GATE
ALL BRANCHES

ENGINEERING MATHEMATICS

Single Variable Calculus

Lecture No. 04

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01 How to calculate limits ✓ Limits examples.

02 Questions on continuity

Q. Questions

Value of the function $\lim\limits_{x \to a} (x-a)^{x-a}$ is____.

(a) 1

(b) 0

(c) ∞

(d) a

$L = \lim\limits_{x \to a} (x-a)^{(x-a)}$

Taking log both sides

$\log L = \lim\limits_{x \to a} (x-a) \log(x-a)$

$\log L = \lim\limits_{x \to a} \dfrac{\log(x-a)}{\frac{1}{(x-a)}}$

$\log L = \lim\limits_{x \to a} \dfrac{\frac{1}{(x-a)}}{\frac{-1}{(x-a)^2}}$

$= \lim\limits_{x \to a} \dfrac{-(x-a)^{\cancel{2}}}{\cancel{(x-a)}} = 0$

$\log L = 0 \quad \boxed{L = e^0 = 1}$

Q. Questions

Limit of the function $f(x) = \dfrac{1-a^4}{x^4}$ as $x \to \infty$ is given by

$\underset{x\to\infty}{Lt} \dfrac{1-a^4}{x^4}$ $\boxed{Ans = 0}$

(a) 1

(b) $e^{-a/4}$

(c) ∞

(d) 0

Q. Questions

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin 2\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}} = \_\_\_\_.$$

(a) 0

(b) 1/2

(c) 1

(d) 2

$$\underset{x\to\frac{\pi}{4}}{Lt} \frac{\sin 2\left(x-\frac{\pi}{4}\right)}{\left(x-\frac{\pi}{4}\right)}$$

Using Template $\underset{x\to\frac{\pi}{4}}{lt} \boxed{\frac{\sin 2\left(x-\frac{\pi}{4}\right)}{2\left(x-\frac{\pi}{4}\right)}}\, 2$

$= 2$ Ans

$$\underset{x\to\frac{\pi}{4}}{lt} \frac{\sin 2\left(x-\frac{\pi}{4}\right)}{\left(x-\frac{\pi}{4}\right)}$$

$$\frac{\sin 2\left(\frac{\pi}{4}-\frac{\pi}{4}\right)}{\left(\frac{\pi}{4}-\frac{\pi}{4}\right)} = \left(\frac{\to 0}{\to 0}\right)$$

Q. Questions

$$\lim_{x\to 0}\frac{\sin^2 x}{x} = \_\_\_\_.$$

(a) 0

(b) ∞

(c) 1

(d) –1

$$\lim_{x\to 0}\frac{\sin^2 x}{x}\left(\frac{\to 0}{\to 0}\right)$$

$$= \lim_{x\to 0}\frac{2\sin x\cos x}{x}$$

$$= \lim_{x\to 0}\sin 2x$$

$= 0$ Ans

Q. Questions

$$\lim_{x\to 0}\frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3}=\_\_\_\_.$$

(a) 0

(b) 1/6

(c) 1/3

(d) 1

$$\operatorname*{Lt}_{x\to 0}\frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3}\quad\left(\frac{\to 0}{\to 0}\right)$$

Using L-Hospital Rule.

$$=\operatorname*{lt}_{x\to 0}\frac{e^x-0-1-x}{3x^2}$$

$$=\operatorname*{lt}_{x\to 0}\frac{e^x-1}{6x}$$

$$\to\frac{1}{6}\operatorname*{lt}_{x\to 0}\frac{e^x-1}{x}=\frac{1}{6}$$

Q. Questions

What is the value of $\lim\limits_{x \to \pi/4} \dfrac{\cos x - \sin x}{x - \pi/4}$ $\left(\dfrac{\to 0}{\to 0}\right)$

= using L-Hospital Rule

(a) $\sqrt{2}$

(b) 0

(c) $-\sqrt{2}$

(d) Limit does not exist

$\Rightarrow \lim\limits_{x \to \frac{\pi}{4}} \dfrac{-\sin x - \cos x}{1}$

$= -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = -\sqrt{2}$

Q. Questions

$$\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\theta/2} \text{ is } \cdot \frac{1}{2} = \frac{1}{2}$$

(a) 0.5

(b) 1

(c) 2

(d) Not defined

Q. Questions

$$\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x} = \_\_\_\_.$$

(a) 1

(b) –1

(c) ∞

(d) $-\infty$

$$\underset{x \to \infty}{Lt} \frac{\cancel{x}\left[1 - \frac{\sin x}{x}\right]}{\cancel{x}\left[1 + \frac{\cos x}{x}\right]} \quad \frac{\sin x}{x} \to 0, \quad \frac{\cos x}{x} \to 0$$

$$= 1$$

Q. Questions

$\displaystyle \mathop{Lim}_{x \to \infty} \frac{\sin x}{x}$ is

(a) Indeterminate

(b) 0

(c) 1

(d) $\propto$

Q. Questions

The value of $\underset{x \to 8}{Lim} \dfrac{x^{1/3}-2}{x-8}$ is

Using L-Hospital Rule

$\left(\dfrac{\to 0}{\to 0}\right)$

$\underset{x \to 8}{lt} \dfrac{x^{\frac{1}{3}}-2}{(x-8)}$

$= \underset{x \to 8}{lt} \dfrac{\frac{1}{3}x^{-2/3}}{1}$

$= \frac{1}{3} \times (8)^{-2/3} = \frac{1}{12}$

Q. Questions

The value of the expression $\lim_{x \to 0}\left[\frac{\sin(x)}{e^x x}\right]$ is

$$\underset{x \to 0}{Lt}\left[\frac{\sin x}{x e^x}\right]$$

(a) 0

Using L-Hospital Rule

(b) 1/2

$$= \underset{x \to 0}{lt} \frac{\cos x}{(x+1)e^x}$$

(c) 1

$$= 1$$

(d) $1/1 + e$

Q. Questions

What is the value of $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$?

$\rightarrow (\to 1)^{\to\infty}$

Ans $= e^{A}$

$A = \lim\limits_{n\to\infty}\left(\cancel{1}-\frac{1}{\cancel{x}}-\cancel{1}\right)2\cancel{x}$

$= -2$

Ans $= e^{-2} = \frac{1}{e^{2}}$

(a) 0

(b) e^{-2}

(c) $e^{-1/2}$

(d) 1

Q. Questions

The $\lim\limits_{x \to 0} \dfrac{\sin\left(\frac{2}{3}x\right)}{x}$ is

Do yourself

(a) 2/3

(b) 1

(c) 3/2

(d) ∞

Q. Questions

$\lim_{x \to \infty} x^{1/x}$ is

(a) ∞

(b) 0

(c) 1

(d) Not defined

$$\log L = \lim_{x \to \infty} \frac{1}{x} \log x$$
$$= \lim_{x \to \infty} \frac{\log x}{x}$$
$$= \lim_{x \to \infty} \frac{1}{x} = 0$$

$$\log L = 0$$
$$L = e^0$$
$$\boxed{L = 1}$$

Q. Questions

The value of $\lim_{x\to\infty}\left(1+x^2\right)^{e^{-x}}$ is

(a) 0

(b) 1/2

(c) 1 ✓

(d) ∞

$\underset{x\to\infty}{Lt}\,(1+x^2)^{e^{-x}}$

Taking log both sides

$\log L = e^{-x}\log(1+x^2)$

L-Hospital Rule $= \left\{ \underset{x\to\infty}{lt} \dfrac{\log(1+x^2)}{e^{x}} \right. \quad \left(\dfrac{\to\infty}{\to\infty}\right)$

$= \underset{x\to\infty}{lt} \dfrac{1}{(1+x^2)}\dfrac{2x}{e^{x}}$

Again

$= \underset{x\to\infty}{lt} \dfrac{2}{e^{x}+2xe^{x}+e^{x}x^{2}}$

$= 0$

$\log L = 0$

$L = e^0 = 1$

Q. Questions

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2+1}\right) \text{ is } \_\_\_\_.$$

Do yourself

Q. Questions

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\log_e(1+4x)}{e^{3x-1}}$$ is equal to

Do yourself

(a) 0

(b) 1/12

(c) 4/3

(d) 1

Q. Questions

$\underset{x\to\infty}{\mathrm{Lt}}\left(\sqrt{x^2+x-1}-x\right)$ is

(a) 0

Do yourself

(b) ∞

(c) 1/2

(d) $-\infty$

Q. Questions

$\lim_{x \to 0} \left(\frac{e^{5x} - 1}{x} \right)^2$ is equal to ____.

Do yourself

Q. Questions

$$\lim_{x \to 4} \frac{\sin(x-4)}{x-4} = \_\_\_\_.$$

Do yourself

Q. Questions

The value of $\lim_{x \to 0} \left(\frac{x^3 - \sin(x)}{x} \right)$ is

Do yourself

(a) 0

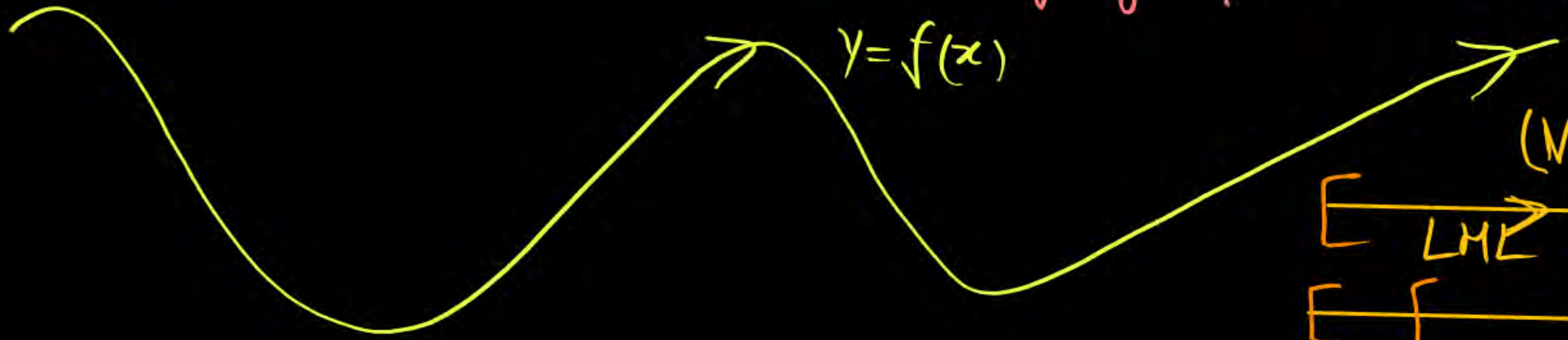
(b) 3

(c) 1

(d) –1

Continuity of a Function :

without lifting my Pen — continuity



Property of continuity

$$\boxed{LHL = RHL = f(a) = \text{finite value}}$$

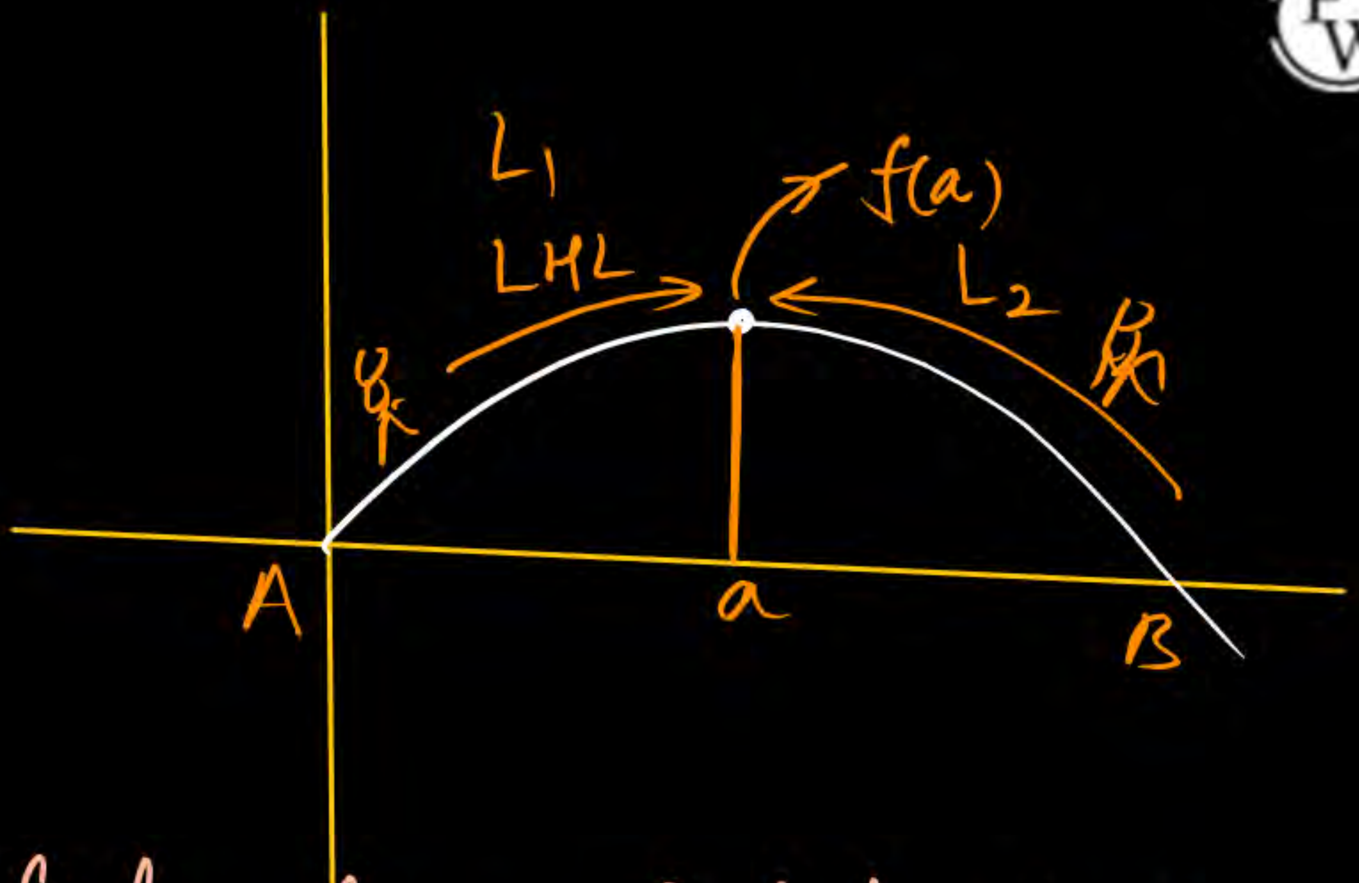
If $y = f(x)$ is continuous $[A, B]$

$$LHL = RHL = f(a)$$

$$LHL = \lim_{h \to 0} f(a-h)$$

$$RHL = \lim_{h \to 0} f(a+h)$$

$$\boxed{\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) = f(a)}$$



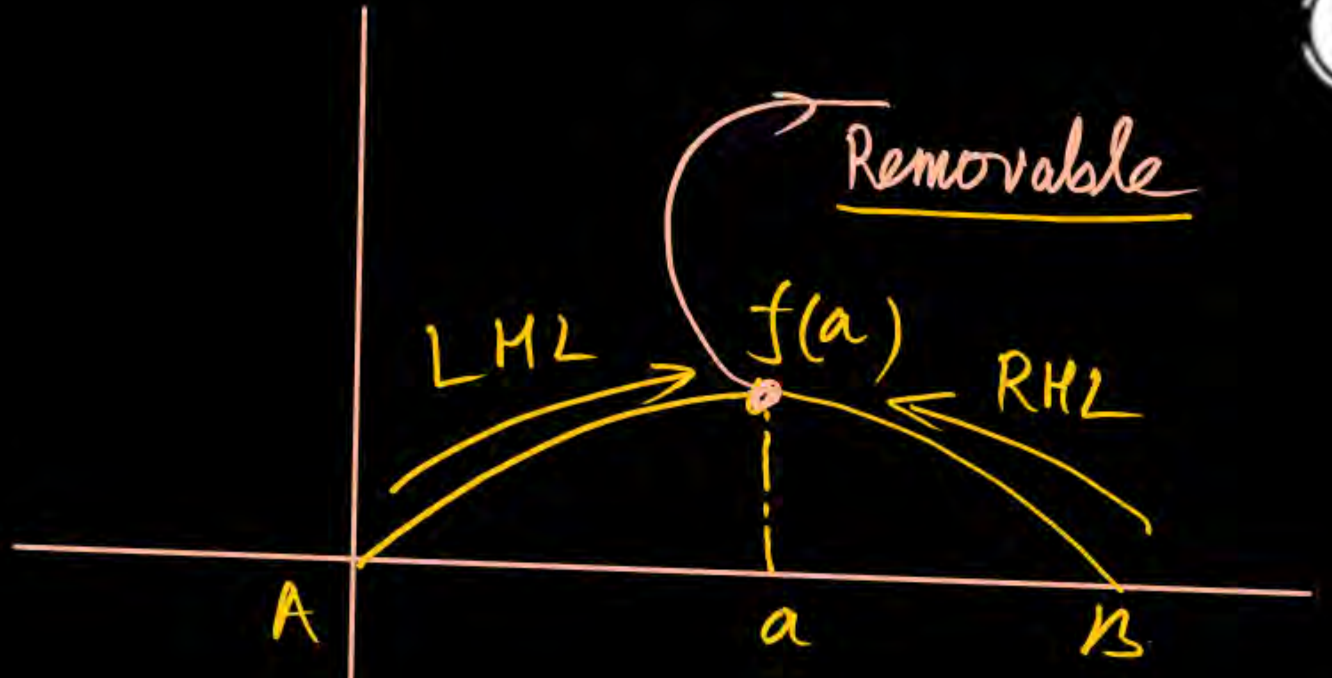
$\Rightarrow$ finite value = Property of conti

"Disconti of a Function"

Removable Discontinuity

$LHL = RHL \neq f(a)$

$$\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) \neq f(a)$$



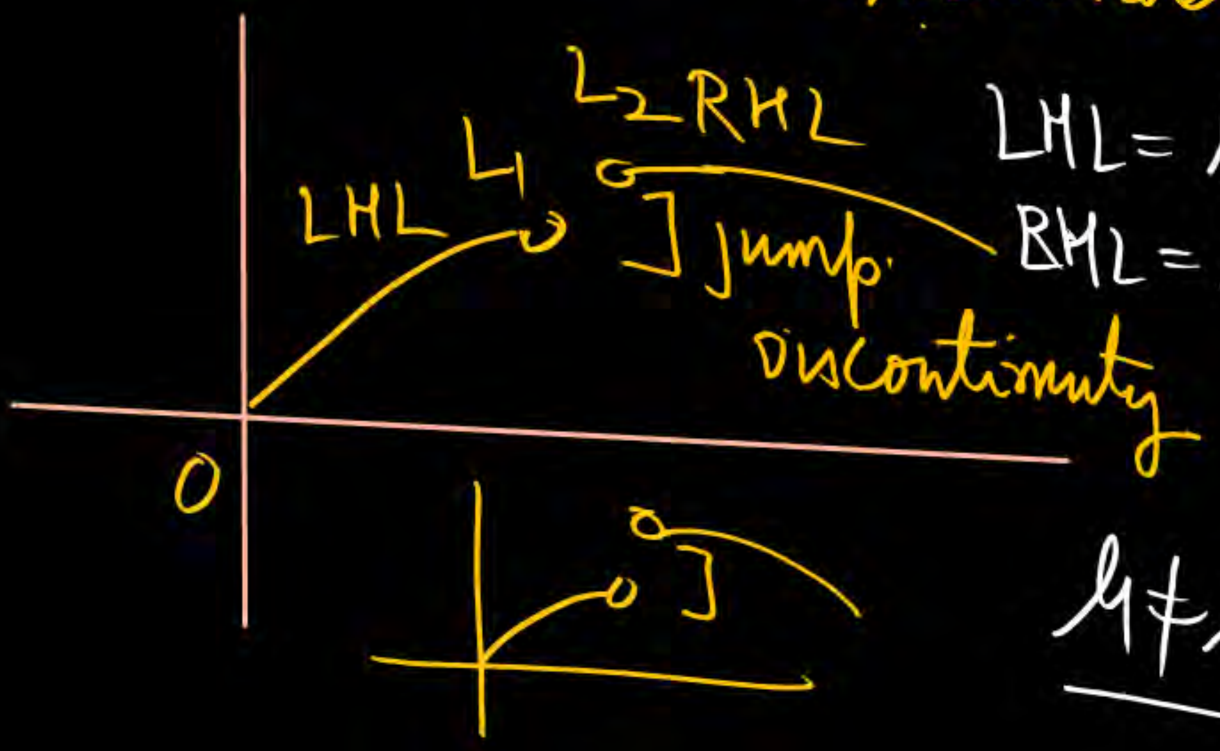
Jump Discontinuity

$LHL \neq RHL$

$$= \lim_{h \to 0} f(a-h) \neq \lim_{h \to 0} f(a+h)$$

(Irremovable Discontinuity)

Irremovable



$LHL = l_1$

$RHL = l_2$

$l_1 \neq l_2$

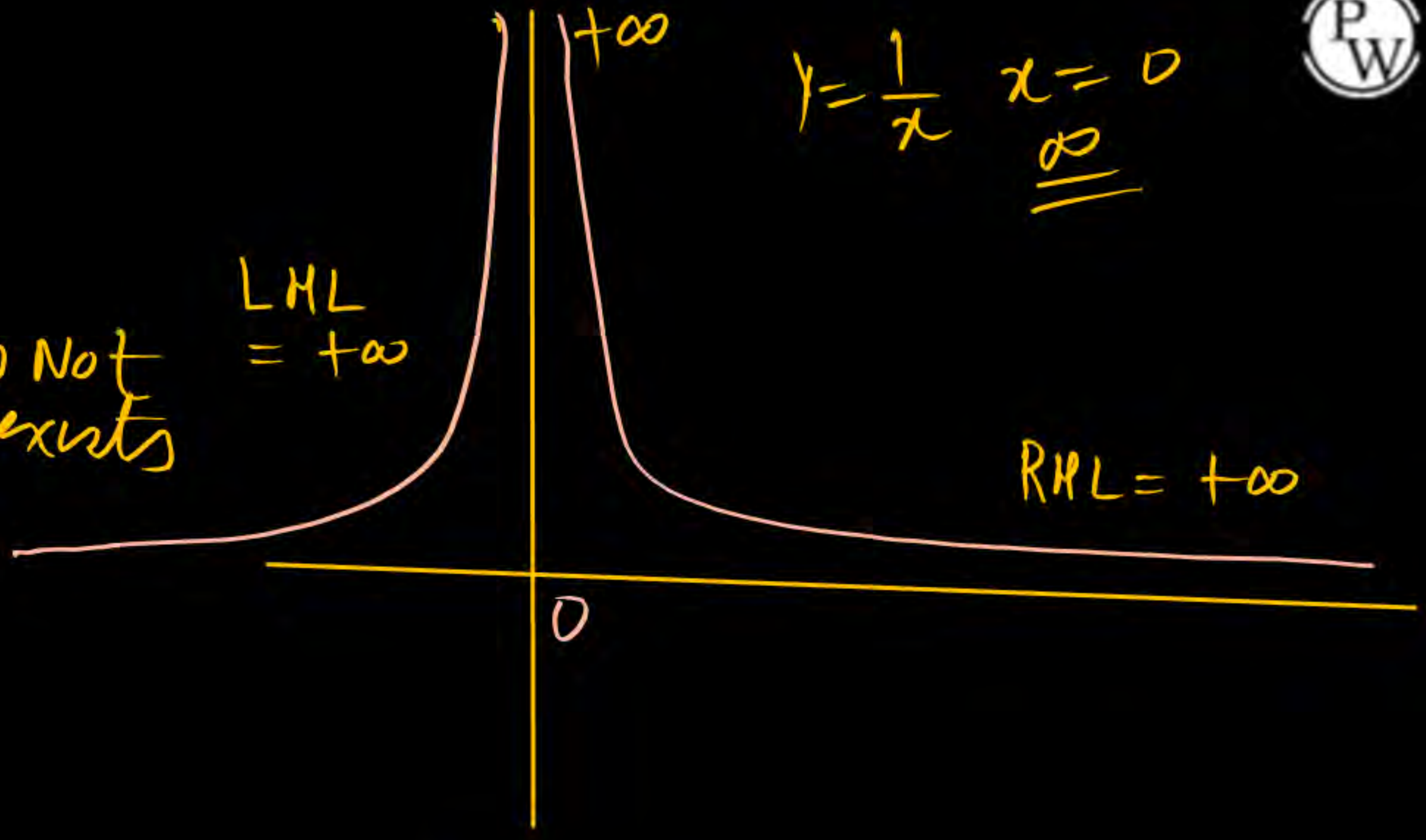
$LHL = \pm\infty$
$RHL = \pm\infty$

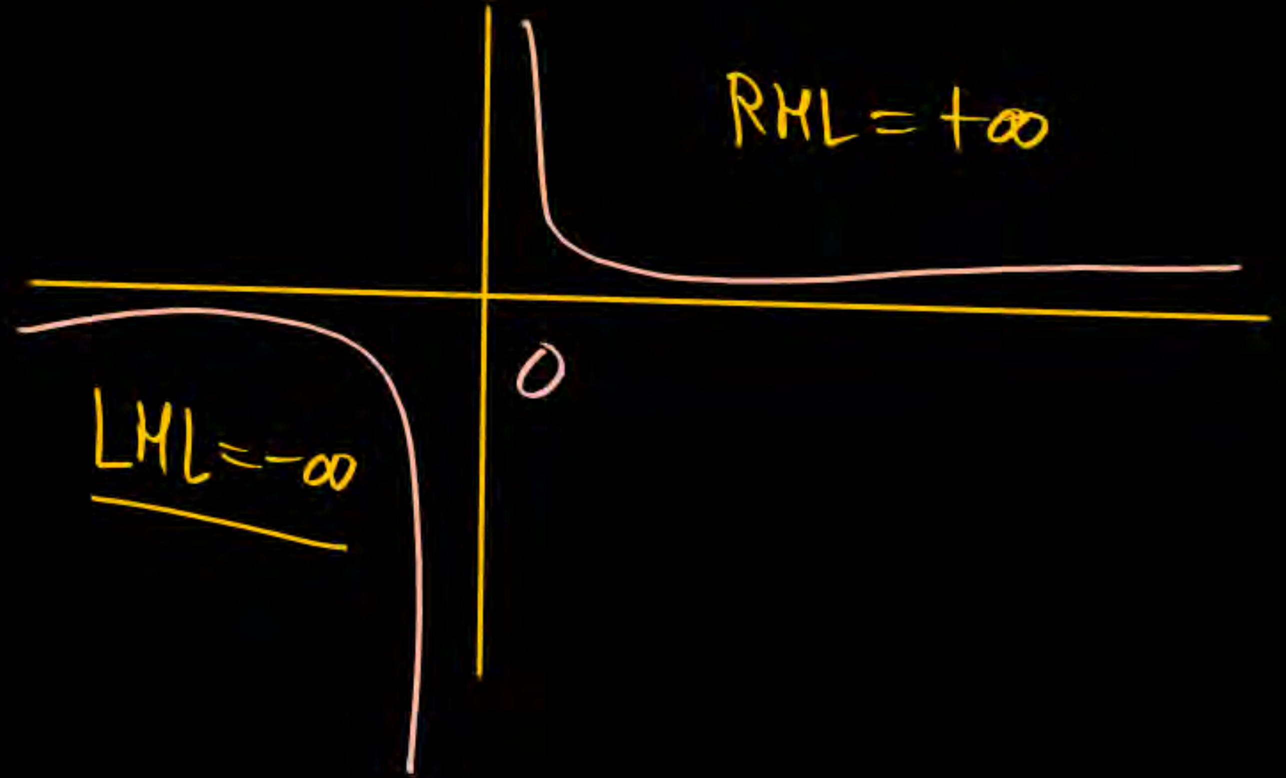
RHL and LHL Are does Not exists

✓ Infinite Discontinuity

$$\left.\begin{array}{l} LHL = -\infty \\ RHL = +\infty \end{array}\right]$$

(Irremovable Discontinuity)





Q. Questions

The values a and b so that the function:

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x & ; \quad 0\le x<\frac{\pi}{4} \\ 2x\cot x+b & ; \quad \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x & ; \quad \frac{\pi}{2}<x\le\pi \end{cases}$$

is continuous $x\in[0,\pi]$ is:

(a) $a=\frac{\pi}{6}, b=\frac{-\pi}{12}$

(b) $a=\frac{\pi}{3}, b=\frac{-\pi}{12}$

(c) $a=\frac{\pi}{6}, b=\frac{\pi}{12}$

(d) None of these

Check $x = \frac{\pi}{4}$

$x + a\sqrt{2}\sin x$

$$LHL = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f\left(\frac{\pi}{4} - h\right) = \lim_{h \to 0}\left[\left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right)\right]$$

$$= \frac{\pi}{4} + a\sqrt{2}\sin\frac{\pi}{4}$$

$$\boxed{LHL = \frac{\pi}{4} + a}$$

$2x\cot x + b$

$$RHL = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f\left(\frac{\pi}{4} + h\right) = \lim_{h \to 0}\left[2\left(\frac{\pi}{4} + h\right)\cot\left(\frac{\pi}{4} + h\right) + b\right]$$

$$RHL = \frac{\pi}{2} + b$$

$$\frac{\pi}{4} + a = \frac{\pi}{2} + b$$

$$a - b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$a - b = \frac{\pi}{4}$$

check at $x = \frac{\pi}{2}$

LHL = RHL

$$\boxed{a + 2b = 0}$$

$a - b = \frac{\pi}{4}$ — (1)

$a + 2b = 0$ — (2)

find the value of

$a = \frac{\pi}{6}$

$b = -\frac{\pi}{12}$

Q. Questions

Let $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & ; \quad \frac{-\pi}{6}<x<0 \\ b & ; \quad x=0 \\ e^{\left(\frac{\tan 8x}{\tan 3x}\right)} & ; \quad 0<x<\frac{\pi}{6}\end{cases}$

The value a and b such that $f(x)$ is continuous at $x=0$ is:

(a) $a=8, b=e^{8}$

(b) $a=\frac{8}{3}, b=e^{-8}$

(c) $a=\frac{8}{3}, b=e^{8/3}$

(d) None of these

$-\frac{\pi}{6}$ ① LHL 0 ② RHL $\frac{\pi}{6}$

$\text{RHL} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$= \lim_{h\to 0} e^{\frac{\tan 8h}{\tan 3h} \times \frac{3h}{3h} \times \frac{8h}{8h}} = e^{\frac{8}{3}}$

$\boxed{\text{LHL} = \text{RHL} = b}$

$\text{LHL} = e^{a}$

$\text{RHL} = e^{8/3}$

$a = \frac{8}{3} \quad b = e^{8/3}$

Check at $x=0$

Step 1) $LHL = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(-h)$

$f(x) = \left[1+|\sin x|\right]^{\frac{a}{|\sin x|}}$

Step 2) $= \lim_{h\to 0}\left[1+|\sin(-h)|\right]^{\frac{a}{|\sin(-h)|}}$

Step 3) $= \lim_{h\to 0}\left[1+|-\sin h|\right]^{\frac{a}{|-\sin h|}}$

Step 4) $= \lim_{h\to 0}\left[\underbrace{1+\sin h}_{f(h)}\right]^{\underbrace{\frac{a}{\sin h}}_{g(h)}}$ $\quad (\longrightarrow 1)^{\to\infty}$

$\text{Ans} = e^{A}$

Where $A = \lim_{h\to 0}\left[\cancel{1}+\sin h-\cancel{1}\right]\frac{a}{\cancel{\sin h}} = a$

$\boxed{\text{Ans} = e^{a}}$

Q. Questions

$x<0$ | $x>0$, 0

LHL =

$$\text{Let } f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2} & ; \boxed{x<0} \text{ LHL} \\ \boxed{a \quad ; \quad x=0} & \text{value of function} \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; x>0 \quad \text{RHL} \end{cases}$$

The value of a, if possible, so that the **function is continuous** at $x=0$ is:

LHL = RHL = $f(0)$

LHL = RHL = a

(a) 6 (b) 8

(c) -6 (d) None of these

Left Hand limit $\lim\limits_{h\to 0} f(a-h)$ $\quad$ $LHL = \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(-h)$

$$f(x) = \frac{1-\cos 4x}{x^2}$$

$$f(-h) = \frac{1-\cos 4(-h)}{(-h)^2}$$

$$= \frac{1-\cos 4h}{h^2}$$

$$= \lim_{h\to 0} \frac{1-\cos 4h}{h^2} \left(\frac{\to 0}{\to 0}\right)$$

$$= \lim_{h\to 0} + \boxed{\frac{\sin 4h}{2h\times 2}}\, 4\times 2$$

$$= 8 \lim_{h\to 0} \frac{\sin 4h}{4h}$$

$$\boxed{LHL = 8}$$ Ans

$$RHL = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4} = 8$$

$$\boxed{LHL = RHL = f(0)}$$

$$8 = 8 = a$$

$a = 8$

Q. Questions

If $f(x)=\dfrac{1}{1+e^{1/x}}, x\neq 0,$ discuss the continuity of $f(x)$ at $x = 0$.

$e^{-\infty}=0$

$e^{\infty}=\infty$

$f(x)=\dfrac{1}{1+e^{1/x}}$

$\text{RHL}=\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(0+h)$

$=\lim_{h\to 0} f(h)$

$=\lim_{h\to 0}\dfrac{1}{1+e^{1/h}}$

$=\lim_{h\to 0}\dfrac{1}{1+e^{1/h\to\infty}}=\underline{0}$

$e^{\frac{1}{h}}=e^{\frac{1}{0}}=e^{\infty}=\to\infty$

$e^{-1/h}=e^{-\infty}=0$

LHL

$=\lim_{h\to 0}\dfrac{1}{1+e^{-1/h}}$

$=\lim_{h\to 0}\dfrac{1}{1+e^{-1/h}}$

$=\underline{1}$

$\text{LHL}\neq\text{RHL}$

Jump.

Thank You!

GW Soldiers