GATE

ALL BRANCHES

ENGINEERING MATHEMATICS

Probability and Statistics

Lecture No. 08

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TOPICS TO BE COVERED

01 Problems based on Random Variables

✓ cdf [Cumulative Distribution Function]

$F_X(x_i) = P[X \leq x_i]$ [Discrete Random Vari]

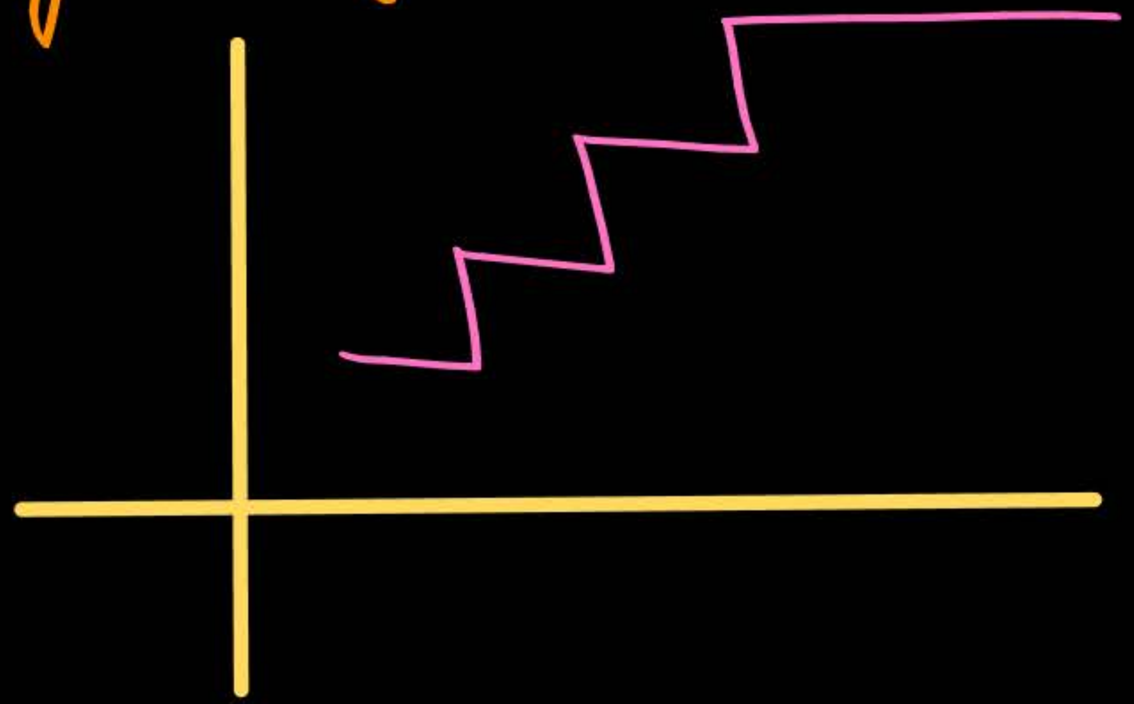
A) $F_X(-\infty) = 0$ $F_X(\infty) = 1$

B) Staircase function / montonic Increasing.

C) always Non-Negative function

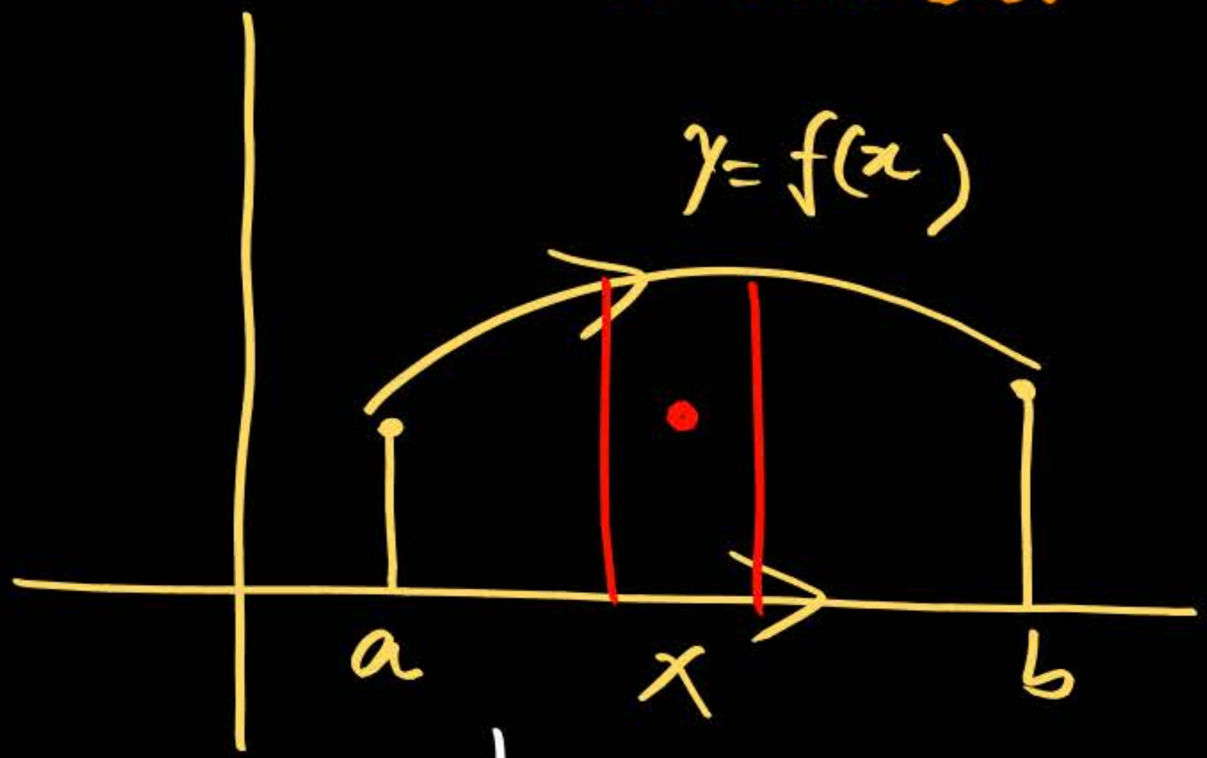
$F_X(x_i) \geq 0$

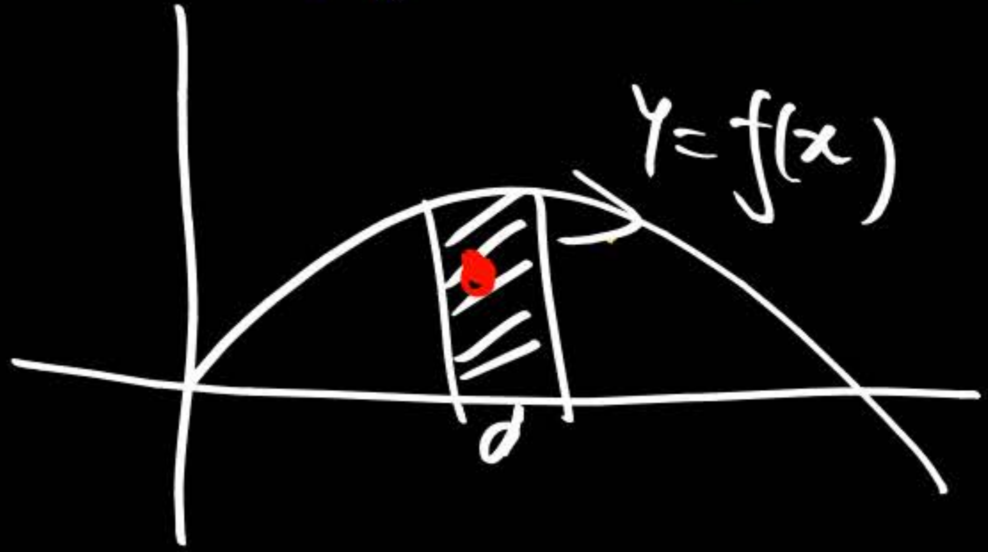
$P_X(x_i) \geq 0$

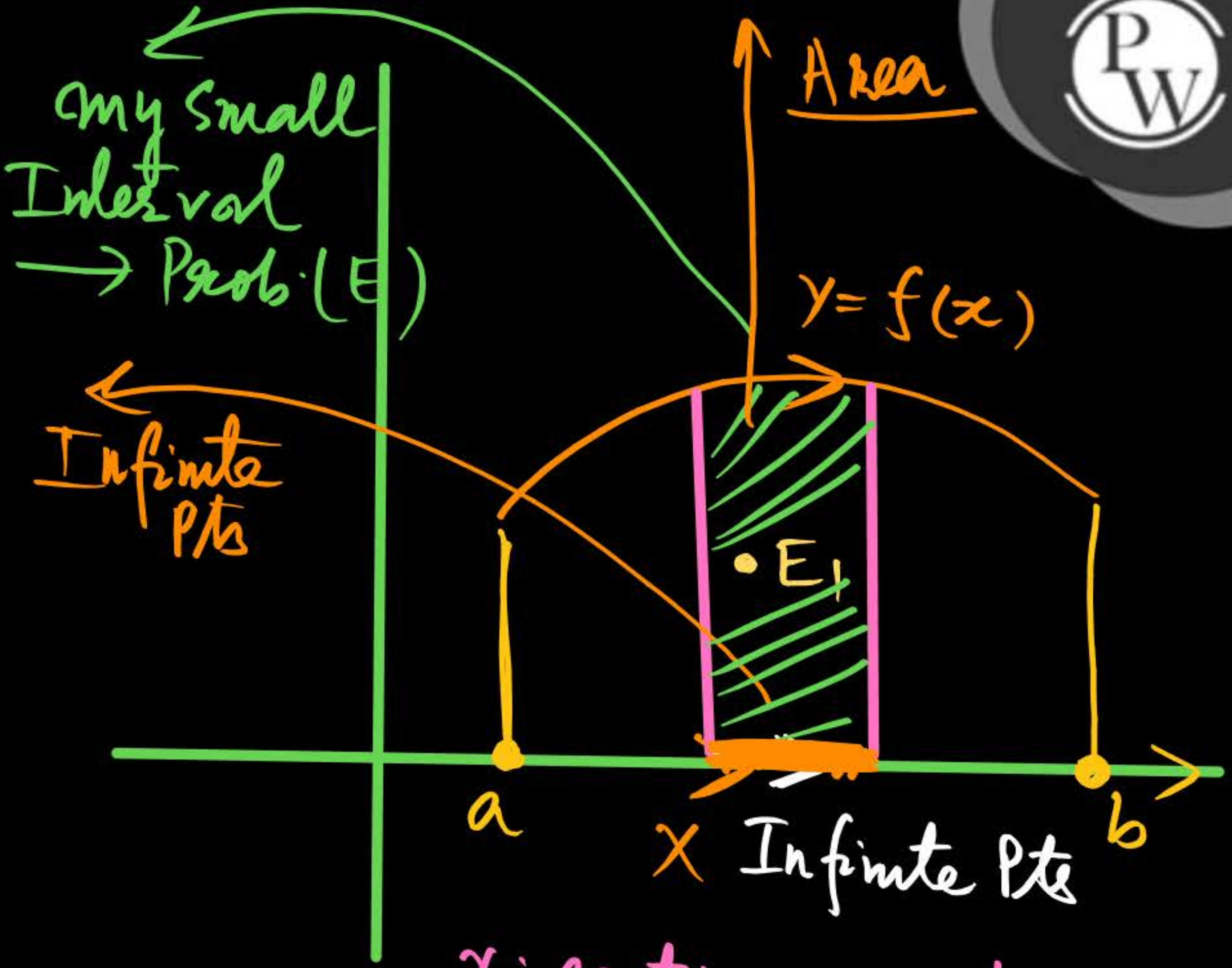


✓ For continuous:

X is a continuous Random variables.







X : continuous random variable.

$P(x \le X \le x+dx) =$ AREA of Rectangle

$\Rightarrow P[x \le X \le x+dx) = f(x) \cdot dx$

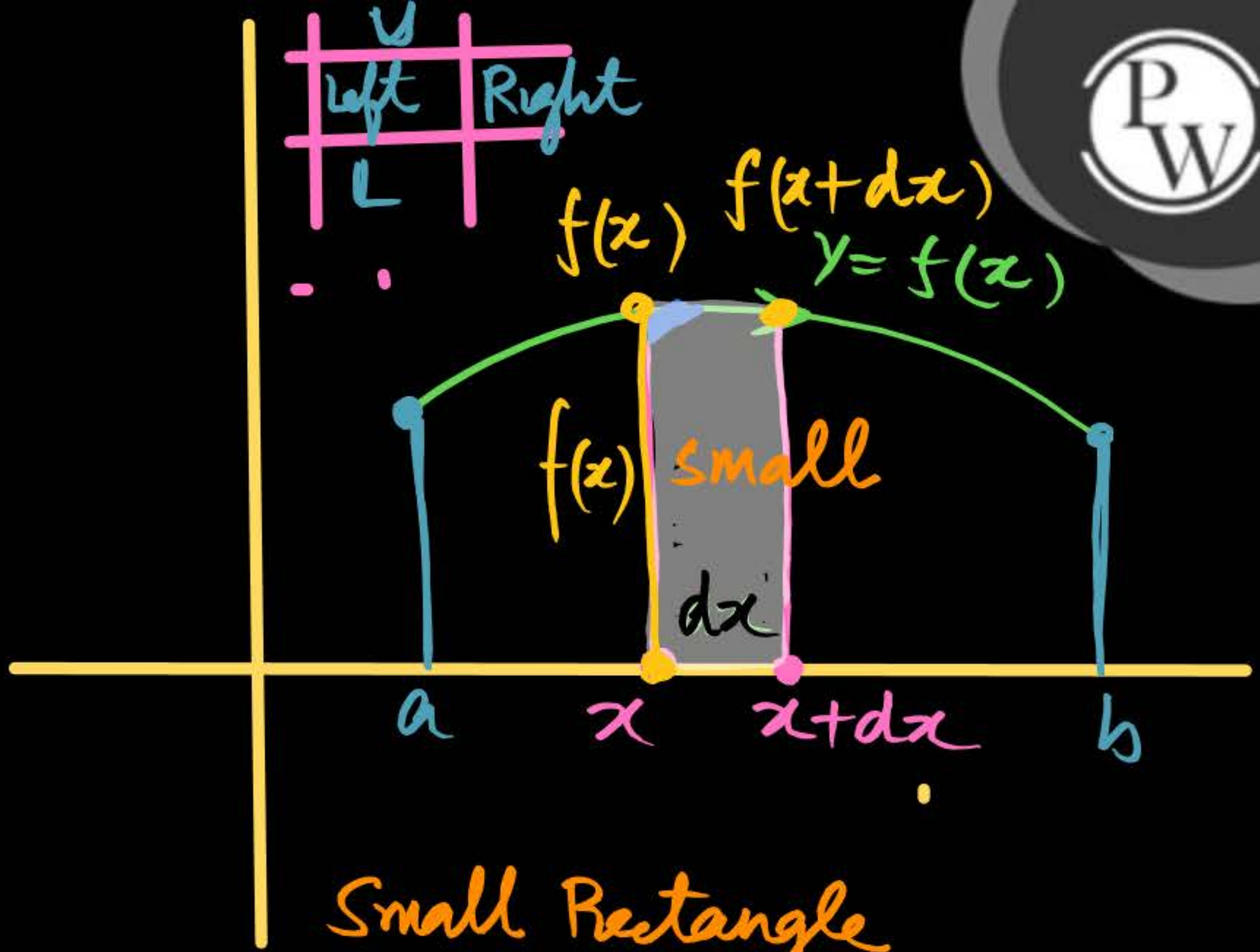
Using cdf defination

$F(x_i) = P(X \le x_i)$

$\Rightarrow F_X(x+dx) - F_X(x) = f(x)\,dx$

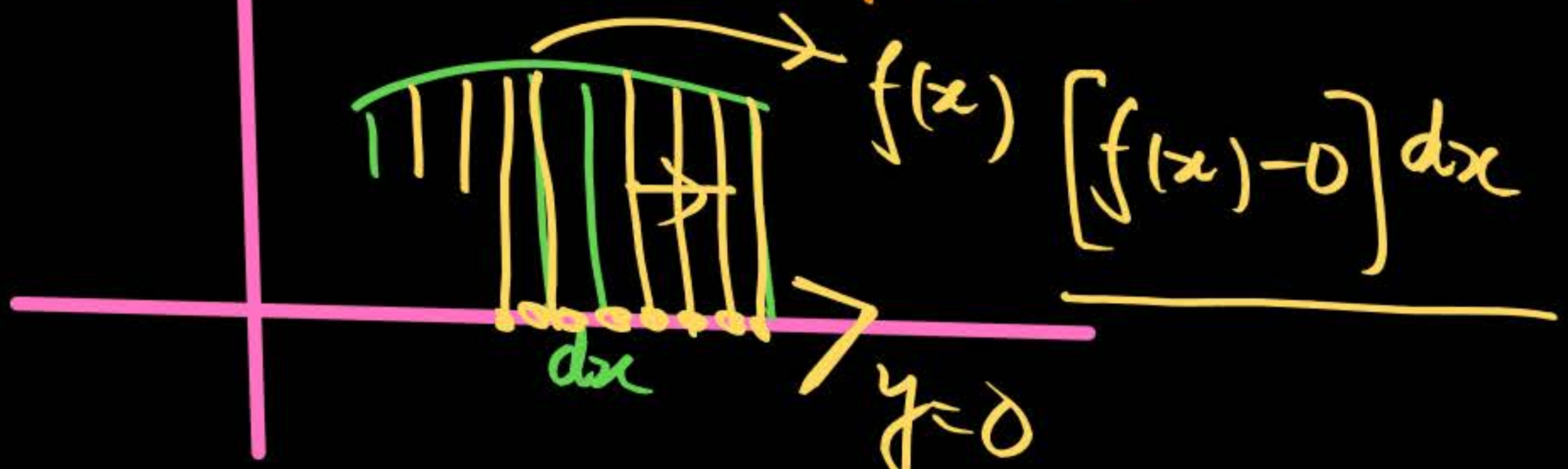
divide via dx

$$\Rightarrow \lim_{dx \to 0} \frac{F_X(x+dx) - F_X(x)}{dx} = f(x)$$



Small Rectangle

Area = $f(x)\,dx$



$$\underset{dx\to 0}{Lt} \frac{F_X(x+dx)-F_X(x)}{dx} = f(x)$$

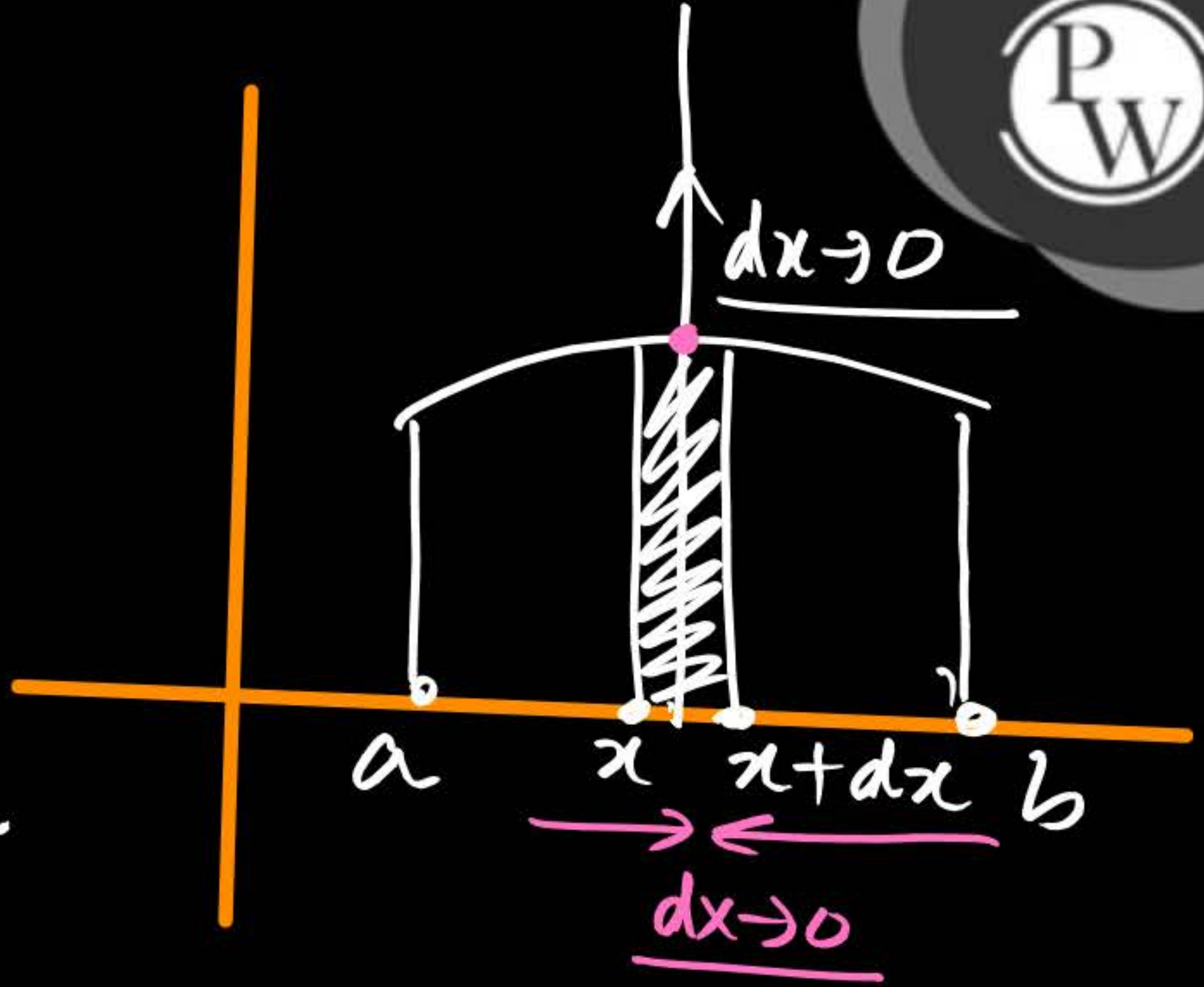
$=$ derivative of $F_X(x)$

$$\Rightarrow \quad \frac{d}{dx}\left[F_X(x)\right] = f(x)$$

Where $f(x)$ = prob. density Function

$$\boxed{F_X(x) = \int_{\text{Region}} f(x)\,dx}$$

cdf — continuous Random var



$$\underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$= f'(x)$$

$$F_X(x) = \int_{\text{Region}} f(x)\,dx$$

$$\underbrace{F_X(x)}_{\text{cdf}} = \int_{-\infty}^{x} \overbrace{f(x)}^{} dx \longrightarrow \text{pdf}$$

(A) $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$ (prob. density function]

(B) $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

C) If X is a continuous RV

$$P[a \le X \le b] = F_X(b) - F_X(a) = \int_a^b f(x)\,dx$$

$$P[1 \le X \le 5] = F_X(5) - F_X(1) = \int_1^5 f(x)\,dx$$

(D) $P(X \ge a)$

X is conti Random var $\Rightarrow \int_a^{\infty} f(x)\,dx$

Q. Questions

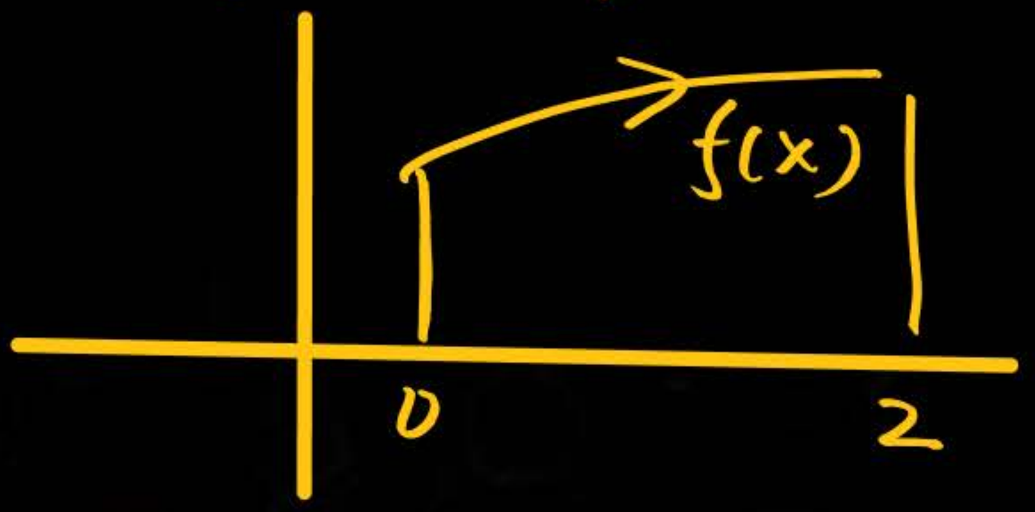
Interval $\to$ cont RV

If X is a continuous random variable whose probability density function is given by

X is a continuous R.V

$$f(x)=\begin{cases} k\left(5x-2x^2\right), & 0\le x\le 2 \\ 0, & \text{Otherwise} \end{cases}.$$

$f(x) = K(5x-2x^2) \quad 0 \le x \le 2$



Then P ($x \ge 1$) is

(a) 3/14

(b) 4/5

(c) 14/17

(d) 17/28

$P(x \ge 1) = \int_1^2 f(x)\,dx = \int_1^2 K(5x-2x^2)\,dx$

x is conti RV

0 to 1

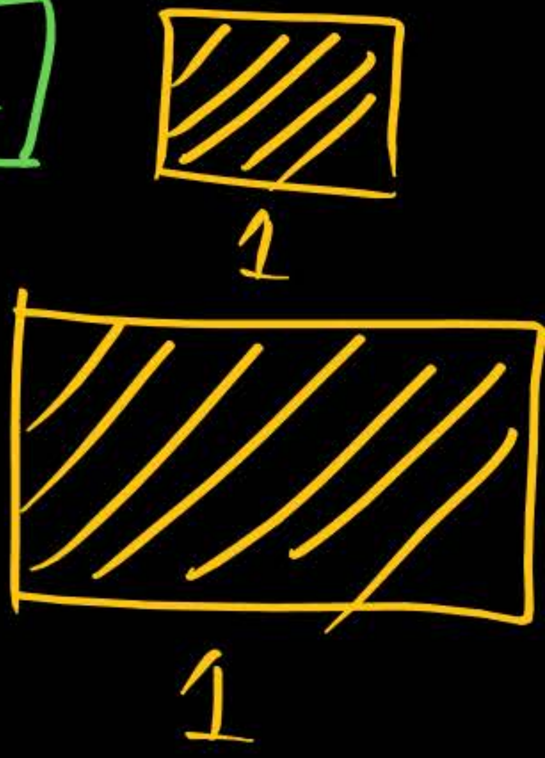
$=$ K-form, remove k

Constant + Strategy

If $f(x)$ is valid prob. Density Function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Total Area = 1



$$\Rightarrow \int_0^2 K(5x-2x^2)\,dx = 1$$

$$= K\int_0^2 (5x-2x^2)\,dx = 1$$

$$= K\left[\frac{5x^2}{2} - \frac{x^3}{3}2\right]_0^2 = 1$$

$$= \boxed{K = \frac{3}{14}}$$

$$P(X \geq 1) = \int_1^2 \frac{3}{14}(5x-2x^2)\,dx$$

$$= \frac{17}{28}$$

Q. Questions

$f(x) = Me^{-2|x|} + Ne^{-3|x|}$

$f_x(X) = Me^{(-2|x|)} + Ne^{(-3|x|)}$ is the probability density function for the real random variable X, over the entire x-axis. M and N are both positive real numbers. The equation relating M and N is

(a) $M + \frac{2}{3}N = 1$

(b) $2M + \frac{1}{3}N = 1$

(c) $M + N = 1$

(d) $M + N = 3$

$-\infty \quad X \quad \infty$

(contu RV)

$$\boxed{\int_{-\infty}^{\infty} Me^{-2|x|} + Ne^{-3|x|}\, dx = 1}$$

This is valid prob. Density function

$x \rightarrow$ simple function

$|x|$ — compound function

$|x| \rightarrow$ compound Function

$$|x| = \begin{cases} x & x \geq 0 \quad \text{Region A} \\ -x & x < 0 \quad \text{Region B} \end{cases}$$

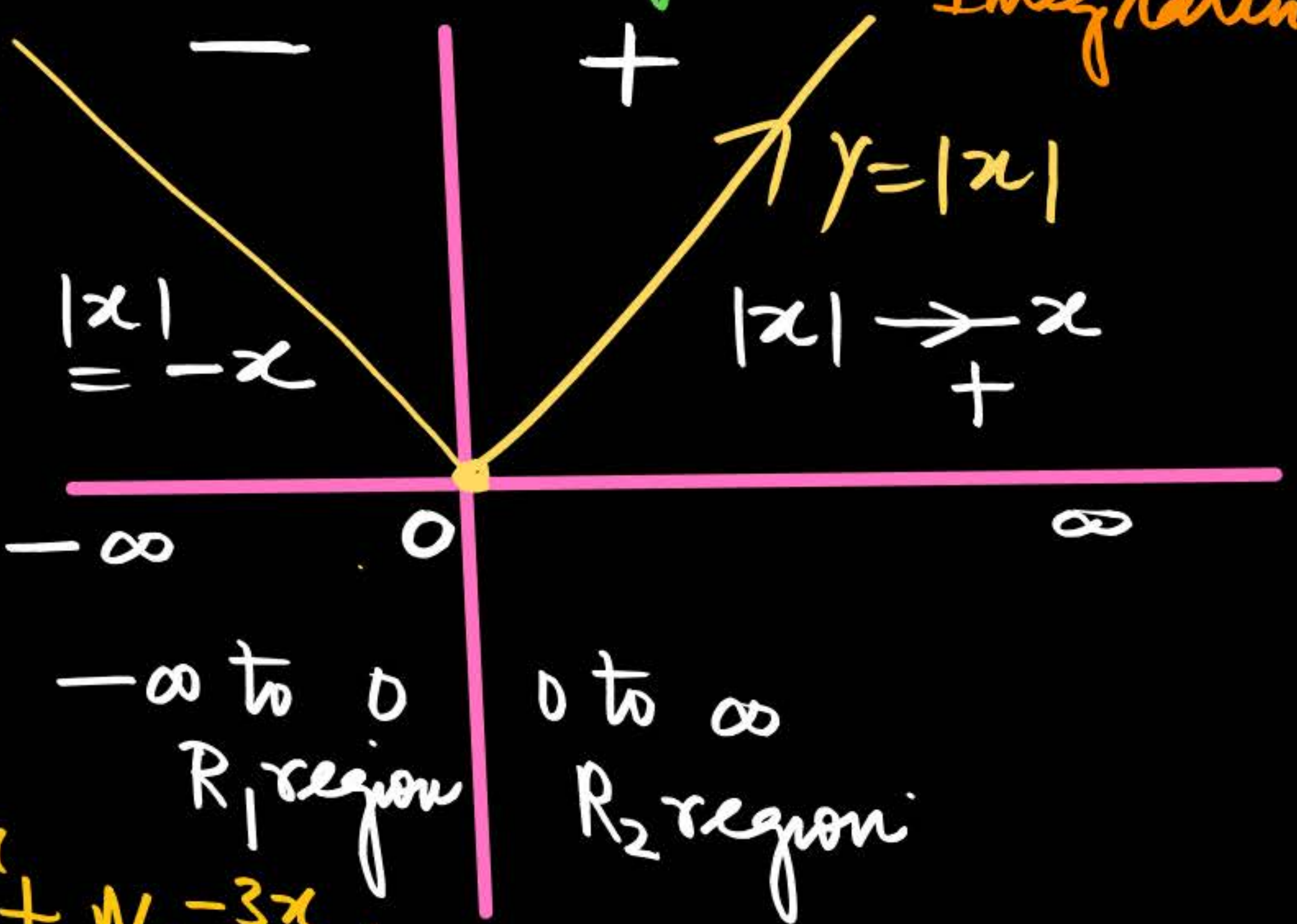
Integrating

$$\int_{-\infty}^{\infty}\left[Me^{-2|x|} + Ne^{-3|x|}\right]dx = 1$$

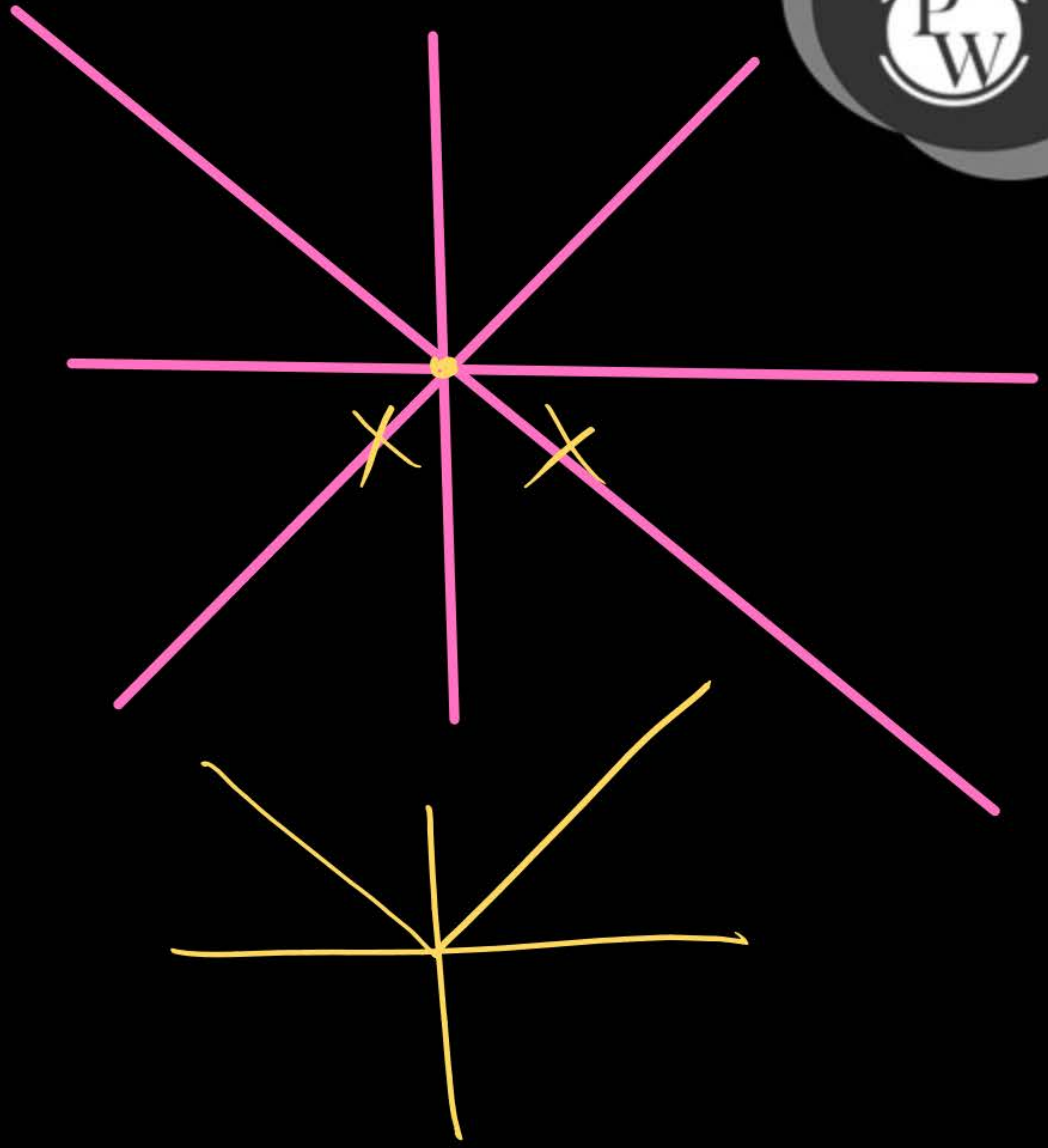
$$\Rightarrow \int_{-\infty}^{0}\left[Me^{-2(-x)} + Ne^{-3(-x)}\right]dx + \int_{0}^{\infty} Me^{-2(x)} + Ne^{-3(x)}\,dx = 1$$

$$\int_{-\infty}^{0}\left[Me^{2x} + Ne^{3x}\right]dx + \int_{0}^{\infty} Me^{-2x} + Ne^{-3x}\,dx = 1$$

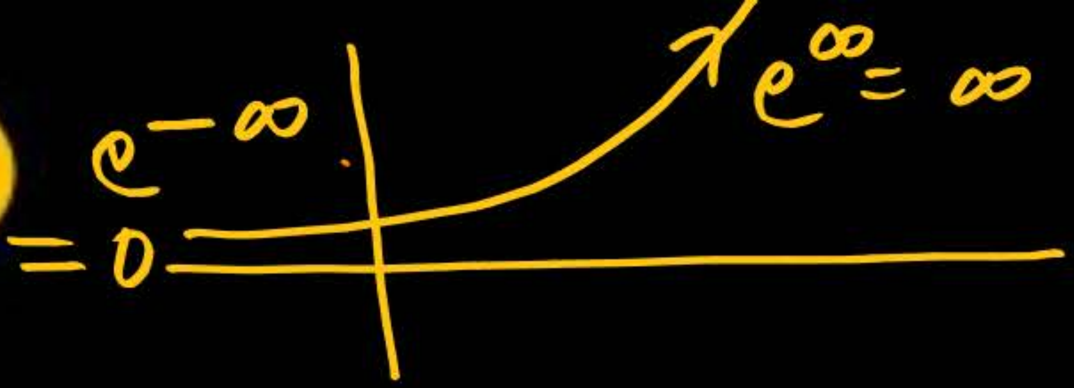
$$= \left[\frac{Me^{2x}}{2} + \frac{Ne^{3x}}{3}\right]_{-\infty}^{0} + \left[\frac{Me^{-2x}}{-2} + \frac{Ne^{-3x}}{-3}\right]_{0}^{\infty} = 1$$



$$\boxed{M + \frac{2N}{3} = 1}$$



Q. Questions



A continuous random variable X has a probability density function

$f(x) = e^{-x}$, $0 < x < \infty$. Then P{X > 1} is

(a) 0.368

(b) 0.5

(c) 0.632

(d) 1.0

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$P(x > 1) = \int_1^\infty f(x)\,dx = \int_1^\infty e^{-x}\,dx = \frac{1}{e} \text{ Ans}$$

$$P(X \geq 1) = \int_1^\infty f(x)\,dx$$

$$= \left[-e^{-x}\right]_1^\infty$$

$$\Rightarrow -e^{-\infty} + e^{-1}$$

$$= 0 + e^{-1} = \frac{1}{e} \text{ Ans}$$

Q. Questions

constant

Find the value of λ such that the function f(x) is a valid probability density function ____.

$f(x) = \lambda(x-1)(2-x) \quad \boxed{1 \le x \le 2}$

X is a Continuous Random variable

$$f(x) = \lambda(x-1)(2-x) \quad \boxed{for\, 1 \le x \le 2}$$
$$= 0 \quad \boxed{\text{otherwise}}$$

$0 \quad 3 \le x \le 4$

$0 \quad 4 \le x \le 5$

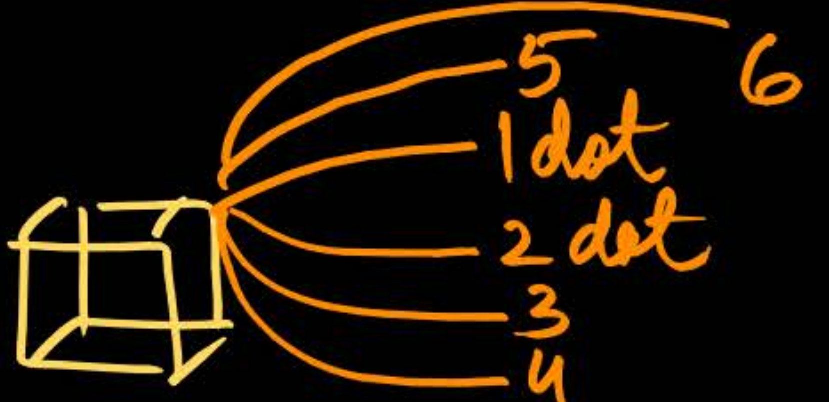
$$\int_1^2 \lambda(x-1)(2-x)\,dx + 0 = 1$$

$$= \int_1^2 \lambda(2x - x^2 - 2 + x)\,dx = 1$$

$$= \lambda \int_1^2 (3x - x^2 - 2)\,dx = 1$$

$\boxed{\lambda = 6}$ Ans

Q. Questions



'n' Dots

Consider a die with the property that the probability of a face with '*n*' dots

$n \propto n$

showing up is proportional to '*n*'. The probability of the face with three dots showing up is____.

X	1	2	3	4	5	6
$P(X=x)$	K	$2K$	$3K$	$4K$	$5K$	$6K$

Prob.

$1 \propto 1 \Rightarrow 1 = K$

$2 \propto 2 \Rightarrow 2 = 2K$

$3 \propto 3 \Rightarrow 3 = 3K$

$4 \propto 4 \Rightarrow 4 = 4K$

$5 \propto 5 \Rightarrow 5 = 5K$

$6 \propto 6 \Rightarrow 6 = 6K$

$P(3\text{dots}) = 3K$

This is Discrete Random variable.

$K + 2K + 3K + 4K + 5K + 6K = 1 \qquad K = \frac{1}{21}$

$$P(3\ \text{Dots}) = 3K$$

$$K = \frac{1}{21} \qquad = 3 \times \frac{1}{21} = \frac{1}{7}$$

$$P(3\ \text{dots}) = \frac{1}{7} \checkmark$$

Q. Questions

Let X be a random variable with probability density function

$$f(x)=\begin{cases}0.2 & \text{for}\ |x|\le 1\\ 0.1 & \text{for}\ 1<|x|\le 4\\ 0 & \text{otherwise}\end{cases}$$

The probability $P(0.5 < x < 5)$ is ____.

Q. Questions

ME

Lifetime of an electric bulb is a random variable with density $f(x) = kx^2$, where x is measured in years. If the minimum and maximum lifetimes of bulb are 1 and 2 years respectively, then the value of k is ____.

$f(x) = kx^2$

$\int_1^2 Kx^2 dx = 1$

$\boxed{K = \frac{3}{7}}$

Q. Questions

$[0,\infty] \rightarrow$ open/closed

Given that x is a random variable in the range $[0, \infty]$ with a probability density function $\boxed{\frac{e^{-\frac{x}{2}}}{K}}$, the value of the constant K is

$$f(x) = \begin{cases} \frac{e^{-\frac{x}{2}}}{K} & 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

If $f(x)$ is a valid pdf $\int_a^b f(x)\,dx = 1$

$$\int_0^\infty \frac{e^{-\frac{x}{2}}}{K}\,dx = 1$$

$$= \frac{1}{K}\int_0^\infty e^{-\frac{x}{2}}\,dx = 1 \qquad \boxed{K = 2}$$

Q. Questions

A normal random variable X has the following probability density function

$$f_x(x) = \frac{1}{\sqrt{8\pi}} e^{-\left\{\frac{(x-1)^2}{8}\right\}}, -\infty < x < \infty$$

Then $\int_1^{\infty} f_x(x)dx =$

(a) 0

(b) $\frac{1}{2}$

(c) $1 - \frac{1}{e}$

(d) 1

$f(x) = \frac{1}{\sqrt{8\pi}} e^{-\left(\frac{(x-1)^2}{8}\right)}$

Then $\int_1^{\infty} f(x)\,dx$

Imp:

$$f(x) = \frac{1}{\sqrt{8\pi}} e^{-\frac{(x-1)^2}{8}} \qquad x-1 = \sqrt{8t}$$

using substitution $\frac{(x-1)^2}{8} = t$ both sides Diff I t

$$\frac{2(x-1)}{8} dx = dt \qquad dx = \frac{8\,dt}{2(x-1)} = \frac{4\,dt}{(x-1)} = \frac{4\,dt}{\sqrt{8t}}$$

$x = 1$

$\frac{(1-1)^2}{8} = t \quad t = 0$

$x = \infty \quad (\infty) = t \quad t = \infty$

$$\int_1^{\infty} f(x)\,dx = \int_0^{\infty} \frac{1}{\sqrt{8\pi}} e^{-t} \cdot \frac{4\,dt}{\sqrt{8t}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \int_0^{\infty} e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \boxed{\int_0^\infty e^{-t}\, t^{-1/2}\, dt}$$

GAMMA Function

Compare It

$n - 1 = -\frac{1}{2}$

$\boxed{n = \frac{1}{2}}$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{\cancel{\sqrt{\pi}}} \times \cancel{\sqrt{\pi}} \times \frac{1}{2} = \frac{1}{2}$$

Ans

A) $\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx}$

B) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

C) $\Gamma(n) = (n-1)!$

$\Gamma(7) = 6!$ $\quad \Gamma(4) = 3!$

$\Gamma(5) = 4!$

$\Gamma(1) = 0!$

D) $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \, \frac{3}{2} \, \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

E) $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$

$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)$

Hydrology
Distr.
Rainfall

#

$$I = \int_0^\infty e^{-x^2}\,dx$$

$$I = \int_0^\infty e^{-t}\cdot\frac{dt}{2\sqrt{t}} \Rightarrow \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt$$

$x^2 = t$

$2x\,dx = dt$

$dx = \dfrac{dt}{2\sqrt{t}}$

$(0)^2 = t$

$\boxed{t = 0}$

$\infty = t$

$\boxed{t = \infty}$

$$\Rightarrow \frac{1}{2}\int_0^\infty \underline{e^{-t}}\,t^{-1/2}\,dt$$

Compare It

$$= \frac{1}{2}\times\Gamma\frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2}\ \checkmark$$

$$\underline{\int_0^\infty e^{-t}\,t^{n-1}\,dt}$$

$n-1 = -\frac{1}{2}$

$\underline{n = \frac{1}{2}}$

Q. Questions

EC

The graph of function $f(x)$ is shown in figure

Total AREA = 1

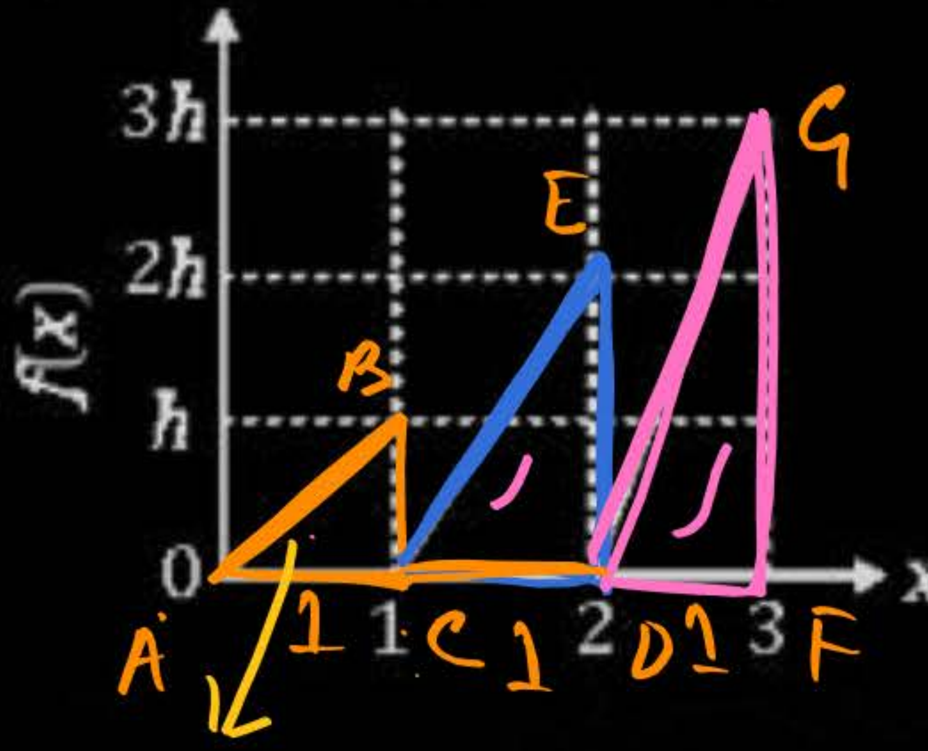
For $f(x)$ to be a valid probability density function, the value of h is

(a) 1/3

(b) 2/3

(c) 1

(d) 3



Area $\Delta ABC + \Delta CDE + \Delta DFG = 1$

$= \frac{1}{2} \times B_1 \times H_1 + \frac{1}{2} \times B_2 H_2 + \frac{1}{2} B_3 H_3 = 1$

$= \frac{1}{2} \times 1 \times h + \frac{1}{2} \times 2h \times 1 + \frac{1}{2} \times 3h \times 1 = 1$

$= \frac{h}{2} + h + \frac{3h}{2} = 1$

$= \frac{h+2h+3h}{2} = 1$

$\frac{3\cancel{6}h}{\cancel{2}} = 1 \quad 3h = 1$

$\boxed{h = \frac{1}{3}}$

Q. Questions

The random variable X takes on the values 1, 2 (or) 3 with probabilities $\frac{2+5P}{5}$, $\frac{1+3P}{5}$ and $\frac{1.5+2P}{5}$ respectively the values of P [redacted] are respectively

(a) 0.05, 1.87

(b) 1.90, 5.87

(c) 0.05, 1.10

(d) 0.25, 1.40

1	2	3
$\frac{2+5p}{5}$	$\frac{1+3p}{5}$	$\frac{1.5+2p}{5}$

$$\frac{2+5p}{5}+\frac{1+3p}{5}+\frac{1.5+2p}{5}=1$$

$$p=\frac{1}{20}$$

Q. Questions

The function p(x) is given by $p(x) = A/x^{\mu}$ where A and μ are constants with $\mu > 1$ and $1 \le x < \infty$ and $p(x) = 0$ for $-\infty < x < 1$. For p(x) to be a probability density function, the value of A should be equal to

(a) $\mu - 1$

(b) $\mu + 1$

(c) $1/(\mu - 1)$

(d) $1/(\mu + 1)$

Thank You!

GW Soldiers