GATE
ALL BRANCHES

ENGINEERING MATHEMATICS

Probability and Statistics

Lecture No. 10

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TOPICS TO BE COVERED

01 Problems based on Random Variables

Discrete
+
continuous Random variable

$\boxed{\text{Statistical Averages}}$

Q. Questions

A random variable X has probability density function $f(x)$ as given below:

$$f(x)=\begin{cases} a+bx & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value E[X] = 2/3, then Pr[X < 0.5] is____.

$$f(x)=\begin{cases} a+bx & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X]=\frac{2}{3}$$

$$P[X<0.5]=?$$

$$P[X<0.5]=\int_0^{0.5} \underline{f(x)\,dx} = \underline{\int_0^{0.5}(a+bx)\,dx} \rightarrow \underline{a \text{ or } b}$$

If X is continuous Random variable

$$f(x) = \begin{cases} a+bx & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Total pdf = 1

$$\boxed{\int_0^1 (a+bx)\,dx = 1}$$

Total Area = 1 (valid pdf)

$$= \left[ax + \frac{bx^2}{2}\right]_0^1 = 1$$

$$\Rightarrow a + \frac{b}{2} = 1$$

$$\boxed{2a + b = 2} \quad \text{—— } 1$$

$$E[X] = \frac{2}{3}$$

$$\int_a^b x f(x)\,dx = \frac{2}{3}$$

$$\int_0^1 x(a+bx)\,dx = \frac{2}{3}$$

$$= \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 = \frac{2}{3}$$

$$= \frac{a}{2} + \frac{b}{3} = \frac{2}{3}$$

$$= \frac{3a+2b}{\cancel{6}} = \frac{2}{\cancel{3}}$$

$$\boxed{3a+2b=4}$$

$$2a+b=2 \quad \text{— ①}$$

$$3a+2b=4 \quad \text{— ②}$$

Solve The Equ^n ① and ②

$$a=0 \qquad b=2$$

$$P[X<0.5] = \int_0^{0.5}(a+bx)\,dx$$

$$= \int_0^{0.5}(0+2\cdot x)\,dx$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^{0.5}$$

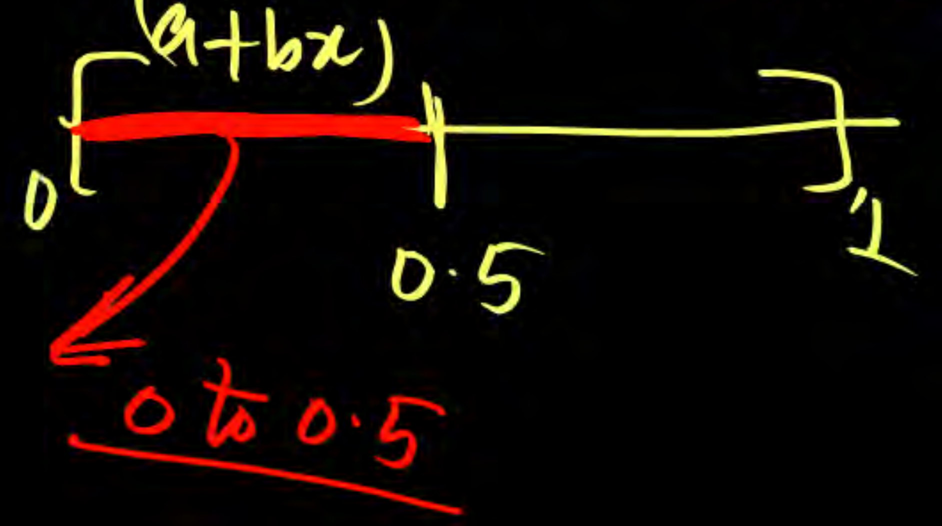
$$= 0.25 = \frac{1}{4} \text{ Ans}$$

Key

$$\int_a^b f(x)\,dx = 1 \checkmark \text{ Conti}$$

$$\sum_{i=0}^{n} P[X=x_i] = 1 \text{ Discrete}$$

$0<x<1 \rightarrow a+bx$



Q. Questions

X = Integer value - Discrete

x = 0, 1 (discrete Random var)

Consider the following probability mass function (p.m.f) of a random variable X.

f(x) = continuous R.V

$$p(x,q)=\begin{cases} q & \text{if } X=0 \\ 1-q & \text{if } X=1 \\ 0 & \text{otherwise} \end{cases}$$

If q = 0.4, the variance of X is ____.

Prob MASS Table

X	0	1
$P(X=x)$	q	$(1-q)$

$Var(X) = E[X^2] - [E[X]]^2$

$E[X^2] = (0)^2 \times q + (1)^2(1-q)$

$E[X^2] = (1-q)$

$E[X] = \text{mean} = 0 \times q + 1 \times 1 - q = 1 - q$

$= [1-q] - [1-q]^2$

$= [1-0.4] - [1-0.4]^2$

$= 0.6 - (0.6)^2 = \underline{0.24}$

8 questions

Q. Questions

A fair die with faces {1, 2, 3, 4, 5, 6} is thrown repeatedly till ‘3’ is observed for the first time. Let X denote the number of times the dice is thrown. The expected value of X is ____.

X = No. of trials / No. of Die thrown

3 – success

$\bar{3}$ = failure

$P(3) = \frac{1}{6}$

$P\{1, 2, 4, 5, 6\} = \frac{5}{6}$

1, 2, 3, 4, 5, 6

$\frac{1}{6}$

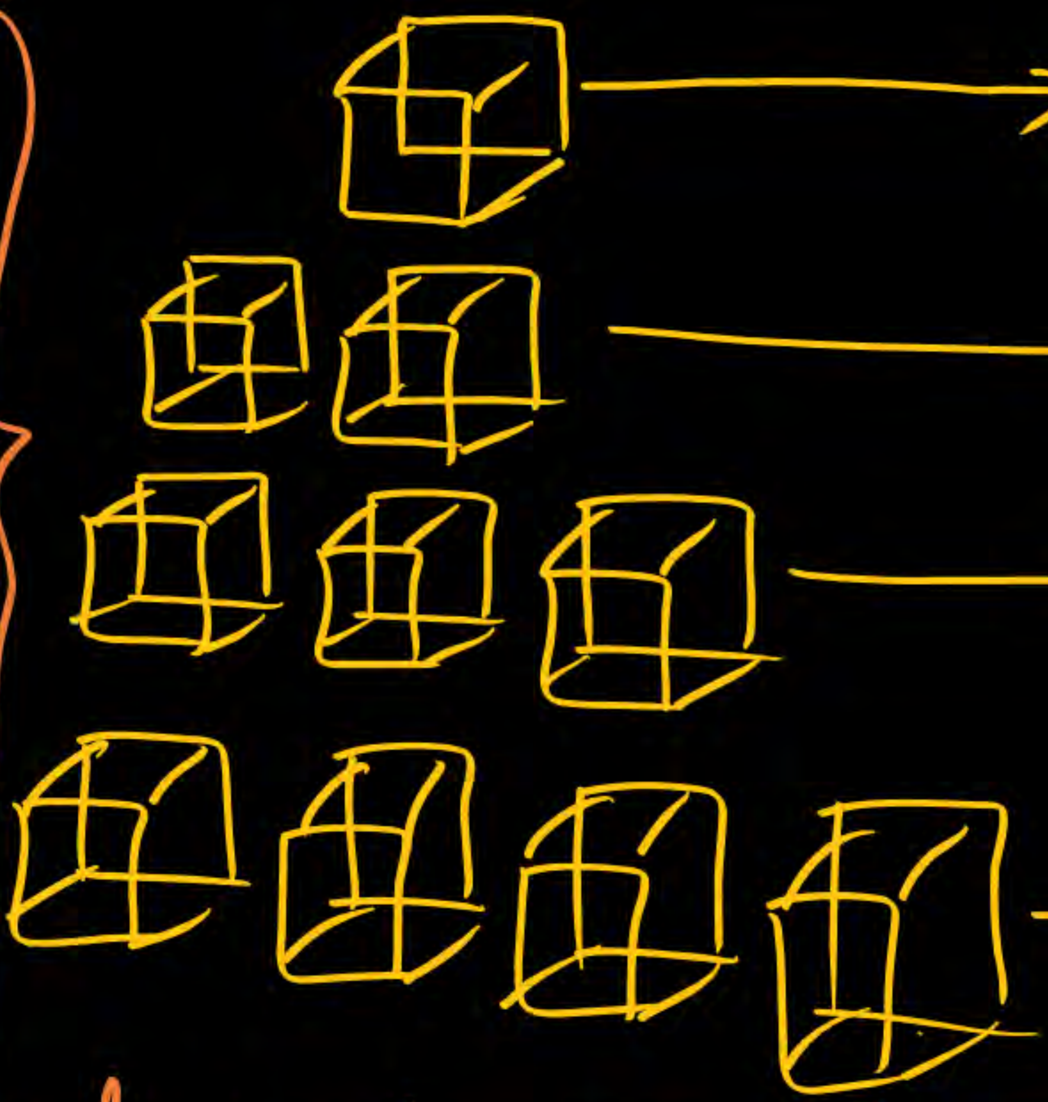
$P(\bar{3}) = 1 - \frac{1}{6} = \frac{5}{6}$

$x = 1$

$x = 2$

$x = 3$

$x = 4$



$\frac{1}{6} = 3$

$\bar{3}\,3 = \frac{5}{6}\,\frac{1}{6}$

$\bar{3}\,\bar{3}\,3 = \frac{5}{6}\,\frac{5}{6}\,\frac{1}{6}$

$\bar{3}\,\bar{3}\,\bar{3}\,3 = \left(\frac{5}{6}\right)^3 \frac{1}{6}$

S S

S FS

S FFS

S FFFS

S FFFFS

only one success

Discrete random variable $x = 1, 2, 3, 4 \ldots$

$E[X]$ A.G.P. 1, 2, 3, 4 – A.P

$$E[X] \Rightarrow 1\cdot\frac{1}{6} + 2\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + 3\left(\frac{5}{6}\right)^2\frac{1}{6} + 4\left(\frac{5}{6}\right)^3\frac{1}{6} + \cdots$$

$$\frac{5}{6}E[X] = \quad + 1\cdot\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + 2\left(\frac{5}{6}\right)^2\frac{1}{6} + 3\left(\frac{5}{6}\right)^3\frac{1}{6} + \cdots$$

$-$

$$\frac{E(X)}{6} = 1\cdot\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^3\frac{1}{6} + \cdots$$

$$\frac{E[X]}{6} = \frac{1}{6} + \frac{1}{6}\left[\frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots\right]$$

$$\frac{E[X]}{6} = \frac{1}{6} + \frac{1}{6}\left[\frac{5/6}{1-5/6}\right] \qquad \boxed{E[X] = 6}$$

X	$P(X=x)$	$q^{x-1}p$
1 (3)	$1/6$	p
2 ($\bar{3}$ 3)	$(5/6)(1/6)$	qp
3 ($\bar{3}\bar{3}$ 3)	$(5/6)^2 \frac{1}{6}$	
4 ($\bar{3}\bar{3}\bar{3}$ 3)	$(5/6)^3 \frac{1}{6}$	
5 ($\bar{3}\bar{3}\bar{3}\bar{3}$ 3)	$\left(\frac{5}{6}\right)^4 \frac{1}{6}$	

$$S_\infty = \frac{a}{1-r}$$

A.G.P
arithmetic
geometric prog.

Q. Questions

$f(x) = \frac{1}{2}|x|e^{-|x|}$ variance

The variance of the random variable X with probability density function

$Var(x) = E[x^2] - [E[x]]^2$ $\quad -\infty < x < \infty$

$f(x) = \frac{1}{2}|x|e^{-|x|}$ is ____.

$f(x) = \frac{1}{2}|x|e^{-|x|}$ → compound Function

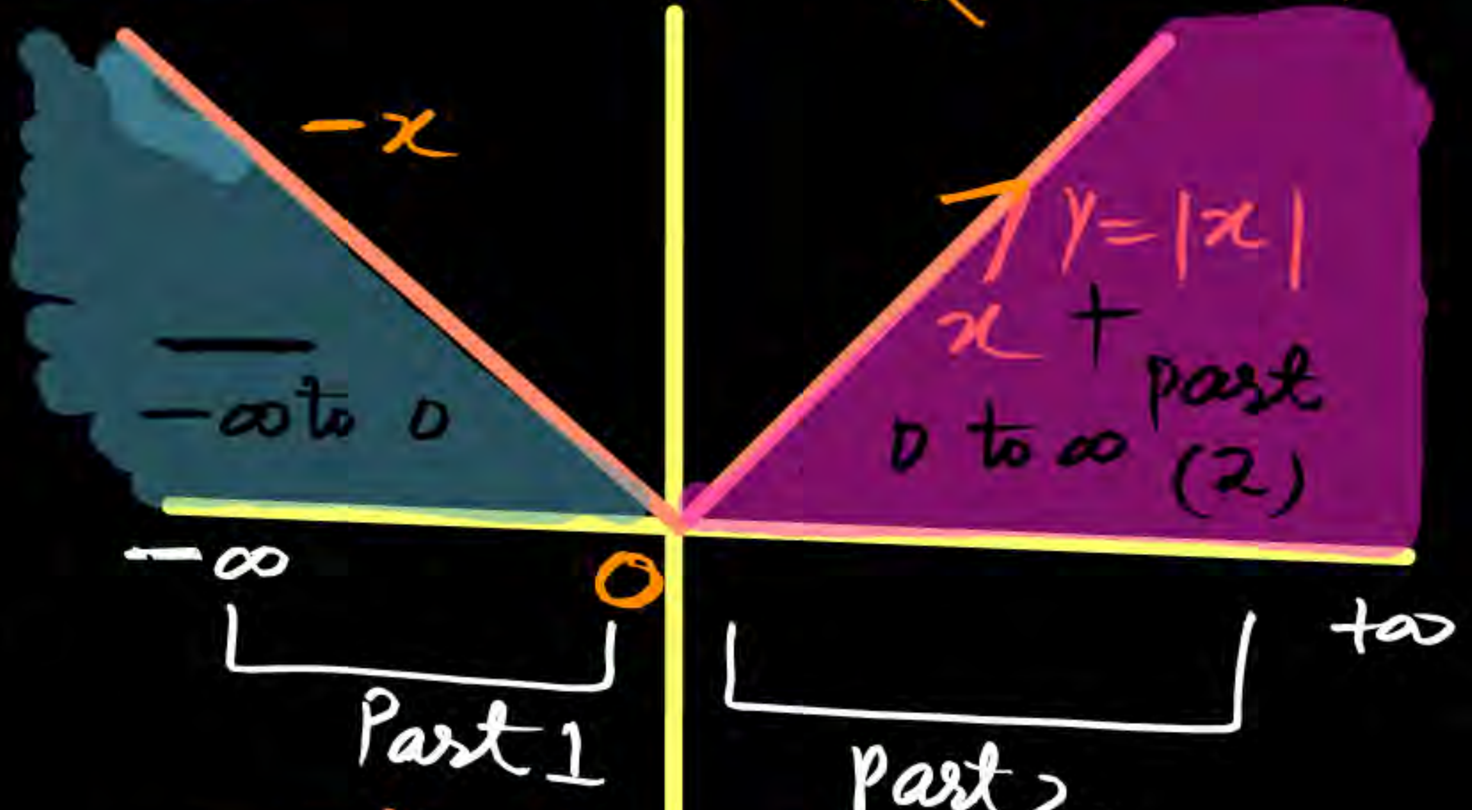
$V(X) = E[X^2] - [E[X]]^2$

$|x| = \text{mod}$ ⟨ x, $-x$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx \qquad -\infty < x < \infty$

$\Rightarrow \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx$

$f(x) = \frac{1}{2}e^{-|x|}|x|$



$-\infty$ to 0

$|x| = -x$

0 to ∞

$|x| = x$

$\Rightarrow \int_{-\infty}^{0} x^2 \cdot \frac{1}{2} x e^{x} + \int_{0}^{\infty} x^2 (x) \frac{1}{2} e^{-x}\,dx$

$= \int_{-\infty}^{0} \frac{x^3}{2} e^{x}\,dx + \int_{0}^{\infty} \frac{x^3}{2} e^{-x}\,dx$

$-\infty$ to 0

$|x| = -x \qquad \frac{1}{2}(-x)e^{-(-x)} = -\frac{1}{2}xe^{x}$

$$= \int_{-\infty}^{0} -\frac{x^3}{2} e^{x}\, dx + \int_{0}^{\infty} \frac{x^3}{2} e^{-x}\, dx \Rightarrow 6$$

$$\boxed{f(x) = \frac{1}{2}|x| e^{-|x|}}$$

$$E[X] = \int_{-\infty}^{0} x f(x)\, dx + \int_{0}^{\infty} x f(x)\, dx$$

$-\infty$ to 0 $\quad f(x) = \frac{1}{2}(-x)e^{x}$

0 to ∞ $\quad f(x) = \frac{1}{2}(x)e^{-x}$

$$\Rightarrow \int_{-\infty}^{0} x \cdot \frac{1}{2}(-x) e^{x}\, dx + \int_{0}^{\infty} x \cdot \frac{1}{2}(x) e^{-x}\, dx$$

$$\Rightarrow \int_{-\infty}^{0} -\frac{x^2}{2} e^{x}\, dx + \int_{0}^{\infty} \frac{x^2}{2} e^{-x}\, dx \Rightarrow 0$$

$$V(X) = E[X^2] - \left[E[X]\right]^2 = 6 - 0 = 6$$

$$\int x^3 e^x \, dx$$

If one Function is algebraic
Then use the Tabular Integration method.

Algebraic
D
$\downarrow$
0 — STOP.

D		I
x^3	$+$	e^x
$3x^2$	$-$	e^x
$6x$	$+$	e^x
6	$-$	e^x
0		e^x

$$\boxed{I = x^3e^x - 3x^2e^x + 6xe^x - 6e^x}$$

Algebraic function = diff.
Other function — Integ.

$$I = \int x^2 \sin x \, dx$$

D		I
x^2	$+$	$\sin x$
$2x$	$-$	$-\cos x$
2	$+$	$-\sin x$
0		$\cos x$

$$\boxed{I = -x^2\cos x + 2x\sin x + 2\cos x}$$

$$I = \int e^x \sin x \text{ or } \int e^x \cos x \, dx$$

Diff.		Integ.
e^x	$+$	$\boxed{\sin x}$
e^x	$-$	$-\cos x$
$\boxed{e^x}$		$-\boxed{\sin x}$ (Repeat - stop)

multiply $\int -e^x \sin x$

(1) Periodic Function

sin|cos

= Integrate It

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

I

$$2I = -e^x \cos x + e^x \sin x$$

$$I = \frac{-e^x \cos x + e^x \sin x}{2}$$

$$I = \int e^{2x} \sin 3x \, dx$$

$$I = \frac{9}{13}\left[-\frac{e^{2x}}{3}\cos 3x + \frac{2}{9} e^{2x} \sin 3x \right]$$

D	I
e^{2x}	$\sin 3x$
$2e^{2x}$	$-\frac{\cos 3x}{3}$
$4e^{2x}$	$-\frac{\sin 3x}{9}$

$+$, $-$

$$= -\int \frac{4}{9} e^{2x} \sin 3x$$

multiply + Integral sign

$$I = -\frac{e^{2x}\cos 3x}{3} + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \underbrace{\int e^{2x} \sin 3x \, dx}_{I}$$

$$I + \frac{4}{9} I = \frac{-e^{2x}}{3} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

(3) $I = \int \log x \, dx = \int \underline{t \cdot e^t \, dt}$

Trick

Rule: $\begin{cases} \log x = t \\ x = e^t \end{cases}$

$\underline{dx = e^t \, dt}$

$$= \begin{array}{ccc} t & + & e^t \\ 1 & - & e^t \\ 0 & & e^t \end{array}$$

$= te^t - e^t$

$= e^t(t-1)$

$= e^{\log x}(\log x - 1)$

$= \underline{x(\log x - 1)}$

Thank You!

GW Soldiers