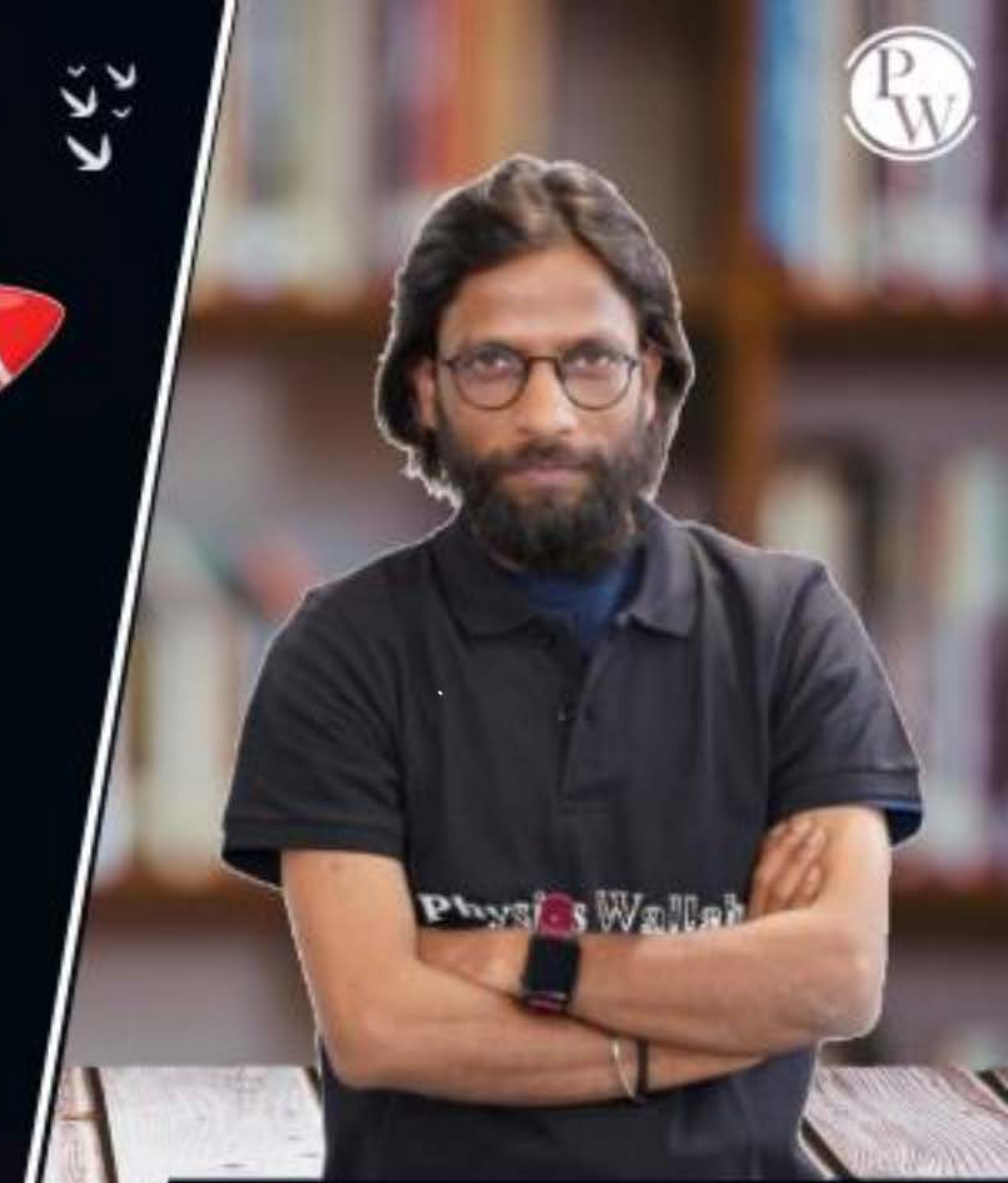
GATE-ALL BRANCHES
ENGINEERING MATHEMATICS

SINGLE VARIABLE CALCULUS

Lecture No.- 11



By- Rahul sir

Recap of Previous Lecture

Topic Problems based on definite integrals

Topics to be covered

Topic Concepts based on definite integrals

Topic Area based problems

Topic Improper integral

Topic : Definite Integrals

#Q. If for non zero $x, af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ $\times a$

$x \to \frac{1}{x}$ $af\left(\frac{1}{x}\right) + bf(x) = x - 5$ $\times b$

where $a \neq b$, then $\int_1^2 f(x)dx = \ldots\ldots\ldots$

$a^2 f(x) + \cancel{ab f\left(\frac{1}{x}\right)} = a\left(\frac{1}{x} - 5\right)$

$-\;\cancel{ab f\left(\frac{1}{x}\right)} + b^2 f(x) = b(x-5)$

$f(x) = \frac{1}{(a^2-b^2)}\left[a\left(\frac{1}{x} - 5\right) - b(x-5)\right]$

$$= \frac{1}{(a^2-b^2)}\int_1^2 \left(\frac{a}{x} - 5a - bx + 5b\right)$$

$$= \frac{1}{(a^2-b^2)}\left[a\ln x - 5ax - b\frac{x^2}{2} + 5bx\right]_1^2$$

MSQ IIT-advanced

#Q. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

$\int y\,dx$ ↑

$\int x\,dy$ ←

(A) $\cancel{e - 1}$

(B) $\int_1^e \ln(e + 1 - y)dy$

(C) $\boxed{e - \int_0^1 e^x dx}$

(D) $\int_1^e \ln y\,dy$

Area bounded via region

$y = e,\ y = e^x$

$e = e^x$

$(x = 1)\ (y = e)$

Area Bounded region

$$= \int_0^1 (e - e^x)\,dx = \int_0^1 e - \int_0^1 e^x\,dx$$

$$= e - \int_0^1 e^x\,dx$$

C

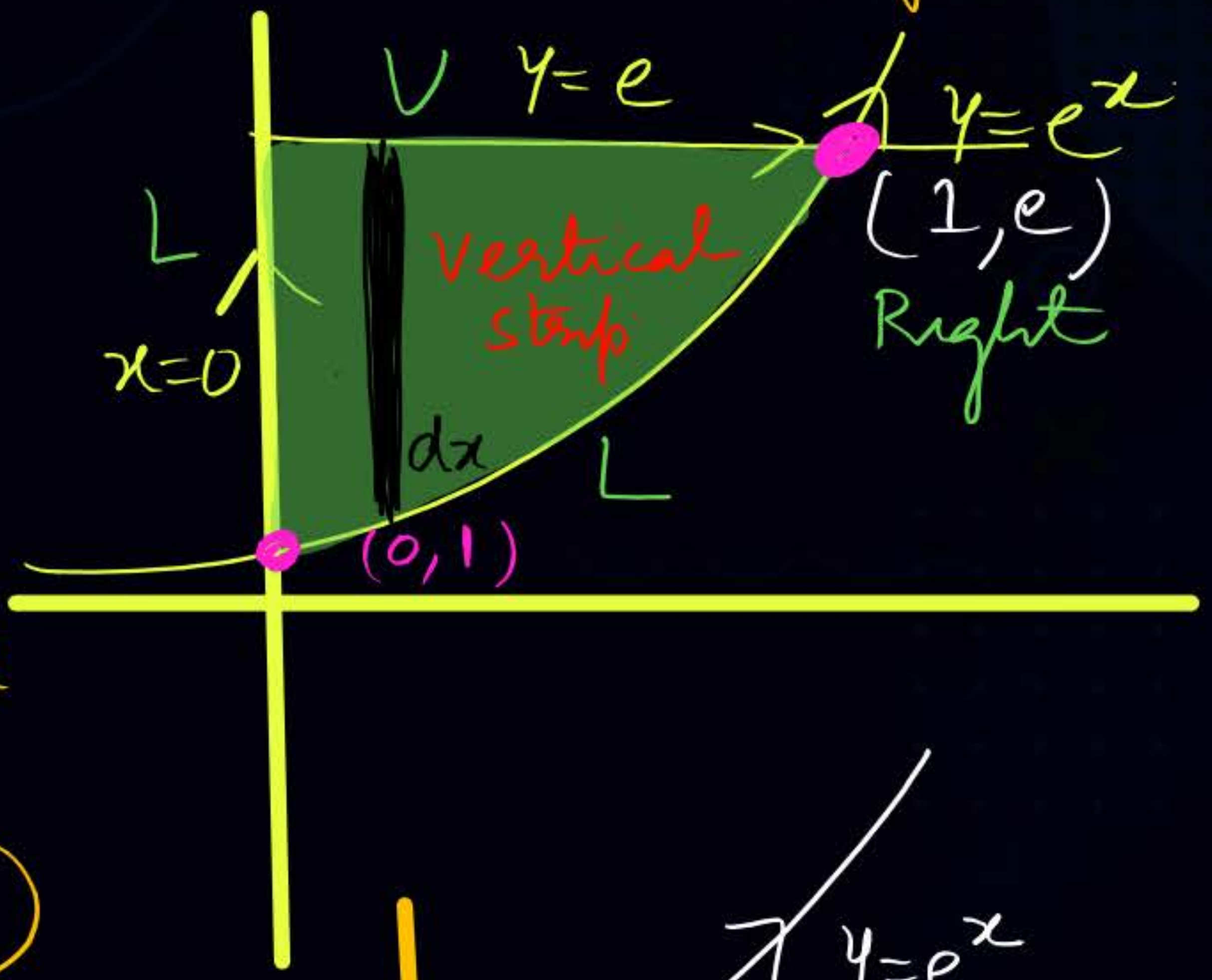
correct

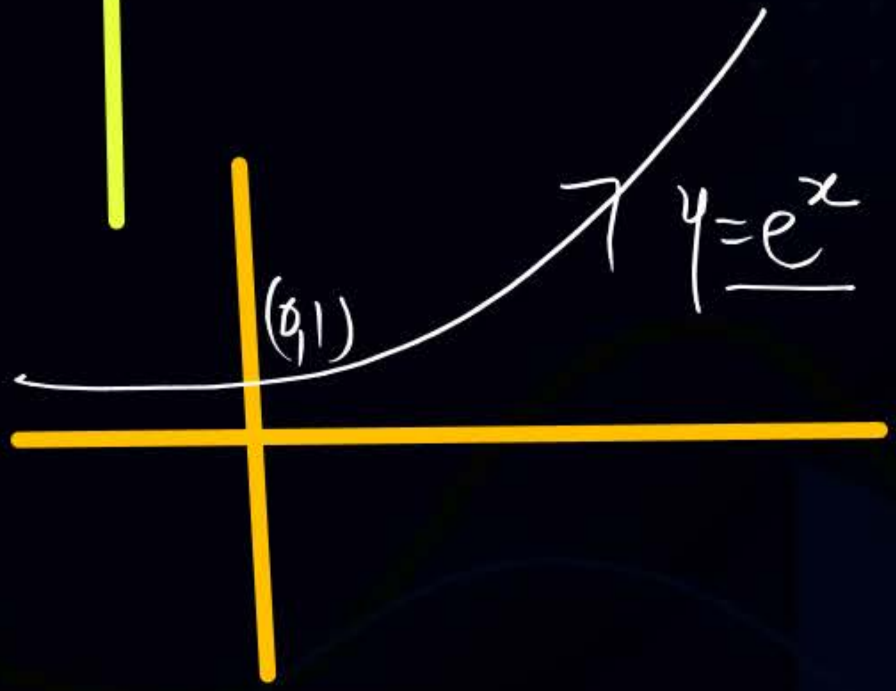
$$\text{Area} = e(1-0) - \left[e^x\right]_0^1$$

$$= e - \left[e^1 - e^0\right]$$

$$= e - e + 1 = \underline{1 \text{ Sq. unit}}$$

$y = e^x \quad x = 0 \quad y = e$

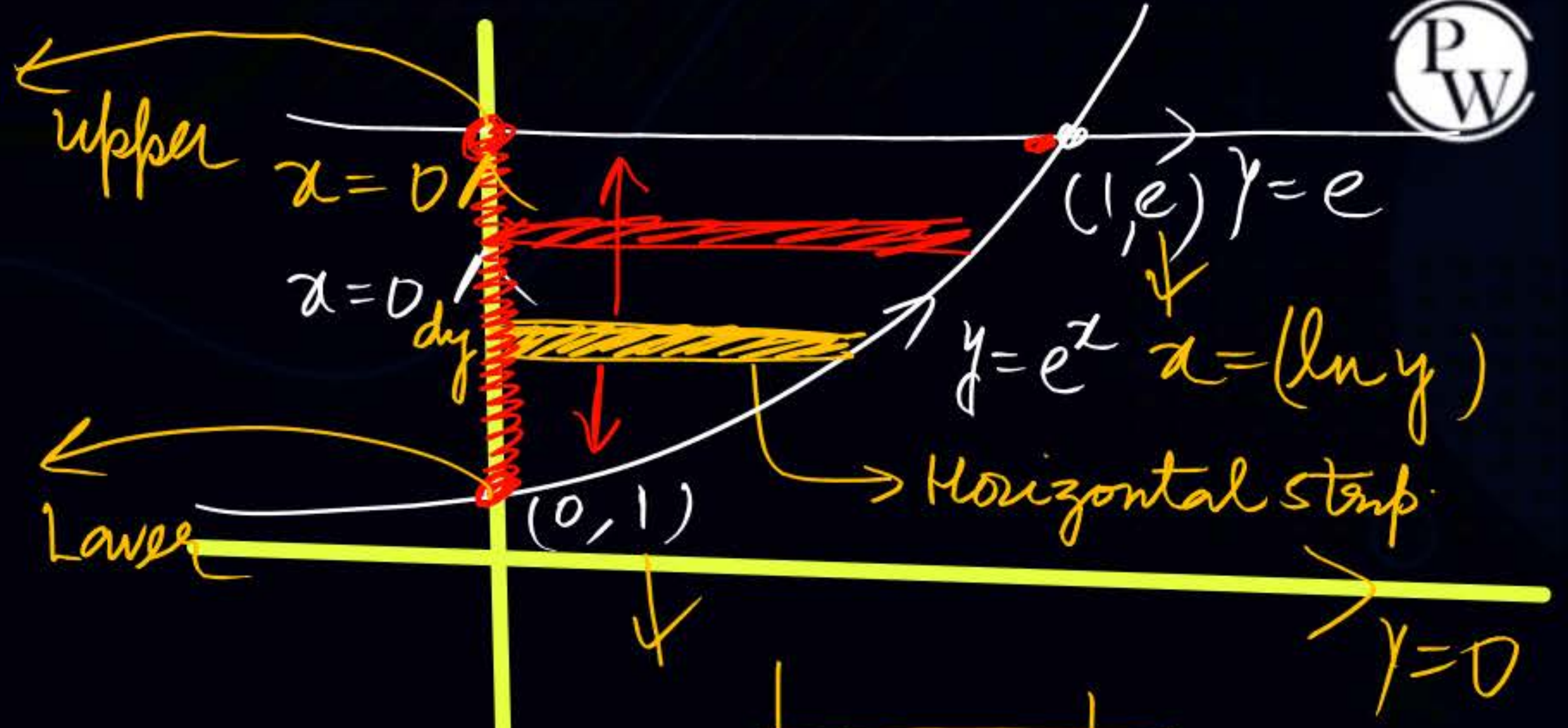




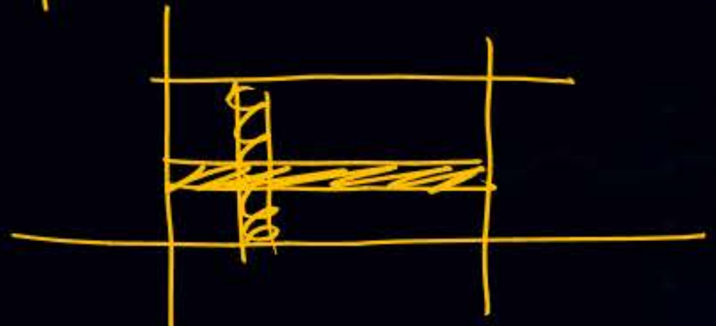
$e^x = y \quad x = \ln y$

Area bounded regions

$$A = \int_{y=1\,(\text{Lower})}^{e\,(\text{upper})} (\text{Right} - \text{Left})\, dy$$



$$A = \int_1^e [\ln y - 0]\, dy = \int_1^e \ln y\, dy \Rightarrow (B) \text{ correct}$$



Horizontal strip

y fixed – constant

x limit – variable

Vertical strip

x – constant

y – variable

$$A = \int_1^e \ln y\, dy = \int_1^e \ln(1+e-y)\, dy \quad (D) \text{ correct}$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Topic : Definite Integrals

#Q. The value of the integral is $I = \int_0^{\pi/4} \cos^2 x\,dx$

A. $\frac{\pi}{8} + \frac{1}{4}$ ✓

B. $\frac{\pi}{8} - \frac{1}{4}$

C. $\frac{-\pi}{8} - \frac{1}{4}$

D. $\frac{-\pi}{8} + \frac{1}{4}$

$I = \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2x}{2}\right) dx$

$\cos 2x = 2\cos^2 x - 1$

$2\cos^2 x = 1 + \cos 2x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{\pi}{8} + \frac{1}{4}$

Ans

Convergent – bounded
Divergent – unbounded

#Q. Which of the following integrals is unbounded?

(A) $\int_0^{\pi/4} \tan x\, dx$

bounded $= [\log \sec x]_0^{\pi/4} = \log \sec\frac{\pi}{4} - \log \sec 0$
convergent $= \log \sqrt{2}$

(B) $\int_0^{\infty} \frac{1}{1+x^2} dx$

Convergent-bounded

$= \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2}$

$= \lim_{t\to\infty} \tan^{-1} t - \tan^{-1} 0$

$= \tan^{-1}(\infty) = \frac{\pi}{2}$

(C) $\int_0^{\infty} x.e^{-x} dx$

convergent bounded

(D) $\int_0^{1} \frac{1}{1-x} dx$

$$I = \int_0^\infty x e^{-x}\,dx$$

$x \quad + \quad e^{-x}$

$1 \quad - \quad -e^{-x}$

$0 \quad \quad +e^{-x}$

$$= \left[-xe^{-x} - e^{-x}\right]_0^\infty$$

$$= \left[-\underline{\infty \times e^{-\infty}} - e^{-\infty}\right] - \left[0 - e^0\right]$$

$$= \left[0 - 0\right] - \left[0 - 1\right]$$

$$= 1$$

D) $\int_0^1 \frac{1}{(1-x)}\,dx$

$$= \left[-\log(1-x)\right]_0^1$$

$$= -\log(1-1) + \log 1$$

$$= \boxed{\text{Unbounded}}$$

$= \text{Divergent}$

D option correct

H.W

#Q. The value of the integral $\int_{-\pi/2}^{\pi/2} (x\cos x)\,dx$ is

A. 0

B. $\pi - 2$

C. π

D. $\pi + 2$

Topic : Definite Integrals

$e^{ix} = \cos x + i\sin x$

#Q. Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral

$$\int_0^{\pi/2} \frac{(\cos x + i\sin x)}{\cos x - i\sin x}\,dx?$$

$$= \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}\,dx$$

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi$

$e^{2\pi i} = 1$ Real

Irrational

Imaginary

Irrational

A 0

B 2

C $-i$

D i

$$I = \int_0^{\frac{\pi}{2}} \frac{e^{ix}}{e^{-ix}}\, dx$$

$$= \int_0^{\pi/2} e^{2ix}\, dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{e^{\cancel{2}i \times \frac{\pi}{\cancel{2}}}}{2i} - \frac{e^{0}}{2i}$$

$$= \frac{1}{2i}\left[e^{i\pi} - 1\right]$$

$$= \frac{1}{2i}\left[(\cos\pi + i\,\cancel{\sin\pi}^{\,0}) - 1\right]$$

$$= \frac{1}{2i}\left[-1 - 1\right]$$

$$= \frac{-\cancel{2}}{\cancel{2}i}$$

$$= -\frac{1}{i} \times \frac{i}{i}$$

$$= \frac{-i}{i^2} = \frac{\cancel{-}i}{\cancel{-}1}$$

$$= i$$

$$\frac{\to 0 \times \to \infty}{\text{Undefined}}$$

$$\frac{0 \times \infty}{0}$$

Topic : Definite Integrals

I.W

$\ln x = t$

#Q. The value of the definite integral $\int_1^e \sqrt{x}\ln(x)dx$ is

A $\frac{4}{9}\sqrt{e^3}+\frac{2}{9}$

B $\frac{2}{9}\sqrt{e^3}-\frac{4}{9}$

C $\frac{2}{9}\sqrt{e^3}+\frac{4}{9}$

D $\frac{4}{9}\sqrt{e^3}-\frac{2}{9}$

$$I = \int_1^e \sqrt{x}\ln x\,dx$$

$$= \int_0^1 e^{t/2}\, t\, e^t\, dt$$

$$\Rightarrow \int_0^1 e^{3t/2}\cdot t\,dt$$

$\ln x = t$

$x = e^t$

$dx = e^t\,dt$

$\ln 1 = t$

$t = 0$

$\ln_e e = t$

$1 = t$

$t \quad + \quad e^{3t/2}$

$1 \quad - \quad \frac{2}{3}e^{3t/2}$

$0 \quad \quad \frac{4}{9}e^{3t/2}$

$$= \left[\frac{2}{3}t\,e^{3t/2} - \frac{4}{9}e^{3t/2}\right]_0^1$$

$$= \frac{2}{9}e^{3/2} + \frac{4}{9}$$

#Q. The value of the integral $\int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2+\cos(x-1)}dx$ is

$x-1=t$ $\quad 0-1=t$

$dx=dt$ $\quad -1=t$

$2-1=t$

$t=1$

A 3

B 0

C -1

D -2

$$=\int_{-1}^{1}\frac{t^2\sin t}{t^2+\cos t}dt=0$$

$$f(t)=\frac{t^2\sin t}{t^2+\cos t}$$

$$f(-t)=\frac{(-t)^2\sin(-t)}{(-t)^2+\cos(-t)}$$

$$=\frac{-t^2\sin t}{t^2+\cos t}$$

$$=-f(t)$$ odd Function

H.W

#Q. $\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2} dx =$ ____.

Topic : Definite Integrals

#Q. Consider the following definite integral $I=\int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$

The value of the integral is

A $\frac{\pi^3}{24}$

B $\frac{\pi^3}{12}$

C $\frac{\pi^3}{48}$

D $\frac{\pi^3}{64}$

$\sin^{-1}x = t$

$\frac{1}{\sqrt{1-x^2}}dx = dt$

$= \int_0^{\pi/2} t^2\, dt$

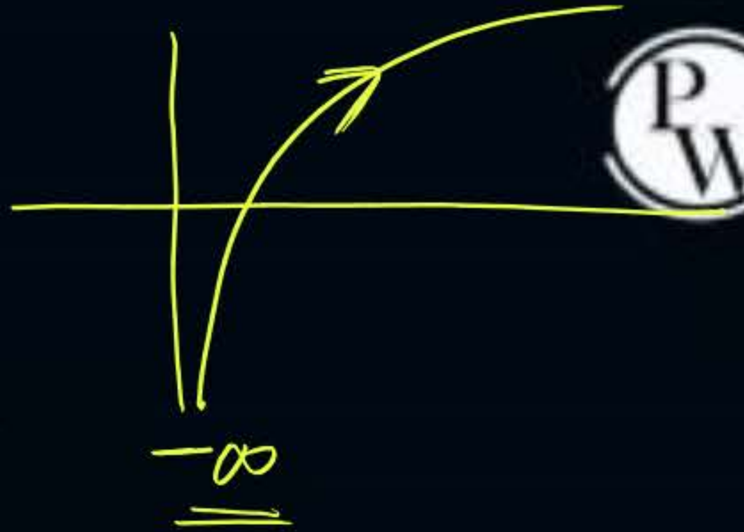
$= \left[\frac{t^3}{3}\right]_0^{\pi/2}$

$= \frac{\pi^3}{8\times 3} = \frac{\pi^3}{24}$

H.W

#Q. The value of $\int_0^{\pi/4} x\cos\left(x^2\right)dx$ correct to three decimal place (assuming that $\pi = 3.14$) is____.

Questions



Let f be a real valued function of a real variable defined as f(x) = x – [x] where [x] denotes the largest less than or equal to x . The value of

$$\int_{0.25}^{1.25} f(x)dx$$ is ____________ (up to 2 decimal places).

$$\int_{0.25}^{1.25}\left(x-[x]\right)dx = \int_{0.25}^{1}(x-0)\,dx + \int_{1}^{1.25}(x-1)\,dx$$

$$\Rightarrow \frac{1}{2} \ \underline{\text{Ans}}$$

$$[0.75] = 0$$

Slide

$\int_0^4 [x]\,dx$ always defined in Integer

$$\Rightarrow \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx$$

Questions

GATE

The value (round off to one decimal place) if $\int_{-1}^{1} xe^{|x|}dx$ is

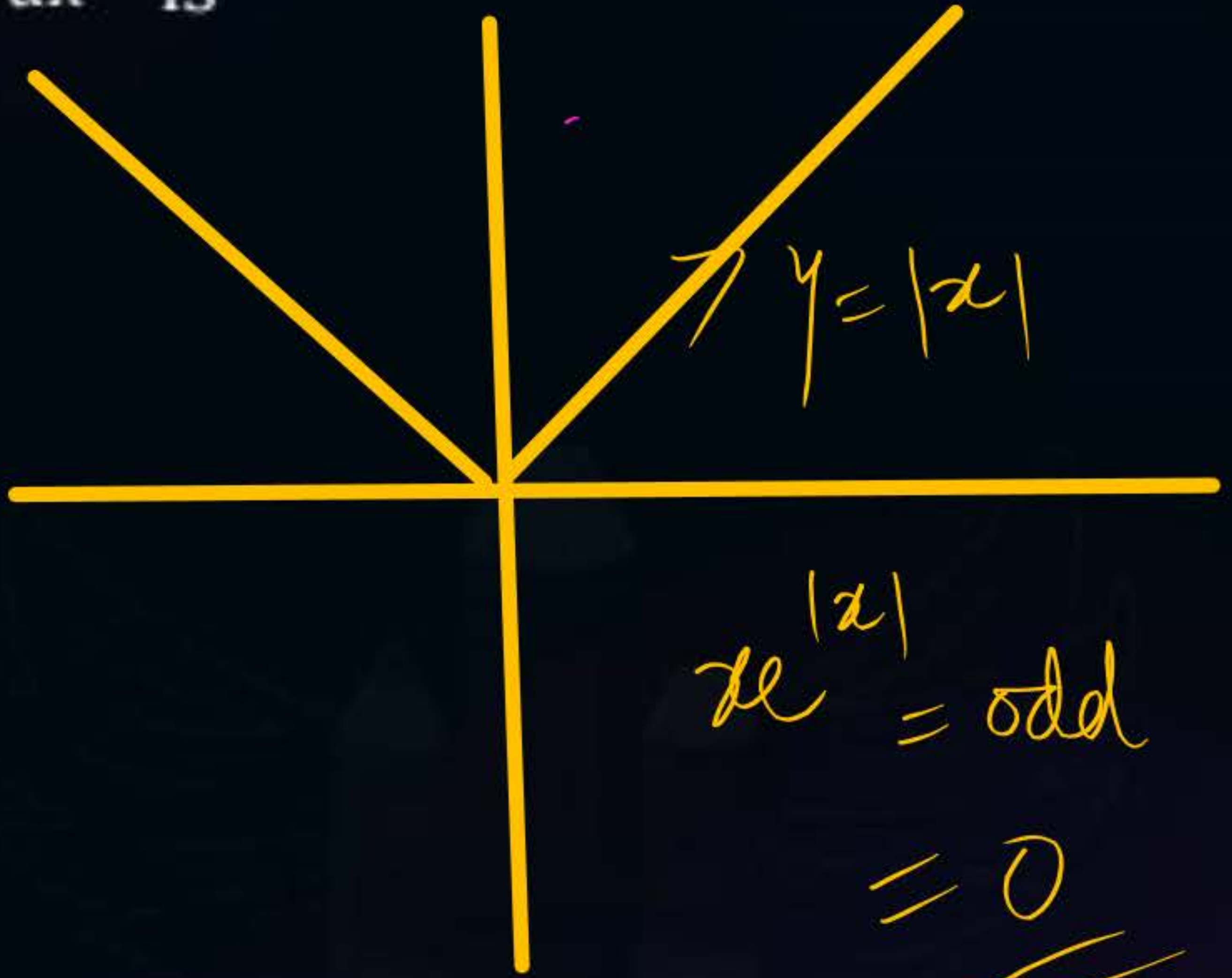
$$\int_{-a}^{a} f(x)\,dx = |x|$$

$$\Rightarrow \int_{-1}^{0} xe^{-x} + \int_{0}^{1} xe^{x}\,dx \Rightarrow 0 \quad \text{Ans}$$

$$f(x) = xe^{|x|}$$

$$f(-x) = -xe^{|-x|}$$

$$= -f(x) \text{ odd function}$$



$$xe^{|x|} = \text{odd}$$

$$= 0$$

Slide

THANK - YOU