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Families of Rotationally-Symmetric Plane Graphs with Constant Metric Dimension*

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Abstract. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(\mathrm{G})$ is finite and does not depend upon the choice of G in $\mathcal{G}$. The metric dimension of some classes of plane graphs has been determined in [1, 2, 3, 4, 11, 12, 13, 16, 23]. In this paper, we extend this study by considering some classes of plane graphs which are rotationally-symmetric. We prove that the metric dimension of these classes of plane graphs is finite and only two or three vertices chosen appropriately suffice to resolve all the vertices of these plane graphs.

Keywords: Metric dimension; Basis; Rotationally-symmetric; Resolving set; Plane graph.

1. Notation and Preliminary Results

If G is a connected graph, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$

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is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *locating set* for G [1]. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21].

For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.

A useful property in finding $dim(G)$ is the following lemma [22]:

Lemma 1.1. *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [20, 21] and studied independently by Harary and Melter in [10]. Applications of this invariant to the navigation of robots in networks are discussed in [17] and applications to chemistry in [4] while applications to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [18].

A graph G is said to be *plane* if it is drawn on the Euclidean plane such that edges do not cross each other except at vertices of the graph.

By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a fan is $f_n = K_1 + P_n$ for $n \geq 1$ and $Jahangir\ graph$ $J_{2n}, (n \geq 2)$ (also known as $gear\ graph$) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [1] determined the dimension of *wheel* W_n, Caceres *et al.* [3] the dimension of fan f_n and Tomescu and Javaid [23] the dimension of $Jahangir\ graph$ J_{2n}.

Theorem 1.2. [1, 3, 23] *Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*
(i) *For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;*
(ii) *For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;*
(iii) *For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.*

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.

On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$. In [4] it was shown that a graph has metric dimension

1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices n. In [2] it was proved that

$$dim(P_m \times C_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* D_n (also called *circular ladders*) are the trivalent plane graphs obtained by the cross product of path P_2 with a cycle C_n, this implies that

$$dim(D_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, *prisms (circular ladders)* constitute a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [13] that the plane graph *antiprism* A_n constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [15].

A *Cartesian product* of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (x, x') and (y, y') are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. A *grid* G_n^m is obtained by the cartesian product of two paths P_n by P_m. In [16], it was shown that $dim(P_n \times P_m) = 2$, so grids constitute a family of plane graphs with constant metric dimension. The metric dimension of Cartesian product of graphs has been studied in [2] and [19].

It was shown in [11] and [12] that some families of plane graphs generated by *convex polytopes* constitute the families of plane graphs with constant metric dimension. Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [8]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [17] and it was shown in [4, 17, 18, 19] that the metric dimension of *trees* can be determined efficiently. It appears unlikely that significant progress can be made in determining the dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.

In this paper, we study the metric dimension of some classes of *plane graphs that are rotationally symmetric*. We prove that the metric dimension of these classes of plane graphs is finite and does not depend upon the number of vertices in these graphs and only two or three vertices appropriately chosen suffice to resolve all the vertices of these classes of plane graphs.

2. The Plane Graph (Pentagonal Circular Ladder) R_n

The *prism* D_n (*circular ladder*), $n \geq 3$ is a cubic graph which can be defined as the cartesian product $P_2 \times C_n$ on a path on two vertices with a cycle on n vertices. Prism D_n, $n \geq 3$ consists of an outer n-cycle $y_1y_2...y_n$, an inner n-cycle

$x_1x_2...x_n$, and a set of n spokes $x_iy_i, i = 1, 2, ..., n$. $|V(D_n)| = 2n$, $|E(D_n)| = 3n$ and $|F(D_n)| = n + 2$.

The *plane graph (pentagonal circular ladder)* R_n is obtained from the graph of a prism D_n (circular ladder) by adding a new vertex z_i between y_i and y_{i+1} for $i = 1, 2, ..., n$.

The *plane graph (pentagonal circular ladder)* R_n (Fig. 1) has

$$V(R_n) = \{x_i; y_i; z_i : 1 \le i \le n\}$$

and

$$E(R_n) = \{x_ix_{i+1} : 1 \le i \le n\} \cup \{x_iy_i; y_iz_i; y_{i+1}z_i : 1 \le i \le n\}.$$

The plane graph (pentagonal circular ladder) R_n has $3n$ vertices and $4n$ edges. In the next theorem, we prove that only two vertices appropriately chosen suffice to resolve all the vertices of plane graph (pentagonal circular ladder) R_n. For our purpose, we call the cycle induced by $\{x_i : 1 \le i \le n\}$, the inner cycle, cycle induced by $\{y_i : 1 \le i \le n\} \cup \{z_i : 1 \le i \le n\}$, the outer cycle. Note that the choice of appropriate basis vertices (also refereed to as landmarks in [16]) is crucial.

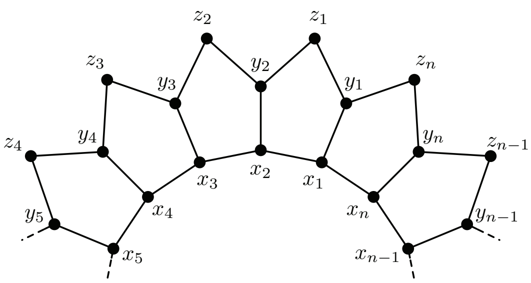

Figure 1: The plane graph R_n

Theorem 2.1. *Let the plane graph R_n be defined as above; then $dim(R_n) = 2$ for every $n \ge 6$.*

Proof. We will prove the above equality by double inequalities.
Case 1. When n is even.

In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{y_1, y_{k-1}\} \subset V(R_n)$, we show that W is a resolving set for R_n in this case. For this we give representations of all vertex of $V(R_n)\backslash W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i, k-i), & 1 \le i \le k-1; \\ (k, 2), & i = k; \\ (k+1, 3), & i = k+1; \\ (2k-i+3, i-k+2), & k+2 \le i \le 2k-1; \\ (2, k), & i = 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(y_i|W) = \begin{cases} (2,k+1), & i=1; \\ (i+1,k-i+1), & 2 \le i \le k-3; \\ (k-1,2), & i=k-2; \\ (k+1,2), & i=k; \\ (2k-i+1,i-k+3), & k+1 \le i \le 2k-1; \\ (2,k+1), & i=2k. \end{cases}$$

and

$$r(z_i|W) = \begin{cases} (1,k), & i=1; \\ (3,k-1), & i=2; \\ (i+2,k-i+1), & 3 \le i \le k-4; \\ (k-1,3), & i=k-3; \\ (k,1), & i=k-2; \\ (k+1,1), & i=k-1; \\ (k+2,3), & i=k; \\ (2k-i+3,i-k+4), & k+1 \le i \le 2k-2; \\ (3,k+2), & i=2k-1; \\ (1,k+1), & i=2k. \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(R_n) \le 2$. On the other hand, since the plane graph (pentagonal circular ladder) R_n is not a path, so the result from [4] implies that $dim(R_n) = 2$ in this case.

Case 2. When n is odd.

In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{y_1, y_{k-1}\} \subset V(R_n)$, again we show that W is a resolving set for R_n in this case. For this we give representations of all vertices of $V(R_n) \backslash W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i,k-i), & 1 \le i \le k-1; \\ (k,2), & i=k; \\ (k+1,3), & i=k+1; \\ (2k-i+3,i-k+2), & k+2 \le i \le 2k-1; \\ (3,k+1), & i=2k; \\ (2,k), & i=2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(y_i|W) = \begin{cases} (2,k-1), & i=2; \\ (i+1,k-i+1), & 3 \le i \le k-3; \\ (k-1,2), & i=k-2; \\ (k+1,2), & i=k; \\ (k+2,4), & i=k+1; \\ (2k-i+4,i-k+3), & k+2 \le i \le 2k-1; \\ (4,k+2), & i=2k; \\ (2,k+1), & i=2k+1. \end{cases}$$

and

$$r(z_i|W) = \begin{cases} (1,k), & i=1; \\ (3,k-1), & i=2; \\ (i+2,k-i+1), & 3 \le i \le k-4; \\ (k-1,3), & i=k-3; \\ (k,1), & i=k-2; \\ (k+1,1), & i=k-1; \\ (k+2,3), & i=k; \\ (2k-i+4,i-k+4), & k+1 \le i \le 2k-1; \\ (3,k+2), & i=2k; \\ (1,k+1), & i=2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representation which implies that $dim(R_n) \le 2$. On the other hand, by the same reasoning as in Case 1 it implies that $dim(R_n) = 2$ in this case, which completes the proof.∎

The plane graph antiprism A_n can be obtained from the graph of a prism D_n (circular ladder) by adding new edges $x_{i+1}y_i$ and having the same vertex set. i.e., $V(A_n) = V(D_n)$ and $E(A_n) = E(D_n) \cup \{x_{i+1}y_i : 1 \le i \le n\}$. It was proved in [13] that by adding new edges in prism D_n affects its metric dimension as $dim(A_n) = 3$ for every $n \ge 5$.

In the next section, we extend this study by considering the plane graph S_n which is obtained from the plane graph (pentagonal circular ladder) R_n by adding new edges y_iy_{i+1} and having the same vertex set. We prove that if we add new edges y_iy_{i+1} to the plane graph (pentagonal circular ladder) R_n, the metric dimension of the resulting graph is increased.

3. The Plane Graph S_n

The *plane graph* S_n (Fig. 2) is obtained from the plane graph (pentagonal circular ladder) R_n by adding new edges y_iy_{i+1} and having the same vertex set. The plane graph S_n has

$$V(S_n) = \{x_i; y_i; z_i : 1 \le i \le n\}$$

and

$$E(S_n) = \{x_ix_{i+1}; y_iy_{i+1} : 1 \le i \le n\} \cup \{x_iy_i; y_iz_i; y_{i+1}z_i : 1 \le i \le n\}.$$

The plane graph S_n has $3n$ vertices and $5n$ edges. In the next theorem, we prove that the metric dimension of the plane graph S_n is 3. For our purpose, we call the cycle induced by $\{x_i : 1 \le i \le n\}$, the inner cycle, cycle induced by $\{y_i : 1 \le i \le n\}$, the outer cycle and set of vertices $\{z_i : 1 \le i \le n\}$, the outer vertices.

Theorem 3.1. *Let the plane graph S_n be defined as above; then $dim(S_n) = 3$ for every $n \ge 6$.*

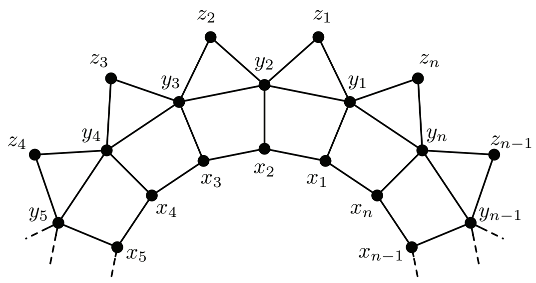

Figure 2: The plane graph S_n

Proof. We will prove the above equality by double inequalities.
Case 1. When n is even.

In this case, we can write $n = 2k$, $k \geq 2$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(S_n)$, we show that W is a resolving set for S_n in this case. For this we give representations of all vertices of $V(S_n)\backslash W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(y_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of outer vertices are

$$r(z_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(S_n) \leq 3$.

On the other hand, we show that $dim(S_n) \geq 3$. Suppose on contrary that $dim(S_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is x_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(x_n|\{x_1, x_t\}) = r(y_1|\{x_1, x_t\}) = (1, t)$ and for $t = k+1$, we have $r(x_2|\{x_1, x_{k+1}\}) = r(x_n|\{x_1, x_{k+1}\}) = (1, k-1)$, a contradiction.

(2) Both vertices are in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is y_1. Suppose that the second resolving vertex is y_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k+1$, we have $r(x_1|\{y_1, y_t\}) = r(y_n|\{y_1, y_t\}) = (1, t)$, a contradiction.

(3) Both vertices are in set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is z_1. Suppose that the second resolving vertex is z_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k-1$, we have $r(x_1|\{z_1, z_t\}) = r(z_n|\{z_1, z_t\}) = (2, t+1)$, if $t = k$, we have $r(x_1|\{z_1, z_{k+1}\}) = r(x_2|\{z_1, z_{k+1}\}) = (2, k+1)$ and for $t = k+1$ we have $r(y_1|\{z_1, z_t\}) = r(y_2|\{z_1, z_t\}) = (1, k)$ a contradiction.

(4) One vertex is in the inner cycle and the other in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is y_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(x_n|\{x_1, y_t\}) = r(x_2|\{x_1, y_t\}) = (1, 2)$ and if $2 \leq t \leq k+1$, $r(x_2|\{x_1, y_t\}) = r(y_1|\{x_1, y_t\}) = (1, t-1)$, a contradiction.

(5) One vertex is in the inner cycle and the other in the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is z_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(y_n|\{x_1, z_t\}) = r(z_n|\{x_1, z_t\}) = (2, t+1)$, if $t = k$, we have $r(z_1|\{x_1, z_t\}) = r(y_n|\{x_1, z_t\}) = (2, k)$ and for $t = k+1$, $r(x_{n-1}|\{x_1, z_t\}) = r(y_n|\{x_1, z_t\}) = (2, k-1)$, a contradiction.

(6) One vertex is in the outer cycle and the other in the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is y_1. Suppose that the second resolving vertex is z_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(x_1|\{y_1, z_t\}) = r(y_n|\{y_1, z_t\}) = (1, t+1)$, if $t = k$, $r(x_1|\{y_1, z_t\}) = r(z_n|\{y_1, z_t\}) = (1, k+1)$ and for $t = k+1$, $r(x_1|\{y_1, z_t\}) = r(z_1|\{y_1, z_t\}) = (1, k+1)$ a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for $V(S_n)$ implying that $dim(S_n) = 3$ in this case.

Case 2. When n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(S_n)$, again we show that W is a resolving set for S_n in this case also. For this we give representations of all vertex of $V(S_n) \backslash W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(y_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (k, k+1, 2), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of outer vertices are

$$r(z_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(S_n) \leq 3$.

On the other hand, suppose that $dim(S_n) = 2$, then there are the same possibilities as in Case 1 and contradictions can be deduced analogously. This implies that $dim(S_n) = 3$ in this case, which completes the proof. ■

It has been proved in recent work that some plane graphs generated by convex polytopes have constant metric dimension [11, 12]. It has been shown in [12] that some classes of graph of convex polytopes G' (which are obtained from graph of convex polytope G by adding new edges in G such that $V(G') = V(G)$) have the same metric dimension as graph of convex polytope G. In the next section, we show that by adding new edges $x_{i+1}y_i$ does not affect the metric dimension of the plane graph S_n.

4. The Plane Graph T_n

The *plane graph* T_n (Fig. 3) is obtained from the plane graph S_n by adding new edges $x_{i+1}y_i$ and having the same vertex set. The plane graph T_n has

$$V(T_n) = \{x_i; y_i; z_i : 1 \leq i \leq n\}$$

and

$$E(T_n) = \{x_i x_{i+1}; y_i y_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i; y_i z_i; x_{i+1} y_i; y_{i+1} z_i : 1 \leq i \leq n\}.$$

Clearly, the plane graph T_n has $3n$ vertices and $6n$ edges. In the next theorem, we prove that the metric dimension of the plane graph T_n is 3. For our purpose, we call the cycle induced by $\{x_i : 1 \leq i \leq n\}$, the inner cycle, cycle induced by $\{y_i : 1 \leq i \leq n\}$, the outer cycle and set of vertices $\{z_i : 1 \leq i \leq n\}$, the outer vertices.

Theorem 4.1. *Let the plane graph T_n be defined as above; then $dim(T_n) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities.
Case 1. When n is even.

In this case, we can write $n = 2k$, $k \geq 2$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(T_n)$, we show that W is a resolving set for T_n in this case. For this we give representations of all vertices of $V(T_n) \backslash W$ with respect to W.

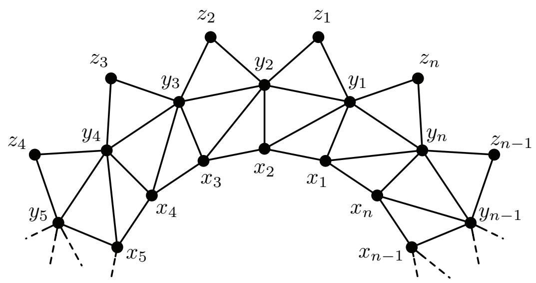

Figure 3: The plane graph T_n

Representations of the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(y_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of outer vertices are

$$r(z_i|W) = \begin{cases} (2, 2, k), & i = 1; \\ (i+1, i, k-i+1), & 2 \leq i \leq k-1; \\ (k+1, k, 2), & i = k; \\ (2k-i+1, 2k-i+2, i-k+1), & k+1 \leq i \leq 2k-1; \\ (2, 2, k+1), & i = 2k. \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(T_n) \leq 3$.

On the other hand, we show that $dim(T_n) \geq 3$. Suppose on contrary that $dim(T_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is x_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(x_n|\{x_1, x_t\}) = r(y_n|\{x_1, x_t\}) = (1, t)$ and for $t = k+1$, we have $r(x_2|\{x_1, x_{k+1}\}) = r(x_n|\{x_1, x_{k+1}\}) = (1, k-1)$, a contradiction.

(2) Both vertices are in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is y_1. Suppose that the second resolving vertex is y_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k+1$, we have $r(x_1|\{y_1, y_t\}) = r(y_n|\{y_1, y_t\}) = (1, t)$, a contradiction.

(3) Both vertices are in set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is z_1. Suppose that the second resolving vertex is z_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we

have $r(x_1|\{z_1, z_t\}) = r(y_n|\{z_1, z_t\}) = (2, t+1)$ and for $t = k+1$, we have $r(x_1|\{z_1, z_{k+1}\}) = r(x_2|\{z_1, z_{k+1}\}) = (2, k+1)$, a contradiction.

(4) One vertex is in the inner cycle and the other in the outer cycle. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is y_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(x_n|\{x_1, y_t\}) = r(x_2|\{x_1, y_t\}) = (1, 2)$ and if $2 \le t \le k+1$, $r(x_2|\{x_1, y_t\}) = r(y_1|\{x_1, y_t\}) = (1, t-1)$, a contradiction.

(5) One vertex is in the inner cycle and the other in the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that the second resolving vertex is z_t $(1 \le t \le k+1)$. Then for $t = 1$, we have $r(x_2|\{x_1, z_t\}) = r(y_n|\{x_1, z_t\}) = (1, 2)$, if $2 \le t \le k$, we have $r(x_1|\{x_1, z_t\}) = r(y_1|\{x_1, z_t\}) = (1, t)$, for $t = k+1$, $r(x_n|\{x_1, z_{k+1}\}) = r(y_1|\{x_1, z_{k+1}\}) = (2, k+1)$, a contradiction.

(6) One vertex is in the outer cycle and the other in the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is y_1. Suppose that the second resolving vertex is z_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(x_1|\{y_1, z_t\}) = r(y_n|\{y_1, z_t\}) = (1, t+1)$, for $t = k+1$, $r(x_2|\{y_1, z_{k+1}\}) = r(y_n|\{y_1, z_{k+1}\}) = (1, k+1)$, a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for $V(T_n)$ implying that $dim(T_n) = 3$ in this case.

Case 2. When n is odd.

In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{x_1, x_2, x_{k+1}\} \subset V(T_n)$, again we show that W is also a resolving set for T_n in this case also. For this we give representations of all vertices of $V(T_n)\backslash W$ with respect to W.

Representations of the vertices on inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(y_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k+1. \end{cases}$$

Representations of outer vertices are

$$r(z_i|W) = \begin{cases} (2, 2, k), & i = 1; \\ (i+1, i, k-i+1), & 2 \le i \le k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \le i \le 2k; \\ (2, 2, k+1), & i = 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representation which implies that $dim(T_n) \le 3$.

On the other hand, suppose that $dim(T_n) = 2$, then there are the same possibilities as in Case 1 and contradiction can be deduced analogously. This implies that $dim(T_n) = 3$ in this case, which completes the proof. ∎

5. Concluding Remarks

In this paper, we have studied the metric dimension of some classes of plane graphs that are rotationally-symmetric. We prove that the metric dimension of these classes of rotationally symmetric plane graphs is finite and does not depend upon the number of vertices in these graphs and only two or three vertices appropriately chosen suffice to resolve all the vertices of these classes of plane graphs. It is natural to ask for the characterizations of families of plane graphs with constant metric dimension.

Note that in [18] Melter and Tomescu gave an example of infinite regular graphs (namely the digital plane endowed with city-block and chessboard distances, respectively) having no finite metric basis. We close this section by raising some questions that naturally arise from the text.

Open Problem 1: Characterize the families of rotationally-symmetric plane graphs G' obtained from G by adding new edges in G such that such that $V(G') = V(G)$ and $dim(G') > dim(G)$.

Open Problem 2: Characterize the families of rotationally-symmetric plane graphs G' obtained from G by adding new edges in G such that such that $V(G') = V(G)$ and $dim(G') = dim(G)$.

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