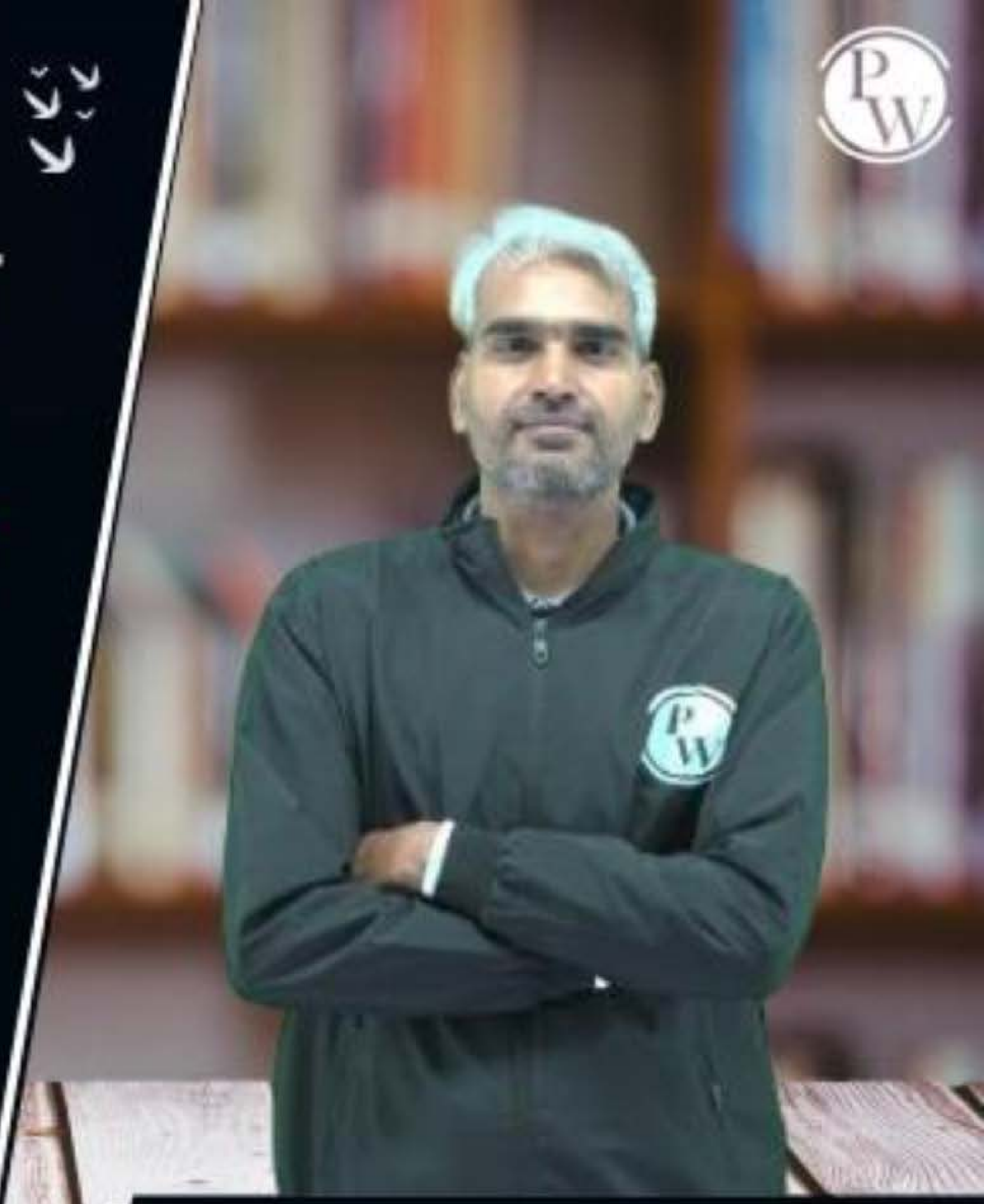
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 19



By- Vishal Sir

Recap of Previous Lecture

Topic Identical Functions

Topic Function composition

Slide

Topics to be Covered

- **Topic** Algebraic structure
- **Topic** Semi-group
- **Topic** Monoid
- **Topic** Group
- **Topic** Abelian group / Commutative group

Group Theory

Slide

Topic : Special Sets

N = Set of all natural numbers

Z = Set of all integers

Q = Set of all rational numbers

Q^* = Set of all non-zero rational numbers

Slide

Topic : Algebraic Structure

A non-empty set S w.r.t. binary opⁿ '$*$' is called an algebraic structure if

$$\boxed{a * b \in S, \ \forall a, b \in S}$$

Closure property

{ i.e. Set S is closed w.r.t. binary opⁿ $*$ }

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	Algebraic Structure	Semi-Group	Monoid	Group	Abelian Group
(N,+)	✓				
(N,·)	✓				
(N,−)	$2-3=-1 \notin N$ ✗				
(N,÷)	✗ $2 \div 5 \notin N$				
(Z,+)	✓				
(Z,·)	✓				
(Z,−)	✓				
(Z,÷)	✗				
(Q,+)	✓				
(Q,·)	✓				
(Q,−)	✓				
(Q,÷)	$0 \in Q, \therefore$ ✗				
(Q^*,+)	$\frac{p}{q}+\left(\frac{-p}{q}\right)=0 \notin Q^*$				
(Q^*,·)	✓				
(Q^*,−)	$\left(\frac{p}{q}\right)-\left(\frac{p}{q}\right)=0 \notin Q^*$				
(Q^*,÷)	✓				

Multiplication

$\frac{\text{number}}{0} \notin Q$

Slide

Topic : Semi-group

* An algebraic structure $(S, *)$ is called a semi-group if

$$\boxed{(a*b)*c = a*(b*c), \forall a, b, c \in S}$$

{ i.e. Binary op^n '$*$' is associative }

Note:- Associativity depends only on the operation, it has nothing to do with the type of elements on which op^n is performed

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Multiplication

	Algebraic Structure	Semi-Group	Monoid	Group	Abelian Group
(N,+)	✓	✓			
(N,·)	✓	✓			
(N,−)	$2-3=-1\notin N$ ✗	✗			
(N,÷)	✗ $2\div 5\notin N$	✗			
(Z,+)	✓	✓			
(Z,·)	✓	✓			
(Z,−)	✓	✗			
(Z,÷)	✗	✗			
(Q,+)	✓	✓			
(Q,·)	✓	✓			
(Q,−)	✓	✗			
(Q,÷)	$0\in Q$, ∴ ✗	✗			
(Q*,+)	$\frac{p}{q}+\left(\frac{-p}{q}\right)=0\notin Q^*$	✗			
(Q*,·)	✓	✓			
(Q*,−)	$\left(\frac{p}{q}\right)-\left(\frac{p}{q}\right)=0\notin Q^*$	✗			
(Q*,÷)	✓	✗			

number $\frac{\ }{0}\notin Q$

Slide

Topic : Monoid

A semi-group $(S, *)$ is called a monoid if and only if there exists an element $e \in S$, such that

$$\boxed{a * e = a, \ \forall a \in S}$$

Where 'e' is called identity element.

$$\left\{ \begin{array}{l} \text{identity w.r.t. addition} = 0 \\ \text{identity w.r.t. multiplication} = 1 \end{array} \right\}$$

Slide

	Algebraic Structure	Semi-Group	Monoid	Group	Abelian Group
(N,+)	✓	✓	identity=0 ∉ N ✗		
(N,·)	✓	✓	identity=1 ∈ N ✓		
(N,−)	2−3=−1 ∉ N ✗	✗	✗		
(N,÷)	✗ 2÷5 ∉ N	✗	✗		
(Z,+)	✓	✓	0 ∈ Z ∴ ✓		
(Z,·)	✓	✓	1 ∈ Z ∴ ✓		
(Z,−)	✓	✗	✗		
(Z,÷)	✗	✗	✗		
(Q,+)	✓	✓	0 ∈ Q ∴ ✓		
(Q,·)	✓	✓	1 ∈ Q ∴ ✓		
(Q,−)	✓	✗	✗		
(Q,÷)	0 ∈ Q, ∴ ✗	✗	✗		
(Q*,+)	$\frac{p}{q}+\left(\frac{-p}{q}\right)=0 \notin Q^*$	✗	✗		
(Q*,·)	✓	✓	$1 \in Q^*$, ∴ ✓		
(Q*,−)	$\left(\frac{p}{q}\right)-\left(\frac{p}{q}\right)=0 \notin Q^*$	✗	✗		
(Q*,÷)	✓	✗	✗		

Multiplication

$\frac{\text{number}}{0} \notin Q$

Slide

Topic : Group

Monoid $(S, *)$ is called a group if and only if for each element $a \in S$ there exists an element $b \in S$ such that

$$\boxed{a * b = e \text{ (identity)}}$$

where a & b are called inverse of each other

i.e $a^{-1} = b$ and $b^{-1} = a$

Slide

Multiplication

	Algebraic Structure	Semi-Group	Monoid	Group	Abelian Group
(N,+)	✓	✓	identity $=0 \notin N$ ✗	✗	
(N,·)	✓	✓	identity $=1 \in N$ ✓	✗ inverse does not exist for any element except for '1'	
(N,−)	$2-3=-1 \notin N$ ✗	✗	✗	✗	
(N,÷)	✗ $2 \div 5 \notin N$	✗	✗	✗	
(Z,+)	✓	✓	$0 \in Z$ ∴ ✓	✓	
(Z,·)	✓	✓	$1 \in Z$ ∴ ✓	✗	
(Z,−)	✓	✗	✗	✗	
(Z,÷)	✗	✗	✗	✗	
(Q,+)	✓	✓	$0 \in Q$ ∴ ✓	✓	
(Q,·)	✓	✓	$1 \in Q$ ∴ ✓	✗ $0 \in Q$, and inv(0) does not exist w.r.t. multiplication	
(Q,−)	✓	✗	✗	✗	
(Q,÷)	$0 \in Q$, ∴ ✗	✗	✗	✗	
(Q*,+)	$\frac{p}{q}+\left(\frac{-p}{q}\right)=0 \notin Q^*$	✗	✗	✗	
(Q*,·)	✓	✓	$1 \in Q^*$, ∴ ✓	✓	
(Q*,−)	$\left(\frac{p}{q}\right)-\left(\frac{p}{q}\right)=0 \notin Q^*$	✗	✗	✗	
(Q*,÷)	✓	✗	✗	✗	

$\frac{\text{number}}{0} \notin Q$

Slide

Topic : Abelian group / Commutative group

A group $(S, *)$ is called an abelian group if and only if

$$\boxed{a * b = b * a, \quad \forall a, b \in S}$$

i.e. op^n $*$ must be Commutative on the elements of the set

Note:- Commutative Property depends on the Operation as well as on the type of elements of the set.

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Multiplication

	Algebraic Structure	Semi-Group	Monoid	Group	Abelian Group
$(N,+)$	✓	✓	identity $=0 \notin N$ ✗	✗	
$(N,\cdot)$	✓	✓	identity $=1 \in N$ ✓	✗ inverse does not exist for any element except for '1'	
$(N,-)$	$2-3=-1 \notin N$ ✗	✗	✗	✗	
$(N,\div)$	✗ $2 \div 5 \notin N$	✗	✗	✗	
$(Z,+)$	✓	✓	$0 \in Z$ $\therefore$ ✓	✓	✓
$(Z,\cdot)$	✓	✓	$1 \in Z$ $\therefore$ ✓	✗	
$(Z,-)$	✓	✗	✗	✗	
$(Z,\div)$	✗	✗	✗	✗	
$(Q,+)$	✓	✓	$0 \in Q$ $\therefore$ ✓	✓	✓
$(Q,\cdot)$	✓	✓	$1 \in Q$ $\therefore$ ✓	✗ $0 \in Q$, and inv(0) does not exist w.r.t. multiplication	
$(Q,-)$	✓	✗	✗	✗	
$(Q,\div)$	$0 \in Q$, $\therefore$ ✗	✗	✗	✗	
$(Q^*,+)$	$\frac{p}{q}+\left(\frac{-p}{q}\right)=0 \notin Q^*$	✗	✗	✗	
$(Q^*,\cdot)$	✓	✓	$1 \in Q^*$, $\therefore$ ✓	✓	✓
$(Q^*,-)$	$\left(\frac{p}{q}\right)-\left(\frac{p}{q}\right)=0 \notin Q^*$	✗	✗	✗	
$(Q^*,\div)$	✓	✗	✗	✗	

number $\frac{}{0} \notin Q$

Slide

A.S.	Semi group	Monoid	Group	Abelian group
1 Closed	1 closed	1 closed	1 closed	1
	2 Associative	2 Associative	2 Associative	2
		3 identity element $\in$ Set	3 identity element $\in S$	3
			4 Inverse of every element of the set must be present in the set	4
				5 Commutative

Topic : Note

For a group $(G, *)$ following properties holds true

(1) Identity element w.r.t. binary op^n '$*$' is unique

(2) If $inv(a) = b$, then $inv(b) = a$ $\left\{ \text{i.e. } (a^{-1})^{-1} = a \right\}$

(3) $(a * b)^{-1} = b^{-1} * a^{-1}$

(4) inverse of identity element always exists, and it is identity element itself

$(e)^{-1} = e$ (Always true)

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#Q. Let $A = \{0, \pm 2, \pm 4, \pm 6, ...\}$

$B = \{0 \pm 1, \pm 3, \pm 5 .....\}$

Which of the following is not a semi- group

Semi group = Closed + Associative

A (A, +) Closed ✓ Associative

B (A, •) Closed ✓ Associative ✓

C (B, +) → not closed because odd + odd = even ∉ B

∴ Not an algebraic structure & hence not a semi-group.

D (B, •) → Closed ✓ → Associative ✓

(e) All are semi-group

Σ^* is the set of all binary strings.

We know Null string $\in \Sigma^*$

#Q. Consider the set Σ^* of all strings over the alphabet $\Sigma = \{0, 1\}$. Σ^* with the concatenation operator for strings

$$\underbrace{01011}\oplus\underbrace{101} = \underbrace{01011}\,\underbrace{101}$$

A. Not a semigroup

B. Semi group but not a monoid

C. Monoid but not a group.

D. A group

inverse does not exist for any binary string, except for NULL string.

① Closure :- Concatenation of any two binary strings of 0 & 1 is also a binary string of 0 & 1 and it will belong to Σ^*.
(∴ A.S.)

② Associative: String concatenation is associative
(∴ Semi-group)

③ identity: NULL string $\in \Sigma^*$
s.t. (binary string) $\oplus$ NULL string = binary string
∴ (Monoid).

H.W

of order $n \times n$

#Q. Let A be the set of all non-singular matrices over real number and let * be the matrix multiple operation. Then

A) A is closed under * but $\langle A, * \rangle$ is not a semigroup

B) $\langle A, * \rangle$ is a semigroup but not a monoid.

C) $\langle A, * \rangle$ is a monoid but not a group.

D) $\langle A, * \rangle$ is a group but not an abelian group.

H.W

#Q. Let S be any finite set, and F(s) is defined as set of all function on set S. Then F(s) with respect to function composition operation (ie., o) is.

A Not a semigroup

B Semi group but not a monoid

C Monoid but not a group.

D A group

H.W ✓

#Q. Let Z is the set of all integers. The binary operation * is defined as a*b = max (a, b) then the structure (Z,*) is

A. Not a semigroup

B. Semi group but not a monoid

C. Monoid but not a group.

D. A group

H.W ✓

#Q. Let Q^* be the set of all positive rational numbers. The binary operation * is defined as $a * b = \frac{ab}{3}\, \forall a, b, \in Q^*$ If $(Q^*, *)$ is a group then find

(i) identity element of the group

(ii) inverse of any element $a, \forall, \in$ Group

H.W.

#Q. Which of the following statement is/are not true.

A $\{0, \pm 2k, \pm 4k,, \pm 6k, ...\}$ is a group with respect to addition where any fixed positive integer

B $\{x \mid x$ is real number and $0 < x \leq 1\}$ is a group with respect to multiplication

C $\{2^n \mid n$ is an integer$\}$ is a group with respect to multiplication

D None of these

2 mins Summary

- **Topic** Algebraic Structure ✓
- **Topic** Semi-group ✓
- **Topic** Monoid ✓
- **Topic** Group ✓
- **Topic** Abelian group ✓

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THANK - YOU

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