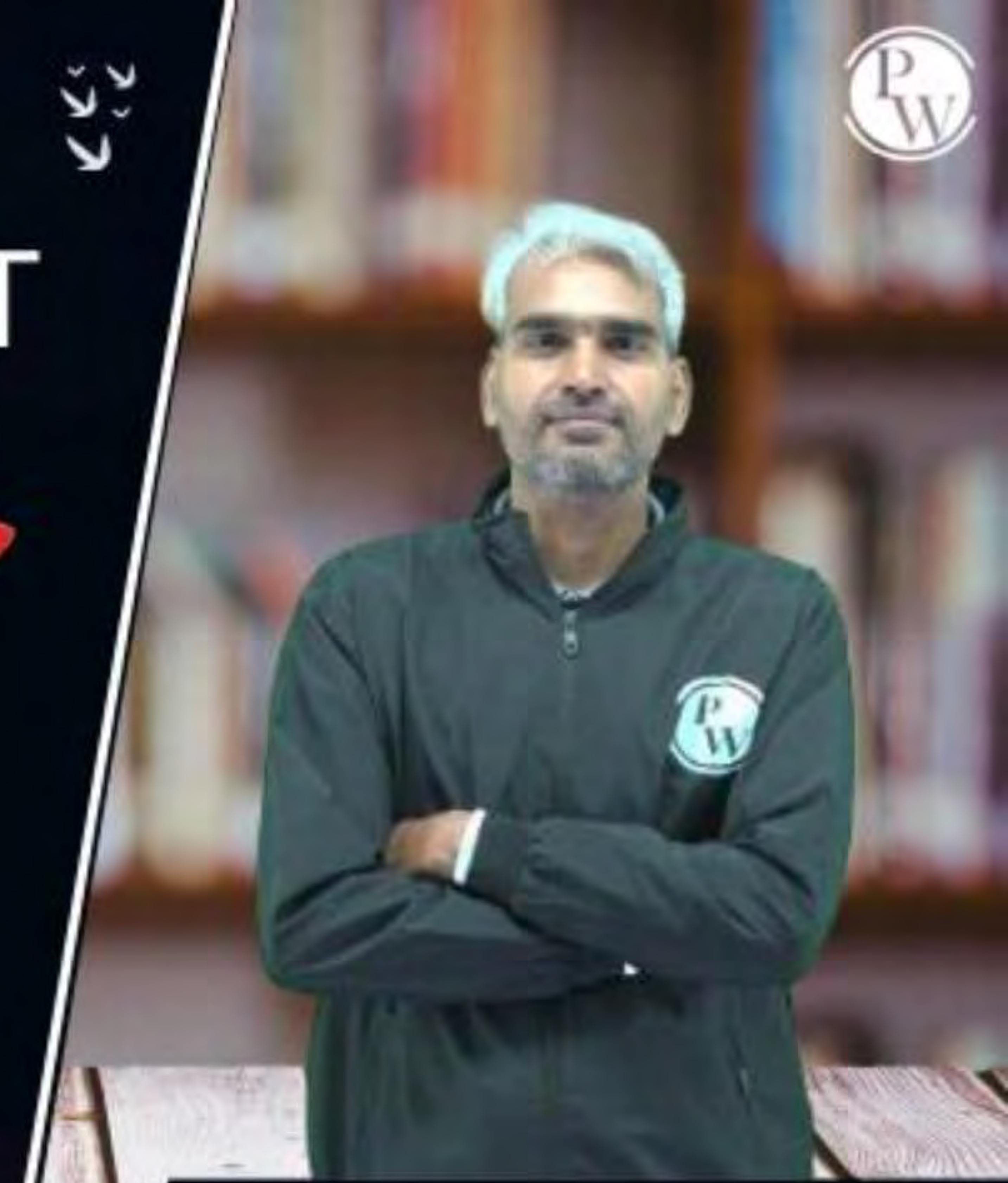
# Computer Science & IT

## Discrete Mathematics

**Mathematical Logic**

**Lecture No. 01**



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# Recap of Previous Lecture

**Topic** Euler trail , Euler Circuit and Traversable graph

**Topic** Hamiltonian path, and Hamiltonian Circuit

**Topic** Distance, Eccentricity, Diameter, Radius and Girth

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# Topics to be Covered

Topic — Propositions and their types

Topic — Connectives

Topic — Tautology, Contradiction

Topic — Contingency and Satisfiable propositional functions

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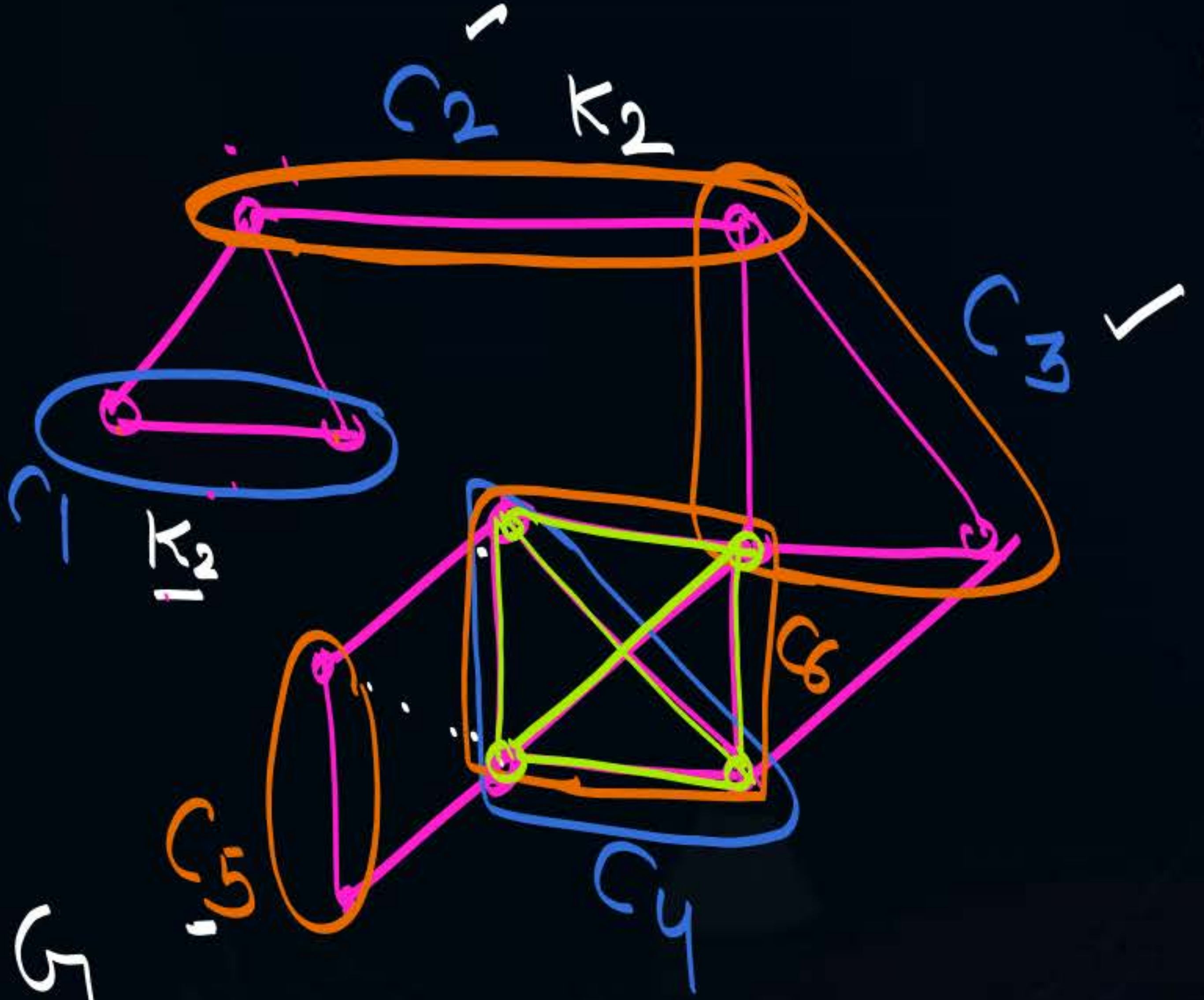
# Topic : Clique

A complete subgraph in graph G is called a clique in graph G.




Maximal clique:- A clique in graph G which can not grow further as a clique even on including some or all adjacent vertices

in the given example

$C_2$, $C_3$, $C_5$ & $C_6$ are some maximal Cliques

**Maximum Clique :-** A clique in graph $G$ with maximum number of vertices is called maximum clique.

In the above example $C_6$ is maximum clique

**Clique No :-** Clique number of graph is defined as number of vertices in maximum Clique in graph $G$

For the above eg: Clique No. = 4

# Topic : Hypercube graph

Hypercube graph $Q_n$ is a graph with $2^n$ vertices, such that each vertex can be lebeled using n-digit binary string.

- Two vertices u & v are adjacent to each other if and only if their lebels are at 1-bit difference. {i.e., lebel of one vertex can be converted into lebel of another vertex by flipping exactly one bit in any one of the lebel}

$Q_1$

$\downarrow$

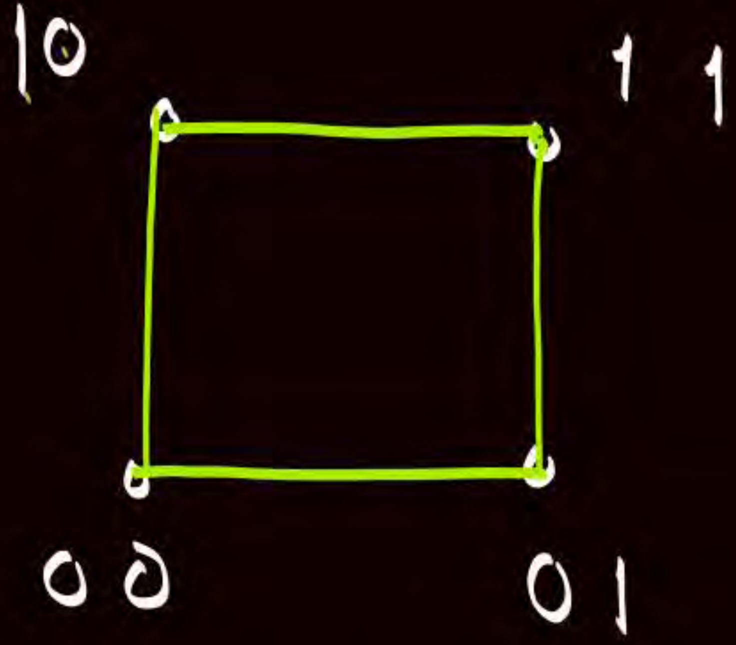
No. of vertices : $2^1 = 2$


$Q_1$

$Q_2$

$\downarrow$

# vertices : $2^2 = 4$

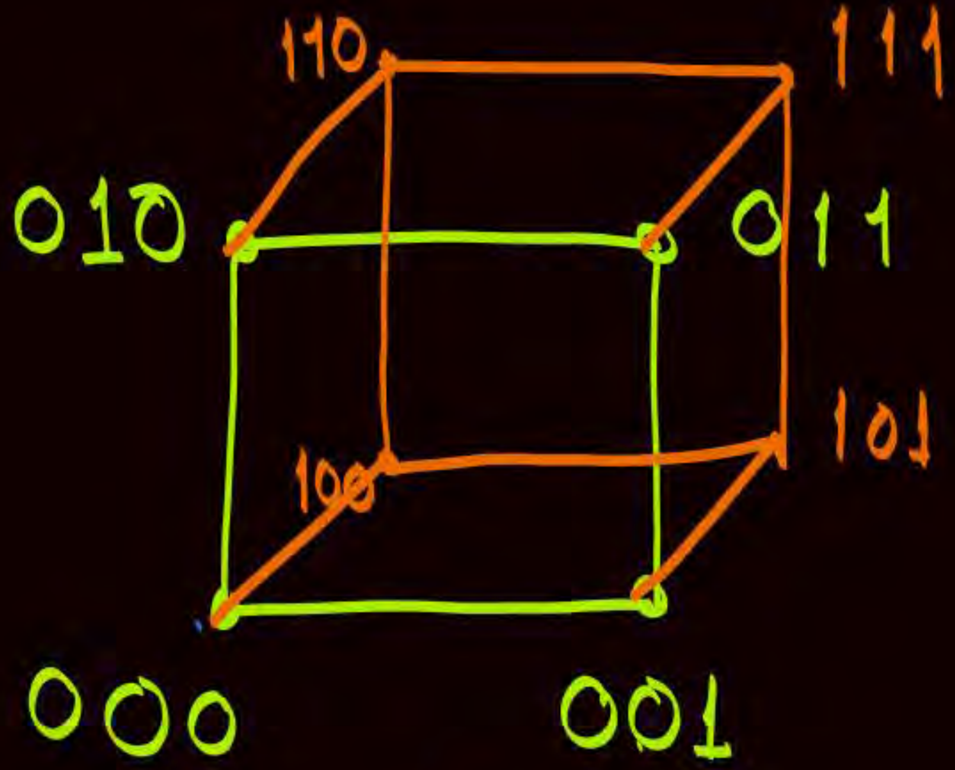



$Q_2$

$Q_3$

$\downarrow$

# vertices : $2^3 = 8$




$Q_3$

- In a Hypercube $Q_n$,
  - there are exactly $2^n$ vertices
  - & there are exactly $n \cdot 2^{n-1}$ edges.
- In hypercube $Q_n$, all cycles are of even length.
  and hence chromatic number of $Q_n = 2$

Diameter of hypercube $Q_n$ is $= n$

$\downarrow$

flip all $n$ bits one at a time.

# Mathematical Logic

- Propositional Logic
- Predicate logic {First-order logic}

# Topic : Propositional Logic

A declarative sentence to which we can assign only one of the truth values (i.e. TRUE/FALSE) is called a propositional statement or proposition.

eg: India's capital is Agra : False

India's capital is New Delhi : True

$9 < 6$ : False

all are Proposition

it is not a Proposition

Pleas shut the door :

## Topic : Assumptions of proposition

Law of excluded middle : If a proposition is not true then it will be false, and similarly if a proposition is not false then it will be true.

Law of contradiction : A proposition can not be true as well as false simultaneously.

## Topic : Atomic propositions

Atomic propositions are simple propositions which can not be divided further.

eg: $\underbrace{\text{India's capital is New Delhi}}_{\text{Atomic Proposition}}$ : True

# Topic : Compound Propositions

Compound propositions are constructed by combining atomic propositions with the help of connectives.

Compound propositions are also known as propositional formula, or propositional function.

eg. $\underbrace{\overset{\text{True}}{\text{India's capital is New Delhi}}}_{\text{one atomic proposition}}$ $\overset{\vee}{\text{OR}}$ $\underbrace{\overset{\text{False}}{\text{India's capital is Agra}}}_{\text{Another Atomic Proposition}}$ = True

Connective

it is a Compound proposition

## Topic : Connectives

1. Negation ( $\neg$/~) / Not
2. AND ($\wedge$) / Conjunction
3. OR ($\vee$) / Disjunction
4. Implication ($\rightarrow$)
5. Biconditional ($\leftrightarrow$)

## Topic : Negation ( ¬/~)

If p is any proposition, then negation of p / not(p) / ¬p is also a proposition whose truth value is true only when p is false and it is false only when p is true.

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

# Topic : AND (∧) / Conjunction

If p and q are any two propositions, then "p AND q" / p ∧ q is also a proposition whose truth value is true only when both p as well as q are true.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

## Topic : OR (V) / Disjunction

If p and q are any two propositions, then "p OR q" / p V q is also a proposition whose truth value is false only when both p as well as q are false.

| p | q | p V q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

## Topic : Implication (→)

V.V.V. Imp.

If p and q are any two propositions, then p implies q (i.e. $p \rightarrow q$ / if p then q ) is also a proposition whose truth value is false only when p is true, and q is false.

$$\underbrace{p}_{\text{antecedent}} \rightarrow \underbrace{q}_{\text{consequent}}$$

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

$p \to q \equiv \sim p \vee q$

$q \to p \equiv \sim p \to \sim q$

same as $p \to q$

| $\sim q$ | $\sim p$ | $p$ | $q$ | $p \to q$ | $\sim p \vee q$ | $q \to p$ | $\sim p \to \sim q$ | $\sim q \to \sim p$ |
|---|---|---|---|---|---|---|---|---|
| F | F | T | T | T | T | T | T | T |
| T | F | T | F | F | F | T | T | F |
| F | T | F | T | T | T | F | F | T |
| T | T | F | F | T | T | T | T | T |

$\therefore p \to q \equiv \sim q \to \sim p$

$q \to p$ & $\sim p \to \sim q$ are not same as $p \to q$

## Topic : Implication ($\rightarrow$)

- If p is true, then q must be true for $p \rightarrow q$ to be true.
- If p is false, then $p \rightarrow q$ is ~~also~~ always true whatever is the truth value of q.
- If q is true, then $p \rightarrow q$ is ~~also~~ always true whatever is the truth value of p.
- Truth table of $p \rightarrow q$ and truth table of $\neg p \vee q$ are exactly same, therefore both are equivalent.
- Converse of $p \rightarrow q$ is $q \rightarrow p$.
- Opposite or Inverse of $p \rightarrow q$ is "$\neg p \rightarrow \neg q$"
- Contra positive for $p \rightarrow q$ is "$\neg q \rightarrow \neg p$"

$$p \to q \equiv \sim p \vee q$$

$$p \to q \equiv \sim q \to \sim p$$

$$p \to q \not\equiv q \to p$$

$$p \to q \not\equiv \sim p \to \sim q$$

If p and q are any two propositions, then "p $\leftrightarrow$ q" (i.e. p *if and only if* q ) is also a proposition whose truth value is true only when both p and q have same truth value.

| $p$ | $q$ | $p \leftrightarrow q$ | $p \to q$ | $q \to p$ | $(p \to q) \wedge (q \to p)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | F |
| F | T | F | T | F | F |
| F | F | T | T | T | T |

$\therefore (p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)$

# Topic : Biconditional ($\leftrightarrow$)

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$p \leftrightarrow q$ is true if and only if both $p \to q$ as well as $q \to p$ are true.

## Topic : Tautology

{ Tautology is defined w.r.t. propositions }

A propositional function which is always true is called tautology.

A propositional function which is always true is also called a valid propositional function.

In propositional logic,

Valid propositional formula & Tautology are the same thing.

eg

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

Always true
$\therefore$ $p \vee \sim p$ is a tautology.

eg

| $p$ | $q$ | $q \to p$ | $p \to (q \to p)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | F | T |
| F | F | T | T |

all true
$\therefore$ Tautology

| $p_1$ | $p_2$ | $p_3$ | $\cdots$ | $p_n$ | $f(p_1, p_2, p_3, \ldots p_n)$ |
|---|---|---|---|---|---|
| | | | $\cdots$ | | T |
| | | | | | T |
| | | | | | T |
| | | | | | T |
| | | | | | $\vdots$ |
| | | | $\cdots$ | | T |

if all true, then propositional function $f(p_1, p_2, \ldots p_n)$ is a tautology.

| $P_1$ | $P_2$ | $P_3$ | ... | $P_n$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | ... |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | ... | 0 | T | F | T | F | |
| | | | | | T | T | F | F | |
| | | | | | T | T | T | T | |
| | | | | | ⋮ | ⋮ | T | T | |
| ⋮ | | | | | ⋮ | ⋮ | ⋮ | ⋮ | |
| 1 | 1 | 1 | ... | 1 | T | T | T | T | |

$2^n$ rows

Number of such functions possible

$$2 \times 2 \times 2 \times 2 \times 2 \times \cdots \times 2 = 2^{(2^n)}$$

How many propositional functions can be defined using 'n' Propositional variable ?

# Topic : Contradiction

A propositional function which is always false is called Contradiction

eg:

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

false for all cases ∴ Contradiction

* $\boxed{\text{If a propositional function is false for at least one case then that propositional function is said to be invalid propositional function}}$

## Topic : Contingency

A propositional function which is neither a tautology nor a contradiction is called a contingency.

i.e., A contingency is true for at least one case as well as false for at least one case.

eg:- $p \to q$ is a Contingency

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

True for at least one case
false for at least one case
} i.e. Contingency

# Topic : Satisfiable

A propositional function which is true for at least one case is called a satisfiable propositional function.

## Topic : NOTE

- Every propositional function which is not a contradiction is satisfiable.

- Every tautology is satisfiable but not vice-versa.

  ie; Every satisfiable function need not be a tautology

- Every contingency is satisfiable, but every satisfiable function need not be contingency.

  eg: Tautology : It is satisfiable, but not a contingency.

## Topic : NOTE

| Tautology | Contradiction | Contingency |
|---|---|---|
| Always True | Always False | Neither always true nor always false |
| Valid | Invalid | Invalid |
| Satisfiable | Not satisfiable | Satisfiable |

# 2 mins Summary

**Topic** Propositions and their types

**Topic** Connectives

**Topic** Tautology, Contradiction

**Topic** Contingency and Satisfiable propositional functions

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# THANK - YOU

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