), Likes(z,FOL)) = {x/z, y/FOL}
  4. UNIFY(Likes(Tom,y), Likes(z,FOL)) = {z/Tom, y/FOL}
  5. UNIFY(Likes(Tom,y), Likes(y,FOL)) = fail
  6. UNIFY(Likes(Tom,Tom), Likes(x,x)) = {x/Tom}
  7. UNIFY(Likes(Tom,Fred), Likes(x,x)) = fail

Unification (3)

Q. 20 Find the MGU for Unify {King(x), King(John)}

Answer:

- Let p = King(x), q = King(John),
- Substitution σ = {John/x} is a unifier for these atoms and applying this substitution, and both expressions will be identical.

Q. 21 Find the MGU for Unify { P(x,y), P(a,f(z)}

Answer:

- n this example, we need to make both above statements identical to each other. For this, we will perform the substitution: 1) P(x, y), 2) P(a, f(z)).
- Substitute x with a, and y with f(z) in the first expression, and it will be represented as a/x and f(z)/y.
- With both the substitutions, the first expression will be identical to the second expression and the substitution set (MGU) will be: [a/x, f(z)/y].

Unification (4)

- Conditions for Unification:
  - Predicate symbol must be same, atoms or expression with different predicate symbol can never be unified.
  - Number of arguments in both expressions must be identical.
  - Unification will fail if there are two similar variables present in the same expression.
  - All variables should be Universally Quantified. This allows full freedom in computing substitutions.
  - Existentially quantified variables may be eliminated from sentences in the database by replacing them with the constants that make the sentence true.

- Implementation of the Algorithm:

Step 1: Initialize the substitution set to be empty.

Step 2: Recursively unify atomic sentences:

a) Check for Identical expression match.

b) If one expression is a variable v_i and the other is a term t_i which does not contain variable v_i, then:

   i. Substitute t_i / v_i in the existing substitutions.

   ii. Add t_i / v_i to the substitution setlist.

   iii. If both the expressions are functions, then function name must be similar, and the number of arguments must be the same in both the expression.

Unification (5)

- Exact variable names used in sentences in the KB should not matter.
  - But if Likes(x, FOL) is a formula in the KB, it does not unify with Likes(John, x) but does unify with Likes(John, y).
- To avoid such conflicts, one can standardize apart one of the arguments to UNIFY to make its variables unique by renaming them.
  - Likes(x, FOL) → Likes(x1, FOL)
  - UNIFY(Likes(John, x), Likes(x1,FOL)) = {x1/John, x/FOL}
- There are many possible unifiers for some atomic sentences.
  - UNIFY(Likes(x,y), Likes(z,FOL)) = {x/z, y/FOL}
  - {x/John, z/John, y/FOL}
  - {x/Fred, z/Fred, y/FOL}
- UNIFY should return the most general unifier which makes the least commitment to variable values.
- Inference using unification:

$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

  - For universally quantified variables, find MGU {x/A} and proceed as in propositional resolution.

Skolemization

Skolemization (1)

- Conversion of sentences FOL to CNF requires skolemization.
- Skolemization: remove existential quantifiers by introducing new function symbols.
- Special case: introducing constants (trivial functions: no previous universal quantifier). Variables bound by existential quantifiers which are not inside the scope of universal quantifiers can simply be replaced by a Skolem term or constants. E.g. $\exists x\,[x < 3]$ can be changed to $c < 3$, with c a suitable constant.
- When the existential quantifier is inside a universal quantifier, the bound variable must be replaced by a Skolem function of the variables bound by universal quantifiers. Replace any existentially quantified variable $\exists x$ that is in the scope of universally quantified variables $\forall y_1 \ldots \forall y_n$ with a new function $F(y_1,\ldots,y_n)$ (a Skolem function). Thus $\forall x\,[x=0 \lor \exists y\,[x = y+1]]$ becomes $\forall x[x=0 \lor x= f(x)+1]$.
- For each existentially quantified variable introduce a n-place function where n is the number of previously appearing universal quantifiers.
- In general, the functions and constants symbols are new ones added to the language for the purpose of satisfying these formulas, and are often denoted by the formula they realize, for instance $c_{\exists x\varphi(x)}$.

Skolemization (2)

- $\forall x\ \forall y\ \exists w\ \forall z\ Q(x, y, w, z, G(w, x))$ is equivalent to $\forall x\ \forall y\ \forall z\ Q(x, y, P(x, y), z, G(P(x, y), x))$ where P is the Skolem function for w.

- Examples:
  - Every philosopher writes at least one book.
    $\forall x[Philo(x) \rightarrow \exists y[Book(y) \wedge Write(x, y)]]$
  - Eliminate Implication:
    $\forall x[\neg Philo(x) \vee \exists y[Book(y) \wedge Write(x, y)]]$
  - Skolemize: substitute y by $g(x)$
    $\forall x[\neg Philo(x) \vee [Book(g(x)) \wedge Write(x, g(x))]]$

  - All students of a philosopher read one of their teacher's books.
    $\forall x \forall y[Philo(x) \wedge StudentOf(y, x) \rightarrow \exists z[Book(z) \wedge Write(x, z) \wedge Read(y, z)]]$
  - Eliminate Implication:
    $\forall x \forall y[\neg Philo(x) \vee \neg StudentOf(y, x) \vee \exists z[Book(z) \wedge Write(x, z) \wedge Read(y, z)]]$
  - Skolemize: substitute z by $h(x, y)$
    $\forall x \forall y[\neg Philo(x) \vee \neg StudentOf(y, x) \vee [Book(h(x, y)) \wedge Write(x, h(x, y)) \wedge Read(y, h(x, y))]]$

Soundness and Completeness

Soundness and Completeness

- Given a certain proof system, the soundness theorem says that only valid sentences are provable in it.
- The completeness theorem says that every valid sentence is provable in it.

- Together, soundness and completeness theorems tell us that provability in that proof system exactly coincides with validity. The provable sentences and the valid ones are exactly the same sentences.

- What soundness shows is that ideal logical reasoning does not lead us astray. When we prove something logically, the actual meaning of the logical words agrees with what we've proved.

- Meanwhile, completeness shows that ideal logical reasoning allows us to prove everything that really is true by the meaning of logical words alone.

- In other words, there's nothing reasoning misses out on. Together, they tell us that, as far as logic goes, proof and meaning are two paths to the same destination—the purely logical truths.

Decidability

Decidability

- A logic is decidable if there exists a method such that, for each formula α, the method will tell us whether or not α is satisfiable.
- Since validity of α is equivalent to nonsatisfiability of $\neg\alpha$, decidability is equivalent to being able to determine if a formula is valid.
- If a proof method is sound and complete, it will be able to prove the validity of any valid formula and will never "prove" the validity of an invalid formula.
- However, there is no guarantee that the proof method will be able to tell that a formula is not valid. How can this be?
  - Certainly, if the proof method terminates, we have an answer one way or the other.
  - But it's possible that, for some invalid formula, the proof method will not terminate.

Decidability

- Semidecidability of FOL:
- FOL is not decidable; that is, it's impossible to construct an algorithm that will determine whether or not a given formula in FOL is valid. (Or, equivalently, it's impossible to construct an algorithm to determine satisfiability.) However, there exist proof methods for FOL that are sound and complete. This situation is described by saying that FOL is semidecidable.

- Why do we care about decidability?
- When using FOL, it might be useful to know that a formula is not valid. In some forms of "nonmonotonic" reasoning, it's essential.
- As a result, the nonexistence of such algorithms in FOL implies the nonexistence of sound and complete proof methods for some nonmonotonic logics.

Thank you!