Analysis of the Benveniste & Goursat Blind Adaptive Algorithm in Comparison to the Statistical Wiener Filter-based Equalizer

EE4140 Course Project

Ayush Jamdar

Department of Electrical Engineering
Indian Institute of Technology Madras

December 12, 2023

Table of Contents

Problem Statement

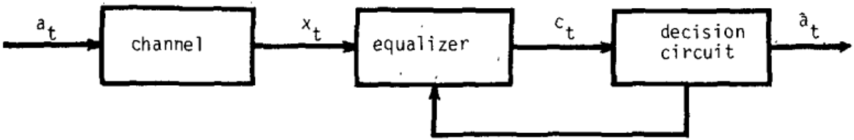

Figure: A single carrier amplitude modulation system.

- a_t is the amplitude modulated data signal that travels through a noisy ISI channel.
- The receiver captures x_t, the distorted signal.
- We desire an equalizer that will recover the message signal $\hat{a}_t$ from x_t.

Mathematically Speaking...

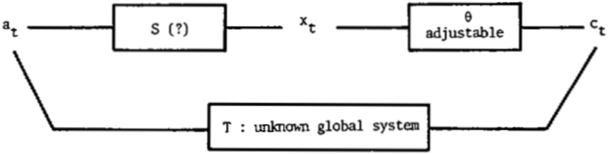

Figure: The channel FIR response S is unknown. θ is the equalizer.

$$x_t = \sum_{k \in Z} s_k a_{t-k}, \quad S = (s_k)_{k \in Z} \tag{1}$$

$$c_t = \sum_{k \in Z} h_k x_{t-k}, \quad \theta = (h_k)_{k \in Z} \tag{2}$$

$$T = S * \theta \qquad c_i = \sum_{k \in Z} t_k a_{i-k}$$

Mathematically Speaking...

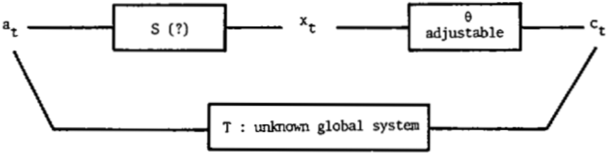

Figure: The channel FIR response S is unknown. θ is the equalizer.

- We aim to construct θ such that the global system $T = S * \theta$ is $\pm$ identity.
- Depending on where unity appears in T, there will be a decoding delay.

The Approach

The following two approaches shall be considered in the sequel –

1. Leverage the distribution of the message signal and noise to build a Statistical Wiener Filter. This requires the channel response S be known.
2. Recover the transmitted message using only the received x_t without any preamble for identification of the unknown channel, hence the *blind* equalizer.

It shall be assumed that the characteristics of the channel are time-invariant.

Statistical Wiener Filter | Linear FIR Equalizer

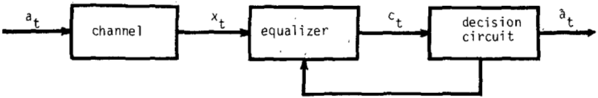

Figure: A single carrier amplitude modulation system.

- Here we compare $\hat{a}_t$ with a delayed input $a_{t-\Delta}$ to calculate decoding error e_t.
- We minimize the MSE $\mathbb{E}[|e_t|^2]$.
- The message signal is characterized by variance σ_a^2 and white Gaussian noise σ_v^2.

Statistical Wiener Filter

$$R \times \theta = P \tag{3}$$

where R denotes the autocorrelation matrix and P, cross-correlation.

$$R = \mathbb{E}[x_t \cdot x_t^T] \tag{4}$$

$$P = \mathbb{E}[x_t \cdot a_{t-\Delta}] \tag{5}$$

- We find R and P using the known statistics of the model, without explicit averaging, hence the *statistical* filter.
- However, this calculation requires knowledge of the channel taps.

Statistical Wiener Filter - Given System

$$G(z) = \frac{1}{C}\begin{pmatrix} (0.9 + j0.8) + (0.95 - j0.6)z^{-1} + (-0.4 + j0.5)z^{-2} + \\ (0.15 + j0.25)z^{-3} + (-0.2 - 0.1)z^{-4} + (0.1 - j0.05)z^{-5} \end{pmatrix}$$

- 4-QAM / QPSK constallation
- Average symbol energy $= 1$
- 6-tap channel
- Wiener Filter order $= 20$
- Decoding delay $\Delta = 8$
- SNR $=$ 10dB, 20 dB

Statistical Wiener Filter - Results

- Wiener Filter order = 20
- Decoding delay $\Delta = 8$
- SNR = 10dB $J_{min} = 0.21$ $SER = 12.5\%$
- SNR = 20dB $J_{min} = 0.06$ $SER = 0.19\%$

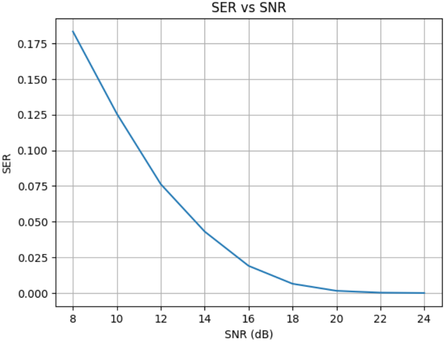

Figure: LMMSE SER for varying SNR

Blind Equalizers

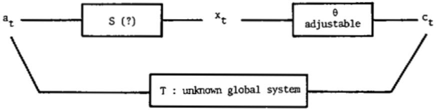

Figure: The channel FIR response S is unknown. θ is the equalizer.

- Do not require any known training sequence for the startup period, but can rather perform at any time the equalization directly on the data stream.
- We analyze the **Benveniste and Goursat Algorithm**.
- Here, we aim to recover the transmitted message using only the received x_t without any preamble for the identification of the unknown channel S.
- To do this, we must have $\theta = S^{-1}$.
- Ideally, the channel inverse is an infinite IR filter. However, practically, an FIR filter satiates.

Blind Equalizers - The Unicity Result

We had

$$x_t = \sum_{k \in Z} s_k a_{t-k}, \quad S = (s_k)_{k \in Z}$$

- Since a_t follows a symmetric distribution, both $+S$ and $-S$ give the same result.
- Hence, at the decoding end too, S^{-1} cannot distinguish a_t from $-a_t$.
- More on this later.

Gain and Phase Recovering Cost Function

Since a_t follows sub-Gaussian distribution, we use the Sato Cost Function.

$$J(\theta) = \mathbb{E}[\psi(Re(c_t(\theta, \phi))) + \psi(Im(c_t(\theta, \phi)))] \tag{6}$$

$$\psi(x) = \frac{1}{2}x^2 - \alpha|x| \tag{7}$$

- This is the cost function we wish to minimize.
- The purpose of θ is to remove ISI slowly while ϕ tracks phase jitter and offset (redundant for a time-invariant system).
- In the case of QPSK, $\alpha = d_{min}$.
- We state without explicit proof that the Sato Cost Function admits $\pm S^{-1}$ as the only local minima.
- Intuitive $\nabla J = 0$ when $c_t(\theta) = \alpha \cdot sgn(c_t(\theta))$ on an average $\Rightarrow$ decoded symbol (coarse estimate) equals equalized symbol.

Gradient of the Cost Function

$$\text{grad}_\theta\, J \triangleq \frac{\partial}{\partial(\text{Re}\,\theta)} J + i\,\frac{\partial}{\partial(\text{Im}\,\theta)} J,$$

we get by a straightforward calculation

$$\text{grad}_\theta\, J(\theta, \phi) = -\mathbb{E}(X_t{}^*(\theta, \phi)\epsilon_t(\theta, \phi)),$$

$$X_t(\theta, \phi) \triangleq \frac{\partial}{\partial(\text{Re}\,\theta)} (\text{Re}\, c_t(\theta, \phi)) + i\,\frac{\partial}{\partial(\text{Im}\,\theta)} (\text{Im}\, c_t(\theta, \phi))$$

$$\epsilon_t(\theta, \phi) = -(\psi'(\text{Re}\, c_t(\theta; \phi)) + i\,\psi'(\text{Im}\, c_t(\theta, \phi))),$$

whereas

$$\frac{\partial}{\partial\phi} J(\theta, \phi) = -\mathbb{E}(\text{Im}\,(c_t(\theta, \phi)\epsilon_t{}^*(\theta, \phi))).$$

The B&G Algorithm

Using a stochastic gradient on the Sato Cost with the gradients obtained, we have the algorithm -

$$\theta_{t+1} = \theta_t + \gamma X_t^* e^{+\phi_t} \cdot \epsilon_t(\theta_t, \phi_t)$$

$$\phi_{t+1} = \phi_t + \mu Im(c_t(\theta_t, \phi_t) \cdot \epsilon^*(\theta_t, \phi_t))$$

$$X_t^T = (x_{t+N}, \ldots, x_{t-N})$$

$$c_t(\theta, \phi) = X_t^T \cdot \theta_t e^{-i\phi_t}$$

- Here, $\epsilon_t(\theta, \phi)$ is called the pseudo-error function.
- The convergence to $+S^{-1}$ or $-S^{-1}$ depends crucially on the initial value for θ. It is recommended to start with θ equal to $+$ identity.

Choice of the Pseudo-Error Function

We use

$$\epsilon_t^G = k_1 e_t + k_2 |e_t| \epsilon_t^S$$

where

$$e_t = \hat{a}_t - c_t$$

$$\epsilon_t^S = c_t - \hat{c}_t$$

$$\hat{c}_t = \alpha(sgn(Re(c_t)) + sgn(Im(c_t))$$

Here

- The Sato error ϵ_t^S is robust but noisy around the solution
- The customary error e_t is zero at the solution but not robust in switching.

The Final Blind Equalizer Setup

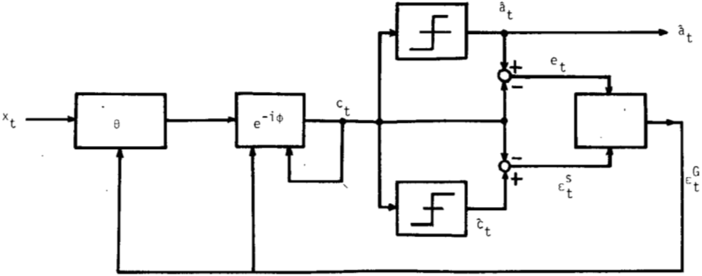

Figure: The theoretical analysis summarised.

Simulation - The Process

1. Generate a symbol sequence a_t.
2. Process the sequence through the channel filter S (invisible to the equalizer) and add noise to get x_t.
3. Initialize θ.
4. Calculate $c_t = X_t^T \cdot \theta \exp^{-i\phi}$.
5. Calculate the G-pseudo error ϵ_t^G.
6. Calculate gradients for θ and ϕ.
7. Perform a descent.
8. Go to 4.

Note: The algorithm starts as soon as the first symbol is received (start-up period) and continues into the normal receiving period after *enough* symbols are received. In case the channel behaviour changes abruptly, the process smoothly shifts back to the start-up mode.

Simulation Setup

1. $N = 30,000$ QPSK Symbols
2. SNR $= 20$dB
3. $N_\theta = 21$ Equalizer taps
4. $k_1 = 3$
5. $k_2 = 1$
6. $\gamma = \mu = 10^{-3}$
7. $\theta_0 = [0 \ldots 0]$ except one $\theta_{0i} = 1$
8. ϕ is chosen randomly

Blind Equalizer - Simulated

When the algorithm converges to $-S^{-1}$

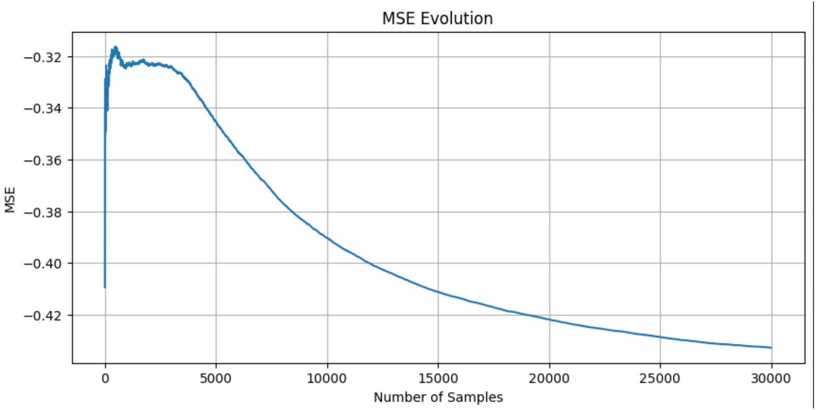

Figure: Evolution of the Sato Cost Function, computed cumulatively.

Blind Equalizer - Simulated

When the algorithm converges to $-S^{-1}$, total average SER $= 99.7\%$

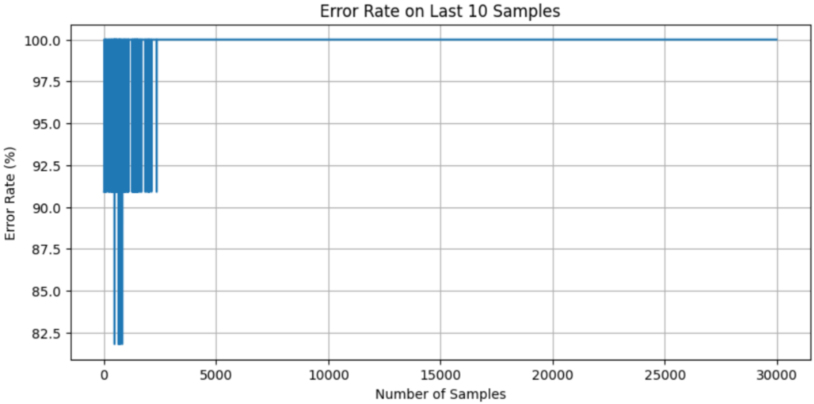

Figure: Error rate calculated on the last ten samples received.

Blind Equalizer - Simulated

When the algorithm converges to $+S^{-1}$

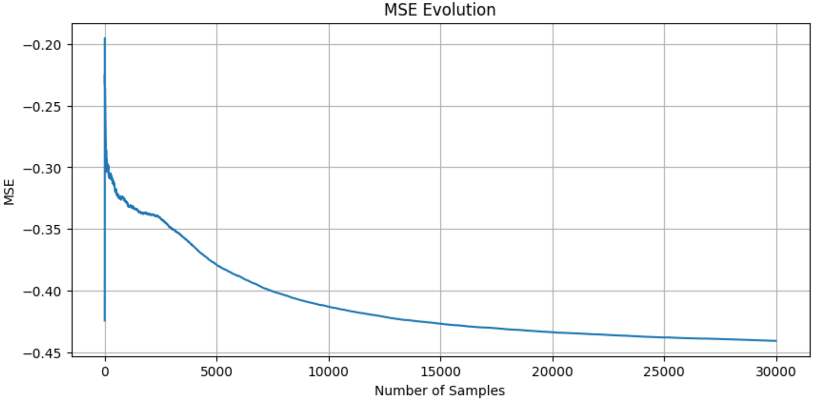

Figure: Evolution of the Sato Cost Function, computed cumulatively.

Blind Equalizer - Simulated

When the algorithm converges to $+S^{-1}$, total average SER $= 1.8\%$

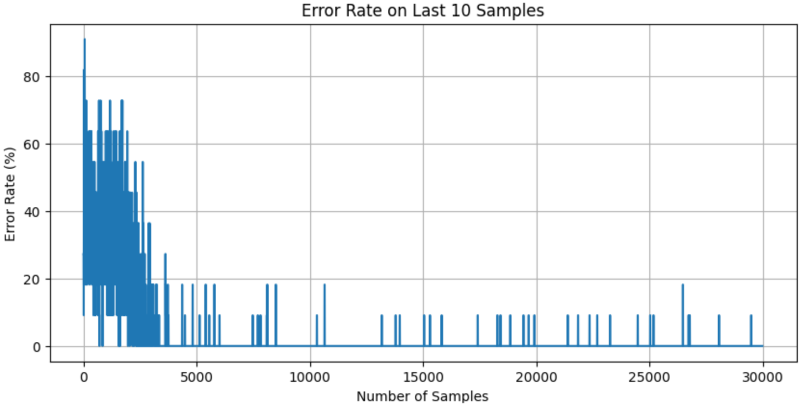

Figure: Error rate calculated on the last ten samples received.

References

- A. Benveniste and M. Goursat, "Blind Equalizers," in IEEE Transactions on Communications, vol. 32, no. 8, pp. 871-883, August 1984, doi: 10.1109/TCOM.1984.1096163.
- A. Benveniste, M. Goursat and G. Ruget, "Robust identification of a nonminimum phase system: Blind adjustment of a linear equalizer in data communications," in IEEE Transactions on Automatic Control, vol. 25, no. 3, pp. 385-399, June 1980, doi: 10.1109/TAC.1980.1102343.