Introductory Exercise – Oscillations of a Loop

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1 Abstract

The acceleration due to gravity is a common and well defined value, g = 9.81 m/s^2, rooted in the fundamentals of physics. We look to obtain an accurate measurement of g through a self-constructed experiment in which loops are used as simple harmonic oscillators for the small angle approximation. From our data we compute the acceleration due to gravity to be, g = 9.66 ± 0.22 m/s^2 or g = 9.64 ± 0.22 m/s^2 based on two slightly varying approaches and this result matches the known value of g within the given uncertainty. The uncertainty stems mainly from miniscule reading errors and innate uncertainties of measurement and still provides a sufficiently accurate result. In addition to reasonably replicating previous results of this measurement, the method used is easy to replicate for further studies into the subject.

2 Introduction

This experiment aims to determine, within a reasonable uncertainty, the acceleration due to gravity. The location of the experiment (Toronto) is only 76 meters above sea level, which is negligible. Thus, a successful experiment would replicate the known result of g = 9.81 m/s^2 within a given uncertainty. We took measurements of diameter and period of small oscillations of four metal loops in a pendulum motion. Specifically, the theory is that these loops can be approximated to be rigid body simple harmonic oscillators with a small angle assumption. Furthermore, by computing the moment of inertia of these loops with a rigid body assumption and the parallel axis theorem, one can determine a good estimation of the acceleration due to gravity at their location.

3 Methods and Materials

The materials utilized for this experiment were: four hoops of varying sizes, a stand with knife support, one stop watch, one ruler, measuring tape and the Jupyter Notebooks software.

4 Experimental Procedure

First the four loops were labelled ($Loop_1$, $Loop_2$, $Loop_3$ and $Loop_4$) in terms of increasing diameter, i.e. $Loop_1$ has the smallest diameter and $Loop_4$ has the largest diameter. A ruler was used to accurately measure the diameter of each loop; this was measured from the outer ring from one end to the inner ring at the other end to get the mean diameter of the loop. To ensure these loops were round, six measurements of diameter were taken per loop by turning the object a set amount after each measurement, thus checking if the diameter was roughly constant around the loop. This procedure was repeated until six diameter measurements were collected for each of the four loops.

Secondly, to determine the period of small oscillations, $Loop_1$ was placed on the stand with knife support and pushed gently to generate a small angle oscillation. Since the timescale of one period was too short, the stopwatch was used to measure the time taken to complete five periods instead. This measurement was taken six times for the loop and the procedure was repeated for each loop. The measurements were later divided by five to determine the period of one oscillation.

Thirdly, the diameter and period data was put into a custom python script (included in the Appendix section) to compute the mean and standard deviation of both parameters. These measurements were then used in computations to achieve the results in the following section.

5 Results

Based on the resulting measurements (Table 1) for diameter it was determined that the mean diameter of each loop was: 5.82 ± 0.05 cm ($Loop_1$), 11.23 ± 0.05 cm ($Loop_2$), 21.48 ± 0.05 cm ($Loop_3$) and 44.63 ± 0.28 cm ($Loop_4$)[1]. Here the errors were taken to be the larger magnitude between the reading error and standard deviation. Additionally, based on these measurements it was concluded that the diameter for each hoop was approximately constant around the loop and within the error margin. Thus it is a good estimation that each loop is round.

	$Loop_1$	$Loop_2$	$Loop_3$	$Loop_4$
Trial #1	5.80 cm	11.20 cm	21.45 cm	44.20 cm
Trial #2	5.80 cm	11.20 cm	21.40 cm	44.50 cm
Trial #3	5.85 cm	11.20 cm	21.50 cm	44.40 cm
Trial #4	5.80 cm	11.25 cm	21.55 cm	44.90 cm
Trial #5	5.80 cm	11.20 cm	21.50 cm	44.90 cm
Trial #6	5.85 cm	11.30 cm	21.50 cm	44.90 cm
Mean Diameter	5.82 cm	11.23 cm	21.48 cm	44.63 cm
Reading Error	± 0.05 cm	± 0.05 cm	± 0.05 cm	± 0.05 cm
Standard Deviation	0.02 cm	0.04 cm	0.05 cm	0.28 cm

Table 1*: Diameter measurements for all four loops with reading errors, computed mean diameter and standard deviation. The reading error was taken to be half of the last precision unit (millimeters).*

Similarly, the mean period of small oscillations was determined for each loop: 0.48 ± 0.02 s ($Loop_1$), 0.68 ± 0.02 s ($Loop_2$), 0.95 ± 0.01 s ($Loop_3$) and 1.35 ± 0.03 s ($Loop_4$)[1].

	$Loop_1$	$Loop_2$	$Loop_3$	$Loop_4$
Trial #1	2.44 s	3.34 s	4.78 s	6.75 s
Trial #2	2.44 s	3.56 s	4.75 s	6.72 s
Trial #3	2.34 s	3.31 s	4.78 s	7.01 s
Trial #4	2.43 s	3.31 s	4.81 s	6.56 s
Trial #5	2.13 s	3.41 s	4.68 s	6.84 s
Trial #6	2.50 s	3.44 s	4.66 s	6.59 s

Table 2: Measurements of time taken to complete five periods for each loop.

	$Loop_1$	$Loop_2$	$Loop_3$	$Loop_4$
Trial #1	0.49 s	0.67 s	0.96 s	1.35 s
Trial #2	0.49 s	0.71 s	0.95 s	1.34 s
Trial #3	0.47 s	0.66 s	0.96 s	1.40 s
Trial #4	0.49 s	0.66 s	0.96 s	1.31 s
Trial #5	0.43 s	0.68 s	0.94 s	1.37 s
Trial #6	0.50 s	0.69 s	0.93 s	1.32 s
Mean Period	0.48 s	0.68 s	0.95 s	1.35 s
Reading Error	± 0.005 s	± 0.005 s	± 0.005 s	± 0.005 s
Standard Deviation	0.02 s	0.02 s	0.01 s	0.03 s

Table 3: Time taken for one period computed from Table 2 by dividing each measurement by five. Done for all four loops with reading errors, computed mean diameter and standard deviation. The reading error was taken to be half of the last precision unit (milliseconds).

To determine g (the acceleration due to gravity) we first make the assumption that our pendulum system is a rigid body system. This means that the loops are a continuous distribution of mass and the object does not deform over time unless outside forces are applied to it. The period for a rigid body pendulum is given by:

$$T = 2\pi\sqrt{\frac{I}{mgR}}$$

Here T is the period of oscillation, I is the moment of inertia, m is the mass of the loop, g is the acceleration due to gravity and R is the radius of the loop. However, since the object is not rotating around its center of gravity and instead around a parallel axis, we apply the parallel axis theorem which states:

$$I = I_{cm} + md^2$$

In this case I_{cm} is the moment of inertia around the center of gravity and d is the distance between the two axes. In this case $d = R$ and for a loop we find that $I = MR^2 + MR^2 = 2MR^2$. Therefore, the period equation simplified and rearranged to isolate for g is given as:

$$T = 2\pi\sqrt{\frac{2R}{g}} = 2\pi\sqrt{\frac{D}{g}}$$

$$\therefore\ g = \frac{4\pi^2 D}{T^2}$$

Using this, the acceleration due to gravity can be solved for each loop using their mean diameter and period that were computed earlier. This gives the acceleration due to gravity for each loop as: 9.97 ± 0.22 m/s^2 ($Loop_1$), 9.59 ± 0.11 m/s^2 ($Loop_2$), 9.40 ± 0.06 m/s^2 ($Loop_3$) and 9.68 ± 0.15 m/s^2 ($Loop_4$)[1]. Taking the average of these results and the highest uncertainty gives, g = 9.66 ± 0.22 m/s^2. Here the uncertainties for g were computed using the rules for error propagation.

Another method to achieve this result is to plot the relationship between T^2 and D (Figure 1). Using this information we can equate the factor, $4\pi/g$, to the slope of the line to solve for g. This method produces the result, g = 9.64 ± 0.22 m/s^2. This result matches very well with the previous method and the uncertainties were computed identically in both cases.

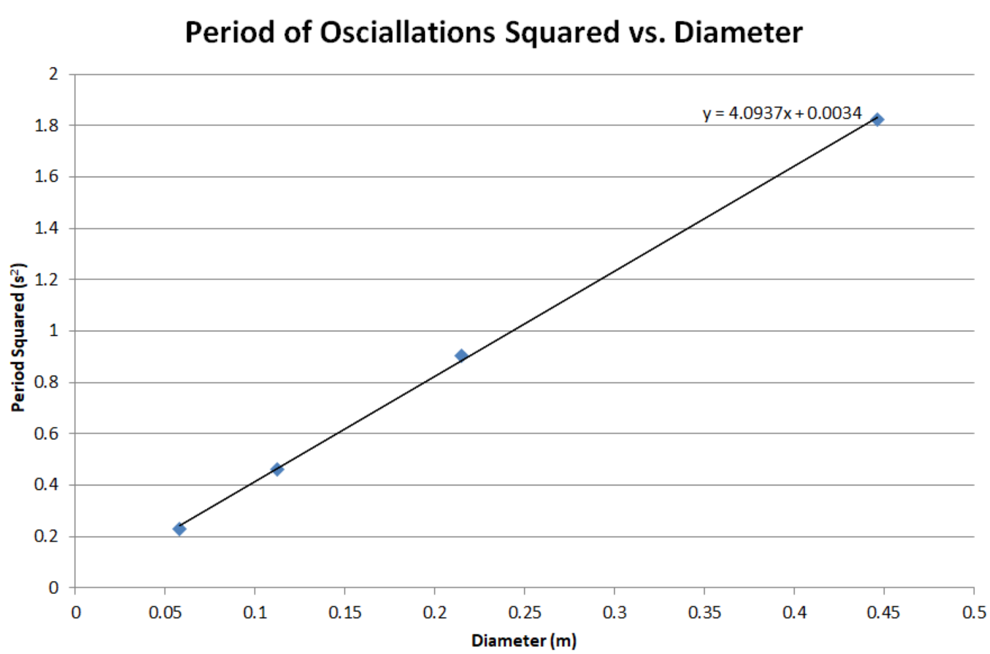

Figure 1: The linear relationship between the squared period and diameter of each loops represented by a line of best fit based on measured values. Plot generated by the author.

6 Discussion

The expected value for the acceleration due to gravity in the given location (Toronto) is g = 9.81 m/s^2. This expected value is within the uncertainty of both of our obtained results in the previous section (g = 9.66 ± 0.22 m/s^2 and g = 9.64 ± 0.22 m/s^2). The slight variation is caused by the innate uncertainty in our method and equipment. The ruler and measuring tape for example only allowed for a reasonable estimation of the loops' diameter to within half of a millimeter. Similarly, the stopwatch only gave measurements up to the millisecond. Additionally, the uncertainty of measurement (standard deviation) was significant for measuring the diameter of the largest loop as well as timing measurements for period of oscillation. In general the uncertainty of measurement dominated the experimental error except for the diameter measurements of $Loop_1$, $Loop_2$ and $Loop_3$, where reading error dominated. Based on this error analysis the experiment could be improved if the period of oscillations and diameter measurements were made using more precise equipment and done using software instead of manually. This would both reduce reading error as well as remove a lot of the uncertainty of measurement from the data. On the other hand the primary benefit of this experimental design is that it is easy to recreate and repeat for anyone aiming to verify or falsify the results of the experiment.

The results have proved to successfully illustrate the theory of simple harmonic oscillators and rigid body pendulums for the small angle approximation. Applying the parallel axis theorem we are able to analyze the experiment as a simple harmonic oscillator with the moment of inertia $I = 2MR^2$ as derived earlier. Based on the measurements and results this small angle oscillation seems to be a very good approximation of the theoretical behaviour of the system as we are able to measure the acceleration due to gravity accurately within our uncertainty.

7 Conclusion

From our results we know that g = 9.66 ± 0.22 m/s^2 (or g = 9.64 ± 0.22 m/s^2 for the alternative method), which is a good estimation of the acceleration due to gravity at our location. This is because the experimental method serves as a good approximation of a theoretical simple harmonic oscillator for small angle oscillations and based on the equations provided in the Results section, the period of such motion can be accurately determined. The uncertainty in this result is based primarily on minuscule uncertainties of measurement and reading errors and thus leads to a high confidence in the outcome of the experiment.

8 References

[1] Pandhi, A., University of Toronto, Toronto, ON. "PHY224 Laboratory Notes: Introductory Exercise – Oscillations of a Loop", September 2018.

9 Appendices

```python
#Importing required modules
import numpy as np

#Data for Loop 1
D1 = [5.80, 5.80, 5.85, 5.80, 5.80, 5.85]
D1_mean = np.mean(D1)
D1_std = np.std(D1)
Tsqr1 = [2.44/5, 2.44/5, 2.34/5, 2.43/5, 2.13/5, 2.50/5]
Tsqr1_mean = np.mean(Tsqr1)
Tsqr1_std = np.std(Tsqr1)
print('Loop 1:')
print('-------------------------------------------------------')
print('Mean diameter: ' + str(D1_mean))
print('Standard deviation in diameter: ' + str(D1_std))
print('Mean period: ' + str(Tsqr1_mean))
print('Standard deviation in period: ' + str(Tsqr1_std))
print(' ')

#Data for Loop 2
D2 = [11.20, 11.20, 11.20, 11.25, 11.20, 11.30]
D2_mean = np.mean(D2)
D2_std = np.std(D2)
Tsqr2 = [3.34/5, 3.56/5, 3.31/5, 3.31/5, 3.41/5, 3.44/5]
Tsqr2_mean = np.mean(Tsqr2)
Tsqr2_std = np.std(Tsqr2)
print('Loop 2:')
print('-------------------------------------------------------')
print('Mean diameter: ' + str(D2_mean))
print('Standard deviation in diameter: ' + str(D2_std))
print('Mean period: ' + str(Tsqr2_mean))
print('Standard deviation in period: ' + str(Tsqr2_std))
print(' ')

#Data for Loop 3
D3 = [21.45, 21.40, 21.50, 21.55, 21.50, 21.50]
D3_mean = np.mean(D3)
D3_std = np.std(D3)
Tsqr3 = [4.78/5, 4.75/5, 4.78/5, 4.81/5, 4.68/5, 4.66/5]
Tsqr3_mean = np.mean(Tsqr3)
Tsqr3_std = np.std(Tsqr3)
print('Loop 3:')
print('-------------------------------------------------------')
print('Mean diameter: ' + str(D3_mean))
print('Standard deviation in diameter: ' + str(D3_std))
print('Mean period: ' + str(Tsqr3_mean))
print('Standard deviation in period: ' + str(Tsqr3_std))
print(' ')

#Data for Loop 4
D4 = [44.20, 44.50, 44.40, 44.90, 44.90, 44.90]
D4_mean = np.mean(D4)
D4_std = np.std(D4)
Tsqr4 = [6.75/5, 6.72/5, 7.01/5, 6.56/5, 6.84/5, 6.59/5]
Tsqr4_mean = np.mean(Tsqr4)
Tsqr4_std = np.std(Tsqr4)
print('Loop 4:')
print('-------------------------------------------------------')
print('Mean diameter: ' + str(D4_mean))
print('Standard deviation in diameter: ' + str(D4_std))
print('Mean period: ' + str(Tsqr4_mean))
print('Standard deviation in period: ' + str(Tsqr4_std))
```

Figure 2*: Python script used to compute the mean and standard deviation in the measured parameters. Script generated by the author.*