GATE

ALL BRANCHES

ENGINEERING MATHEMATICS

Probability and Statistics

Lecture No. 13

BY- RAHUL SIR

01 Problems based on Probability Distributions

Uniform Distribution
[Continuous]

Uniform Distribution

[Continuous Prob. Distribution]

This distribution defined in Interval

$a \le X \le b$

Throwing A Die

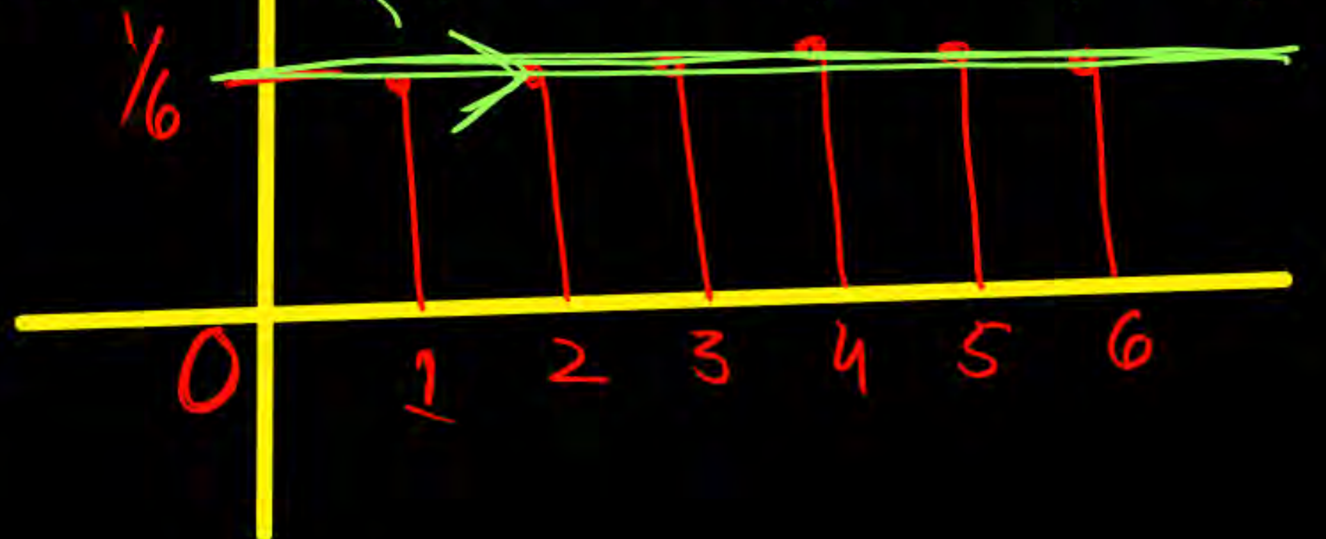
$X =$ No. of dots

$X = 1, 2, 3, 4, 5, 6$

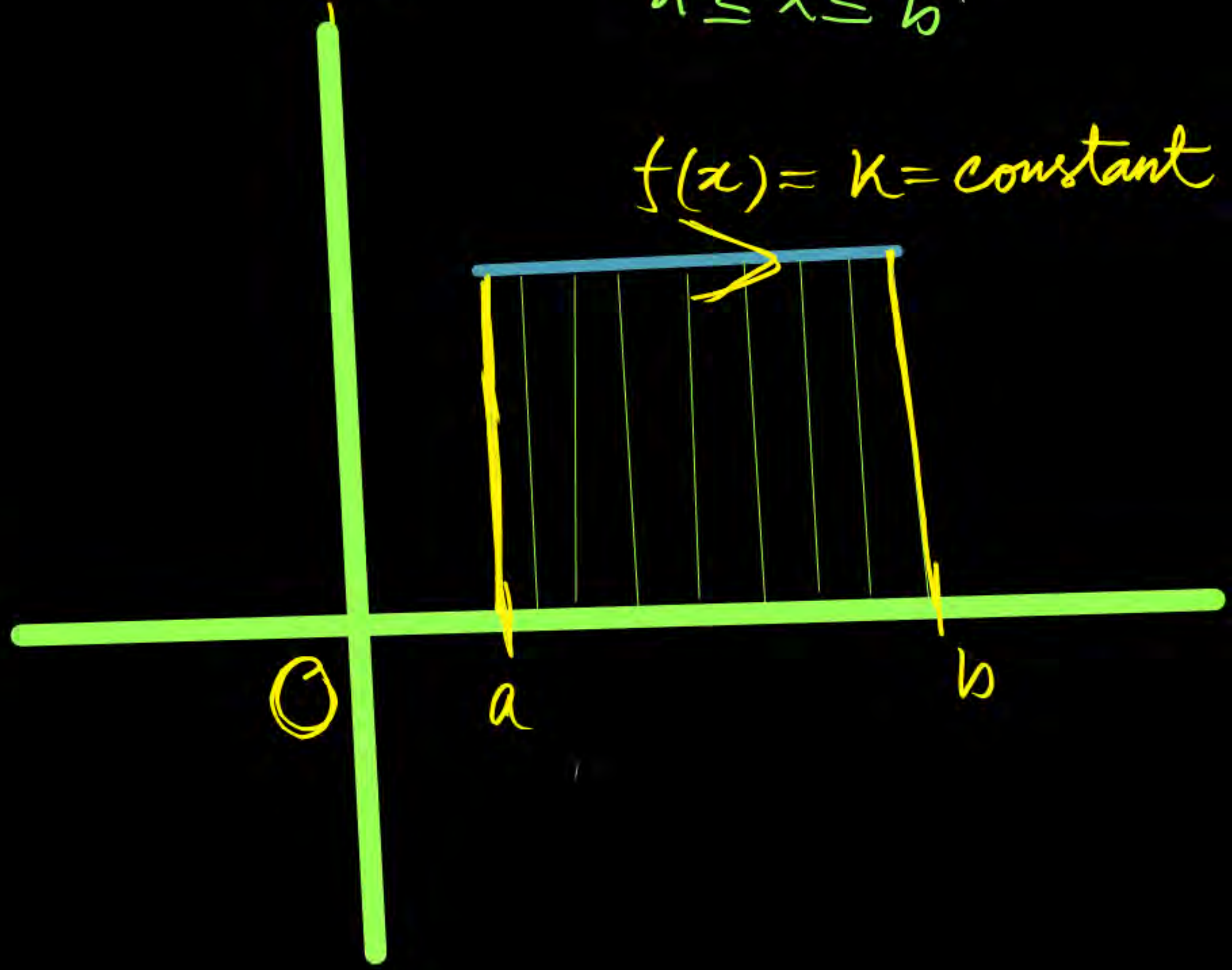
X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

constant

prob are equal $P(X=x) = \frac{1}{6}$



$f(x) = K =$ constant



$U[a,b]$ = Uniform continuous Distribution

$$f(x) = \begin{cases} \underline{K} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

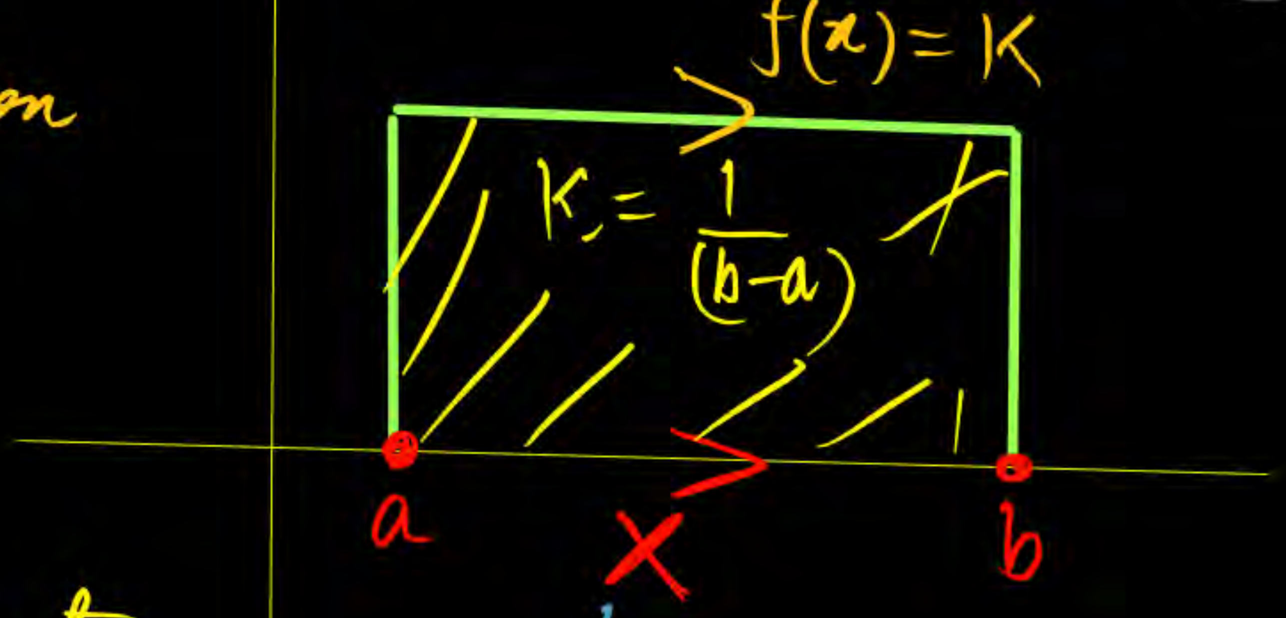
If this is valid Prob. density function

$$\int_a^b f(x)\,dx = 1$$

$$\Rightarrow \int_a^b K\,dx = 1$$

$$\Rightarrow K(b-a) = 1$$

$$\boxed{K = \frac{1}{b-a}}$$



Uniform random variable

$U[a,b]$

$\Downarrow$

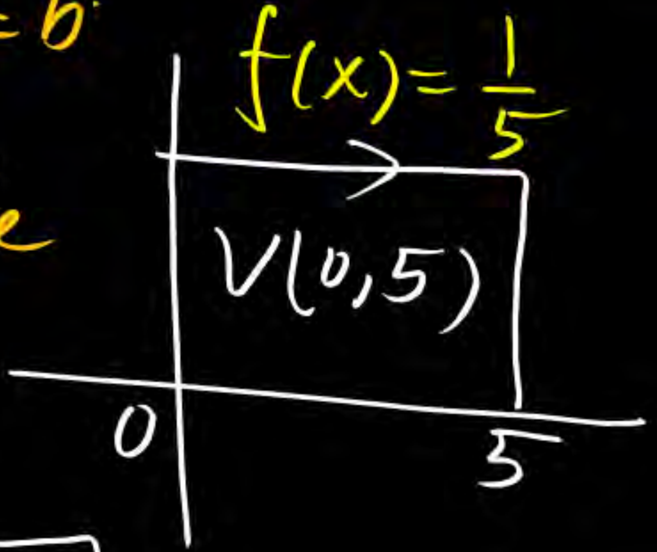
Interval

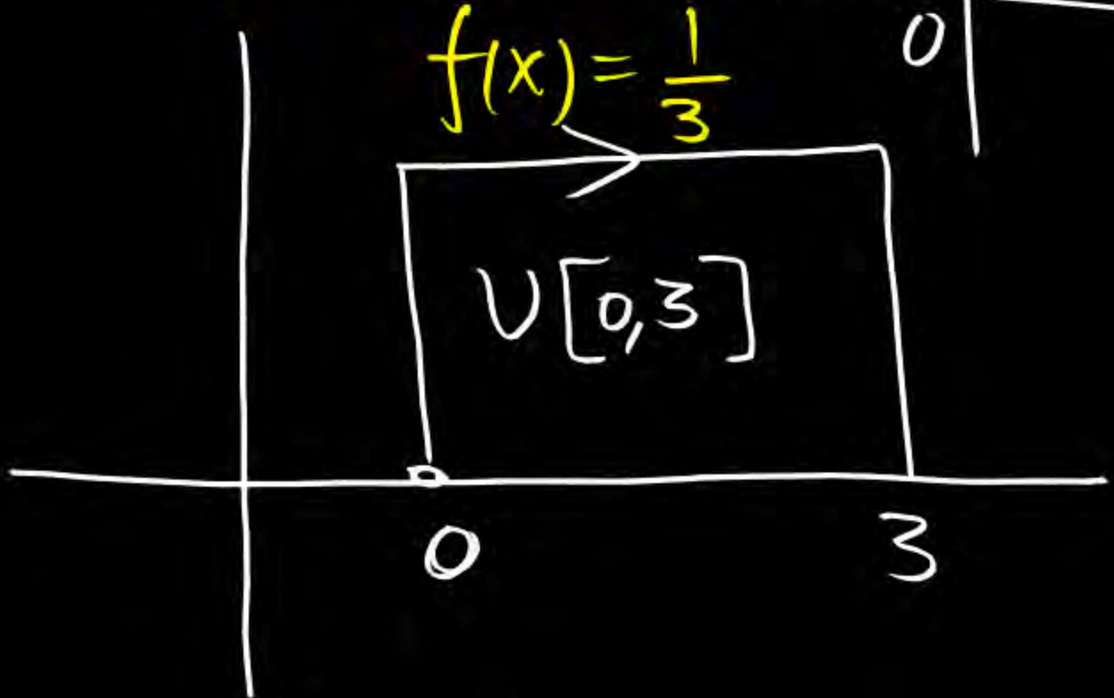
$$\Rightarrow K(b-a) = 1$$

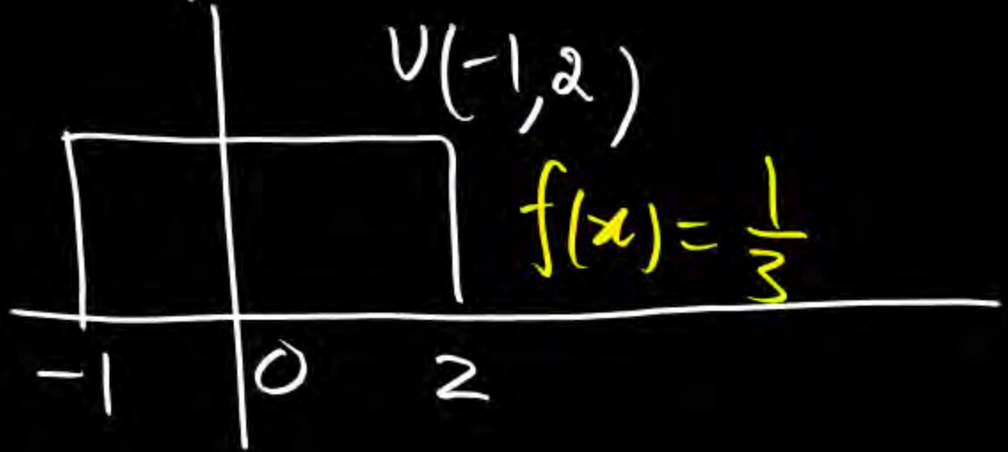
$$K = \frac{1}{(b-a)}$$

$$\begin{cases} f(x) = \dfrac{1}{(5-0)} = \dfrac{1}{5} \\ f(x) = \dfrac{1}{(3-0)} = \dfrac{1}{3} \\ f(x) = \dfrac{1}{2-(-1)} = \dfrac{1}{3} \end{cases}$$

$$V(a,b) = \begin{cases} \dfrac{1}{(b-a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$







Statistical Averages:

$$V(a,b) = \begin{cases} \dfrac{1}{(b-a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$V[a,b] = \begin{cases} \dfrac{1}{(b-a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \mu = \int_a^b x f(x)\,dx = \int_a^b x \cdot \frac{1}{(b-a)}\,dx$$

$$= \frac{1}{(b-a)} \int_a^b x\,dx$$

mean

$$\Rightarrow \frac{(b^2-a^2)}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$\boxed{E[X] = \frac{a+b}{2}}$$

$$E[X] = \frac{(a+b)}{2}$$

$$V(X) = E[X^2] - [E[X]]^2$$

$$= \int_a^b x^2 f(x)\,dx - \left[\int_a^b x f(x)\,dx\right]^2$$

$$Var(X) = \sigma_X^2 = \int_a^b x^2 \cdot \frac{1}{(b-a)}\,dx - \left[\int_a^b x \cdot \frac{1}{(b-a)}\,dx\right]^2$$

$$= \frac{(b^3 - a^3)}{3\,(b-a)} - \left[\frac{(b^2 - a^2)}{(b-a)2}\right]^2$$

$$= \frac{(b-a)(a^2+b^2+ab)}{3\,(b-a)} - \left[\frac{(b-a)(b+a)}{(b-a)2}\right]^2$$

$$\boxed{Var(X) = \sigma_X^2 = \frac{(b-a)^2}{12} \text{ or } \frac{(a-b)^2}{12}}$$

\# Standard deviation $= \sqrt{var(X)}$

Q. Questions

$\frac{9}{16}$

Two independent random variables X and Y are uniformly distributed in the interval [–1, 1]. The probability that max[X, Y] is less than 1/2 is

$P\left[\max[X,Y] \le \frac{1}{2}\right]$

(a) 3/4

(b) 9/16

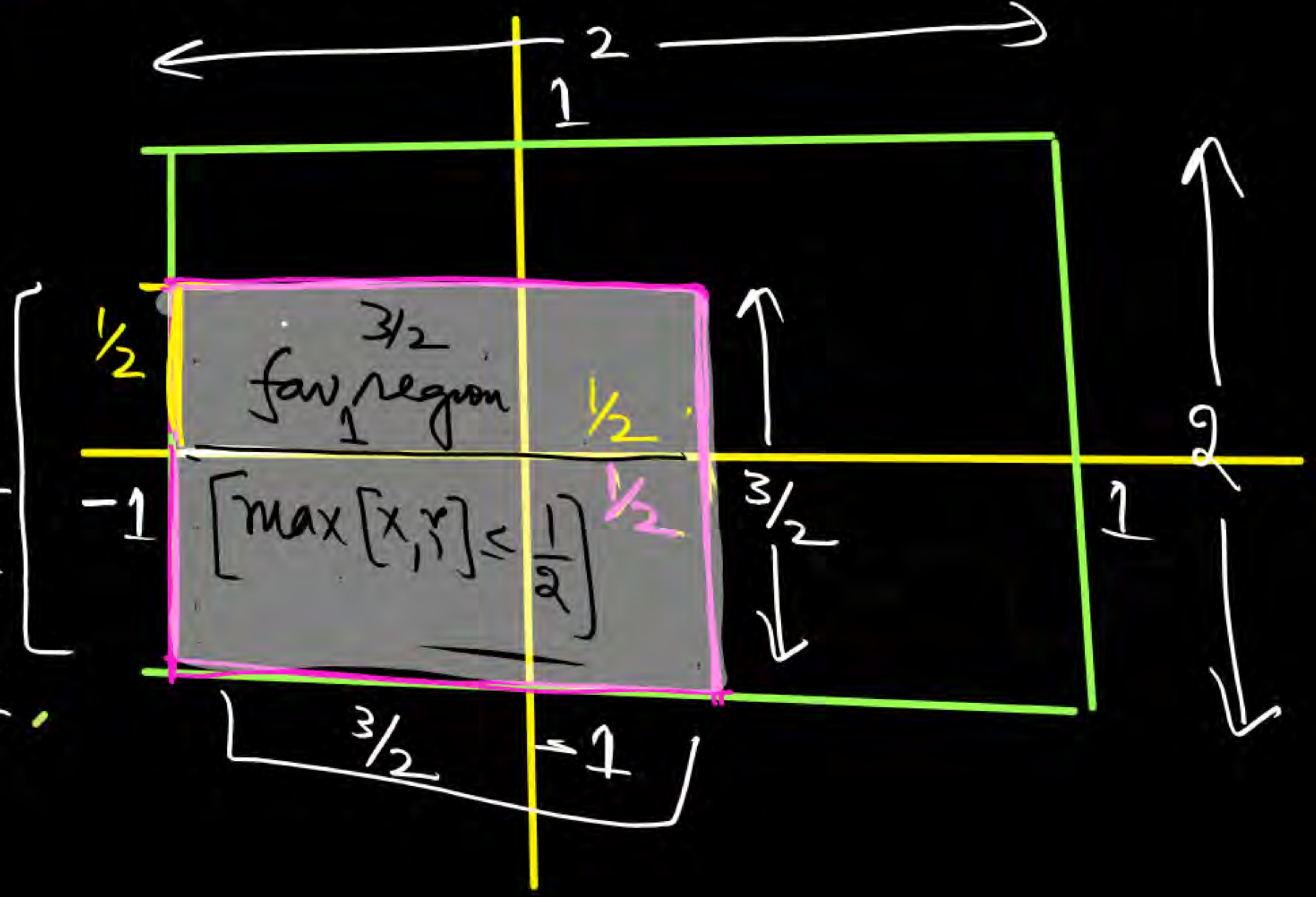
(c) 1/4

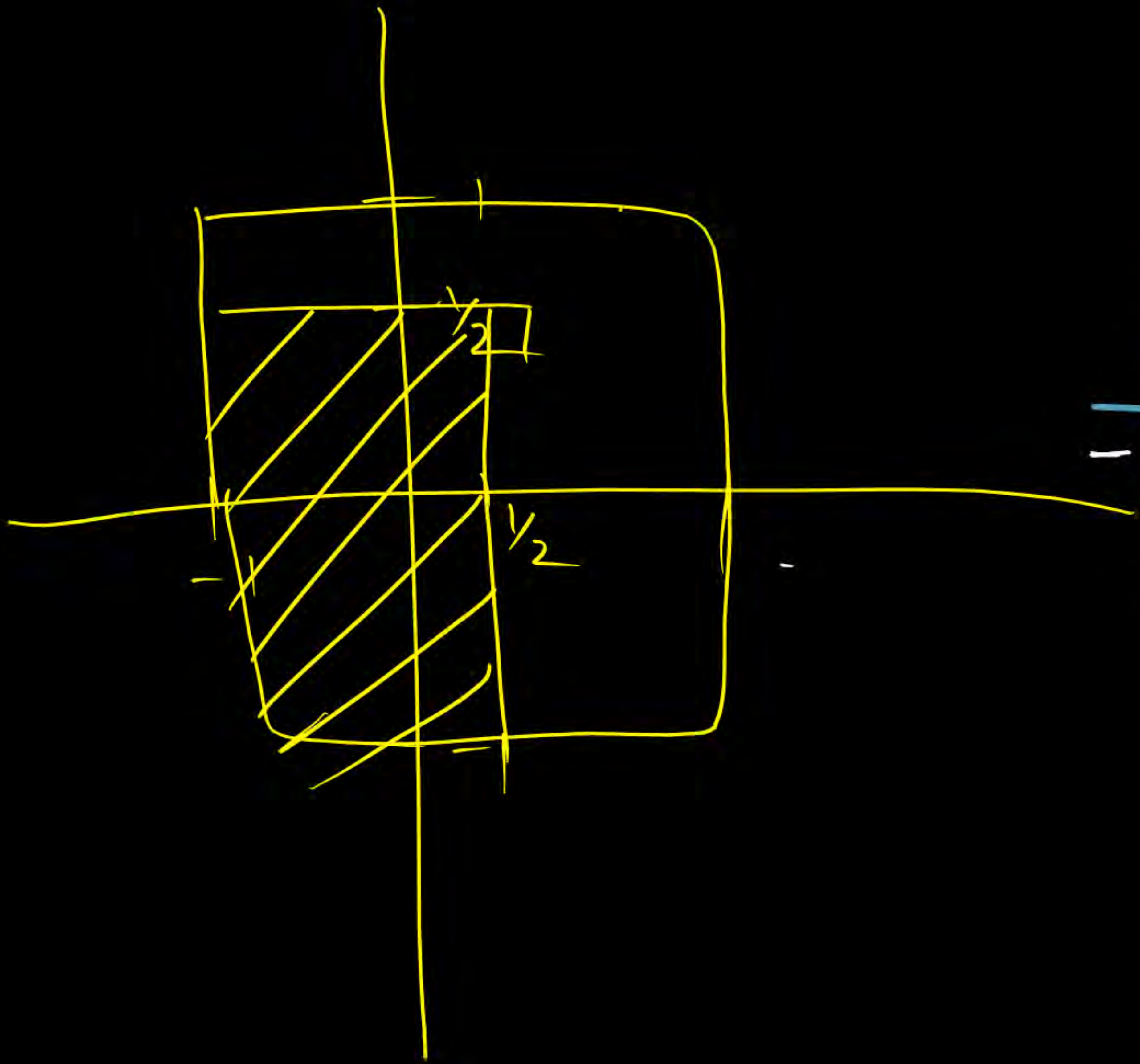
(d) 2/3

$P\left[\max[X,Y] \le \frac{1}{2}\right]$

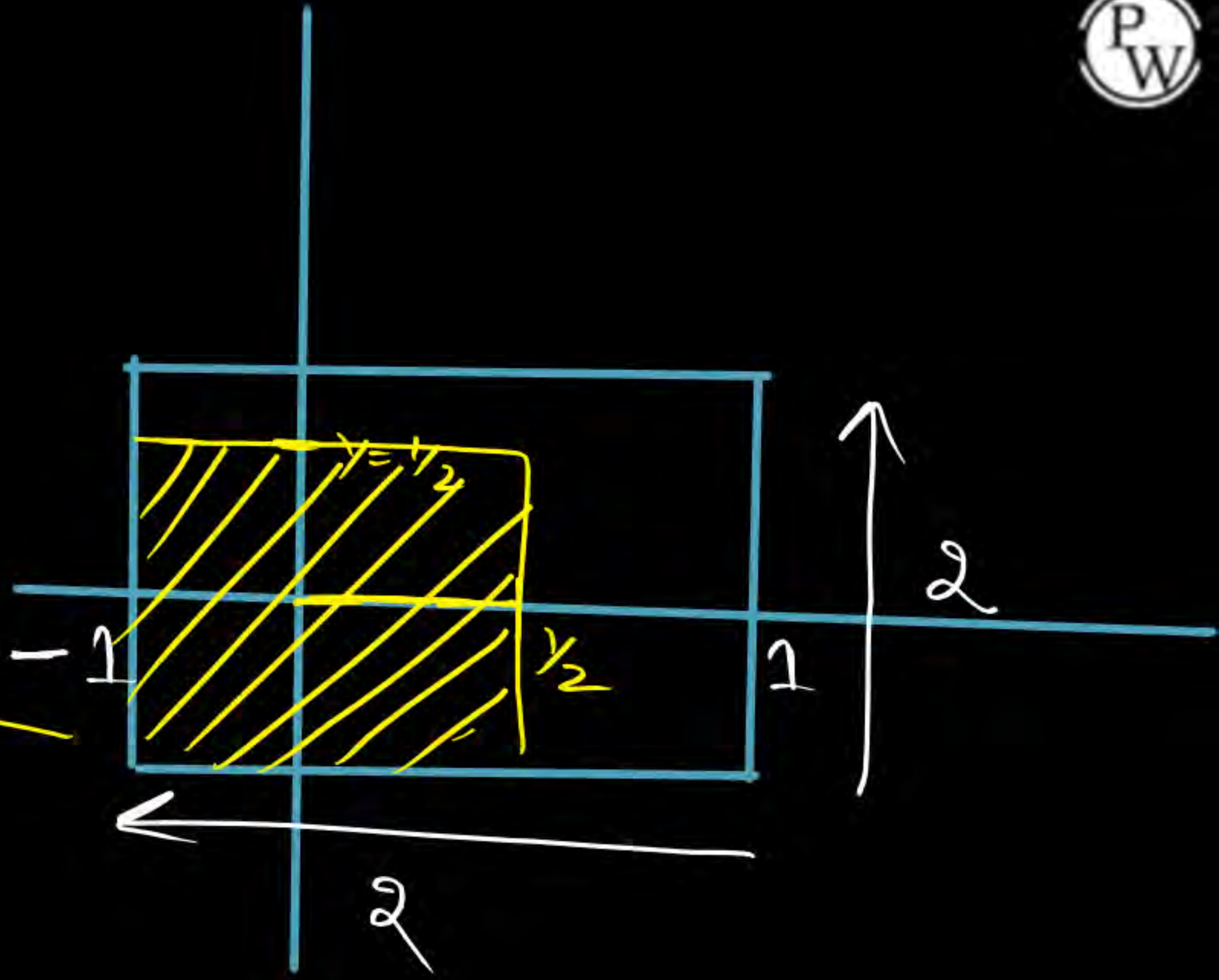
$= \frac{\text{fav rectangle}}{\text{Avg Rectangle}} = \frac{(3/2)^2}{(2)^2}$

$= \frac{9}{16}$ Ans





1/2
1/2
-1



y = 1/2
-1
1/2
1
2
2

Q. Questions

Subway trains on a certain line runs every half hour between midnight and six in the morning. What is the probability that the men entering the station at random time during the period will have to wait at least 20 minutes.

12:00 o'clock to 6:00 mrg

12 to 12:30 — 1
12:30 to 1 —— 1
1 to 1:30 —— 1
1:30 to 2:00 —— 1

2:00 to 2:30
2:30 to 3:00
3:00 to 3:30
3:30 to 4:00
4:00 to 4:30

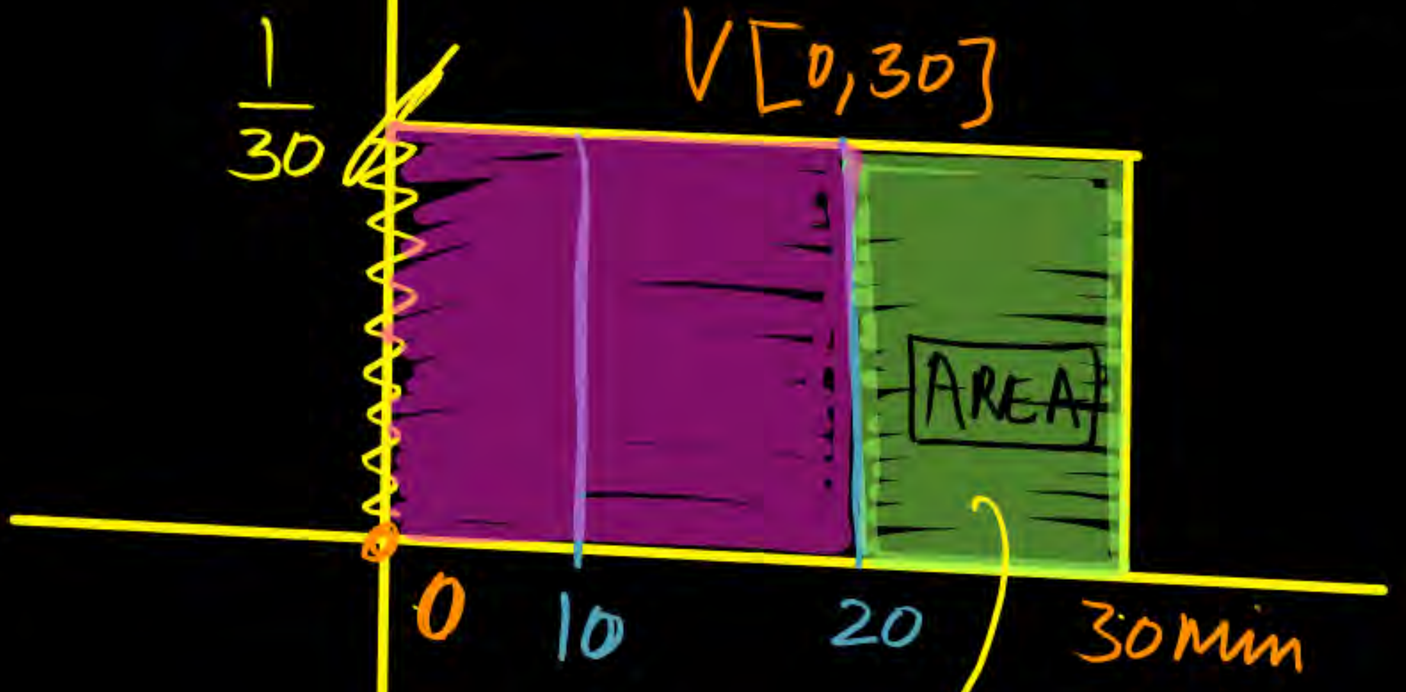
4:30 to 5:00
5:00 to 5:30
5:30 to 6:30

$$P(X \geq 20) = \int_{20}^{30} f(x)\,dx$$

$$= \int_{20}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[30-20\right]$$

$$\boxed{P(X \geq 20) = \frac{1}{3}}$$



$$= 10 \times \frac{1}{30} = \frac{1}{3}$$

$$f(x) = \begin{cases} \dfrac{1}{30-0} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Q. Questions

Assume that in a traffic junction, the cycle of the traffic signal lights is 2 minutes of green (vehicle does not stop) and 3 minutes of red (vehicle stops). Consider that the arrival time of vehicles at the junction is uniformly distributed over 5 minute cycle. The expected waiting time (in minutes) for the vehicle at the junction is _____.

$$f(x) = \begin{cases} \dfrac{1}{(5-0)} & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \dfrac{1}{5} & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Waiting Time

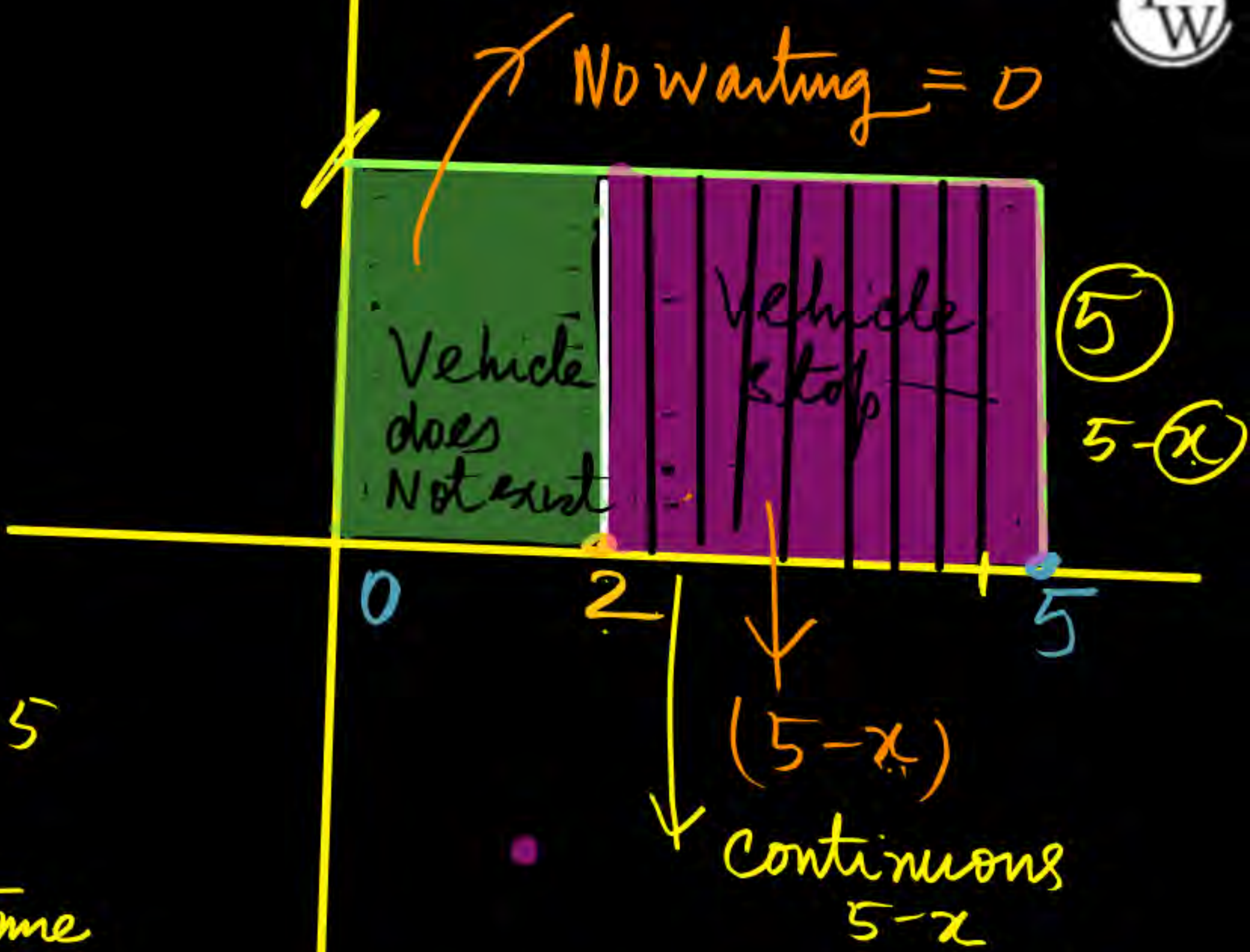
$$W(x) = \begin{cases} 0 & 0 \le x \le 2 \\ (5-x) & 2 \le x \le 5 \end{cases}$$

$E[W(x)]$ = expected waiting Time

$$E[X] = \int_a^b x f(x)\,dx$$

$X = g(x)$

$$E[g(x)] = \int_a^b g(x) f(x)\,dx$$



Waiting Time $2 < x \le 5$

$$W(x) = \begin{cases} 0 & 0 \le x \le 2 \\ 5-\text{some No} & 2 \le x \le 5 \end{cases}$$

Expected waiting Time

$$E_X[W(X)] = \int_a^b f(x)\, w(x)\, dx$$

$= w(x)$

$$E[W(X)] = \int_0^2 \frac{1}{5} \cdot 0 \, dx + \int_2^5 \frac{1}{5} \cdot (5-x)\, dx$$

0

$$= \int_2^5 \frac{1}{5}(5-x)\, dx$$

$$= \frac{1}{5}\left[5x - \frac{x^2}{2}\right]_2^5 = \frac{1}{5}\left[\left(25 - \frac{25}{2}\right) - \left(10 - \frac{4}{2}\right)\right] = \underline{0.9}$$

Q. **Questions**

Do yourself

Suppose Y is distributed uniformly in the open interval (1, 6). The probability that the polynomial $3x^2 + 6xy + 3y + 6$ has only real roots is (rounded off to 1 decimal place)

$\boxed{\text{Ans} = 0.8}$

Q. Questions

If X is a uniformly distributed random variable with mean 1 and variance 4/3. Find $P(X \le 0)$

$X \to U[a,b]$ Mean $= 1$

$\frac{a+b}{2} = 1$

$a+b = 2$ — 1

$\text{vari} = \frac{4}{3}$

$\frac{(b-a)^2}{\cancel{12}\ 4} = \frac{4}{\cancel{3}}$

$(b-a)^2 = 16$

$\boxed{(b-a) = \pm 4}$

$a+b=2$

$(b-a) = +4 \quad (b-a) = -4$

$P(X \le 0) = \int_a^b f(x)\,dx$

1)

$a + b = 2$ —— ①

$(b - a) = 4$

$2b = 6$

$\boxed{b = 3}$

Put The value $b = 3$

$a + b = 2$

$a + 3 = 2$

$\boxed{a = -1}$

$\boxed{b = 3}$

$\left.\begin{array}{l} a = -1 \\ b = 3 \end{array}\right\}$ ✓ interval

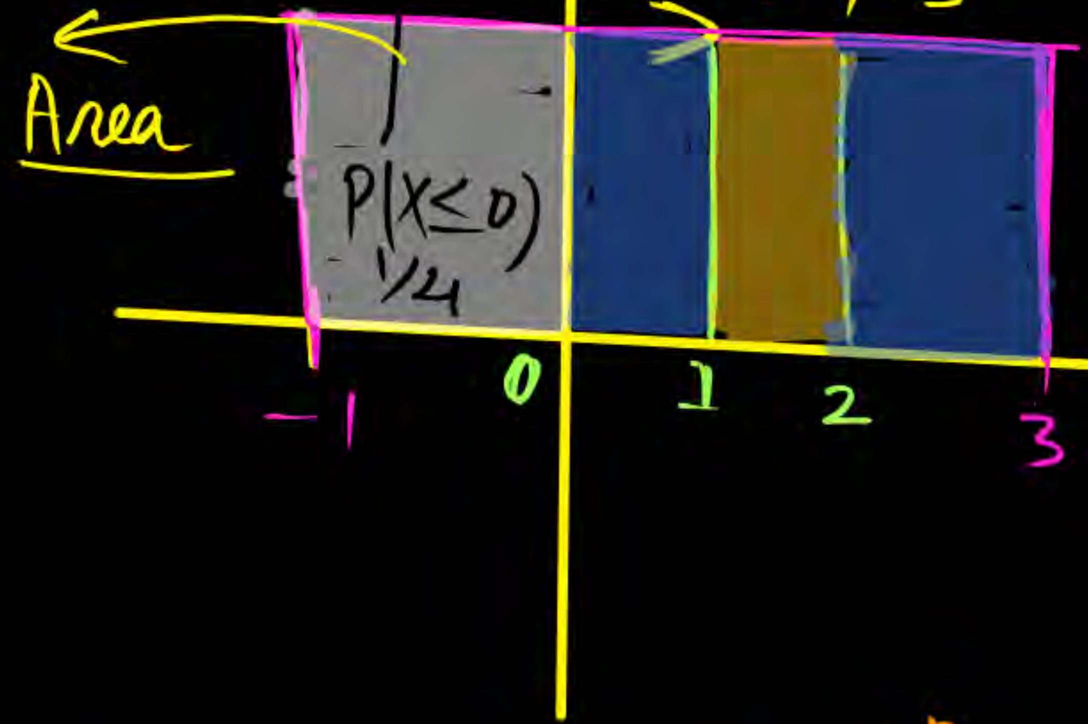
$a + b = 2$

$b - a = 4$

$\left.\begin{array}{l} a = 3 \\ b = -1 \end{array}\right]$

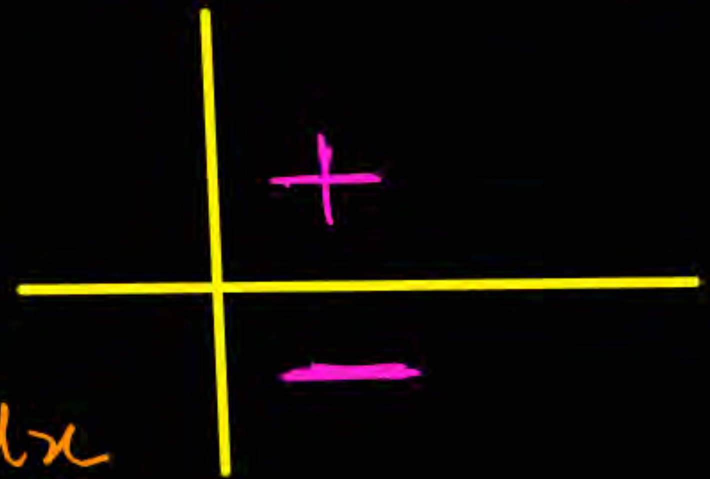
$\left.\begin{array}{l} a = 3 \\ b = -1 \end{array}\right]\Bigg\}$

$U[-1, 3]$

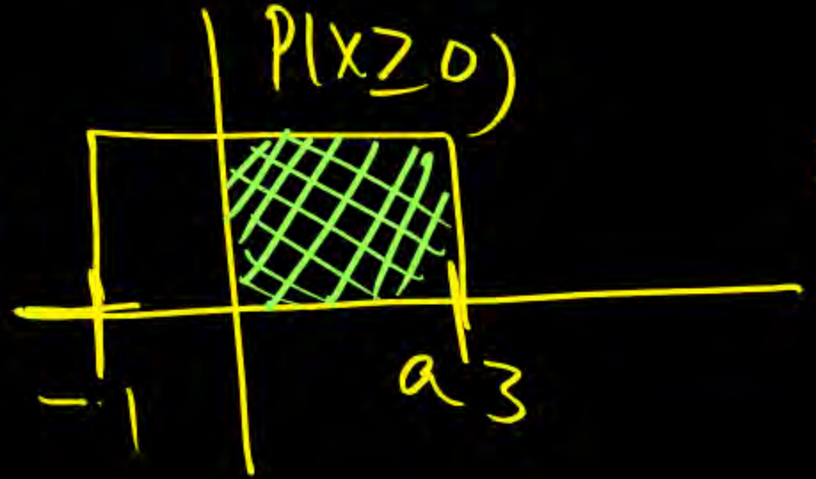


$f(x) = \dfrac{1}{3-(-1)} = \dfrac{1}{4}$

$f(x) = \dfrac{1}{4}$



$P(X \le 0) = \displaystyle\int_{-1}^{0} \frac{1}{4}\, dx$



$= \dfrac{1}{4}[1] = \dfrac{1}{4}$ Ans

Thank You!

GW Soldiers