GATE
ALL BRANCHES

ENGINEERING MATHEMATICS

Single Variable Calculus

Lecture No. 07

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TOPICS TO BE COVERED

01 Integration By parts

02 Partial Fractions

03 Integrals of Trigonometric functions

04 Special kind of Integrals

Integration Using Parts: (Product of two functions)

$$\int f(x)\, g(x)\, dx$$

$g(x)$

$f(x)$

$$\frac{d}{dx}(\mathrm{I} \times \mathrm{II})$$

$\mathrm{II} \times \mathrm{I}$

$$= \mathrm{I}\frac{d}{dx}(\mathrm{II}) + \mathrm{II}\frac{d}{dx}(\mathrm{I})$$

$$\Rightarrow f(x)\int g(x)\, dx - \int\left[\frac{d}{dx} f(x) \int g(x)\, dx\right] dx$$

HECTIC Job.

✓ I = Inverse circular function $\sin^{-1}x \,|\, \cos^{-1}x \,|\, \tan^{-1}x$

L = Logarithmic Function $\log x$ $\log 2x$ - - .

✓ A = Algebraic Function x^2, x^3 - -

✓ T = Trigonometric function $\sin|\cos|\tan|\cot|\mathrm{cosec}|\sec$

E = exponential function, e^x, a^x - - - .

$$\int \overset{\mathrm{I}}{x} \overset{\mathrm{II}}{\cos x}\, dx$$

$$\int \overset{\mathrm{II}}{x^2} \overset{\mathrm{I}}{\tan^{-1}x}\, dx$$

$$\int \log x\, dx$$

(A) $I = \int \log x \, dx = \int \log x \cdot 1 \, dx = \int \underset{\mathrm{I}}{\log x} \cdot \underset{\mathrm{II}}{x^0} \, dx$

$$= \log x \cdot \int x^0 dx - \int \left[\frac{d}{dx} \log x \int x^0 dx \right] dx$$

$$= \log x \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$$

$$= \boxed{\log x \cdot x - x}$$

Put

Rule
$\log x = t$ (every time)
$x = e^t$
$dx = e^t dt$

$$I = \int \underbrace{t \cdot e^t} dt$$

$$I = \int t e^t \, dt$$

$$I = te^t - e^t$$

$$= \log x \, e^{\log x} - e^{\log x}$$

$$\boxed{I = x \log x - x + C}$$

\# $I = \int x^2 \sin x \, dx$

$$\boxed{I = -x^2 \cos x + 2x \sin x + 2\cos x + C}$$

Must : If one function is algebraic

"BECOMES ZERO"

D		I
t	$+$	e^t
1	$-$	e^t
0		e^t

I ✓ any function

D		I
x^2	$+$	$\sin x$
$2x$	$-$	$-\cos x$
2	$+$	$-\sin x$
0		$\cos x$

$$I = \int (\log x)^2 dx \quad = \int t^2 \cdot e^t dt$$

$$\log x = t$$
$$x = e^t$$
$$dx = e^t dt$$

D	I
t^2	e^t
$2t$	e^t
2	e^t
0	e^t

$+$, $-$, $+$

$$I = t^2 e^t - 2te^t + 2e^t$$

$$I = (\log x)^2 e^{\log x} - 2\log x\, e^{\log x} + 2e^{\log x}$$

$$\boxed{I = (\log x)^2 \cdot x - 2\log x \cdot x + 2 \cdot x}$$

(B) $I = \int e^x \sin x \, dx$ OR $\int e^x \cos x \, dx$

$$I = \int e^x \sin x \, dx$$

$$I = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{I}$$

$$I + I = -e^x \cos x + e^x \sin x$$

$$\boxed{I = \frac{-e^x \cos x + e^x \sin x}{2}}$$

Rule : Periodic Function

sin | cos $\longrightarrow$ Integration

$e^x \longrightarrow$ Differentiation

D		I
e^x	+	$\sin x$
e^x	−	$-\cos x$
e^x		$-\sin x$

Repeat (STOP)

multiply with Integral sign

$$I = \int e^{x} \sin 3x \, dx$$

$$= -\frac{e^{x}\cos 3x}{3} + \frac{e^{x}\sin 3x}{9} - \frac{1}{9}\int e^{x}\sin 3x \, dx$$

$$I + \frac{I}{9} = -\frac{e^{x}\cos 3x}{3} + \frac{e^{x}\sin 3x}{9}$$

$$\boxed{I = \frac{9}{10}\left[-\frac{e^{x}\cos 3x}{3} + \frac{e^{x}\sin 3x}{9}\right]}$$

D		I
e^{x}	$+$	$\sin 3x$
e^{x}	$-$	$-\frac{\cos 3x}{3}$
e^{x}		$-\frac{\sin 3x}{9}$

(e) $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + c$

$$\int e^x(\underbrace{\sin x}_{f(x)} + \underbrace{\cos x}_{f'(x)})\,dx = e^x \sin x + c$$

$$\int e^x(\underbrace{\tan x}_{f(x)} + \underbrace{\sec^2 x}_{f'(x)}]\,dx = e^x \tan x + c$$

$$\int e^x\left[\underbrace{\frac{1}{x}}_{f(x)} - \underbrace{\frac{1}{x^2}}_{f'(x)}\right]dx = e^x \cdot \frac{1}{x} + c$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\int e^x [f(x) + f'(x)]\,dx$$

$$= \int e^x[\underbrace{\sin x}_{f(x)} + \underbrace{\cos x}_{f'(x)}]\,dx$$

$$= e^x \sin x + c$$

Q. Questions

Illustration:

$$\int \underbrace{x\cos x}\,dx$$

DONE

Ans.: $\underline{x\sin x + \cos x + c.}$

Q. Questions

M. Imp.

Illustration:

$$\int x\tan^{-1}x\,dx$$

Ans.: $\frac{1}{2}x^2(\tan^{-1}x)-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c$

This Strategy

$$I=\int \underset{\text{II}}{x}\,\underset{\text{I}}{\tan^{-1}x}\,dx$$

$$I=\tan^{-1}x\int x\,dx-\int\left[\frac{d}{dx}\tan^{-1}x\int x\,dx\right]dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{(x^2+1)}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[x-\tan^{-1}x\right]+c$$

$$\int\frac{x^2+1-1}{(x^2+1)}\,dx$$

$$\Rightarrow\int\frac{\cancel{x^2+1}}{\cancel{x^2+1}}\,dx-\int\frac{1}{(x^2+1)}\,dx$$

$$\Rightarrow\int dx-\int\frac{1}{1+x^2}\,dx$$

$$I = \int \tan^{-1}x \, dx$$

$$= \int \tan^{-1}x \cdot 1 \, dx$$

$$= \tan^{-1}x \int 1\, dx - \int \left[\frac{d}{dx}(\tan^{-1}x) \int 1\, dx\right] dx$$

$$= x \tan^{-1}x - \int \frac{1}{1+x^2} \cdot x \, dx$$

$$= x \tan^{-1}x - \frac{1}{2}\int \frac{2x}{(1+x^2)} \, dx$$

$$= x \tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$$

$$\left.\begin{aligned} &\int \sin^{-1}x \, dx \\ &\int \cos^{-1}x \, dx \end{aligned}\right]$$

Q. Questions

Illustration:

$$\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$$

Ans.: $(\sin^{-1} x).\frac{x}{\sqrt{1-x^2}} + \ln\left(\sqrt{1-x^2}\right) + c$

$$\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$$

Put $\underline{\sin^{-1} x = t}$

Integration Using Parts

Do yourself.

Q. Questions

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$

Do yourself

Illustration:

$\int \sin t \,(e^t dt)$

$\int \sin(\ln x)\,dx$ $= \int e^t \sin t \, dt$

Ans. $\frac{1}{2}x(\sin(\ln x)) + \cos(\ln x) + c$

$\ln x = t$

$x = e^t$

$dx = e^t dt$

Q. Questions

Do yourself

Illustration:

$$\int x^2 e^{3x}\, dx$$

Ans.: $\frac{1}{3}x^2e^{3x}-\frac{2}{9}x\,e^{3x}+\frac{2}{27}e^{3x}+c$

Q. Questions

Illustration:

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

Ans.: $e^x \frac{1}{1+x} + c$

$$\int \frac{xe^x}{(1+x)^2}\,dx = e^x\left[\frac{x}{(1+x)^2}\right]$$

$$\int e^x[f(x)+f'(x)]\,dx = \underline{e^x f(x) + c}$$

$$\int e^x\left[\frac{x+1-1}{(1+x)^2}\right]$$

$$= \int e^x\left[\frac{\cancel{(x+1)}}{\underset{(x+1)}{\cancel{(1+x)^2}}} - \frac{1}{(1+x)^2}\right]$$

$$= \int e^x\left[\underbrace{\frac{1}{(x+1)}}_{f(x)} - \underbrace{\frac{1}{(1+x)^2}}_{f'(x)}\right]$$

$$= \frac{e^x}{(x+1)} + c$$

Q. Questions

Illustration:

$$\int \sin(\ln x) + \cos(\ln x)\, dx$$

Ans.: $x \sin(\ln x) + c$

$\ln x = t \quad x = e^t \quad dx = e^t dt$

$\int [\sin t + \cos t] e^t dt$

$=$

✓ Do yourself

Q. Questions

$\ln x = t$

Illustration:

$$\int \frac{e^x}{x}(1 + x.\ln x)\,dx$$

Ans.: $e^x \ln x + c.$

$$= \int e^x \Big[\underbrace{\frac{1}{x}}_{f(x)} + \underbrace{\ln x}_{f'(x)}\Big]\,dx$$

$$= e^x \cdot f(x) + c$$

$$= e^x \ln x + c$$

Q. Questions

Do yourself

Illustration:

$$\int \frac{x^2 e^x}{(x+2)^2} dx$$

Ans.: $\int e^x \left(\frac{(x-2)}{(x+2)} \right) + c$

Q. Questions

Do yourself

H.W

Illustration:

$$\int (\sin x + x\cos x)\, dx$$

Ans.: $x \sin x + c$

Q. Questions

Illustration:

$$\int (2\ln x + (\ln x)^2)\,dx$$

Ans.: $x \cdot (\ln n)^2 + c$

Do yourself.

H.W

Q. Questions

$\ln x = t$

Do yourself

Illustration:

$$\int\left(\ln(\ln x)+\frac{1}{\ln^2 x}\right)dx$$

Ans.: $x\left[\ln(\ln x)-\frac{1}{\ln x}\right]+c$

Q. Questions

$\ln x = t$

$$\int (\ln x)^2 dx = Ax(\ln x)^2 - Bx \ln x + cx + D.$$

$A + B + C$

Find value of A + B + C?

Ans

Do yourself

Ans.: 5

Q. Questions

$$\int \frac{\ln(x^2+a^2)}{x^2}dx$$

$\ln(x^2+a^2)=t$

Do yourself

Ans.: $\frac{-\ln(x^2+a^2)}{x}+\frac{2}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

Q. Questions

$$\int \frac{x}{1+\sin x} dx$$

$$\int \frac{x}{1+\sin x} \frac{(1-\sin x)}{(1-\sin x)} dx$$

Do yourself

Ans.: $-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+2\ln\left(\ln\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|\right)+C$

Q. Questions

$$\int \frac{e^{\tan^{-1}x}(1+x+x^2)}{1+x^2}dx$$

$\tan^{-1}x = t$

Ans.: $xe^{\tan^{-1}x}$

Do yourself

Q. Questions

$$\int \frac{x+2}{\sqrt{x-3}}\,dx$$

$x-3=t^2$

Put

solve

It

Ans.: $\frac{2}{3}(x-3)^{3/2}+10\sqrt{x-3}+C$

Integration by parts

Illustration:

$$\int \frac{\sqrt{x}}{a\sqrt{x}+b}\,dx$$

Ans.: $\frac{1}{a^3}\left(a\sqrt{x}+b\right)^2-\frac{4b}{a^3}\left(a\sqrt{x}+b\right)+\frac{2b^2}{a^3}\ln\left|a\sqrt{x}+b\right|+C$

Special type of Integrals:

$$\int \frac{1}{(x^4+1)}, \int \frac{x^2}{x^4+1}, \int \frac{1}{x^4+x^2+1}, \int \frac{x^2}{x^4+x^2+1}$$

Completing The Square

Special kind of Integrals

Illustration:

$$\int \frac{(x^2+1)dx}{x^4+1}$$

$$I = \int \frac{x^2+1}{(x^4+1)} dx$$

"divide via x^2 both Num and deno"

$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+2-2}$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}$$

$$\frac{d}{dx}\left(x-\frac{1}{x}\right) = 1+\frac{1}{x^2}$$

$$\frac{d}{dx}\left(x+\frac{1}{x}\right) = 1-\frac{1}{x^2}$$

forms

$$x-\frac{1}{x} = t$$

$$1+\frac{1}{x^2}\, dx = dt$$

$$\Rightarrow \int \frac{dt}{(t^2+2)} = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + c$$

$$(x^2+a^2)$$

$$\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}} + c$$

Q. Questions

Illustration:

$$\int \frac{(x^2-1)dx}{x^4+1}$$

$$I = \int \frac{(x^2-1)}{(x^4+1)} dx = \int \frac{1-\frac{1}{x^2}}{\underbrace{x^2+\frac{1}{x^2}+2}-2} dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right) dx}{\left(x+\frac{1}{x}\right)^2 - 2}$$

$$x + \frac{1}{x} = t$$

$$1 - \frac{1}{x^2} dx = dt$$

$$= \int \frac{dt}{t^2 - (\sqrt{2})^2} = \underline{\frac{1}{2\sqrt{2}} \ln \left| \frac{t-\sqrt{2}}{t+\sqrt{2}} \right| + c}$$

$$\underline{\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c}$$

$$= \underline{\frac{1}{2\sqrt{2}} \ln \left| \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right| + c}$$

Q. Questions

Illustration:

$$\int \frac{dx}{x^4+1}$$

$$2 = (x^2+1)-(x^2-1)$$

$$= \cancel{x^2}+1-\cancel{x^2}+1$$

$$2 = 2$$

$$I = \int \frac{1}{(x^4+1)}\,dx \qquad 2 = (x^2+1)-(x^2-1)$$

$$= \frac{1}{2}\int \frac{2}{(x^4+1)}\,dx$$

$$= \frac{1}{2}\left[\int \frac{(x^2+1)-(x^2-1)}{(x^4+1)}\right]dx$$

$$= \frac{1}{2}\left[\underbrace{\int \frac{x^2+1}{(x^4+1)}\,dx}_{I_1} - \underbrace{\int \frac{(x^2-1)}{(x^4+1)}}_{I_2}\right]dx$$

previous one | previous one

Q. Questions

Illustration:

$$\int \sqrt{\tan x}\,dx$$

put $\tan x = t^2$

$\sec^2 x\,dx = 2t\,dt$

$$dx = \frac{2t}{\sec^2 x}dt$$

$$= \frac{2t}{1+\tan^2 x}dt$$

$$= \frac{2t}{1+t^4}dt$$

$$= \int \sqrt{t^2}\cdot\frac{2t}{1+t^4}dt = \int \frac{2t^2}{1+t^4}dt$$

$$= \int \frac{(t^2+1)+(t^2-1)}{(t^4+1)}dt$$

$$= \int \frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1}dt$$

Thank You!

GW Soldiers