GATE-All BRANCHES
Engineering Mathematics

Linear Algebra

Lecture No.- 09

By- Rahul sir

Recap of previous lecture

Topic Question based on system of equations

Topic Span of vector space

Topics to be Covered

- **Topic** Properties of matrices
- **Topic** Question based on properties of the matrix
- **Topic** Concept of eigen values and eigen vectors
- **Topic** Problems based on eigen values and eigen vectors

CS ✓ ME

Eigen Vectors :

$$A\boxed{X} = \lambda \boxed{X}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$A \longrightarrow n \times n$ (only square matrix)

$\lambda \longrightarrow$ eigen value (scaler quantity)

$X \longrightarrow$ corresponding Eigen value $\rightarrow$ eigen vector

How to Calculate eigen vector :

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

step 01 $|A - \lambda I| = 0$

Calculate The eigen value

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

Step · 01

$$\boxed{|A - I\lambda| = 0}$$

$$\begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\text{Trace } A)\lambda + \det A = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + (4-10) = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + \lambda - 6 = 0$$

$$\Rightarrow \lambda(\lambda - 6) + 1(\lambda - 6) = 0$$

$$\Rightarrow \boxed{\lambda = -1 \quad \lambda = 6}$$

$$\left.\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 6 \end{aligned}\right] \text{Eigen values}$$

Corresponding eigen value $\lambda_1 = -1$ $\lambda_2 = 6$

$\lambda = \lambda_1 \quad AX_1 = \lambda_1 X_1$

$\lambda = \lambda_1 \rightarrow [A - I\lambda_1]\, X_1 = [0]$

$\lambda = \lambda_2 \rightarrow [A - I\lambda_2]\, X_2 = [0]$

using eigen vector Equations

$$[A - I\lambda_1]\, X_1 = [0]$$

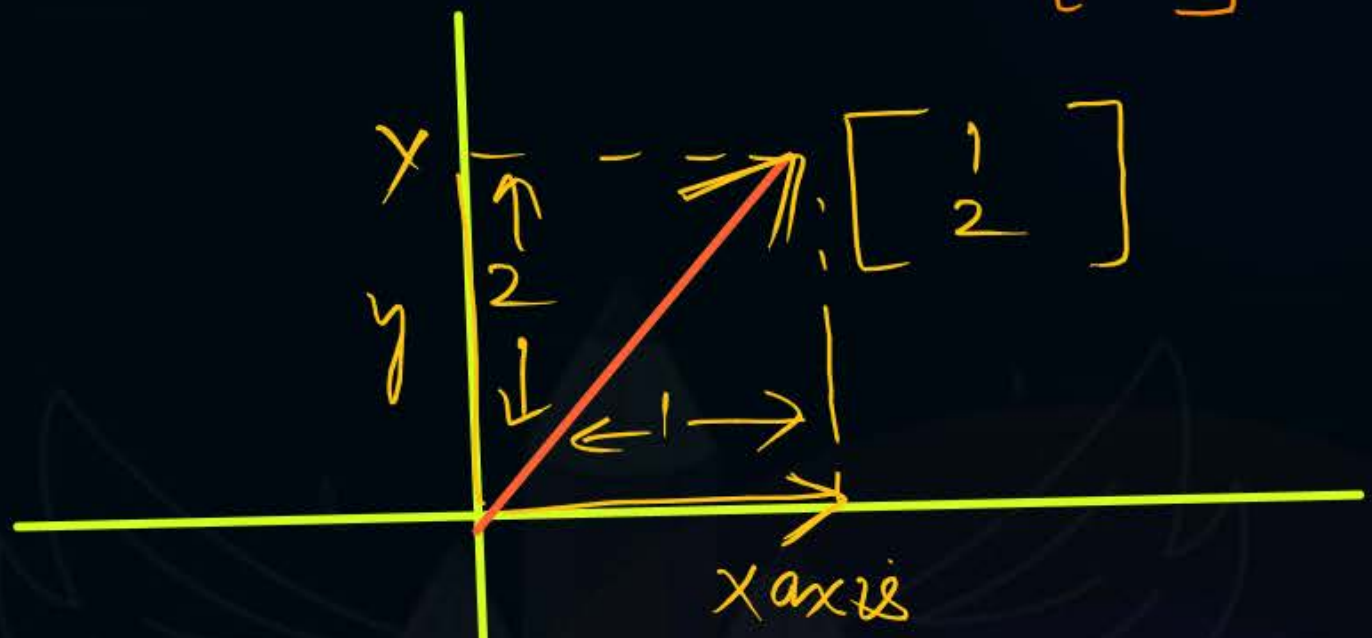
$$\lambda_1 = -1 \quad \begin{bmatrix} 4-\lambda_1 & 5 \\ 2 & 1-\underset{\lambda_1}{(-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4-(-1) & 5 \\ 2 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\left.\begin{matrix} X_1 \\ X_2 \end{matrix}\right]$ eigen vector column matrix

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow$ vector

$$\begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Infinite eigen vectors $\Rightarrow \left\{\begin{array}{l} 5x_1 + 5x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{array}\right]$ SAME

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\boxed{\begin{array}{l} x_2 = K \\ x_1 = -K \end{array}}$$

$$X_1 = \begin{bmatrix} -K \\ K \end{bmatrix} = K\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

first eigenvector $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Normalization of Eigen vector

$$X_1 = \begin{bmatrix} -K \\ K \end{bmatrix}$$

$$X_1 = \frac{1}{\sqrt{(-K)^2 + K^2}}\begin{bmatrix} -K \\ K \end{bmatrix}$$

$$= \frac{1}{K\sqrt{2}}\begin{bmatrix} -K \\ K \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \text{first Normalized eigenvector}$$

$$\begin{bmatrix} 2K \\ 3K \end{bmatrix} = \frac{1}{\sqrt{(2K)^2 + (3K)^2}}\begin{bmatrix} 2K \\ 3K \end{bmatrix}$$

Properties of Eigen vector :

Matrix		
M or M^n	SAME	
A or A^{-1}	SAME	
A or KA	SAME	
A or A^T	Different	Row $\rightleftarrows$ column
A or $A^n + a_0 A^{n-1} + a_1 A^{n-2} + \dots + a_{n-1} = 0$	SAME	

Normalization :

$$A = \begin{bmatrix} 2K \\ K \end{bmatrix}$$

$$= \frac{1}{\sqrt{(2K)^2 + (K)^2}} \begin{bmatrix} 2K \\ K \end{bmatrix}$$

$$= \frac{1}{\sqrt{5K^2}} \begin{bmatrix} 2K \\ K \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

eigen value $\rightarrow$ 3 eigen value

$\underbrace{x_1 \quad x_2}_{}$

unique

For $\lambda = 6$

$$\begin{bmatrix} 4-6 & 5 \\ 2 & 1-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \left.\begin{aligned} -2x_1+5x_2 &= 0 \\ 2x_1-5x_2 &= 0 \end{aligned}\right\} \text{Infinite solutions}$$

$$\boxed{2x_1 = 5x_2}$$

$x_2 = K$

$2x_1 = 5K$

$x_1 = \frac{5}{2}K$

$$\hat{X_2} = \begin{bmatrix} \frac{5}{2}K \times 2 \\ K \times 2 \end{bmatrix}$$

[Remove the constants]

$$\hat{X_2} = \begin{bmatrix} 5K \\ 2K \end{bmatrix} = K\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Normalized eigen vector

$$= \frac{1}{\sqrt{(5K)^2+(2K)^2}}\begin{bmatrix} 5K \\ 2K \end{bmatrix}$$

$$X_{2N} = \begin{bmatrix} \frac{5}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \end{bmatrix}$$

Topic: Eigen values

$A = \begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}_{2\times 2}$ $\frac{3}{1} = \frac{\lambda_1}{\lambda_2}$

for $p = 2$

Trace(A) $(p+2)$

#Q. The two eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3:1 for p=2.

$p \longrightarrow$ Different value $\rightarrow \frac{\lambda_1}{\lambda_2} = \frac{3}{1}$

What is another value of "p" for which the eigen values have the same ratio of 3:1?

$A = \begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}_{2\times 2}$

$\boxed{\text{Trace}(A) = p+2}$

$\boxed{\text{Det } A = (2p-1)}$

$\boxed{\begin{aligned} \lambda_1 + \lambda_2 &= p+2 \\ \lambda_1 \lambda_2 &= (2p-1) \end{aligned}}$

$\frac{\lambda_1}{\lambda_2} = \frac{3}{1}$

$\lambda_1 = 3\lambda_2$

$\lambda_1 + 3\lambda_2 = p+2$ $\lambda_2 = \frac{p+2}{4}$

$3\lambda_2 \cdot \lambda_2 = (2p-1)$

$3\lambda_2^2 = (2p-1)$

(a) -2

(b) 1

(c) 7/3

(d) 14/3

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$$3\lambda_2^2 = (2p-1)$$

$$= 3 \times \left(\frac{p+2}{4}\right)^2 = (2p-1)$$

$$= 3(p+2)^2 = 16(2p-1)$$

$$= 3(p^2+4+4p) = 32p-16$$

$$= 3p^2 - 20p + 28 = 0$$

$$\Rightarrow p = 2$$

$$p = \frac{14}{3}$$

Topic : Eigen values

Eigen values

#Q. The eigen values of the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$

A. -1,5,6

B. 1,-5,+6i,-6i

C. 1,5,+6i,-6i

D. 1,5,5

$$\lambda^3 - (\text{Trace})\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det A = 0$$

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$$

$$= \lambda^3 - (1+5+5)\lambda^2 + (\quad 71 \quad)\lambda - 61 = 0$$

$$= \underline{\lambda^3 - 11\lambda^2 + 71\lambda - 61 = 0}$$

Roots $\longrightarrow$ $\underline{1,\ 5 \pm 6i}$

#Q. The eigen values of a 2× 2 matrix X are -2 and -3.The eigen values of matrix $\underline{(X+I)}^{-1}\underline{(X+5I)}$ are

$2\times 2 \longrightarrow -2, -3$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow 1, 1$

A -3,-4

B -1,-2

C $\boxed{-1,-3}$

D -2,-4

$$(X+I)^{-1}(X+5I) = (X+I)^{-1}(\boxed{X+I}+4I)$$

$$= (X+I)^{-1}(X+I) + 4(X+I)^{-1}$$

$$= I + 4(X+I)^{-1}$$

$$= 1 + 4 \times -1 = -3$$

$$= 1 + -\frac{1}{2} \times 4 = -1$$

$I \to 1, 1$

$X \to -2, -3$

$X+I = -2+1, -3+1$

$= -1, -2$

$(X+I)^{-1} = -1, -\frac{1}{2}$

Topic : Linear Algebra

M.W

#Q. Consider the 5 × 5 matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$. It is given that A has only real eigen value.

Then the real eigen value of A is:

A. -2.5

B. 0

C. 15

D. 25

Topic : Linear Algebra

#Q. Two eigenvalues of a 3×3 real matrix P are $(2+\sqrt{-1})$ and 3. The determinant of P is ___.

$P \longrightarrow 2+i,\ 2-i,\ 3$

det of P = Product of The eigen values

$= (2+i)(2-i) \times 3$

$= 15$

Topic : Linear Algebra

$A \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\lambda_1 = -1$, $\lambda_2 = -2$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

#Q. A matrix has eigen values –1 and –2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. Then matrix is-

Trace = −3
det = 2

$AX = \lambda X$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$a - b = -1$
$c - d = 1$
$a - 2b = -2$
$c - 2d = 4$

$a = 0$
$b = 1$
$c = -2$
$d = -3$

A $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

C $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

D $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $= 0 - (-2) = 2$

Trace = −3
det = 2

#Q. Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigenvectors corresponding to eigen values λ_1 and λ_2 are $x_1 = \begin{bmatrix} 70 \\ \lambda_1 \quad -50 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 \quad -80 \\ 70 \end{bmatrix}$, respectively. The value of $x_1^T x_2$ is______.

Trace = 130

Det = 4000 − 4900

= $\underline{-900}$

$\left.\begin{array}{l} \lambda_1 + \lambda_2 = 130 \\ \lambda_1 \lambda_2 = -900 \end{array}\right]$

$\begin{bmatrix} 70 & \lambda_1 - 50 \end{bmatrix}_{1\times 2} \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}_{2\times 1} = \begin{bmatrix} \quad \end{bmatrix}_{1\times 1}$

$= \boxed{0}$

Topic : Linear Algebra

#Q. Let the eigenvalues of a 2×2 matrix A be 1,–2 with eigenvectors x_1 and x_2 respectively. Then the eigenvalues and eigenvectors of the matrix $A^2 - 3A + 4I$ would respectively, be

$2\times 2 \rightarrow 1, -2$

$A^2 - 3A + 4I \longrightarrow \overset{\text{SAME}}{x_1, x_2}$

$A^2 - 3A + 4I = (1)^2 - 3\times 1 + 4, (-2)^2 - 3(-2) + 4$

$= 2, 14$

A. 2, 14 ; x_1 , x_2

B. 2, 14; $x_1 + x_2$; $x_1 - x_2$

C. 2, 0; x_1, x_2

D. 2, 0; $x_1 + x_2$, $x_1 - x_2$

$-3, -3, 5 \rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

#Q. The matrix $M = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$ has eigen values $\boxed{-3, -3, 5.}$ An eigen vector corresponding to the eigen value 5 is $[1\ 2\ -1]^T$. One of the eigen vector of the matrix M^3 is-

$M^3 = [1\ 2\ -1]^T$

M & M^n eigen vector SAME

A. $[1\ 8\ -1]^T$

B. $[1\ 2\ -1]^T$

C. $[1\ 2^{(3/2)}\ -1]^T$

D. $[1\ 1\ -1]^T$

$M \rightarrow [1\ 2\ -1]^T$

$M^3 \rightarrow [1\ 2\ -1]^T$

#Q. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. The eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is-

$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $X = \begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$AX = \lambda X$

$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$= \begin{bmatrix} 606 \\ 606 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$6 \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \lambda \begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$\lambda = 6$

A 2

B 4

C 6

D 8

THANK - YOU