CS & IT

Discrete Mathematics

Mathematical Logic

DPP: 2

Q1 Let p(x) and q(x) denote the following open statements.
p(x): $x^2 > 0$
q(x): x is odd
for the universe of all integers, determine the truth or falsity of each of the statement.
S_1: $\forall x\,[p(x) \to q(x)]$
S_2: $\exists x\,[p(x) \to q(x)]$
Which of the following is true?
(A) S_1 only
(B) S_2 only
(C) Both S_1 and S_2
(D) Neither S_1 nor S_2

Q2 Consider following two First Order Logic Statements:
S_1: $[\forall x(\sim P(x) \vee Q(x))] \to [\forall x\, P(x)] \to [\forall x\, Q(x)]$
S_2: $[\exists x\, P(x)] \to [\exists x\, Q(x)] \to [\exists x\, (P(x) \to Q(x))]$
Which of the following is valid?
(A) S_1 only
(B) S_2 only
(C) Both S_1 and S_2
(D) None of these

Q3 $P(y) = \sqrt{y}$ is real in the domain of Z^+, then which of the following is / are correct?
(A) $\forall$ P(y)
(B) $\exists y\, P(y)$
(C) $\forall y \sim P(y)$
(D) $\exists y \sim P(y)$

Q4 Which of the following is not valid logical expression?
(A) $\forall x\,[P(x) \to Q(x)] \to [\forall x\, P(x)] \to [\forall x\, Q(x)]$
(B) $\forall x\,[P(x) \vee Q(x)] \to [\forall x\, P(x)] \vee [\forall x\, Q(x)]$
(C) $\exists x\,[P(x) \wedge Q(x)] \to [\exists x\, P(x)] \wedge [\exists x\, Q(x)]$
(D) $\forall x\,[P(x) \leftrightarrow Q(x)] \to [\forall x\, P(x)] \leftrightarrow [\forall x\, Q(x)]$

Q5 Consider following logical expressions:
I: $\forall y[P(y) \to Q] \leftrightarrow [\forall y\, P(y)] \to Q$
II: $\exists y[P(y) \to Q] \to [\exists y\, P(y)] \to Q$
which of the following logical expression is valid?
(A) I only (B) II only
(C) Both I and II (D) None of these

Q6 **Consider**
Actor (x) = x is an actor
Smart (x) = x is smart
and the well-formed formula:
$\exists x\,(\text{Actor}(x) \wedge \text{Smart}(x))$
Choose the correct representation of above in English sentence.
(A) Some Actor is smart
(B) Some Actor is not smart
(C) All actors are smart
(D) All smart are actors

Q7 Consider the following statement
"There is exactly one apple".
Let G(x) : x is an apple.
Now consider the predicate logic statements:
I. $\exists x\, \text{apple}(x) \wedge \forall_y (\text{apple}(y)) \Rightarrow x = y)$
II. $\exists x\, \text{apple}(x)$
The correct representation in predicate logic is ?
(A) Only I
(B) Only II
(C) Both I and II
(D) Neither I and II

Q8 Consider the following statements:
I. $\exists x\, \{p(x) \to \{\forall\, x\, P(x) \to fx\, Q(x)\}$
II. $\exists x\, \forall_y\, P(x, y) \to \forall_y\, \exists x\, P(x, y)]$
The number of valid statements is/are _________.

Q9 Choose among the following that are not equivalent to the given first order logic statement:

$(\exists x)\ (\forall y)\ [p(x, y) \wedge q(x, y) \wedge \neg r(x, y)]$

(A) $(\forall x)\ (\exists y)\ [p(x, y) \wedge q(x, y) \rightarrow r(x, y)]$

(B) $(\exists x)\ (\forall y)\ [p(x, y) \vee q(x, y) \wedge \neg\ r(x, y)]$

(C) $\neg\ (\forall x)\ (\exists y)\ [p(x, y) \vee q(x, y) \rightarrow r(x, y)]$

(D) $\neg\ (\forall x)\ (\exists y)\ [p(x, y) \wedge q(x, y) \rightarrow r(x, y)]$

Q10 Choose the correct representation for the below statement:

"Every player is liked by some coach"

(A) $\forall(x)$ [player (x) $\rightarrow \exists y$ [coach (y) $\wedge$likes (y, x)]]

(B) $\forall(x)$ [player (x) $\rightarrow \exists y$ [coach (y) $\rightarrow$ likes (y, x)]]

(C) $\exists$ (x) [player (x) $\rightarrow \forall y$ [coach (y) $\rightarrow$ likes (y, x)]]

(D) $\exists$ (x) [player (x) $\rightarrow \forall y$ [coach (y) $\wedge$ likes (y, x)]]

Answer Key

Q1	(B)
Q2	(C)
Q3	(A, B)
Q4	(B)
Q5	(D)
Q6	(A)
Q7	(A)
Q8	2
Q9	(A, B, C)
Q10	(A)

Hints & Solutions

Q1 Text Solution:

Statement S_1: $\forall x[p(x) \to q(x)]$

As we know the $\forall x$ connected through '$\wedge$' operator.

So, check the statement for x = 3

$\therefore [p(3) \to q(3)] = [(3^2 > 0) \to (3 \text{ is odd})]$

$\equiv [\text{True} \to \text{True}] \equiv \text{True}$

Now, check the statement for x = 2

$\therefore [p(2) \to q(2)] = [(2^2 > 0) \to (2 \text{ is odd})]$

$\equiv [\text{True} \to \text{False}] \equiv \text{False}$

Here S_1 is false.

Statement S_2: True

If $\exists x$ is true for one value then the overall the truth value of the statement will be true.

So, Check the statement for x = 3

$\therefore [p(3) \to q(3)] = [(3^2 > 0) \to (3 \text{ is odd})]$

$\equiv [\text{True} \to \text{True}]$ ° True

Hence, S_2 is True.

Q2 Text Solution:

S_1: $[\forall x\ (P(x) \to Q(x)] \to [\forall x\ P(x)] \to [\forall x\ Q(x)]$

(Property of Predicate Logic)

$[\forall x\ (\sim P(x) \vee Q(x)] \to [\forall x\ P(x)] \to [\forall x\ Q(x)]$

S_2: $[\exists x\ P(x)] \to [\exists x\ Q(x)] \to [\exists x\ (P(x) \to Q(x)]$

Proof:

$(P_1 \vee P_2) \to (Q_1 \vee Q_2) \to [(P_1 \to Q_1) \vee (P_2 \to Q_2)]$

$(P_1'P_2' + Q_1 \vee Q_2) \to [(P_1' + Q_1) + (P_2' + Q_2)]$

$(P_1 + P_2) \cdot (Q_1'\ Q_2') + P_1' + Q_1 + P_2' + Q_2$

$P_1Q_1'Q_2' + P_2Q_1'Q_2' + P_1' + Q_1 + P_2' + Q_2$

$\boxed{A'B + A = A + B}$

$P_1 + P_2 + P_1' + Q_1 + P_2' + Q_2$

$P_1 + P_1' = 1 \ and \ 1 + anyting = 1$

$1 + P_1 + P_2 + Q_1 + Q_2 + P_2'$

1 True

Hence both are valid.

Q3 Text Solution:

P(y) = $\sqrt{y}$ is real

domain = positive integers (z^+)

(a) $\forall$y P(y) True

For every values of y, $\sqrt{y}$ is real because domain is positive integer

(b) $\exists$y P(y) True

For some values of y, $\sqrt{y}$ is real

(c) $\forall$y ~ P(y) False

(d) $\exists$y ~ P(y) False

Q4 Text Solution:

$(P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \to (P_1 \wedge P_2) \vee (Q_1 \wedge Q_2)$

$(P_1 + Q_1) \wedge (P_2 + Q_2) \to P_1P_2 + Q_1Q_2$

$P_1' + Q_1' + P_2' + Q_2' + P_1P_2 + Q_1Q_2$

$P_1P_2 + Q_1Q_2 + P_1' + Q_1' + P_2' + Q_2'$ (Invalid)

Remaining all are valid.

Q5 Text Solution:

I: $\forall y[P(y) \rightarrow Q] \leftrightarrow [\forall y\, P(y)] \rightarrow Q$

$(P_1 \rightarrow Q) \wedge (P_2 \rightarrow Q) \equiv (P_1 \wedge P_2) \rightarrow Q$

$(P_1' + Q) \wedge (P_2' + Q) \equiv P_1' + P_2' + Q$

$P_1' P_2' + P_1' Q + P_2' Q + Q \equiv P_1' + P_2' + Q$

$P_1' P_2' + Q \neq P_1' + P_1' + Q$ (invalid)

II: $\exists y\, [P(y) \rightarrow Q] \rightarrow [\exists y\, P(y)] \rightarrow Q$

$(P_1 \rightarrow Q) \vee (P_2 \rightarrow Q) \rightarrow (P_1 \vee P_2) \rightarrow Q$

$P_1' + P_2' + Q \rightarrow P_1' P_2' + Q$

$P_1 + P_2\, Q' + P_1' P_2' + Q$

$P_1 P_2 + P_1' P_2' + Q$ (Invalid)

Hence, option (d) is correct

Q6 Text Solution:

- $\exists x$ represents some/any/at least one:
- Actor (x) $\wedge$Smart (x) means x is an actor and smart.
- Therefore $\exists x$ (Actor (x) $\wedge$ smart (x)) represents some actor is smart.

Q7 Text Solution:

I is the correct representation as it reads “there exist an apple x and there exist an apple y and if apple y exists then it is equal to x” that means there is only one apple (exactly one).

II is absolutely incorrect as it says “some apple or at least one apple” instead of exactly one apple.

Q8 Text Solution:

I: $\exists x\, \{P(x) \rightarrow Q(x)\} = \exists x\, \{\neg\, (P(x)) \vee Q(x)\}$

$= \{\exists x\, \neg\, (P(x)) \vee \exists x\, Q(x)\}$

$= \forall x\, P(x) \rightarrow \exists x\, Q(x)\}$ $\therefore$True.

II. $\exists x\, \forall_y\, P(n, y)$

$\forall_y\;\; P(a, y)$ for same a

$P(a, b)$ is true for$\forall_b = \exists x\, P(x, b)$

$\forall_y\; \exists x\, P(x, y)$ is ture.

Q9 Text Solution:

Two points/rules to solve the question:

I. $\exists x\, f(x) \equiv \neg\forall(x)\, \neg f(x)$

II. $\forall x\, f(x) \equiv \neg\exists\, (x)\, \neg f(x)$

The given statement:

$(\exists x)\, (\forall y)\, [p(x, y) \wedge q(x, y) \wedge \neg\, r(x, y)]$

Can be written as: -

$\neg\, (\forall x)\, \exists(y)\, [p(x, y) \wedge q(x, y) \rightarrow r(x, y)]$

NOTE: $[\neg\, (x \wedge y) \vee 3 \equiv x \wedge y \rightarrow 3\,]$

Q10 Text Solution:

We write “Every player” as $\forall x$[player (x) $\rightarrow$]

‘There is some coach who likes x” as $\exists y$ [coach (y) p(x)]

Where P is the property.

Therefore, we can write the first order logic for the given statement as

$\forall(x)$ [player (x) $\rightarrow \exists y$ [coach (y) $\wedge$likes (y, x)]]