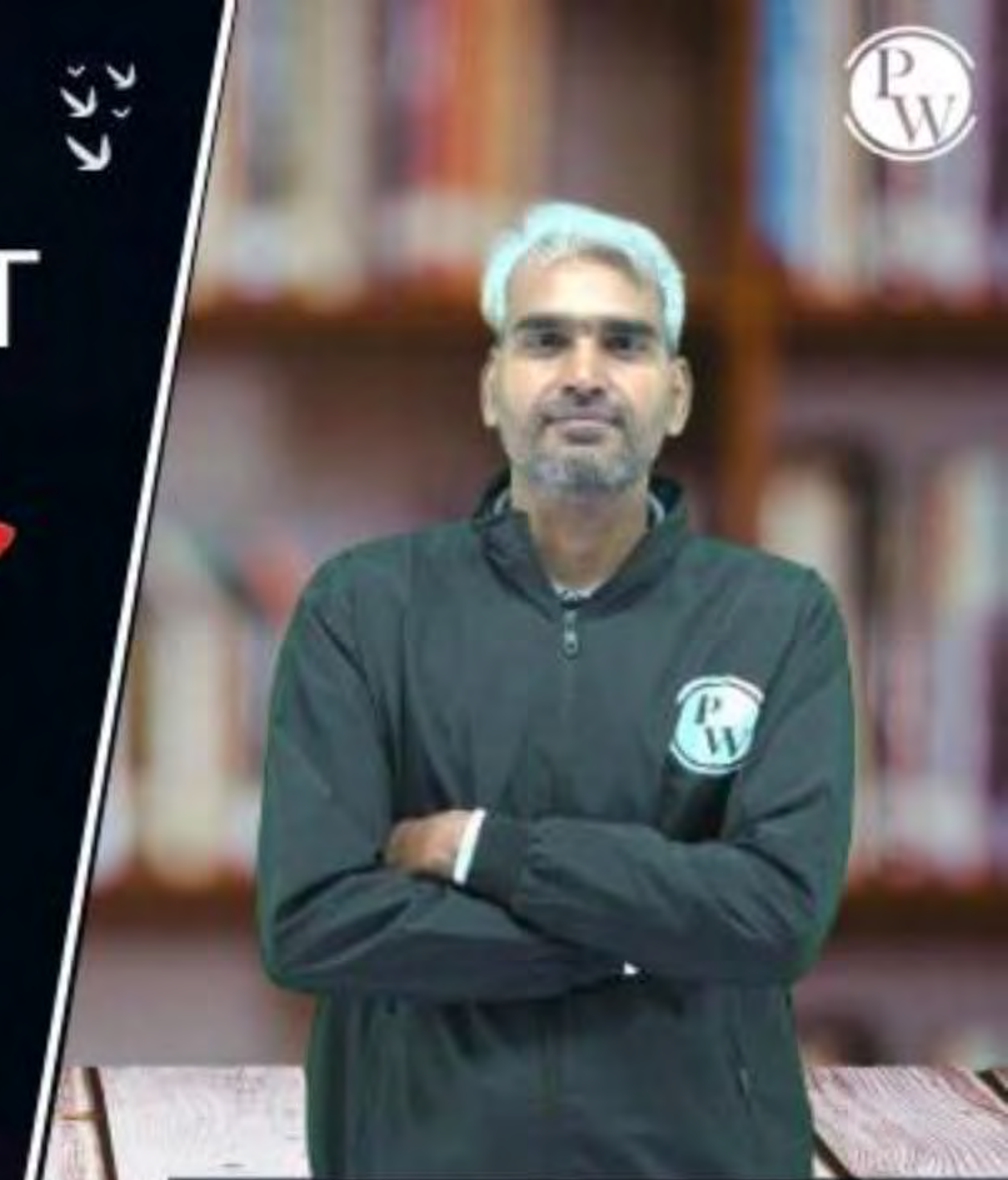
Computer Science & IT

Discrete Mathematics

Mathematical Logic

Lecture No. 06



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Recap of Previous Lecture

Topic Quantifiers

Topic Scope of the quantifier

Topic Negation of a statement formula

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Topics to be Covered

- Topic: Sentences with multiple quantifiers
- Topic: Relationship diagram
- Topic: Rules of inference w.r.t. predicate logic
- Topic: Equivalences and implications

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Topic : Sentences with multiple quantifiers

let Universe: Set of all human being

P: likes , $P(x,y)$: x likes y $\equiv$ y is liked by x

① $\forall_x \forall_y \{P(x,y)\}$: Everybody likes Everybody.
(All x likes All y)

② $\forall_y \forall_x \{P(x,y)\}$: Everybody liked by Everybody
(All y liked by all x)

$$\boxed{\forall_x \forall_y \{P(x,y)\} \equiv \forall_y \forall_x \{P(x,y)\}}$$

Topic : Sentences with multiple quantifiers

let Universe: Set of all human being

P: likes, $P(x,y)$: x likes y $\equiv$ y is liked by x

③ $\exists_x \forall_y \{P(x,y)\}$: Some one likes Everybody
There exist at least one x who likes all y.

④ $\forall_y \exists_x \{P(x,y)\}$: Everybode is liked by somebody.
All y are liked by some x.

$$\forall_x \forall_y \{P(x,y)\} \Rightarrow \boxed{\exists_x \forall_y \{P(x,y)\} \underset{\Leftarrow \times}{\overset{\Rightarrow \checkmark}{\rightleftharpoons}} \forall_y \exists_x \{P(x,y)\}}$$

Topic : Sentences with multiple quantifiers

let Universe: Set of all human being

P: likes , $P(x,y)$: x likes y $\equiv$ y is liked by x

⑤ $\exists_y \forall_x \{P(x,y)\}$: Someone is liked by Everyone.

There is at least one 'y' who is liked by Every x

⑥ $\forall_x \exists_y \{P(x,y)\}$: Every body likes somebody.

Every person likes $\underbrace{\text{at least one Person}}_{\text{need not be same for all } x.}$

$$\boxed{\exists_y \forall_x \{P(x,y)\} \overset{\checkmark}{\Longrightarrow} \forall_x \exists_y \{P(x,y)\} \qquad \exists_y \forall_x \{P(x,y)\} \overset{\times}{\Longleftarrow} \forall_x \exists_y \{P(x,y)\}}$$

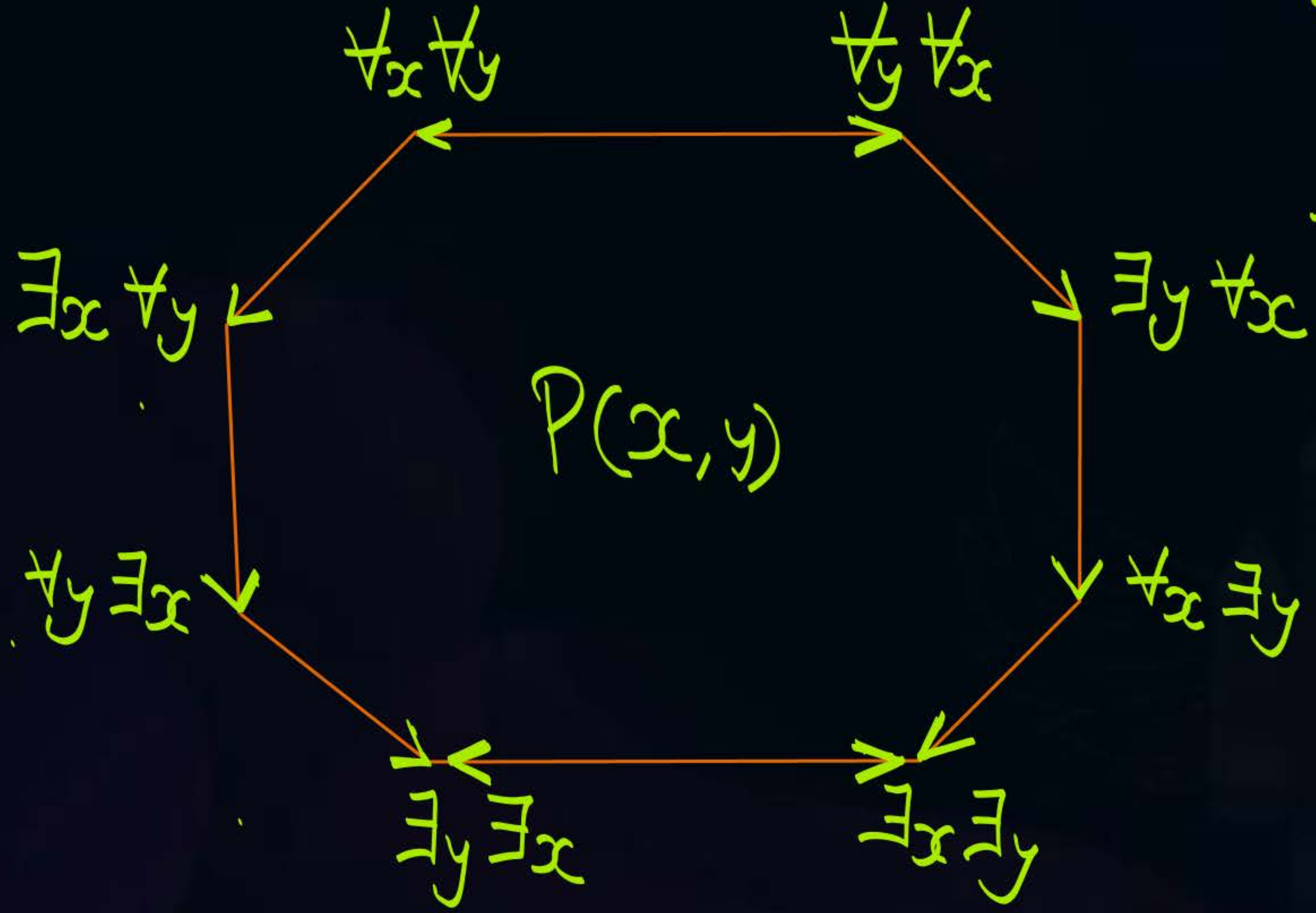
Topic : Sentences with multiple quantifiers

(7) $\exists x \exists y \{P(x,y)\}$: Somebody likes somebody

(8) $\exists y \exists x \{P(x,y)\}$: Somebody is liked by somebody

$$\boxed{\exists x \exists y \{P(x,y)\} \equiv \exists y \exists x \{P(x,y)\}}$$

Topic : Relationship diagram



$\longleftrightarrow$ represent Equivalence

$\longrightarrow$ represent logical implication

[MSQ]

#Q. Which of the following is/are true?

A $\forall x \, \exists y \, P(x,y) \Rightarrow \exists y \, \forall x P(x,y)$

B $\forall x \, \forall y \, P(x,y) \Rightarrow \exists y \, \forall x P(x,y)$

C $\exists x \, \forall y \, P(x,y) \Rightarrow \forall y \, \exists x \, P(x,y)$

D $\exists x \, \exists y \, P(x,y) \Rightarrow \forall y \, \exists x \, P(x,y)$

[MCQ]

#Q. Consider the first order logic sentence $F: \forall x(\exists y\, R(x,y))$. Assuming non-empty logical domains, which of the sentences below are implied by F?

$F \Rightarrow$ I. $\exists y\, (\exists x\, R(x,y))$

$F \not\Rightarrow$ II. $\exists y\, (\forall x\, R(x,y))$

$F \Rightarrow$ III. $\forall y(\exists x\, R(x,y)$

$F =$ IV. $\sim\exists x(\forall y \sim R(x,y))$

A. IV only

B. I and IV only

C. II only

D. II and III only

Topic : Rules of Inferences w.r.t. predicate logic

- In predicate logic, we can use some additional inference rules, along with all the rules of inference we have discussed in propositional logic.

- Additional Rules of Inference w.r.t. Predicate Logic

1. Universal Instantiation (Universal Specification)
2. Universal Generalization
3. Existential Instantiation (Existential Specification)
4. Existential Generalization

Topic : Universal Instantiation

/ Universal specification

$$\forall x \{P(x)\}$$

$\therefore$ $P(c)$ is true for all 'c' in universe of discourse

Topic : Universal Generalization

P(c) is true for $\underbrace{\text{all 'c'}}$ in the universe of discourse

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$\therefore \forall_z \{P(z)\}$

We can use any Variable

Topic : Existential Instantiation

$$\frac{\exists x \{P(x)\}}{\therefore P(c) \text{ is true for some 'c' in the universe of discourse}}$$

Topic : Existential Generalization

$P(c)$ is true for $\underbrace{\text{some 'C'}}$ in the universe of discourse

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$\therefore \exists y \{P(y)\}$

Logical Equivalences & Implications

$$① \quad \forall x \{P(x) \wedge Q(x)\} \underset{②}{\overset{①}{\rightleftarrows}} \forall x\{P(x)\} \wedge \forall x\{Q(x)\}$$

$$① \quad \forall x \{P(x) \wedge Q(x)\} \equiv \forall x\{P(x)\} \wedge \forall x\{Q(x)\}$$

Logical Equivalences & Implications

② $\exists x \{P(x) \vee Q(x)\} \underset{②\ \Longleftarrow}{\overset{①\ \Longrightarrow}{\equiv}} \exists x \{P(x)\} \vee \exists x \{Q(x)\}$

for at least one x
disjunction of P & Q
must be true

i.e. for at least one 'x'
$P(x)$ or $Q(x)$ must be
true.

for R.H.S to be true
Either P should be true
for at least one x
or
Q should be true
for at least one x

Logical Equivalences & Implications

③ $\forall x\{P(x) \vee Q(x)\} \not\equiv \forall x\{P(x)\} \vee \forall x\{Q(x)\}$

① $\not\Rightarrow$

② $\Leftarrow$

$\underbrace{\forall x\{P(x) \vee Q(x)\}}$ → for all x disjunction of P & Q must be true

$F \vee F$

F

$U = \{1, 2\}$

$P(1) = T \quad Q(1) = F \quad P(1) \vee Q(1) = T$

$P(2) = F \quad Q(2) = T \quad P(2) \vee Q(2) = T$

$\forall x\{P(x)\} = F \quad \forall x\{Q(x)\} = F$

$\therefore$ $\forall x\{P(x) \vee Q(x)\}$ is true

R.H.S. Can be true only if P is true for all x <u>or</u> Q is true for all x, in that case disjunction of P & Q will also be true for all x

$\therefore$ L.H.S. will also be true

Logical Equivalences & Implications

(4) $\exists x \{P(x) \wedge Q(x)\} \not\equiv \exists x\{P(x)\} \wedge \exists x\{Q(x)\}$

① $\Longrightarrow$ ✓

② $\Longleftarrow$ ✗

For L.H.S. to be true

for at least one x

Conjunction of P & Q

must be true

i.e., for at least one x, both P & Q should be true simultaneously

$\Rightarrow$ if it is true, then for at least one x P will be true & for at least one x Q will be true

$\therefore$ RHS will also be true

For RHS to be true

P must be true for at least one x, and Q must be true for at least one x

but it is not necessary for P & Q to be true for same x

$\therefore$ LHS may be false

Topic : Equivalences & Implications

1. $\forall x\, [P(x) \wedge Q(x)] \equiv [\forall x\, P(x)] \wedge [\forall x\, Q(x)]$

2. $\exists\, x\, [P(x) \vee Q(x)] \equiv [\exists\, x\, P(x)] \vee [\exists\, x\, Q(x)]$

3. $[\forall x\, P(x)] \vee [\forall x\, Q(x)] \Rightarrow \forall x\, [P(x) \vee Q(x)]$

4. $\exists\, x\, [P(x) \wedge Q(x)] \Rightarrow [\exists\, x\, P(x)] \wedge [\exists\, x\, Q(x)]$

5. $\boxed{\forall x\, [P(x) \rightarrow Q(x)] \Rightarrow [\forall x\, P(x)] \rightarrow [\forall x\, Q(x)]}$

We have already discussed when we were discussing "for all" quantifier

Topic : Some important equivalences

H.W.

1. $\forall x\,[P(x) \wedge Q] \equiv \forall x\, P(x) \wedge Q$

2. $\exists x\,[P(x) \vee Q] \equiv \exists x\, P(x) \vee Q$

3. $\forall x\,[P(x) \vee Q] \equiv \forall x\, P(x) \vee Q$

4. $\exists x\,[P(x) \wedge Q] \equiv \exists x\, P(x) \wedge Q$

Assume,
Q is a predicate formula in which x is not a free variable.

↳ i.e., truth value of Q will not vary with x.

↳ i.e. if Q is true it will remain true for all x, & if Q is false it will remain false for all x

Topic : Some important equivalences

H.W.

5. $\forall x\,[P \to Q(x)] \equiv P \to \forall x\, Q(x)$

6. $\exists x\,[P \to Q(x)] \equiv P \to \exists x\, Q(x)$

7. $\forall x\,[P(x) \to Q] \equiv \exists x\, P(x) \to Q$

8. $\exists x\,[P(x) \to Q] \equiv \forall x\, P(x) \to Q$

[MSQ]

H.W.

#Q. Let $P(x)$ and $Q(x)$ be arbitrary predicates. Which of the following statements is always TRUE?

A $(\forall x(P(x) \vee Q(x))) \Rightarrow ((\forall x P(x)) \vee (\forall x Q(x)))$

B $(\forall x(P(x) \Rightarrow Q(x))) \Rightarrow ((\forall x P(x)) \Rightarrow (\forall x Q(x)))$

C $(\forall x(P(x) \Rightarrow \forall x Q(x))) \Rightarrow (\forall x\,(P(x) \Rightarrow Q(x)))$

D $((\forall x(P(x)) \Leftrightarrow (\forall x Q(x))) \Rightarrow (\forall x(P(x) \Leftrightarrow Q(x)))$

2 mins Summary

- **Topic** Sentences with multiple quantifiers
- **Topic** Relationship diagram
- **Topic** Rules of inference w.r.t. predicate logic
- **Topic** Equivalences and implications

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THANK - YOU

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