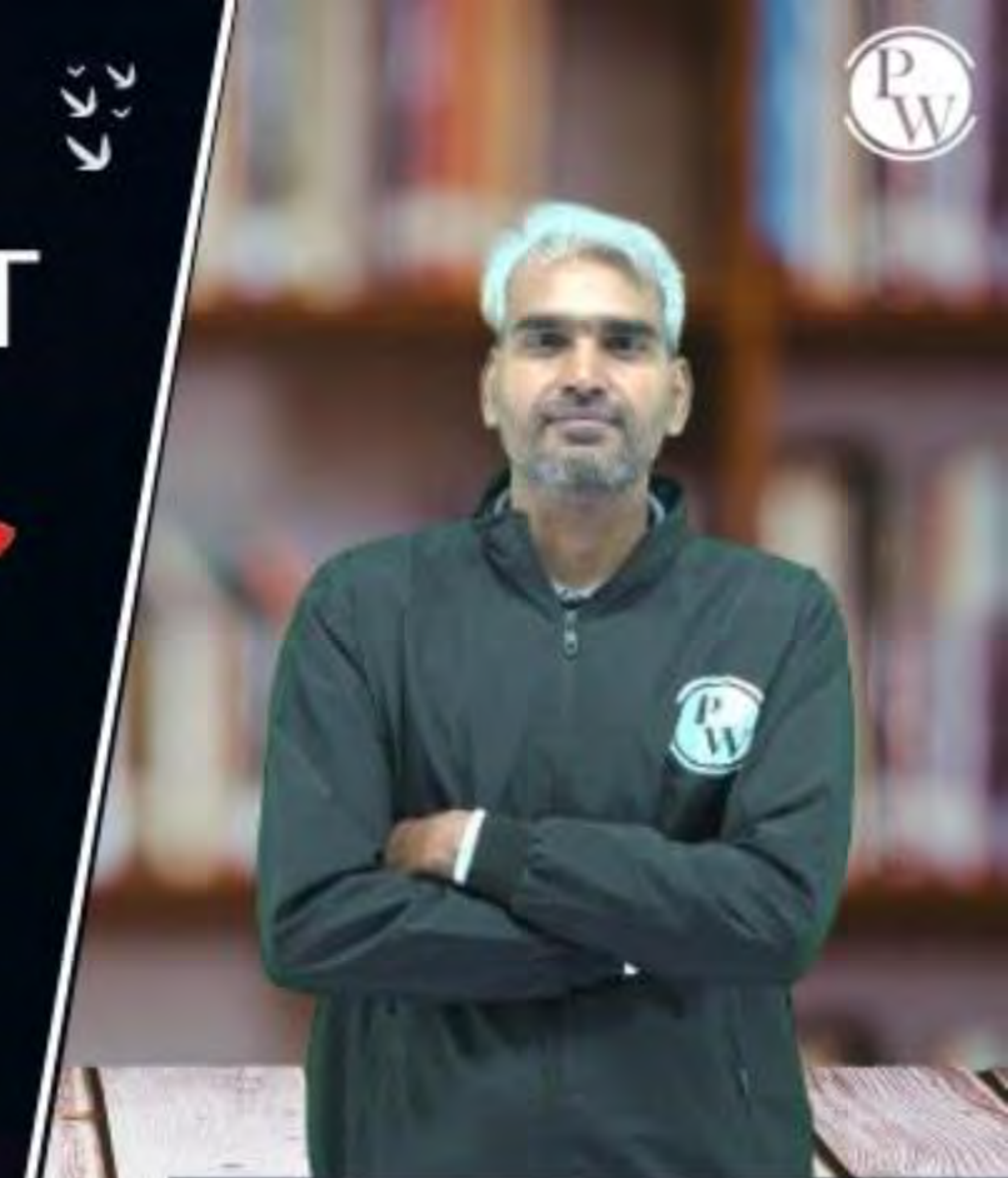
Computer Science & IT

Discrete Mathematics

Mathematical Logic

Lecture No. 04



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Recap of Previous Lecture

Topic Rules of inference

Topic Practice questions

Slide

Topics to be Covered

- **Topic** Practice questions
- **Topic** Proof by contradiction
- **Topic** Conditional proof rule
- **Topic** Predicate
- **Topic** Quantifiers
- **Topic** Scope of the quantifier

Slide

Q:- Which of the following is/are true?

(a) Conclusion R follows from the premises $\{P \to (Q \to R),\ (P \wedge Q)\}$

(b) $\{P \to (R \to S),\ \sim R \to \sim P,\ P\}$ logically implies S.

(c) $\{(P \vee Q),\ \sim P\} \vdash (P \wedge Q)$ is valid

(d) $\{(\sim T \to \sim R) \wedge \sim S \wedge (T \to W) \wedge (R \vee S)\} \to W$ is a tautology.

(a) $\{P \rightarrow (Q \rightarrow R),\ (P \wedge Q)\} \vdash R$

||| Simplification

$P,\ Q$

$$\begin{array}{l} P \rightarrow Q \\ \quad\;\; P \\ \hline \therefore\ Q \end{array}$$

M.P.

$\therefore\ Q \rightarrow R$

M.P.

$\therefore\ R$

Hence argument is valid

i.e. R follows from the premises

(b) $\{ P \to (R \to S),\ \sim R \to \sim P,\ P \}$ logically implies S.

$P \to R$

M.P.

$\therefore R \to S$

MP

$\therefore R$

M.P.

$\therefore S$

(c) $\{(P \vee Q), \sim P\} \vdash P \wedge Q$ is valid.

$\therefore Q$

if $\sim P$ is true then '$P \wedge Q$' can never be true

$\therefore \underline{\underline{\sim P \wedge Q}} \not\equiv P \wedge Q$

(d) $\left.(\sim T \to \sim R) \land \sim S \land (T \to W) \land (R \lor S)\right\} \longrightarrow W$ is a tautology

||| Simplification

$\sim T \to \sim R, \quad \sim S, \quad T \to W, \quad R \lor S$

|||

$R \to T$

Transitivity

$\therefore R \to W$

D.S.

$\therefore R$

M.P.

$\therefore W$

Q: Check whether the following argument is valid or not

(i) If it rains today then cricket match will not be played

(ii) Cricket match is not played

$\therefore$ It rained today.

let P = it rains today
Q = Cricket match is not played

$\Rightarrow$ $\therefore$ given argument becomes

(i) $P \longrightarrow Q$

(ii) Q

$\therefore P$

invalid

fallacy of affirming the Consequent

Q: Check whether the following argument is valid or not

(i) If it rains today then cricket match will not be played

(ii) Cricket match is played

$\therefore$ It did not rain today

let P = it rains today
Q = Cricket match is not played

$\Rightarrow$ $\therefore$ given argument becomes

(i) $P \rightarrow Q$

(ii) $\sim Q$

$\therefore \sim P$

Valid

Modus tollen

Q: Check whether the argument is valid or invalid.

$$\{P \to Q,\ \sim(P \wedge Q)\} \text{ logically implies } Q$$

$$\underbrace{\{\, \underset{\sim P \vee Q}{\underset{|||}{P \to Q}},\ \underset{\sim P \vee \sim Q}{\underset{|||}{\sim P \vee \sim Q}} \,\}}_{\text{if } \sim P = \text{True},}$$

then no constraint on Q.

Q may be true or may be false

∴ Argument is invalid

Topic : Proof by contradiction

If we want to check whether the argument

$$\left\{ \begin{array}{l} P_1 \checkmark \\ P_2 \checkmark \\ \vdots \\ P_n \checkmark \\ \hline \therefore Q \checkmark \end{array} \right. \quad \text{"}\sim Q\text{"}$$

is valid or not

We will assume that the Conclusion of the argument is false {i.e. we will assume that 'Q' is false i.e. '$\sim Q$' is true}
and include that false Conclusion in the set of Premises.

$\therefore$ new set of Premises becomes

$$\{P_1, P_2, P_3, \dots P_n, \sim Q\}$$

Note: If this results in any Contradiction, then our assumption is false. $\therefore$ Conclusion of the argument is true, and hence Argument is valid.

$\leftarrow$ if there is no Contradiction then our assumption may be true $\leftarrow$ Hence, Argument is invalid

Q. Check whether the argument is valid or invalid.

$$\{P \to Q,\ \sim(P \wedge Q)\} \text{ logically implies } Q$$

- Let Conclusion of the argument is false, i.e Q is false, i.e. $\sim Q$ is true
- include this false conclusion in the set of premises.

$\therefore$ Set of premises becomes $\{P \to Q,\ \sim(P \wedge Q),\ \sim Q\}$

$\therefore \sim P$ $\qquad$ $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$

No Contradiction is present

$\therefore$ Argument is invalid.

Q. Check whether the argument is valid or invalid.

$$\{P \to Q, \ \sim(P \wedge Q)\} \text{ logically implies } \sim P$$

$$\{\sim P \vee Q, \ \ \sim P \vee \sim Q\}$$

At least one of Q or ~Q will be false

$\therefore$ ~P must be true

Q. Check whether the argument is valid or invalid.

$\{P \to Q, \ \sim(P \wedge Q)\}$ logically implies $\sim P$

Proof by contradiction

Let Conclusion of the argument is false, i.e. '$\sim P$' is false

i.e. 'P' is true

Include this false Conclusion in set of premises,

∴ Set of Premises becomes $\{P \to Q, \ \sim(P \wedge Q), \ P\}$

$\sim(P \wedge Q) \equiv \ \sim P \vee \sim Q$

M.P.: $P \to Q$, P ⟹ Q

D.S.: Q, $\sim P \vee \sim Q$ ⟹ $\sim P$

$\sim P \qquad P$

P & $\sim P$ can not be true simultaneously,

∴ it is contradiction,

Hence our assumption is wrong

∴ Argument is Valid.

Topic : Conditional proof

Check whether the argument is valid or not ?

$\{P \to (Q \to S), \sim R \vee P, Q\}$ logically implies $R \to S$.

Topic : Conditional proof

Argument

$\{P_1, P_2, P_3, \ldots P_n\} \vdash Q \rightarrow R$

$T \quad T \quad = T$

is valid, if and only if

argument

$\{P_1, P_2, P_3, \ldots P_n, Q\} \vdash R$ is valid.

Topic : Conditional proof

- Check whether the argument is valid or not ?

$\{P \to (Q \to S),\ \sim R \vee P,\ Q\}$ logically implies $R \to S$.

- Checking whether the above argument is valid or not is same as checking whether the argument

$\{P \to (Q \to S),\ \sim R \vee P,\ Q,\ R\} \vdash S$ is valid or not.

D.S.

M.P.

$\therefore P$

$Q \to S$

M.P

$\therefore S$

Topic : Predicate

- It is part of a sentence or clause stating something about the subject.

Ram is a politician → { This statement may be true or may be false }

$\underbrace{\text{Ram}}_{\text{Subject}}$ $\underbrace{\text{is a politician}}_{\text{Predicate}}$

Consider, P : is a politician { i.e., P is used to denote the predicate "is a politician" }

∴ P(Ram) : Ram is a politician

Ram → Subject

Predicate 'P' is applied over the subject 'Ram'

{ P(Ram) may be true
may be false }

let Predicate, S : is a sportsman

S(Ram) : Ram is a sportsman. { May be true or false }

S(Mohan) : Mohan is a sportsman. { May be true or false }

P: is a politician
S: is a sportsman

$P(x)$: x is a politician
$S(x)$: x is a sportsman

$P(x) \longrightarrow S(x)$: if x is a politician then x is a sportsman

{ if 'x' is not a politician, then $P(x)$ will return false, and implication will return true irrespective of the truth value of Predicate $S(x)$.

{ If x is a politician {i.e. $P(x)$ is true}, then x must also be a sportsman for implication to be true }

{ if $P(x) = T$ & $S(x) = F$ then $P(x) \rightarrow S(x)$ is false. }

* $\underbrace{P(x) \longrightarrow S(y)}$

  - For this predicate to be true,
    either x should not be a politician
    (or)
    if x is a politician then y must be sportsman

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* $P(x) \vee S(x)$:
  For this statement to be true,
  either x should be a politician or x should be sportsman

$P(x) \wedge S(x)$

└ x must be politician as well as Sportsman

$P(x) \longleftrightarrow S(y)$: x is a politician iff y is a sportsman

└ It will return true in two cases

(i) $P(x) = T$ & $S(y) = T$

& (ii) $P(x) = F$ & $S(y) = F$

Topic : Predicate with multiple subjects

Let Predicate, F : is a friend of

$F(x, y)$: x is a friend of y

Predicate

applied over x & y

F(Krishna, Sudama).

Consider Predicate, G: is greater than

$G(a, b)$: a is greater than b.

$G(2,3)$: $\underbrace{\text{2 is greater than 3}}_{\text{it is false}}$

$\therefore G(2,3)$ will return false

$G(5,1)$: it returns true.

Let Predicate $F(x, y, t)$ denotes that

Person 'x' can fool, person 'y', at time 't'

Topic : Quantifiers

- In predicate logic, predicates are used alongside quantifiers to express the extent to which a predicate is true over a range of elements.
- There are two types of quantifiers

1. Universal Quantifier
2. Existential Quantifier

Topic : Scope of the Quantifier

- The part of the logical expression to which a quantifier can be applied is called the scope of the quantifier.
- Scope of the quantifier is either represented explicitly using bracket or comma, or The Scope of a quantifier is the shortest full sentence/predicate formula which follows it. Everything inside this shortest full sentence is said to be in the scope of the quantifier.

Topic : Scope of the Quantifier

- A variable whose occurrence is bounded by a quantifier is called a bounded variable. Variables not bounded by any quantifiers are called free variables.

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$
- $\exists y(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$
- $(\forall y\, H(y) \rightarrow \exists z\, G(z)) \rightarrow \exists z\, W(z, y)$
- $(\forall z\, H(y) \rightarrow \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$
- $(\forall y\, H(y) \rightarrow \exists z \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$
- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists x\, G(z)$

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$

Topic : Scope of the Quantifier

- $\exists y(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, G(z)) \rightarrow \exists z\, W(z, y)$

Topic : Scope of the Quantifier

- $(\forall z\, H(y) \rightarrow \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists x\, G(z)$

2 mins Summary

Topic Practice questions

Topic Proof by contradiction

Topic Conditional proof rule

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THANK - YOU

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