PyLab 1 – Numerical Integration Methods

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1 Abstract

Spring-mass systems provide an avenue through which we can study fundamental theories in physics such as oscillating motion and dampening. In this experiment we aim to create and describe a spring-mass system in terms of position, velocity and energy for both the undamped and damped assumptions, as well as testing different numerical integration techniques to find if they provide an approximation of the system that is consistent with the theory. We find, with our initial conditions for both systems, that our results match the theory; the undamped system has a stable oscillation with energy conservation and the damped system has its positon and energy over time decreasing exponentially. Additionally the Symplectic Euler method shows a much closer fit to the theory and physical data than the Forward Euler numerical integration technique.

2 Introduction

The purpose of this lab is to determine the motion of a simple spring-mass system, and the effect that an external damping force has on the system's motion. Additionally, we aim to show that the motion matches or closely follows the theoretical predictions. Theoretically speaking, the undamped system will follow a sinusoidal wave pattern in its motion about the equilibrium position, whereas the underdamped system, though it still preserves the sinusoidal nature, the amplitude of the position and energy will decrease at the rate of $e^{-\gamma t}$, where γ is the damping constant. This is predicted to be much smaller than the natural frequency ω_0 of the undamped system. By means of solving second order differential equations involving damping force, we shall determine the constant γ , the spring constant k and hence the natural frequency ω_0 of the system in an attempt to simulate the ideal conditions of the system and compare them to the physical results.

3 Methods and Materials

The materials utilized for this experiment were: a weighted bob, a dampening disk, a stand holding a hanging spring, a motion sensor, one ruler, a data acquisition device (DAQ), the LabView application and the Jupyter Notebooks software.

4 Experimental Procedure

Before beginning the experiment, the mass of the bob was measured by an electronic scale to be 0.1999 ± 0.00005 kg. The LabView application, MotionSensor.vi, settings were updated to collect at 100 samples/second. Once the application was setup and ready to collect data, the bob was placed on the spring at approximately 0.200 ± 0.0005 m above the motion sensor.

With the experiment now set up, the bob was pulled slightly (a few centimeters) below equilibrium and released to create an oscillating motion for the spring-mass system. The data was then collected through the motion sensor in the MotionSensor.vi application for 10.0 ± 0.05 seconds.

Using the application's "Analyze" panel, the period was recorded to be T = 0.73 ± 0.001 seconds by reading the time taken for 10 oscillations and dividing by 10. Additionally, both the raw position vs. time data and velocity vs. time data from the application were saved as .txt files to be used further in our results and discussion.

Later this experiment was repeated but with the addition of a dampening disk. The setup for this run following exactly from before except it was run over 120.0 ± 0.05 seconds rather than 10.0 ± 0.05 seconds to be able to see the dampening effect on the system in more robust detail.

5 Results

Using the initial conditions and constants measured from the experiment, we were able to set up a simulation of the system over time based on two numerical integration methods. The simulation was set up with: t = 10.0 ± 0.005 seconds (total time), Δt = 0.01 seconds (time step), T = 0.73 ± 0.001 seconds (period of oscillation), ω_0 = 8.61 ± 0.074 seconds^{-1} (initial angular frequency), y_0 = 0.200 ± 0.0005 m (initial height), m = 0.1999 ± 0.00005 kg (mass) and k = 14.8 ± 0.12 N/m (spring constant). Here m, y_0, t, Δt and T were measured directly from the position vs. time data, while ω_0 and k were computed using the formulas:

$$\omega_0 = \frac{2\pi}{T} \quad and \quad k = m{\omega_0}^2$$

The first time-stepping scheme is the Forward Euler method, which follows:

$$y_{i+1} = y_i + \Delta t v_i$$

$$v_{i+1} = v_i - \Delta t {\omega_0}^2 y_i$$

Where y_i, y_{i+1}, v_i and v_{i+1} are the positions and velocities at step i and $i+1$. The second such method is the Symplectic Euler Method, which combines the Forward and Backward Euler methods and follows:

$$y_{i+1} = y_i + \Delta t v_i$$

$$v_{i+1} = v_i - \Delta t \frac{k}{m} y_{i+1}$$

The benefit of the Symplectic Euler Method is that it approximately conserves energy while the Forward Euler Method does not. The energy in this instance is expressed as:

$$E_{tot} = \frac{1}{2} m v^2 + \frac{1}{2} k y^2$$

Where the first term is the kinetic energy and the second term is the potential energy. The following are simulation plots based on these two numerical integration methods:

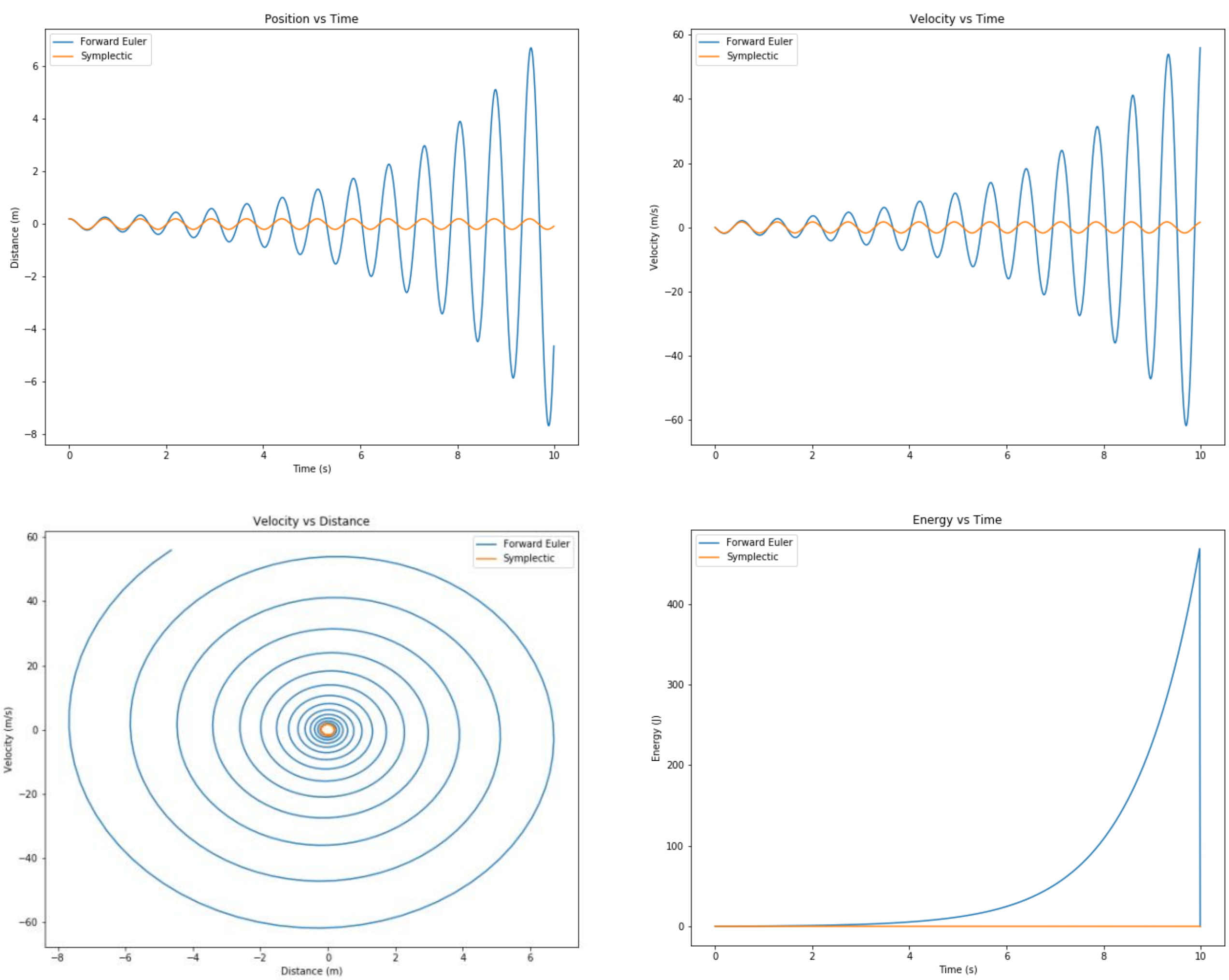


Figure 1: *Simulated plots for both numerical integration methods. Top-left: Position vs Time, Top-right: Velocity vs Time, Bottom-left: Phase plot of Velocity vs Position and Bottom-right: Energy vs Time. Note that for the bottom-left plot the sharp vertical drop should be ignored, it is an unfortunate remnant of the integration method as the final value of energy goes to 0.*

From the top two plots it is clear that the Forward Euler method is incorrect as both the amplitude of position and velocity is increasing over time. This is due to the formula being regressive such that v_{i+1} depends on y_i and so it is accumulating uncertainty at each step and thus becoming a worse approximation over time. This is clear in the phase plot as the expected orbit of a simple pendulum without dampening is an ellipse but it is in fact spiral shaped due to both position and velocity increasing. We also see this reflected in the final plot, where the energy is not conserved; instead it is increasing over time. On the other hand the Symplectic method has v_{i+1} depending on y_{i+1}, thus avoiding the problem of the Forward Euler method. In the plots for Symplectic integration we see a steady (approximately no change in amplitude) position and velocity over time. The phase plot is also as we expected: an ellipse with the energy being conserved throughout. Comparing this to our observed data from the experiment we see that indeed the Symplectic integration method is a much better approximation than the Forward Euler. This is because the observed data's position and velocity amplitude are approximately constant over time and thus the energy over time is also roughly constant.

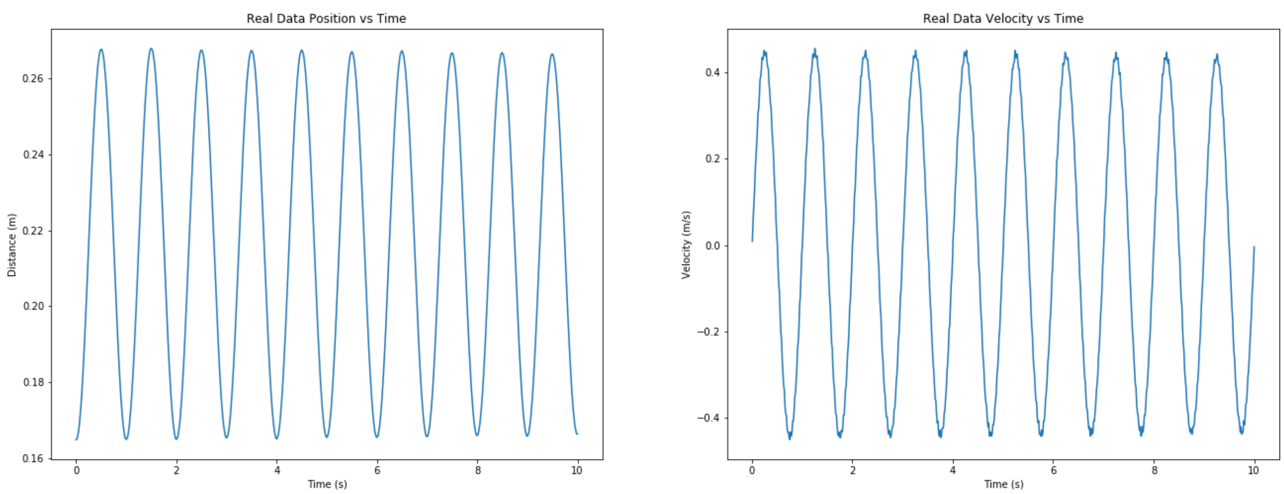


***Figure 2**: Experimental data for position vs. time (left) and velocity vs. time (right) without dampening.*

With the second run of the experiment we were able to obtain data about an analogous system with dampening effects. The initial conditions and constants for this run were: t = 120.0 ± 0.005 seconds (total time), Δt = 0.01 seconds (time step), T = 0.75 ± 0.001 seconds (period of oscillation), ω_0 = 8.38 0.074 seconds^{-1} (initial angular frequency), y_0 = 0.20 ± 0.005 m (initial height), m = 0.2166 ± 0.00005 kg (mass) and k = 15.4 ± 0.12 N/m (spring constant). The primary contributing force to the dampening effect is the drag force; the quadratic drag force is modeled as:

$$F_d = \frac{1}{2} C \rho A |v| v$$

Where C is the drag coefficient (dimensionless), ρ is the density of the medium (kg/m^3), A is the cross-sectional area perpendicular to the flow (m^2) and v is the velocity of the body relative to the medium (m/s). To characterize the flow of the spring-mass in the medium we use the Reynolds number, Re, which is defined as:

$$Re = \frac{\rho v l}{\eta}$$

Where, l is the characteristic length in the direction of flow (m) and η is the dynamic viscosity (Ns/m^2). Using the initial (in the case the max) velocity of the system we find Re = 3050 with negligible error but this quickly drops to under 2300 so we characterize the flow as approximately laminar. Thus we find that the drag force acting on the system is directly proportional to velocity and can be simplified to:

$$F_d = -\gamma v$$

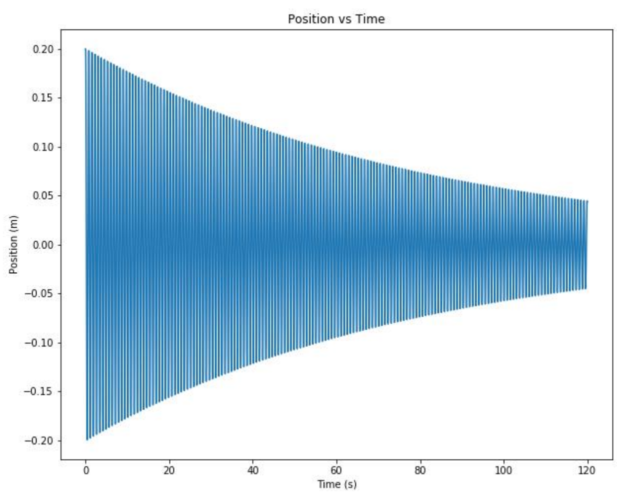
Where γ is the dampening coefficient and using this we can write the equation of motion for the system as:

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + {\omega_0}^2 y = 0$$

For our system we can compute the dampening coefficient, γ, experimentally with the equation:

$$\gamma = \frac{2}{\tau}$$

Where we have defined τ as the time it takes for the amplitude to reach $\frac{A_0}{e}$. This gives γ = 0.025 with negligible error and we can now plot the simulated position vs. time and velocity vs. time for a dampened spring-mass system.



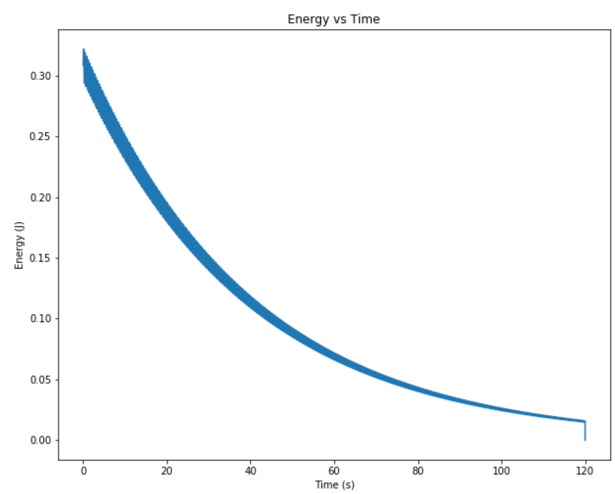


Figure 3: Simulated data for a spring-mass system with dampening. Position vs. time (left) and energy vs. time (right). Note that for the plot on the right, the sharp vertical drop at the end should be ignored; it is an unfortunate remnant of the integration method as the final value of energy goes to 0.

The simulated data shows an exponential decay in position and energy due to the dampening effect, as was expected from our previous equation for the dampening force. Although it is hard to see, there are individual oscillations within the larger structure of the plots in Figure 3. If we zoom into the first 10 seconds of the simulation we can see more robustly the decay in the oscillations on a shorter timescale, where the position and energy decrease ever so slightly with every oscillation.

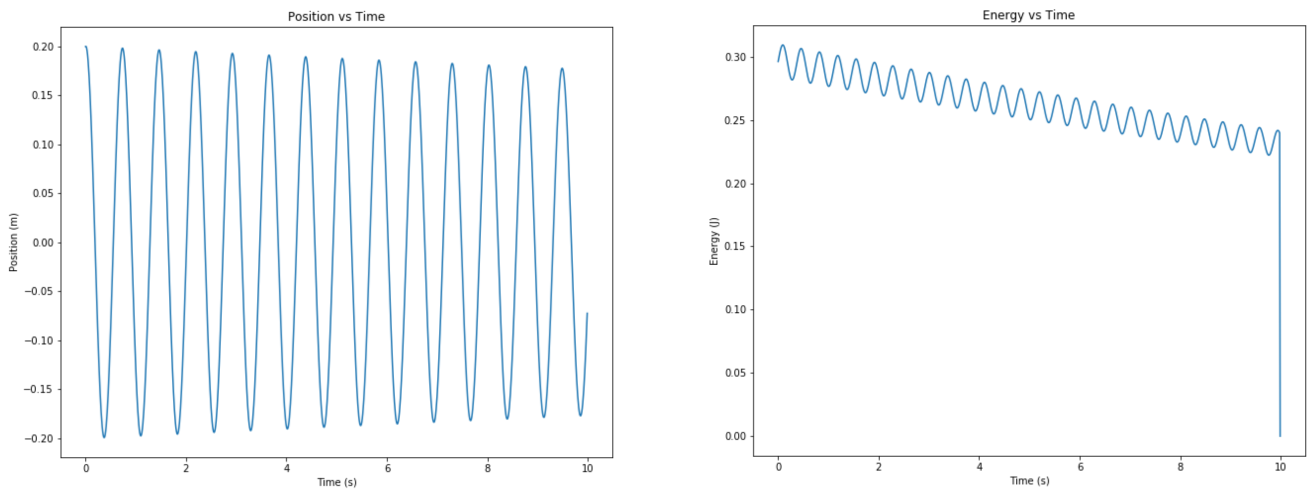


Figure 4: *Identical plots to Figure 3 but plotted in a smaller range of time to more clearly see the decay in individual oscillations. Again, the sharp drop in the energy plot should be ignored for the same reasons as before.*

This simulation data lines up fairly well with the observed data from the experiment. Both have an exponential decay in position and energy due to dampening, however the observed data is not as smooth and idealized as the simulated data. Overall, this shows that the numerical integration method implemented on the dampened system is approximately accurate when compared to the physical experiment.

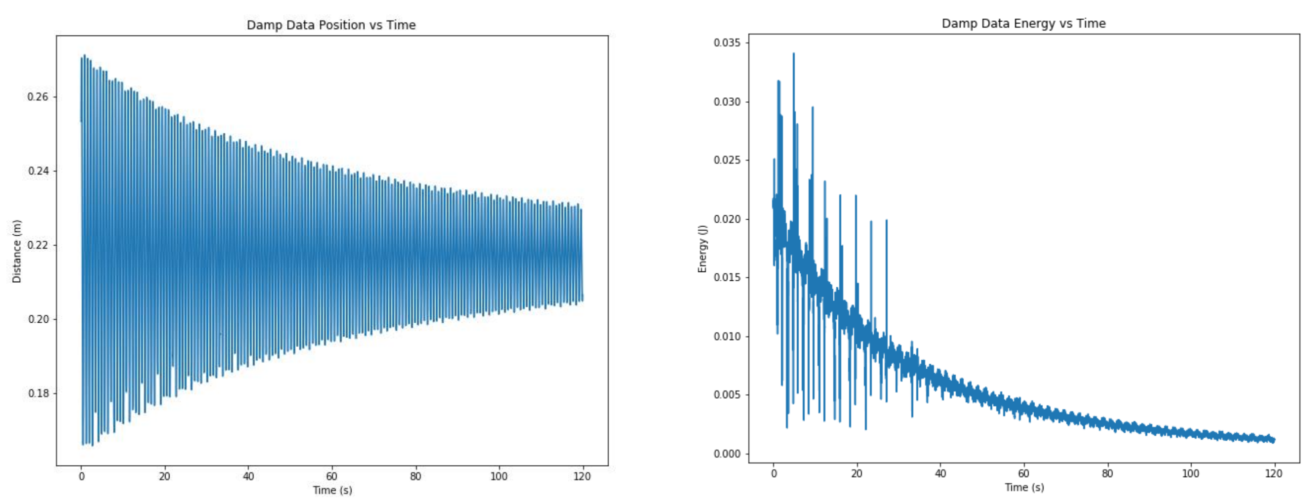


Figure 5: *Experimental data from the experiment of position vs. time (left) and energy vs. time (right).*

6 Discussion

For the first experiment the expected results were that of a simple harmonic oscillator without dampening effects. For position and velocity vs. time we expect an oscillating pattern with almost constant amplitude and thus the phase plot should trace out an elliptical orbit for the object. Throughout this process we also expect that the energy is conserved. From our simulated data we find that the first numerical integration method, Forward Euler, is not very accurate as it shows the position and velocity increasing over time. This means that for this integration scheme, energy is not conserved, instead it is increasing over time and thus the phase plot is a spiral shape rather than an ellipse. On the other hand the Symplectic integration method matches up very well with the experimental data and proves to be an accurate time-stepping scheme in which energy is also conserved.

The experiment successfully demonstrated the description of a spring-mass system in both the undamped and damped case. In the undamped case, the position and velocity displayed a periodic oscillation with almost no change over time. Meanwhile in the damped case it was determined that the system was undergoing an exponential decay in both position an energy over time, which agrees with the equation of motion that was derived earlier. In terms of the numerical integration we found that for the undamped system, the Symplectic Euler method was a much better approximation than the Forward Euler which became increasingly inaccurate over time.

The uncertainty in the experiment first arises from reading errors in t, m, v and y_0, for which we take the error to be one half of the last measurable digit on the instrument. The uncertainty in the period, T, was computed using the error proportion equation and rounded to two significant figures:

$$E_T = |\frac{1}{10}| E_t$$

Here we are dividing by 10 because the period was measured over 10 oscillations and the time taken was thus divided by a factor of 10. Then we computed the error in ω_0 using both the error propagation rules for powers and the above for constants:

$$E_{\omega_0} = |-1||2\pi||\omega_0|\frac{E_T}{T}$$

The uncertainty in γ was also found using this rule. For the spring constant, k, the error was propagated using the rule for powers first to get the error in ω_0^2 and then using the rule for multiplication:

$$E_k = |k|\sqrt{\left(\frac{E_{\omega_0^2}}{\omega_0^2}\right)^2 + \left(\frac{E_m}{m}\right)^2}$$

The uncertainty in our computed Reynold's Number, Re, was also found using this same rule.

The strength of this experimental design is that it is quite simple to set up and replicate the results. However, there are a lot of places where small adjustments can be made. For example the oscillations are generated by physically pulling the system out of equilibrium. This leads to uncertainty in the exact conditions of the system. Additionally, the experiment was repeated again in the next lab session, which means the mass and spring that were used the second time are not necessarily the same as the ones from the first lab session. To improve the experiment it would be better to label your mass and spring so you can use the same system again for the second experiment for the sake of remaining consistent. Also, the experiment would be more idealized if there was an automated oscillation generated by an instrument rather than doing it manually, thus reducing our uncertainties and margin of error.

7 Conclusion

From this experiment, we conclude that the motion of the spring-mass system is indeed sinusoidal as was expected from the theory. Applying an external damping force on the mass causes an exponential decay in amplitude for both position and energy over time, which agrees with the equation of motion for the system given the laminar flow assumption. Additionally, the results suggest that the Symplectic Euler method provides a better numerical description of the physical system than the Forward Euler method.

8 References

[1] Pandhi, A. and Chen, X., University of Toronto, Toronto, ON. "PHY224 Laboratory Notes: PyLab 1 – Numerical Integration Methods", October 2018.

9 Appendices

The code used in this lab can be found in the "PyLab 1 Code.ipynb" Jupyter notebook file. Higher resolution images of all of the figures within the report can also be found in this notebook file. Answers to the specific numbered questions in the lab document can be found in the separate document titled "PyLab 1 Questions", which is both submitted alongside this report and in the Github repository. All of the submitted files can also be found on the author's Github repository:
https://github.com/AyushPandhi/Pandhi_Ayush_PHY224/tree/master/PyLab%201.