Continuous Random Variable :- A random variable X is said to be continuous if it can take all possible values between certain limits.

Examples :- (1) Life time of electric bulb in hours.
(2) Height, weight, temperature, etc.

Probability density function :- A fun. f is said to be probability density function (pdf) of the continuous random variable X if it satisfies the following condition:

(i) $f(x) \geq 0$ for all $x \in R$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Distribution function :- (Cumulative Distribution function)
The fun. $F(X)$ is said to be the distribution fun. of the random variable X, if $\boxed{F(X) = P(X \leq x)}$; $-\infty \leq x \leq \infty$

The distribution fun. F is also called cumulative distribution fun.

Note :- (1) If X is a discrete random variable then from the definition it follows that $F(X) = \Sigma p(x_i)$, where the summation is overall x_i, such that $x_i \leq x$.
If X is a continuous random variable, then from the def. it follows that

$$f(x) = y = Ae^{x} + Be^{2x} + \frac{e^{-4x}}{10} + 4xe^{2x}$$

where $f(x)$ is the value of the probability density fun. of X at x.

Q: Verify that $f(x) = \begin{cases} \frac{2x}{9} & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$ is a probability density function.

Sum of two Independent Random Variables :-

If X and Y are independent random variables then $Z = X+Y$ is the sum of two independent random variables.

Mean of Z (or Expected value of Z) :-

$$\boxed{E(Z) = E[X+Y] = E(X) + E(Y) = \mu_x + \mu_y}$$

Variance of Z : $Var(Z) = E[Z - E(Z)]^2$

$$= E[X+Y-(\mu_x+\mu_y)]^2$$
$$= E[(X-\mu_x) + (Y-\mu_y)]^2$$
$$= E[(X-\mu_x)^2 + (Y-\mu_y)^2 + 2(X-\mu_x)(Y-\mu_y)]$$
$$= E(X-\mu_x)^2 + E(Y-\mu_y)^2 + 2E[(X-\mu_x)(Y-\mu_y)]$$
$$= E[X-E(X)]^2 + E[Y-E(Y)]^2 + 2E[(X-\mu_x)(Y-\mu_y)]$$

$$= \sigma_x^2 + \sigma_y^2 + 2\,Cov.(XY)$$

$$\boxed{Var(Z) = \sigma_x^2 + \sigma_y^2}$$ [$\because Cov.(XY) = 0$ as X and Y are indepent variables]

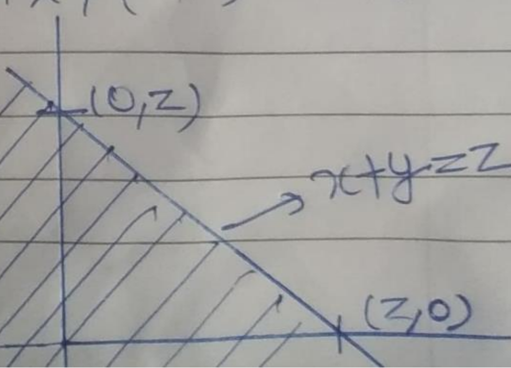
Probability density function :-

We know that cdf of Z is given by

$$F_Z(Z) = P(Z \le z) = P(X+Y \le z)$$
$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,z)\,dz\,dx .$$

$$f_Z(z) = f_X(x) \cdot f_Y(y)$$
$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(z-y)\,dy$$



Characteristic function of $Z = X+Y$,

$$\phi_Z(\omega) = E[e^{i\omega x}] \cdot E[e^{i\omega y}]$$

or $\boxed{\phi_Z(\omega) = \phi_X(\omega) \cdot \phi_Y(\omega)}$

Q: Let X and Y be exponentially distributed random variable with $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

and $Z = X+Y$, then find mean, variance, Pdf and characteristic function.

Soln: We know,

$$\mu_Z = \mu_X + \mu_Y = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$$

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 2/\lambda^2$$

Again, $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$

$$f_X(x) = \lambda e^{-\lambda x} \qquad 0 \leq x < \infty$$

$$f_Y(z-x) = \lambda e^{-\lambda(z-x)} \qquad 0 \leq z-x < \infty$$

$$\text{i.e., } 0 \leq x \leq z < \infty$$

$$f_Z(z) = \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)}\, dx$$

$$= \int_0^z \lambda^2 e^{-\lambda z}\, dx = \lambda^2 e^{-\lambda z}[x]_0^z$$

$$= \lambda^2 e^{-\lambda z} \cdot z$$

$$\therefore f_Z(z) = \begin{cases} \lambda^2 e^{-\lambda z} \cdot z & z \geq 0 \\ 0 & z < 0 \end{cases}$$

We know that $\phi_Z(\omega) = \phi_X(\omega) \cdot \phi_Y(\omega)$

$$= \frac{\lambda}{\lambda - i\omega} \cdot \frac{\lambda}{\lambda - i\omega}$$

$$= \frac{\lambda^2}{(\lambda - i\omega)^2} \quad \text{Ans}$$

Bunk Pages - The Social Notebook

Q: Suppose $Z = X+Y$, where X and Y are independent random variables with Pdf $f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ Determine mean, variance and Pdf of Z.

Sol: $\mu_x = \frac{a+b}{2}$ $\therefore$ $\mu_x = \frac{1}{2}$ and $\mu_y = \frac{1}{2}$

$\therefore \mu_z = \mu_x + \mu_y = \frac{1}{2} + \frac{1}{2} = 1$ ✓

$\sigma_x^2 = \frac{(b-a)^2}{12} = \frac{1}{12}$ and $\sigma_y^2 = \frac{1}{12}$

$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$ ✓

Now, $f_X(x) = 1 \quad 0<x<1$

$f_Y(z-x) = 1 \quad 0 < z-x < 1$ or $z-1 < x < z$

Case (i) If z varies from 0 to 1, lower limit $\to 0$
Upper " $\to z$

$$f_Z(z) = \int_0^z dx = [x]_0^z = z$$

Case (ii) z varies from 1 to 2 lower lt. $\to z-1$
Upper $\to 1$

$$f_Z(z) = \int_{z-1}^{1} dx = [x]_{z-1}^{1} = 1-z+1 = 2-z$$

$$\therefore f_Z(z) = \begin{cases} z & 0<z<1 \\ 2-z & 1<z<2 \\ 0 & \text{otherwise} \end{cases}$$ Ans

Expectation and Variance of Continuous Random Variables :- If X is a continuous random variable having probability density fun. $f(x)$, then expected value of X is given by

$$\boxed{E(X) = \int_{-\infty}^{\infty} x f(x)\, dx}$$

Q:- Find E(X) when the density fun. of X is $f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$

Soln $E(X) = \int x f(x)\, dx = \int_0^1 2x^2\, dx = 2/3.$

Q:- The density fun. of X is $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1. \\ 0 & \text{otherwise} \end{cases}$ find $E[e^X]$.

Soln let $Y = e^X$. We start by determining F_Y, the cumulative distribution function of Y.

Now, for $1 \le x \le e$, $F_Y(x) = P(Y \le x)$

$$= P\{e^X \le x\}$$

$$= P\{X \le \log x\}$$

$$= \int_0^{\log x} f(y)\, dy = \log x$$

By differentiating $F_Y(x)$, we can conclude that the prob. density fun of Y is given by

$$f_Y(x) = 1/x \qquad 1 \le x \le e$$

Hence, $E[e^X] = E[Y] = \int_{-\infty}^{\infty} x f_Y(x)\, dx$

$$= \int_1^e dx = e - 1 \text{ Ans}$$

Imp. Note :- If X is a continuous random variable with pdf $f(x)$, then for any real-valued fun. g.

$$\boxed{E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx}$$

So, for aboved qus. $E[e^X] = \int_0^1 e^x dx = e - 1$

$[\text{as } f(x) = 1,\ 0 < x < 1]$

Q: Suppose that if you are s minutes early for an appointment, then you incur the cost cs, and if you are s minute late, then you incur the cost ks. Suppose also that the travel time from where you presently are to the location of your appointment is a continuous random variable having pdf f. Determine the time at which you should depart if you want to minimize your expected cost.

Soln: Let X denote the travel time. If you leave t minutes before your appointment, then your cost - call $C_t(X)$ is given by

$$C_t(X) = \begin{cases} c(t-X) & \text{if } X \le t \\ k(X-t) & \text{if } X \ge t \end{cases}$$

$$\therefore \quad E[C_t(X)] = \int_0^\infty C_t(x) f(x)\,dx$$

$$= \int_0^t c(t-x) f(x)\,dx + \int_t^\infty k(x-t) f(x)\,dx$$

$$= ct\int_0^t f(x)\,dx - c\int_0^t x f(x)\,dx + k\int_t^\infty x f(x)\,dx - kt\int_t^\infty f(x)\,dx$$

$$\frac{d}{dt} E[C_t(X)] = ctf(t) + cF(t) - ctf(t) - ktf(t) + ktf(t) - k[1-F(t)]$$

$$= (k+c)F(t) - k$$

Equating the rightmost sides to zero shows that the minimal expected cost is obtained when you leave t^* minutes before your appointment, where t^* satisfies $\boxed{F(t^*) = \frac{k}{k+c}}$

Corollary :- If a and b are constants, then

$$\boxed{E[aX+b] = aE[X]+b}$$

Variance :-

$$\boxed{\begin{aligned} &Var(X) = E[(X-\mu)^2] \\ &\text{or } Var(X) = E[X^2] - (E[X])^2 \end{aligned}}$$

Q. Find Var. (X) for X, if $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Sol.: $E(X) = 2/3$, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 2x^3\,dx$
$= 1/2$

$Var(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = 1/18$

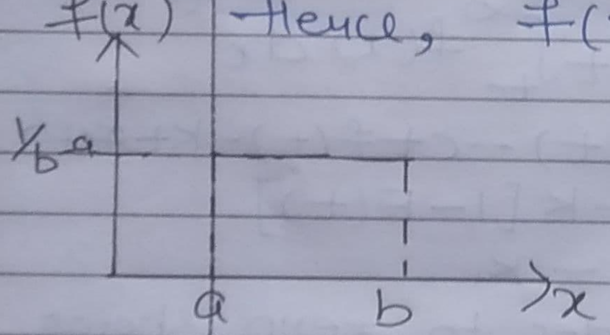
Rectangular or Uniform Distribution :- A continuous Random variable X is said to follow a continuous uniform or rectangular dist. over interval (a,b), if its pdf is given by

$\boxed{f(x) = \begin{cases} k & a < x < b \\ 0 & \text{otherwise} \end{cases}}$, where $f(x)$ is p.d.f.

Now, we know, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_a^b k\,dx = 1 \Rightarrow k[x]_a^b = 1 \Rightarrow \boxed{k = \frac{1}{b-a}}$

Hence, $f(x) = \begin{cases} 1/b-a & a < x < b \\ 0 & \text{otherwise.} \end{cases}$



Mean and Variance of Uniform Distribution :-

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$\Rightarrow E(X) = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left\{\frac{x^2}{2}\right\}_a^b$

$\Rightarrow \frac{1}{b-a}\left\{\frac{b^2-a^2}{2}\right\} = \frac{b+a}{2}$, $\boxed{E(X) = \frac{b+a}{2}}$

Variance :- $E(X^2) - [E(X)]^2$

So, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$

$= \frac{1}{b-a} \cdot \left[\frac{x^3}{3}\right]_a^b = \frac{1}{(b-a)} \frac{(b^3-a^3)}{3}$

$$E(X^2) = \frac{1}{(b-a)}\left\{\frac{(b-a)(b^2+ab+a^2)}{3}\right\} = \frac{b^2+ab+a^2}{3}$$

So, $\text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4} = \frac{a^2-2ab+b^2}{12}$$

$$\boxed{\text{Var}(X) = (a-b)^2/12}$$

Moment Generating fun. of Uniform Distribution:-

Here $M_x(t) = E[e^{xt}]$

$$= \int_{-\infty}^{\infty} e^{xt} f(x)\,dx = \int_a^b e^{xt} \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b e^{xt}\,dx = \frac{1}{b-a}\left[\frac{e^{xt}}{t}\right]_a^b$$

$$= \frac{1}{(b-a)}\left[\frac{e^{bt}-e^{at}}{t}\right] = \boxed{\frac{e^{bt}-e^{at}}{t(b-a)} = M_x(t)}$$

Corollary :-

Mean $E(X) = \frac{d}{dt}[M_x(t)]_{t=0}$

$E(X^2) = \frac{d^2}{dt^2}[M_x(t)]_{t=0}$

$\text{Var} = E(X^2) - [E(X)]^2$

Q:- If X is uniformly dist. with mean 1 and var. 4/3. Find $P(x<0)$.

Sol. We know, $\frac{a+b}{2} = 1$, $\frac{(a-b)^2}{12} = 4/3$

$a+b = 2$, $(a-b)^2 = 16 \Rightarrow a-b = \pm 4$

By solving these, we get, $a = 3,\ b = -1$

$a = -1,\ b = 3$

So, $f(x) = \begin{cases} 1/b-a & a<x<b \\ 0 & \text{otherwise.} \end{cases}$

So, $f(x) = \begin{cases} \frac{1}{4} & -1<x<3 \\ 0 & \text{otherwise} \end{cases}$

$$P(X<0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-1}^{0} f(x)\,dx$$

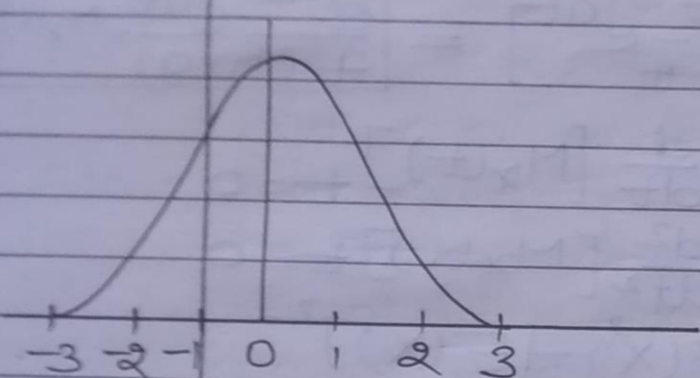
$$= \int_{-1}^{0} 1/4\,dx = \left[\frac{x}{4}\right]_{-1}^{0} = \frac{1}{4}[0-(-1)] = 1/4.$$

Normal Distribution:- We say that X is a normal random variable or normally distributed with parameters μ and σ^2 if the density of X is defined by $\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}}$; $-\infty < x < \infty$

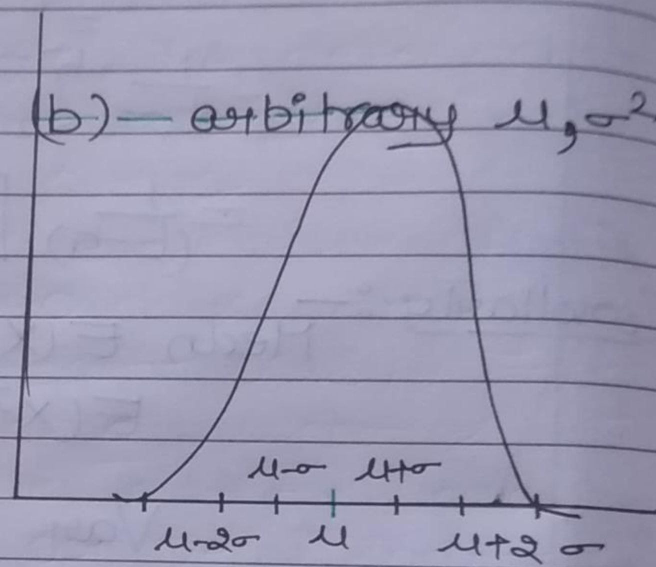
The density fun. is a bell-shaped curve that is symmetric about μ.

Normal density fun.

(a) $\mu = 0$, $\sigma = 1$



(b) arbitrary μ, σ^2



Mean and Variance of a normal random variable X with parameters μ and σ^2:-

Let us start by finding the mean and var. of the standard normal variable $Z = \frac{(X-\mu)}{\sigma}$.

We have

$$E(Z) = \int_{-\infty}^{\infty} x f_Z(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \left\{ e^{-x^2/2} \right\}_{-\infty}^{\infty} = 0$$

Thus, $\text{Var } Z = E(Z^2) - [E(Z)]^2$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2/2}\,dx - 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2/2}\,dx$$

Integrating by parts, we get

$$Var(Z) = \frac{1}{\sqrt{2\pi}}\left[\left(-xe^{-x^2/2}\right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = 1$$

$\because X = \mu + \sigma Z$, then $E[X] = \mu + \sigma E[Z] = \mu$ and $Var[X] = \sigma^2 Var(Z) = \sigma^2$

It is customary to denote the cumulative dist. fn. of a standard normal random variable by $\phi(x)$. i.e., $\boxed{\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy}$

for -ve values of x, $\boxed{\phi(-x) = 1 - \phi(x)}$, $-\infty < x < \infty$

The above equation states that if Z is a standard normal variable, then

$\boxed{P\{Z \le -x\} = P\{Z > x\}}$ $-\infty < x < \infty$

Since $Z = (X-\mu)/\sigma$ is a standard normal random variable whenever X is normally distributed with parameters μ and σ^2, it follows that the dist. fn of X can be expressed as

$$\boxed{F_X(a) = P\{X \le a\} = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = \phi\left(\frac{a-\mu}{\sigma}\right)}$$

Q:- If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find
(a) $P\{2 < X < 5\}$ (b) $P\{X > 0\}$ (c) $P\{|X-3| > 6\}$.

Soln (a) $P\{2 < X < 5\} = P\left\{\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right\}$

$= P\left\{-\frac{1}{3} < Z < \frac{2}{3}\right\} = \phi\left\{\frac{2}{3}\right\} - \phi\left\{-\frac{1}{3}\right\}$

$= \phi\left(\frac{2}{3}\right) - \left[1 - \phi\left(\frac{1}{3}\right)\right] \approx .3779$

(b) $P\{X > 0\} = P\left\{\frac{X-3}{3} > \frac{0-3}{3}\right\} = P\{Z > -1\}$

$= 1 - \phi(-1) = \phi(1) \approx .8413$

(c) $P\{|X-3|>6\} = P\{X>9\} + P\{X<-3\}$

$$= P\left\{\frac{X-3}{3} > \frac{9-3}{3}\right\} + P\left\{\frac{X-3}{3} < \frac{-3-3}{3}\right\}$$

$$= P\{Z>2\} + P\{Z<-2\}$$

$$= 1-\phi(2) + \phi(-2)$$

$$= 2[1-\phi(2)] \approx .0456 \text{ Ans}$$

Q:- An expert witness in a paternity suit testifies that the length (in days) of human gestation is app. normally distributed with parameters $\mu = 270$ and $\sigma^2 = 100$. The defendant in the suit is able to prove that he was out of the country during a period that began 290 days before the birth of the child and ended 240 days before the birth. If the defendant was, in fact, the father of the child, what is the probability that the mother could have had the very long or very short gestation indicated by the testimony?

Sol.n) Let X denote the length of the gestation, and assume that the defendant is the father. Then the prob. that the birth could occur within the indicated period is

$$P\{X>290 \text{ or } X<240\} = P\{X>290\} + P\{X<240\}$$

$$= P\left\{\frac{X-270}{10} > 2\right\} + P\left\{\frac{X-270}{10} < -3\right\}$$

$$= 1-\phi(2) + 1-\phi(3)$$

$$\approx .0241 \text{ Ans}$$

Exponential Distribution :- A continuous random variable X, which has the following pdf

$\boxed{f(x) = \alpha e^{-\alpha x}}$. $\alpha > 0$, $0 < x < \infty$,

is called exponential variable and its distribution is called exponential distribution.

Mean & Variance of Exponential Distribution :-

We know,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_0^{\infty} x e^{-\alpha x}\,dx$$

[By gamma fun $\int_0^{\infty} x^m e^{-ax}\,dx = \frac{\Gamma n}{a^n}$ where $m = n-1$]

$$= \alpha \cdot \frac{\Gamma 2}{\alpha^2} = \frac{1}{\alpha}$$

$\boxed{E(X) = 1/\alpha}$ Again, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_0^{\infty} x^2 \cdot \alpha e^{-\alpha x}\,dx = \alpha \int_0^{\infty} x^2 \cdot e^{-\alpha x}\,dx$$

$$= \alpha \cdot \frac{\Gamma 3}{\alpha^3} = \frac{2}{\alpha^2}, \text{ So,}$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= 2/\alpha^2 - (1/\alpha)^2 = \boxed{1/\alpha^2 = Var[X]}$$

Moment Generating fun. of Exponential Distribution :

$$M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$$

$$= \int_0^{\infty} e^{xt} \cdot \alpha e^{-\alpha x}\,dx = \alpha \int_0^{\infty} e^{-x(\alpha - t)}\,dx$$

$$= \alpha \cdot \left[\frac{e^{-x(\alpha-t)}}{-(\alpha-t)}\right]_0^{\infty} = \alpha\left[0 + \frac{1}{\alpha - t}\right] = \frac{\alpha}{\alpha - t}$$

Characteristic fun :- $\phi_x(t) = E[e^{ixt}]$

$$\boxed{\phi_x(t) = \frac{\alpha}{\alpha - it}}$$

Q8. The length of Telephone conversation is an exponential variate with mean 3 minutes. find probability that call (i) end less than 3 min. (ii) b/w 3 to 5 min.

Sol^n Given, Mean $= \left(\frac{1}{\alpha}\right) = 3 \Rightarrow \alpha = 1/3$

$f(x) = \alpha e^{-\alpha x} = \frac{1}{3} e^{-x/3}$

(i) $P(X<3) = \int_0^3 f(x)\,dx = \int_0^3 \frac{1}{3} e^{-x/3}\,dx$

$= \frac{1}{3}\left\{\frac{e^{-x/3}}{-1/3}\right\}_0^3 = -\{e^{-1} - 1\}$

$= 1 - 1/e$ Ans

(ii) $P(3<X<5) = \int_3^5 f(x)\,dx = \int_3^5 \frac{1}{3} e^{-x/3}\,dx$

$= \frac{1}{3}\left\{\frac{e^{-x/3}}{-1/3}\right\}_3^5 = -\{e^{-5/3} - e^{-1}\}$

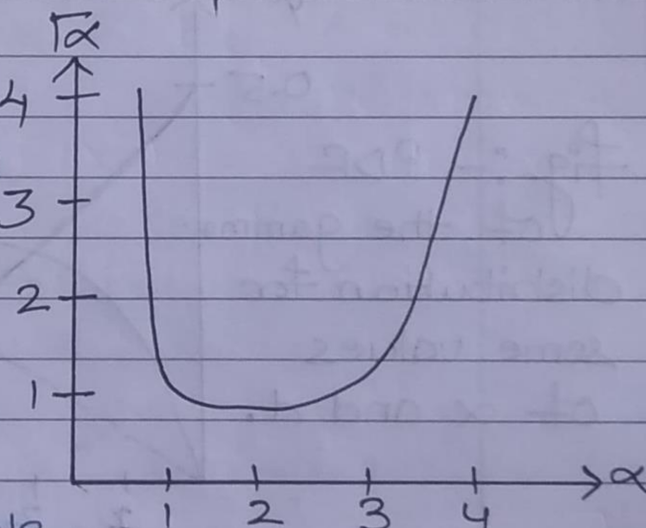
$= e^{-1} - e^{-5/3}$ Ans.

GAMMA DISTRIBUTION :- Before introducing the gamma random variable, we need to introduce the gamma fu^n.

Gamma fu^n. :- The gamma fu^n. is shown by Γx, is an extension of the factorial fu^n. to real (and complex) no. Specially, if $n \in \{1, 2, 3, \ldots\}$, then $\boxed{\Gamma n = (n-1)!}$ More generally, for any positive real number α, $\Gamma\alpha$ is defined as

$$\boxed{\Gamma\alpha = \int_0^\infty x^{\alpha-1} e^{-x}\, dx} \; ; \quad \text{for } \alpha > 0$$

fig. shows the gamma fu^n. for some real values of α.



Note:- for $\alpha = 1$,

$$\Gamma 1 = \int_0^\infty e^{-x}\, dx = 1.$$

Using the change of variable, $x = \lambda y$,

$$\Gamma\alpha = \lambda^\alpha \int_0^\infty y^{\alpha-1} \cdot e^{-\lambda y}\, dy \; ; \quad \text{for } \alpha, \lambda > 0$$

Using Integration by part, $\boxed{\Gamma\alpha+1 = \alpha\Gamma\alpha \; ;}$ $\alpha > 0$

If $\alpha = n$, where n is positive integer, then $\boxed{n! = n \cdot (n-1)!}$

Some properties of gamma fu^n:-

① $\Gamma\alpha = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ ② $\int_0^\infty x^{\alpha-1} e^{-x}\, dx = \frac{\Gamma\alpha}{\lambda^\alpha}$

③ $\Gamma\alpha+1 = \alpha\Gamma\alpha$ ④ $\Gamma n = (n-1)!$ for $n = 1, 2, 3, \ldots$

⑤ $\Gamma\frac{1}{2} = \sqrt{\pi}$

Gamma distribution :- A continuous random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $d > 0$, shown as $X \sim \text{Gamma}(\alpha, d)$, if its PDF is given by

$$\boxed{f_X(x) = \begin{cases} \dfrac{d^{\alpha} x^{\alpha-1} e^{-dx}}{\Gamma\alpha} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}}$$

If we take $\alpha = 1$, $\boxed{f_X(x) = \begin{cases} d e^{-dx} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}}$

Thus, we conclude $\text{Gamma}(\cancel{d}\ 1, d) = \text{Exponential}(d)$

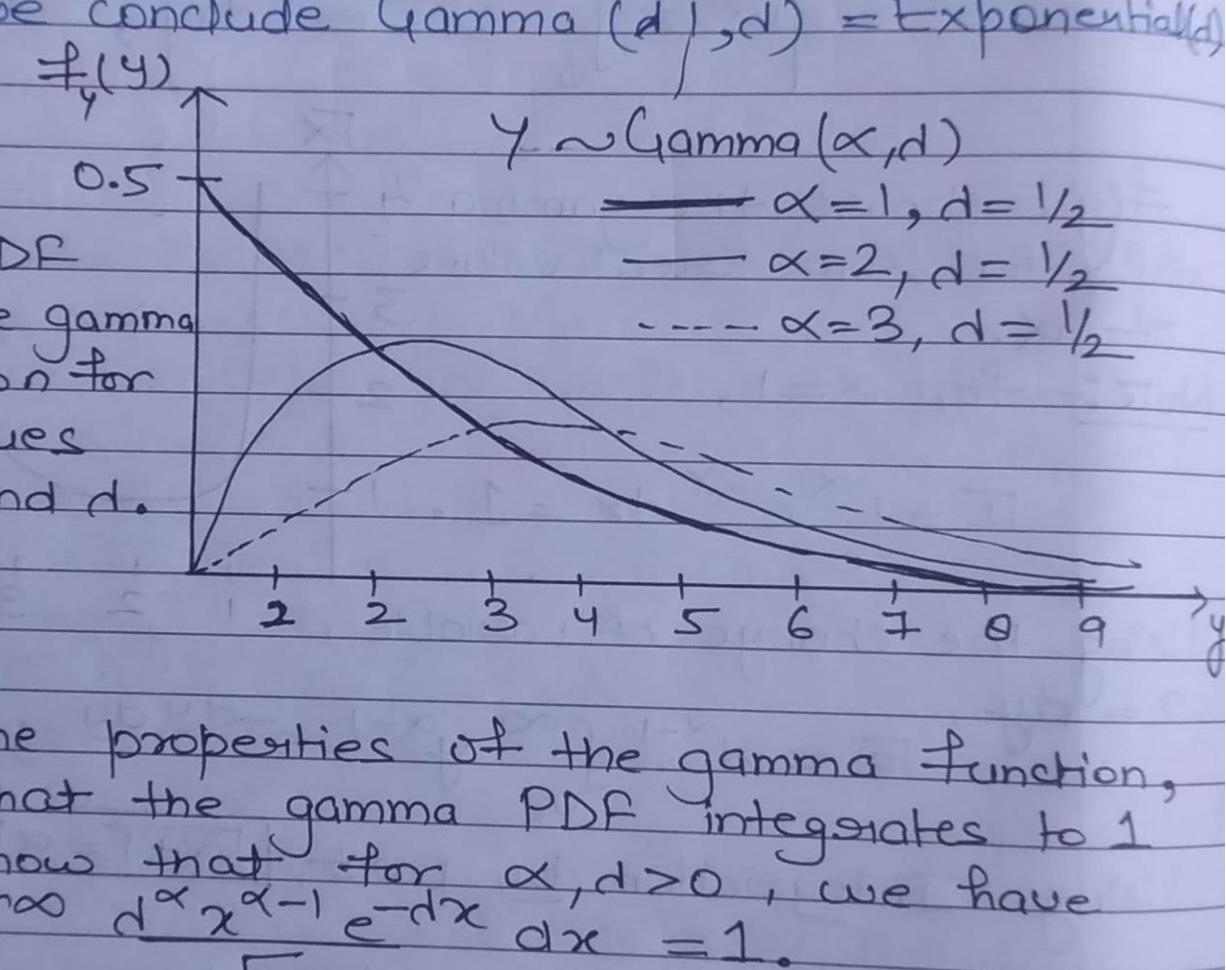


fig :- PDF of the gamma distribution for some values of α and d.

Using the properties of the gamma function, show that the gamma PDF integrates to 1 i.e., show that for $\alpha, d > 0$, we have

$$\int_0^{\infty} \frac{d^{\alpha} x^{\alpha-1} e^{-dx}}{\Gamma\alpha}\, dx = 1.$$

Proof :- $\displaystyle\int_0^{\infty} \frac{d^{x} x^{\alpha-1} e^{-dx}}{\Gamma\alpha}\, dx\, dx = \frac{d^{\alpha}}{\Gamma\alpha} \int_0^{\infty} x^{\alpha-1} . e^{-dx} dx$

$$= \frac{d^{\alpha}}{\Gamma\alpha} \cdot \frac{\Gamma\alpha}{d^{\alpha}} \quad (\text{Using prop. 2})$$

$$= \underline{1}.$$

Mean of Gamma distribution :-

We know, Expected value of a continuous random variable $E[X] = \int_0^\infty x f(x)\,dx$

$$= \int_0^\infty \frac{x \cdot \lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma\alpha} = \frac{\lambda^\alpha}{\Gamma\alpha}\int_0^\infty x^\alpha e^{-\lambda x}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha}\int_0^\infty \left(\frac{t}{\lambda}\right)^\alpha e^{-\lambda} \cdot \frac{dt}{\lambda} \quad \text{Put } \lambda x = t, \ \lambda \cdot dx = dt$$

$$= \frac{\lambda^\alpha}{\lambda^{\alpha+1} \cdot \Gamma\alpha}\int_0^\infty t^\alpha \cdot e^{-\lambda}\,dt$$

$$= \frac{\Gamma(\alpha+1)}{\lambda\,\Gamma\alpha} = \frac{\alpha\,\Gamma\alpha}{\lambda\,\Gamma\alpha} = \frac{\alpha}{\lambda} \Rightarrow \boxed{E[X] = \frac{\alpha}{\lambda}}$$

Variance of Gamma distribution :

$$Var[X] = E[X^2] - \{E[X]\}^2$$

$$= \int_0^\infty x^2 f(x)\,dx - (\alpha/\lambda)^2$$

$$= \int_0^\infty \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha+1} e^{-\lambda x}\,dx - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha}\int_0^\infty x^{\alpha+1} e^{-\lambda x}\,dx - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha}\int_0^\infty \left(\frac{t}{\lambda}\right)^{\alpha+1} \cdot e^{-t}\,\frac{dt}{\lambda} - \frac{\alpha^2}{\lambda^2} \quad \left\{\begin{array}{l}\text{Put } \lambda x = t \\ \lambda dx = dt\end{array}\right\}$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha\,\lambda^{\alpha+2}}\int_0^\infty t^{\alpha+1} \cdot e^{-t}\,dt - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{1}{\Gamma\alpha\,\lambda^2} \cdot \Gamma(\alpha+2) - \frac{\alpha^2}{\lambda^2} = \frac{\Gamma(\alpha+2) - \cancel{\alpha}^2\,\Gamma\alpha}{\lambda^2\,\Gamma\alpha}$$

$$\Rightarrow \frac{(\alpha+1)\alpha\,\Gamma\alpha - \alpha^2\,\Gamma\alpha}{\lambda^2\,\Gamma\alpha} = \frac{\Gamma\alpha(\alpha^2+\alpha-\alpha^2)}{\lambda^2\,\Gamma\alpha}$$

$$\boxed{Var[X] = \alpha/\lambda^2}$$

Moment Generating fun :- We define moment generating fun. of Gamma distribution for random variable X is

$$\boxed{M_X(t) = \left(1-\frac{t}{\lambda}\right)^{-\alpha} ; \quad t<\lambda}$$

We know,

$$M(t) = E[e^{tx}] = \int_0^\infty e^{tx} \cdot \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma\alpha} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha} \int_0^\infty e^{(t-\lambda)x} \cdot x^{\alpha-1}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma\alpha} \int_0^\infty e^{y} \cdot \frac{y^{\alpha-1}}{(t-\lambda)^{\alpha-1}} \cdot \frac{dy}{t-\lambda}$$

Put $(t-\lambda)x = y$

$x = \frac{y}{t-\cancel{\lambda}}$

$dx = \frac{dy}{t-\lambda}$

$$= \frac{\lambda^\alpha}{\Gamma\alpha} \int_0^\infty e^{y} \cdot y^{\alpha-1} \cdot \frac{dy}{(t-\lambda)^\alpha}$$

$$\boxed{M(t) = \left\{1-\frac{t}{\lambda}\right\}^{-\alpha}}$$