GATE
ALL BRANCHES

ENGINEERING MATHEMATICS

Single Variable Calculus

Lecture 06

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01 Concept of Indefinite Integration

02 Problems based on Indefinite Integration

03 Techniques of Substitution and Practice problems

04 Integration By parts and Practice problems

Indefinite Integration:

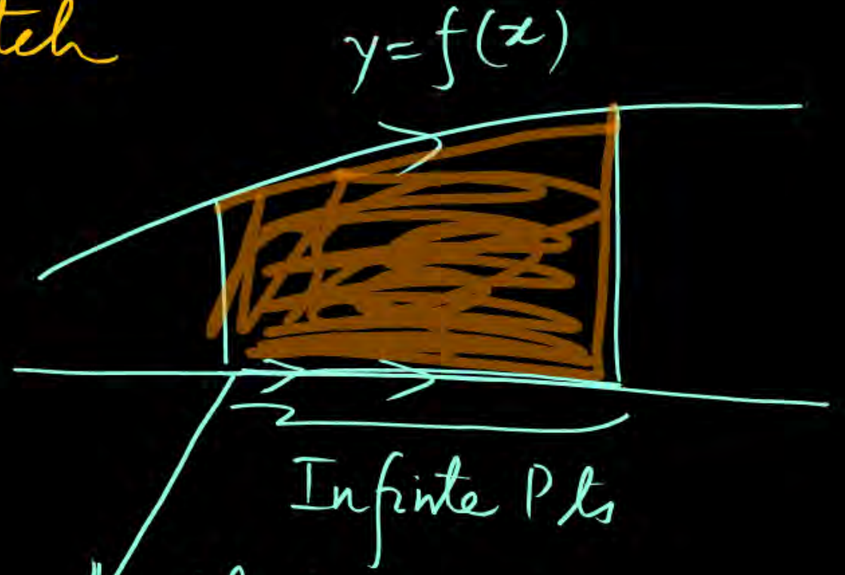
$y = f(x)$ = curve / graph / sketch

$\int$ = sign

$$\int f(x)\,dx = F(x) + C$$

C = Arbitrary constant

Limits Not Identified = algebraic sum OR AREA



Algebraic sum (Infinite Uncountable space)

Some formulae (List)

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$\int e^x \, dx = e^x + c$

{ Linear Function

B) $\int e^{2x+3} \, dx$ ← coefficient

$= \frac{1}{2} e^{2x+3} + c$

C) $\int e^{3x+9} \, dx = \frac{1}{3} e^{3x+9} + c$

, $\int x^2 \, dx = \frac{x^{2+1}}{2+1} = \frac{1}{3} x^3 + c$

A) $\int e^{2x} \, dx$

$2x = t$ (Let us assume)

both sides Diff. It w.r.t t

$2 \cdot \frac{dx}{dt} = 1$

$2 \, dx = dt$

$dx = \frac{dt}{2}$

$= \int e^t \cdot \frac{dt}{2}$

$= \frac{1}{2} e^t + c$

$= \frac{1}{2} e^{2x} + c$

Procedure

Power (Num)

$$\# \int a^x \, dx = \frac{a^x}{\ln a} + c$$

$a \in$ Real No

$$\# \int \sin x \, dx = -\cos x + c$$

$$\# \int \sin x \sin 2x \, dx$$

→ convert formula List

$$\# \int 2^x \, dx = \frac{2^x}{\ln 2} + c$$

$$\# \int 3^x \, dx = \frac{3^x}{\ln 3} + c$$

$$\# \int \sin \boxed{2x} \, dx$$

Convert the List of Formulae

$2x = t$

$2\,dx = dt$

$dx = \frac{dt}{2}$

$$= \int \sin t \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \int \sin t \, dt$$

$$= \frac{1}{2}\left[[-\cos t] + c\right] = \underline{\frac{1}{2}\left[-\cos 2x\right] + c}$$

$$I = \int \sin x \sin 2x \, dx$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$+ \sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \quad (\text{Add})$$
$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B \quad (\text{Subtract})$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B \quad (\text{Add})$$
$$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B \quad (\text{Subtract})$$

$$\frac{\cos(A-B) - \cos(A+B)}{2} = \sin A \sin B$$

$f_1(x) \cdot f_2(x) = \text{Non Linear}$

Product

$xy = 1$

$x + y = 1$

Linear

$\rightarrow$ Linear form

$$I = \int \sin x \sin 2x \, dx$$

$$= \frac{1}{2}\int \left[\cos(2x - x) - \cos(2x + x)\right] dx$$

$$= \frac{1}{2}\int \left[\cos x - \cos 3x\right] dx \Longrightarrow \text{List of Formula}$$

convert to +, −

$$= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right] + C$$

$$\# \int \cos x \, dx = \sin x + C$$

$$\# \int \frac{1}{x} dx = \log_e |x| + C$$

$$\# \int \frac{1}{(x+1)^2} dx$$

$x + 1 = t$

$dx = dt$

$$= \int \frac{dt}{t} = \log_e t + C = \log_e |x+1| + C$$

$\int \sec^2 x\,dx = \tan x + C$

$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$ $\quad \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1}x + C$

$\int \frac{dx}{1+x^2} = \tan^{-1}x + C$ $\quad \int \frac{-1}{1+x^2}dx = \cot^{-1}x + C$

$\int \frac{dx}{x\sqrt{x^2-1}} = \text{SEC}^{-1}x + C$ $\quad \int \frac{-dx}{x\sqrt{x^2-1}} = \operatorname{cosec}^{-1}x + C$

Formula List

Problems based on Indefinite Integral

Illustration:

$$\int 2^x . e^x dx = \int (2e)^x dx$$

Ans.: $\dfrac{(2e)^x}{\ln(2e)} + c = \dfrac{2e^x}{\ln(2e)} + c$

$\int 2^x e^x dx$ (Number)

$\int a^x dx = \dfrac{a^x}{\ln a} + c$

Q. Questions

Illustration:

$$\int \frac{1-\tan^2 x}{1+\tan^2 x} dx$$

$$I = \int \frac{1-\tan^2 x}{1+\tan^2 x} dx$$

$$I = \int \cos 2x \, dx$$

$$= \frac{\sin 2x}{2} + C$$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

Ans.: $\frac{1}{2}\sin 2x + c$

Q. Questions

Illustration:

$$\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}dx$$

Ans.: $\frac{x^2}{2}-x+c$

A^3-B^3
$=(A-B)(A^2+B^2+AB)$

$x^{3/2}=(\sqrt{x})^3-(1)^3$
$=(\sqrt{x}-1)(x+\sqrt{x}+1)$

$$I=\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}$$

$$=\int \frac{(\sqrt{x}+1)\sqrt{x}(x^{3/2}-1)}{\sqrt{x}(x+\sqrt{x}+1)}$$

$$=\int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+\sqrt{x}+1)}{(x+\sqrt{x}+1)}$$

$$=\int (\sqrt{x}+1)(\sqrt{x}-1)\,dx$$

$$=\int (x-1)\,dx = \frac{x^2}{2}-x+c$$

[formula Lust]

$$\int (\sqrt{x}-1)(\sqrt{x}-1)\,dx$$
$$=\int (\sqrt{x}-1)^2\,dx = \text{Ans}$$

Q. Questions

Illustration:

$$\int \frac{(x^2+\sin^2 x)\sec^2 x}{1+x^2}dx$$

Ans.: $\tan x - \tan^{-1} x + c$

$$\int \frac{(x^2+\sin^2 x)\sec^2 x}{1+x^2}dx = \int \frac{x^2\sec^2 x + \sin^2 x \frac{1}{\cos^2 x}}{1+x^2}dx$$

$$= \int \frac{x^2\sec^2 x + \tan^2 x}{1+x^2}dx$$

$$= \int \frac{x^2(1+\tan^2 x)+\tan^2 x}{(1+x^2)}dx$$

$$= \int \frac{x^2 + x^2\tan^2 x + \tan^2 x}{(1+x^2)}dx$$

$$= \int \frac{x^2 + \tan^2 x(1+x^2)}{1+x^2}$$

$= $ Do yourself

$\sec^2 x - \tan^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

Q. Questions

$$\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$$

✓ Do yourself

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

Ans.: $\frac{1}{2}(\tan x + x) + C$

Q. Questions

$$\int \frac{1+\tan^2 x}{1+\cot^2 x}\,dx$$

DO YOURSELF

Ans.: $\tan x - x + C$

Q. Questions

$$\int \frac{x^4+x^2+1}{2(1+x^2)}dx$$

$(x^2+1)^2$

Ans.: $\frac{1}{2}\left[\frac{x^3}{3}+\tan^{-1}x\right]+C$

Do yourself

Techniques of Substitution

$\int \frac{\cos(\ln x)}{x} dx$ $\ln x = t$

We Know That link between D and I

Illustration:

$$\int \frac{\cos(\ln x)}{x} dx$$

$\ln x = t$

$\frac{1}{x} dx = dt$

$= \int \cos t \cdot dt$

$= \sin t + C$

$= \sin(\ln x) + C$

Ans.: $\sin(\ln x) + c$

Q. Questions

Illustration:

$$\int \frac{x^3 dx}{1+x^8}$$

Ans.: $\frac{1}{4}\tan^{-1}(x^4)+c$

$$\int \frac{x^3\ dx}{1+x^8}$$

$$= \int \frac{x^3}{1+(x^4)^2}\, dx$$

$x^4 = t$

$4x^3\, dx = dt$

$x^3\, dx = \frac{dt}{4}$

$$= \frac{1}{4}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{4}\tan^{-1} t + c$$

$$= \frac{1}{4}\tan^{-1}(x^4) + c$$

Q. Questions

Illustration:

$$\int \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$$

Ans.: $\frac{1}{2}\left\{\ln\left(x+\sqrt{1+x^2}\right)\right\}^2+c$

$\ln(x+\sqrt{1+x^2}) = t$

$\frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{1}{2\sqrt{1+x^2}}2x\right]dx = dt$

$\Rightarrow \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = dt$

$= \frac{dx}{\sqrt{1+x^2}} = dt$

$= \int t\cdot dt$

$= \frac{t^2}{2}+c = \frac{[\ln(x+\sqrt{1+x^2}]^2}{2}+c$

$\int \frac{f'(x)}{f(x)}\,dx = \ln[f(x)] + C$

$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$

$\int f'(x)\,f(x)\,dx = \frac{[f(x)]^2}{2} + C$

$$\int \tan x\,dx = -\int \frac{\sin x}{\cos x}\,dx$$

$$\Rightarrow -\log\cos x + C$$

$\int \tan x\,dx = \log \sec x + C$

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$$= \log \sin x + C$$

$$\# \frac{1}{2}\int \frac{2x}{\sqrt{x^2+2}}\,dx$$

$$= \frac{1}{2} \times 2\sqrt{x^2+2}$$

$$= \sqrt{x^2+2}$$

$\int \sin x \cos x\,dx = -\frac{(\cos x)^2}{2} + C$

Q. Questions

Illustration:

$$\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$$

$$\int \frac{x^2 \tan^{-1}(x^3)}{1+(x^3)^2} \quad \tan^{-1}(x^3) = t$$

$$= \frac{1}{1+x^6} \cdot 3x^2 \, dx = dt$$

Ans.: $\frac{1}{6}(\tan^{-1} x^3)^2 + c$

Do yourself

Q. Questions

Illustration:

$$\int \frac{\tan\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx$$

Ans.: $\left(\tan\sqrt{x}\right)^2 + c$

Do yourself.

Q. Questions

Illustration:

$$\int \frac{\text{cosec}(\tan^{-1} x)}{1+x^2} dx$$

Ans.: $\ln\left[\frac{1+x^2}{x} - \frac{1}{x}\right] + c$

Do yourself

Q. Questions

Do yourself

Illustration:

$$\int \frac{\cos 2x}{\sin x} dx$$

Ans.: $\ln(\operatorname{cosec} x - \cot x) + 2\cos x + c$

Q. Questions

$xe^x = t$

Illustration:

$$\int \frac{e^x(1+x)}{\cos(xe^x)}dx$$

Ans.: $\ln(\sec(xe^x)+\tan(xe^x))+c$

Do yourself.

Q. Questions

Do yourself

Illustration:

$$\int \frac{\sin 2x}{\sqrt{9-\sin^4 x}}dx$$

Ans.: $\sin^{-1}\left(\frac{\sin^2 x}{3}\right)+c$

Q. Questions

Do yourself

Illustration:

$$\int \frac{e^x dx}{\sqrt{e^{2x}-1}}$$

Ans.: $\ln\left(e^x+\sqrt{e^{2x}-1}\right)+c$

Q. Questions

Illustration:

$$\int \frac{e^x dx}{4+e^{2x}}$$

Ans.: $\frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right)+c$

Do yourself

Thank You!

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