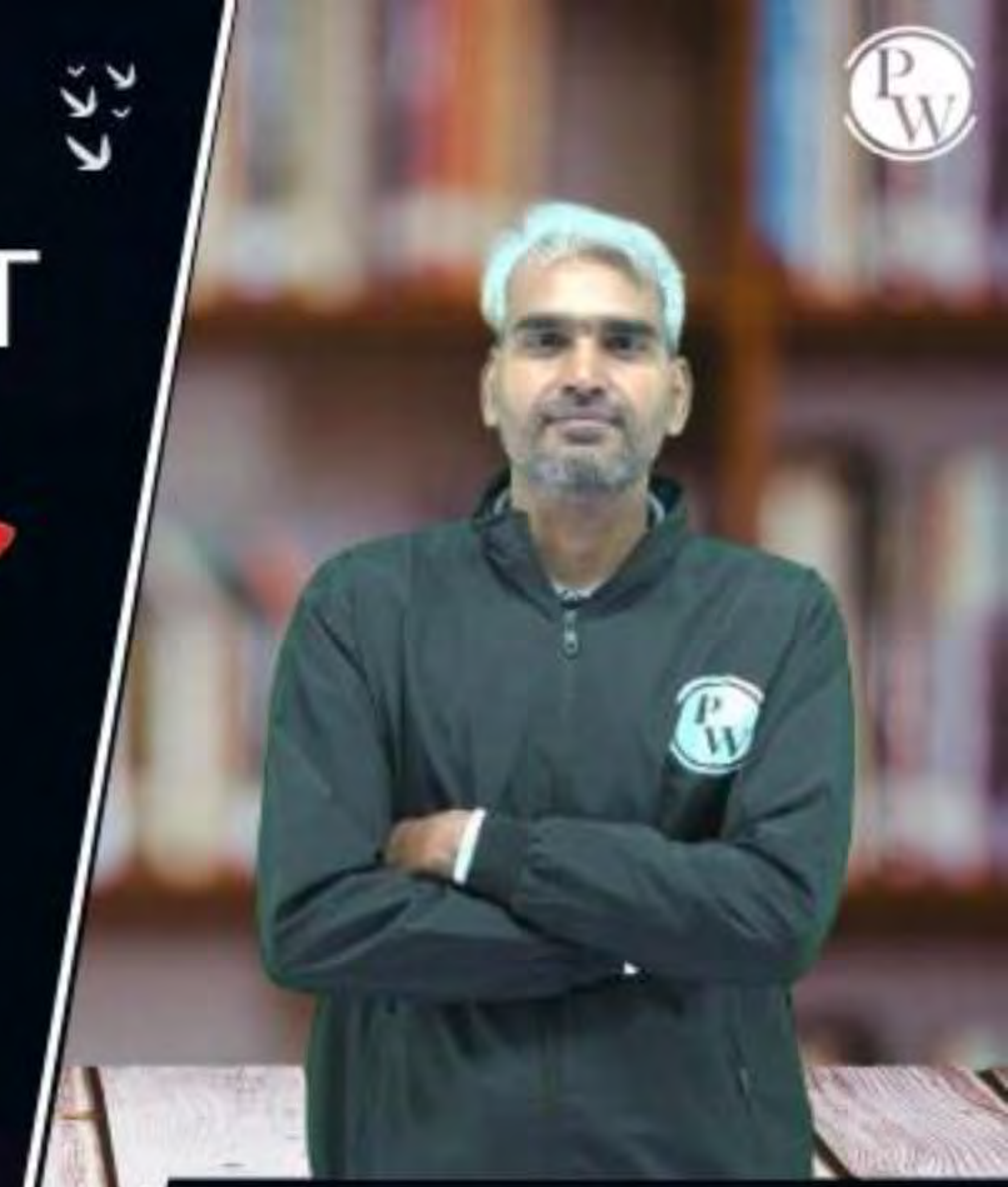
Computer Science & IT

Discrete Mathematics

Mathematical Logic

Lecture No. 07



By- Vishal Sir

Recap of Previous Lecture

- **Topic** Sentences with multiple quantifiers
- **Topic** Relationship diagram
- **Topic** Rules of inference w.r.t. predicate logic
- **Topic** Equivalences and implications

Slide

Topics to be Covered

Topic Equivalences and implications

Topic Practice questions on predicate logic

Topic Tautology in predicate logic

Slide

Topic : Equivalences & Implications

1. $\forall x\,[P(x) \wedge Q(x)] \equiv [\forall x\, P(x)] \wedge [\forall x\, Q(x)]$

2. $\exists\, x\,[P(x) \vee Q(x)] \equiv [\exists\, x\, P(x)] \vee [\exists\, x\, Q(x)]$

3. $[\forall x\, P(x)] \vee [\forall x\, Q(x)] \Rightarrow \forall x\,[P(x) \vee Q(x)]$

4. $\exists\, x\,[P(x) \wedge Q(x)] \Rightarrow [\exists\, x\, P(x)] \wedge [\exists\, x\, Q(x)]$

5. $\forall x\,[P(x) \to Q(x)] \Rightarrow [\forall x\, P(x)] \to [\forall x\, Q(x)]$

Topic : Some important equivalences

1. $\forall x\,[P(x) \land Q] \equiv \forall x\, P(x) \land Q$

2. $\exists x\,[P(x) \lor Q] \equiv \exists x\, P(x) \lor Q$

3. $\forall x\,[P(x) \lor Q] \equiv \forall x\, P(x) \lor Q$

4. $\exists x\,[P(x) \land Q] \equiv \exists x\, P(x) \land Q$

Note: x is not a free variable in predicate 'Q'.
i.e. Value of Q does not vary with x.

Topic : Some important equivalences

1. $\forall x\, [P(x) \wedge Q] \equiv \forall x\, P(x) \wedge Q$

if Q = false, then both L.H.S. & R.H.S. will be false,
∴ we don't need to worry when Q = False

When Q = True.

(i) For L.H.S. to be true conjunction of $P(x)$ & Q must be true for all x, 'Q' is already true, ∴ 'P' must be true for all x, in this case RHS will also be true

(ii) For RHS to be true, 'P' should be true for all x, in that case LHS will also be true

2. $\exists x\, [P(x) \vee Q] \equiv \exists x\, P(x) \vee Q$

3. $\forall x\, [P(x) \vee Q] \equiv \forall x\, P(x) \vee Q$

4. $\exists x\, [P(x) \wedge Q] \equiv \exists x\, P(x) \wedge Q$

Topic : Some important equivalences

1. $\forall x\,[P(x) \land Q] \equiv \forall x\, P(x) \land Q$

2. $\exists x\,[P(x) \lor Q] \equiv \exists x\, P(x) \lor Q$

if Q = True, then both LHS & RHS will be true,
∴ we don't need to worry when Q = True

if Q = False
(i) For LHS to be true, P must be true for at least one x, in that case RHS will also be true.
(ii) For RHS to be true 'P' must be true for at least one x. in that case LHS will also be true

3. $\forall x\,[P(x) \lor Q] \equiv \forall x\, P(x) \lor Q$

4. $\exists x\,[P(x) \land Q] \equiv \exists x\, P(x) \land Q$

Topic : Some important equivalences

1. $\forall x\,[P(x) \wedge Q] \equiv \forall x\, P(x) \wedge Q$

2. $\exists x\,[P(x) \vee Q] \equiv \exists x\, P(x) \vee Q$

3. $\forall x\,[P(x) \vee Q] \equiv \forall x\, P(x) \vee Q$

if Q = True then both LHS & RHS will be true

if Q = False,

(i) For LHS to be true, P must be true for all x, in that case RHS will also be true

(ii) For RHS to be true, P must be true for all x in that case LHS will also be true

4. $\exists x\,[P(x) \wedge Q] \equiv \exists x\, P(x) \wedge Q$

Topic : Some important equivalences

1. $\forall x\,[P(x) \wedge Q] \equiv \forall x\, P(x) \wedge Q$

2. $\exists x\,[P(x) \vee Q] \equiv \exists x\, P(x) \vee Q$

3. $\forall x\,[P(x) \vee Q] \equiv \forall x\, P(x) \vee Q$

4. $\exists x\,[P(x) \wedge Q] \equiv \exists x\, P(x) \wedge Q$

if Q = False, then both LHS & RHS will also be false

if Q = True

(i) for LHS to be true, P must be true for at least one x in that case RHS will also be true.

(ii) For RHS to be true, P must be true for at least one x, in that case LHS will also be true

Topic : Some important equivalences

5. $\forall x\,[P \rightarrow Q(x)] \equiv P \rightarrow \forall x\, Q(x)$

6. $\exists x\,[P \rightarrow Q(x)] \equiv P \rightarrow \exists x\, Q(x)$

'x' is not a free variable in predicate formula 'P'.

7. $\forall x\,[P(x) \rightarrow Q] \equiv \exists x\, P(x) \rightarrow Q$

8. $\exists x\,[P(x) \rightarrow Q] \equiv \forall x\, P(x) \rightarrow Q$

'x' is not a free variable in Q.

Topic : Some important equivalences

5. $\forall x\,[P \rightarrow Q(x)] \equiv P \rightarrow \forall x\, Q(x)$

$\forall_x \{\sim P \vee Q(x)\} \equiv \sim P \vee \forall_x Q(x) \equiv P \rightarrow \forall_x Q(x)$

Predicate with no free variable in LHS of implication

6. $\exists x\,[P \rightarrow Q(x)] \equiv P \rightarrow \exists x\, Q(x)$

$\exists_x \{\sim P \vee Q(x)\} \equiv \sim P \vee \exists_x Q(x) \equiv P \rightarrow \exists_x Q(x)$

7. $\forall x\,[P(x) \rightarrow Q] \equiv \exists x\, P(x) \rightarrow Q$

$\forall_x \{\sim P(x) \vee Q\} \equiv \forall_x \{\sim P(x)\} \vee Q \equiv \sim\{\exists_x P(x)\} \vee Q \equiv \exists_x P(x) \rightarrow Q$

Predicate with no free variable is in RHS of implication

8. $\exists x\,[P(x) \rightarrow Q] \equiv \forall x\, P(x) \rightarrow Q$

$\exists_x \{\sim P(x) \vee Q\} \equiv \exists_x \{\sim P(x)\} \vee Q \equiv \sim\{\forall_x P(x)\} \vee Q \equiv \forall_x P(x) \rightarrow Q$

[MSQ]

#Q. Let $P(x)$ and $Q(x)$ be arbitrary predicates. Which of the following statements is always TRUE?

A. $(\forall x(P(x) \vee Q(x))) \Rightarrow ((\forall x P(x)) \vee (\forall x Q(x)))$

B. $(\forall x(P(x) \Rightarrow Q(x))) \Rightarrow ((\forall x P(x)) \Rightarrow (\forall x Q(x)))$

C. $(\forall x(P(x)) \Rightarrow \forall x(Q(x))) \Rightarrow (\forall x\,(P(x) \Rightarrow Q(x)))$

D. $((\forall x(P(x)) \Leftrightarrow (\forall x Q(x))) \Rightarrow (\forall x(P(x) \Leftrightarrow Q(x)))$

T

$\forall x \{P(x)\} \longleftrightarrow \forall x \{Q(x)\}$

$F \longleftrightarrow F = T$

LHS True

$U = \{1, 2\}$

$P(1) = T$ $\quad Q(1) = F$

$P(2) = F$ $\quad Q(2) = T$

$\forall x \{P(x) \longleftrightarrow Q(x)\}$

$P(1) \longleftrightarrow Q(1)$

$T \longleftrightarrow F$

F

RHS False

[MSQ]

#Q. Which of the following predicate calculus statements is/are valid

A. $(\forall x)P(x) \vee (\forall x)Q(x) \rightarrow (\forall x)\{P(x) \vee Q(x)\}$

B. $(\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)\{P(x) \wedge Q(x)\}$

T

C. $(\exists x)\{P(x) \vee Q(x)\} \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$

$U=\{1,2\}$ $P(1)=T$ $P(2)=F$ $Q(1)=F$ $Q(2)=F$ F $\vee$ F = F

D. $(\exists x)\{P(x) \vee Q(x)\} \rightarrow \{\exists x\, P(x) \vee \forall x\, Q(x)\}$

[MSQ]

#Q. Which of the following predicate calculus statements is/are valid

$\exists x\{P(x) \vee Q(x)\} \equiv \exists x\{P(x)\} \vee \exists x\{Q(x)\}$

A. $(\forall x)P(x) \vee (\forall x)Q(x) \rightarrow (\forall x)\{P(x) \vee Q(x)\}$

B. $(\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)\{P(x) \wedge Q(x)\}$

C. $(\exists x)\{P(x) \vee Q(x)\} \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$

D. $(\exists x)\{P(x) \vee Q(x)\} \rightarrow \{\exists x\, P(x) \vee \forall x\, Q(x)\}$

$U = \{1, 2\}$ $P(1) = F$ $Q(1) = T$ $P(2) = F$ $Q(2) = F$

$F \vee F = F$

[MSQ]

#Q. Which of the following predicate calculus statements is/are valid

A. $\forall(x)(P(x) \Rightarrow Q(x)) \Rightarrow ((\forall(x)P(x)) \Rightarrow (\forall(x)Q(x)))$

B. $\exists x(P(x) \vee Q(x)) \Rightarrow ((\exists xP(x)) \Rightarrow (\exists xQ(x)))$

$\boxed{T \longrightarrow F \equiv F}$

if P is true for at least one 'x' & Q is false for all x, then LHS = True & RHS = False

C. $\exists x(P(x) \wedge Q(x)) \Leftrightarrow ((\exists xP(x)) \wedge (\exists xQ(x)))$

D. $\forall(x)\exists y\, P(x, y) \Rightarrow \exists y\, \forall(x)P(x, y)$

[MSQ]

#Q. Which of the following is a valid first order formula? (Here α and β are first order formulae with x as their only free variable)

$P(x) \equiv \alpha(x)$, $\beta(x) \equiv \varphi(x)$

A. $((\forall x)[\alpha] \Rightarrow (\forall x)[\beta]) \Rightarrow (\forall x)[\alpha \Rightarrow \beta]$

$(\forall x\{P(x)\} \to \forall x\{\varphi(x)\}) \xrightarrow{\times} \forall x\{P(x) \to \varphi(x)\}$

B. $((\forall x)[\alpha] \Rightarrow (\exists x)[\alpha \wedge \beta])$

$\boxed{\forall x\{P(x)\}} \to \boxed{\exists x\{P(x) \wedge \varphi(x)\}}$

if $P=\alpha$ is true for all x & $\varphi = \beta$ is false for all x, then LHS is true but RHS is false

C. $((\forall x)[\alpha \vee \beta] \Rightarrow ((\exists x)[\alpha] \Rightarrow (\forall x)[\alpha]))$

$\forall x\{P(x) \vee \varphi(x)\} \to (\exists x\, P(x) \to \forall x\{P(x)\})$

$U = \{1, 2\}$

$P(1) = T$, $P(2) = F$ | $\varphi(1) = T$, $\varphi(2) = T$

LHS = True | but RHS = False

D. $((\forall x)[\alpha \Rightarrow \beta] \Rightarrow ((\forall x)[\alpha] \Rightarrow (\forall x)[\beta]))$

T F

F

[MSQ]

#Q. Which of the following first order formulae is logically valid? Here $\alpha(x)$ is a first-order formula with x as a free variable and β is a first-order formula with no free variable.

$\equiv \exists x$

A $[\beta \rightarrow (\exists x, \alpha(x))] \rightarrow [\forall x, \beta \rightarrow \alpha(x)]$

B $[\exists x, \beta \rightarrow (\alpha(x))] \rightarrow [\beta \rightarrow (\forall x, \alpha(x))]$

$\equiv$ $\exists x$

C $[(\exists x, \alpha(x)) \rightarrow \beta] \rightarrow [\forall x, \alpha(x) \rightarrow \beta]$

D $[(\forall x, \alpha(x)) \rightarrow \beta] \rightarrow [\forall x, \alpha(x) \rightarrow \beta]$

$\equiv \exists x$

[MSQ]

#Q. Which one of the following predicate formulae is NOT logically valid? Note that W is a predicate formula without any free occurrence of x.

A $\exists x(p(x) \land W) \equiv (\exists x\, p(x)) \land W$ (valid)

B $\forall x(p(x) \rightarrow W) \equiv (\forall x\, p(x)) \rightarrow W$ (invalid)

$\exists x$ (valid)

C $\exists x(p(x) \rightarrow W) \equiv (\forall x\, p(x)) \rightarrow W$ (valid)

D $\forall x(p(x) \lor W) \equiv (\forall x\, p(x)) \lor W$ (valid)

[MSQ]

#Q. What is the correct translation of the following statement into mathematical logic?

"some real numbers are rational"

{ There exist at least one number such that it is real & rational }

A $\exists x\,(real(x) \lor rational(x))$

Some numbers are real or rational

B $\exists x\,(real(x) \land rational(x))$

Some numbers are real & rational

C $\forall x\,(real(x) \rightarrow rational(x))$

for all x, if x is real then x is rational

(All real numbers are rational)

D $\forall x\,(rational(x) \rightarrow real(x))$

for all x, if x is rational then x is real

All rational numbers are real.

Some simple graphs are Connected graphs

|||

Some Connected graphs are simple graphs

$$\forall x \{ real(x) \wedge rational(x) \} \equiv \forall x \{ rational(x) \wedge real(x) \}$$

all numbers are real & rational

$$\forall x \{ rational(x) \rightarrow real(x) \} \not\equiv \forall x \{ real(x) \rightarrow rational(x) \}$$

all rational numbers are real

$$\forall x \{ real(x) \rightarrow rational(x) \}$$

All real numbers are rational

<u>Universe</u>: is a set of all living things.

$$\forall x \{ Human(x) \rightarrow Intelligent(x) \}$$

All human are intelligent.

$\exists x \{ Human(x) \wedge intelligent(x) \}$: Some human are intelligent

$$\forall x \{ Human(x) \wedge intelligent(x) \}$$

All are human & intelligent

$$\neg \exists x \{ Human(x) \rightarrow Intelligent(x) \} \not\equiv \text{Some human are intelligent}$$

In general,

If question is related to "some" then we use "$\wedge$"

& If question is related to "all" then we use "$\longrightarrow$"

[MSQ]

Not atleast one of my friend is perfect

$\sim \{ \exists x \{ friend(x) \wedge Perfect(x) \} \}$

#Q. What is the logical translation of the following statement?

"None of my friends are perfect"

$|||$

All my friends are imperfect

$\forall x \{ friend(x) \rightarrow \sim Perfect(x) \}$

A. $\exists x\,(F(x) \wedge \sim P(x))$

B. $\exists x\,(\sim F(x) \wedge \sim P(x))$

C. $\exists x\,(\sim F(x) \wedge P(x))$

D. $\sim (\exists x\,(F(x) \wedge P(x)))$

$\forall x \{ \sim F(x) \vee \sim P(x) \}$

[MSQ] H.W.

#Q. The CORRECT formula of the sentence “not all rainy days are cold”

A $\forall d\,(rainy(d) \wedge \sim cold(d))$

B $\exists d\,(\sim rainy(d) \rightarrow cold(d))$

C $\forall d\,(\sim rainy(d) \rightarrow cold(d))$

D $\exists d\,(rainy(d) \wedge \sim cold(d))$

[MSQ] H.W.

#Q. Which one the following is the most appropriate logical formula to represent the statement "gold and silver ornaments are precious". The following notations are used: $G(x)$:x is a gold ornament

$S(x)$:x is a silver ornament $P(x)$:x is precious

A $\forall x(P(x) \rightarrow G(x) \wedge S(x))$

B $\exists x\left((G(x) \wedge S(x)) \rightarrow P(x)\right)$

C $\forall x\left((G(x) \wedge S(x)) \rightarrow P(x)\right)$

D $\forall x\left((G(x) \vee S(x)) \rightarrow P(x)\right)$

[MSQ] H.W.

#Q. Which one of the first order predicate calculus statements given below correctly expresses the following English statements? "Tigers and lions attack if they are hungry or threatened".

A $\forall x[(tiger(x) \wedge lion(x)) \rightarrow ((hungry(x) \vee threatened(x)) \rightarrow attacks(x))]$

B $\forall x[(tiger(x) \vee lion(x)) \rightarrow ((hungry(x) \vee threatened(x)) \wedge attacks(x))]$

C $\forall x[(tiger(x) \wedge lion(x)) \rightarrow (attacks(x) \rightarrow (hungry(x) \vee threatened(x)))]$

D $\forall x[(tiger(x) \vee lion(x)) \rightarrow ((hungry(x) \vee threatened(x)) \rightarrow attacks(x))]$

2 mins Summary

Topic Equivalences and implications

Topic Practice questions on predicate logic

Topic Tautology in predicate logic

Slide

THANK - YOU

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