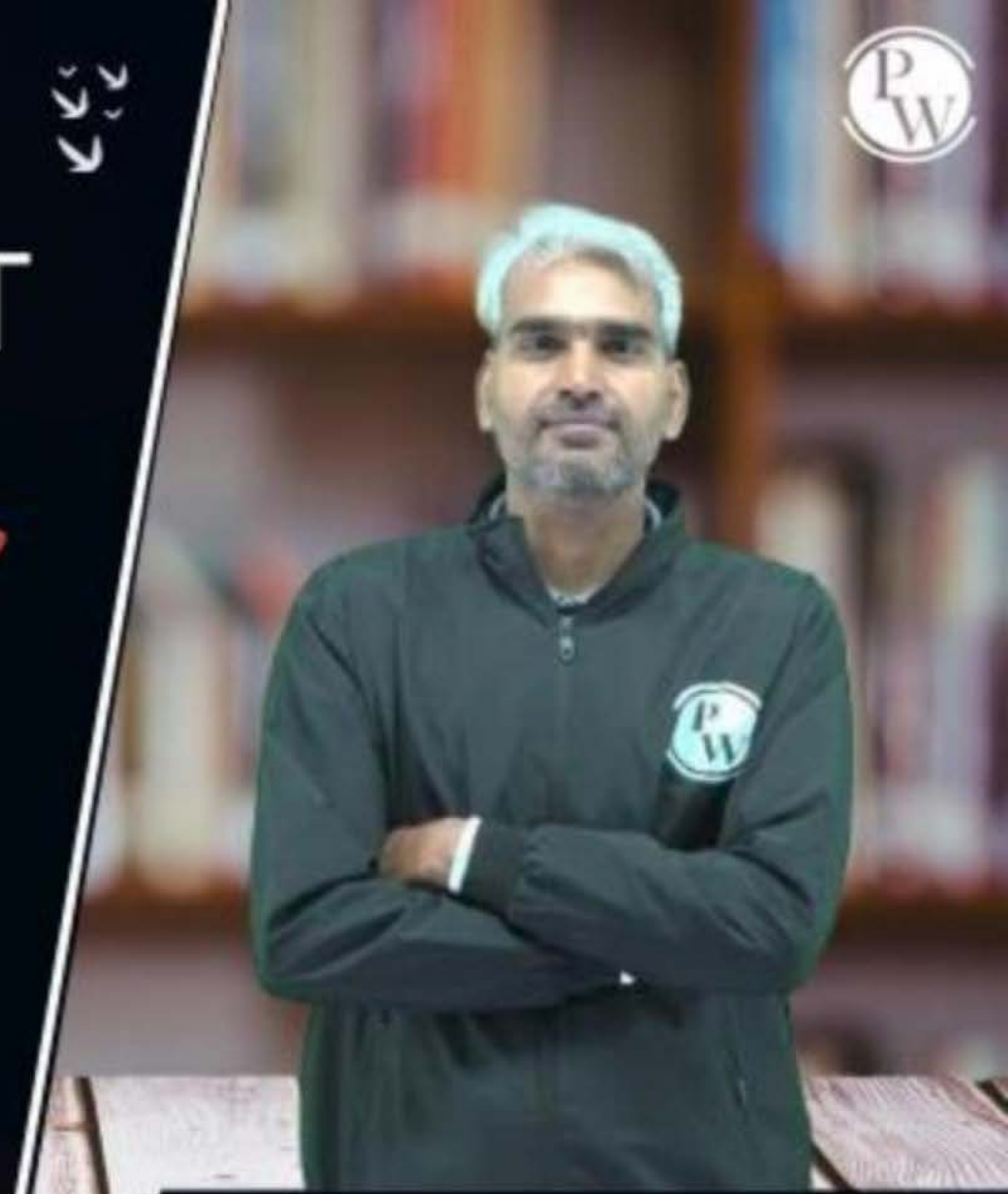
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 12



By- Vishal Sir

Recap of Previous Lecture

Ref + Anti-symm + Trans. (A, R)

Topic Partial Order Relation / Partially Ordered Set

Comparability

Topic Total Order Relation / Totally Ordered Set

Topic Least Upper Bound / Greatest Lower Bound

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Topics to be Covered

Topic — Least Upper Bound / Greatest Lower Bound

Topic — Minimal / Maximal elements in a POSET

Topic — Minimum(Least) element of a POSET

Topic — Maximum(Greatest) element of a POSET

Topic — Lattice

Slide

Topic : Least Upper Bound

Let $(A,\leq)$ be a POSET, for any two elements $a,b\in A$ if there exists an element $c\in A$ such that,

$a\leq c$ and $b\leq c$, then c is called upper bound of $a\ and\ b$,

And if there exists no element $d\in A$ such that

$a\leq d$ and $b\leq d$ and $d\leq c$, then c is called least upper bound of $a\ and\ b$

Slide

Topic : Greatest Lower Bound

Let $(A, \leq)$ be a POSET, for any two elements $a, b \in A$ if there exists an element $c \in A$ such that,

$c \leq a$ and $c \leq b$, then c is called lower bound of a and b,

And if there exists no element $d \in A$ such that

$\boxed{d \leq a \text{ and } d \leq b}$ and $c \leq d$,

then c is called greatest lower bound of a and b

Slide

Topic : Least Upper Bound / Greatest Lower Bound

Note: ① Let A is any set of real numbers,

then $(A, \leq)$ is a POSET,

for any two elements $a, b \in A$

$$lub(a,b) = Max(a,b)$$

$$Glb(a,b) = Min(a,b)$$

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Note: ② Let A is set of all +ve integers

then $(A, \div)$ is a POSET.

for any two elements $a, b \in A$

$$\begin{cases} \text{lub}(a,b) = \text{LCM}(a,b) \\ \text{Glb}(a,b) = \text{GCD}(a,b) \end{cases}$$

$\therefore$ LCM & GCD exists for every pair of elements within the set

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Note (3) :- Let A is any finite set, and P(A) is the power set of set A.

then $(P(A), \subseteq)$ is a POSET.

For any two sets $X, Y \in P(A)$

$lub(X, Y) = X$ Union Y

$glb(X, Y) = X$ intersection Y

$(X \cap Y) \subseteq X$

$(X \cap Y) \subseteq Y$

Slide

Topic : Least Upper Bound / Greatest Lower Bound

Q: Let $A = \{1, 2, 3, 4, 9\}$ and $(A, \div)$ is a POSET.

$\text{lub}(2,3) = ?$

$2 \div x$
$3 \div x$

→ no element 'x' exists in the set A, such that both 2 & 3 divides 'x'

$\therefore \text{lub}(2,3) =$ does not exists

$\text{glb}(2,3) = \underline{\underline{1}}$

Q: Let $A = \{1, 2, 3, 9, 24\}$ and $(A, \div)$ is a POSET.

$\text{lub}(2,3) = \underline{24}$

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Q: Let $S = \{ \underset{A}{\{1\}}, \underset{B}{\{2,3\}}, \underset{C}{\{2,3,4\}}, \underset{D}{\{1,2,3,4,5\}} \}$,

then $(S, \subseteq)$ is a POSET

* $lub\{A, B\} = \{1,2,3,4,5\} = D$

* $glb\{A, B\} =$ does not exists in S

Slide

Q: let $A = \{1, 2, 3\}$

✓ $P(A) = \{\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

then $(P(A), \subseteq)$ is a POSET.

- For any two sets in $P(A)$, their union and intersection will always exists in $P(A)$ itself

Topic : NOTE

In a POSET lub and/or glb need not exist for every pair of elements, but if exists for any pair of elements then it is unique.

May or May not

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Topic : Minimal elements in a POSET

(A, R)

An element x of the POSET is called minimal element of POSET, only if there is no other element on the lower side of x, apart from x itself

Let $(A, \leq)$ be a POSET

An element $x \in A$ is called a minimal element of POSET$(A, \leq)$ if there exists no element $y \in A$ other than x itself, such that $y \leq x$.

i.e. $y R x$

or i.e. $(y, x) \in R$

lower side

upper side

Slide

Topic : Minimal elements in a POSET

eg: Let $A = \{2, 3, 6, 12\}$ and $(A, \div)$ is the POSET.

$\therefore$ 2 & 3 are the minimal elements of the POSET.

eg: let $A = \{1, 2, 3, 6, 12\}$ and $(A, \div)$ is the POSET.

'1' is the only minimal element of the POSET.

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Topic : Maximal elements in a POSET

An element x of the POSET is called maximal element of the POSET, only if there is no other element on the upper side of x, apart from x itself.

(A, R)

Let $(A, \leq)$ be a POSET

An element $x \in A$ is called a maximal element of POSET$(A, \leq)$ if there exists no element $y \in A$ other than x itself, such that $x \leq y$.

(i.e. $x R y$ or $(x, y) \in R$)

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Topic : Maximal elements in a POSET

eg: let $A = \{1, 2, 3, 4, 6\}$ and $(A, \div)$ is the POSET

4 & 6 are maximal elements of the above POSET.

eg: let $A = \{2, 3, 4, 6, 12\}$ and $(A, \div)$ is the POSET.

12 is the only maximal elements of the above POSET.

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Topic : Minimum element of a POSET

Least element of POSET.

An element x of the POSET is called minimum element of the POSET only if 'x' is on the lower side of all other elements

(A, R)

Let $(A,\leq)$ be a POSET

An element $m \in A$ is called a minimum element of POSET$(A,\leq)$

(A,R)

if and only if

$m \leq a, \forall a \in A$

lower upper

i.e. $m R a, \forall a \in A$

or $(m,a) \in R, \forall a \in A$

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Topic : Minimum element of a POSET

eg. $A = \{2, 3, 6, 12\}$ and $(A, \div)$ is the POSET

No minimum element exists in the POSET.

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eg: $A = \{1, 2, 3, 6, 12\}$ and $(A, \div)$ is the POSET.

"1" divides all the elements of the POSET.

$\therefore$ 1 is the minimum element of the POSET.

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Topic : Maximum element of a POSET

Greatest element of POSET

An element x of the POSET is called maximum element of the POSET only if it is on the upper side of all other elements

(A,R)

Let $(A, \leq)$ be a POSET

An element $n \in A$ is called a maximum element of POSET$(A, \leq)$ (A,R)

if and only if

$a \leq n, \forall a \in A$

lower upper

$a \, R \, n, \forall a \in A$

or $(a, n) \in R, \forall a \in A$

Slide

Topic : Maximum element of a POSET

eg: let $A = \{1,2,3,4,6\}$ and $(A, \div)$ is the POSET.

No maximum element exists in the POSET.

eg: let $A = \{1,2,3,6,12\}$ and $(A, \div)$ is the POSET.

"12" is the maximum element of the POSET.

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Topic : NOTE

there exists at least one minimal element

① In a POSET, minimal elements need not be unique

② If there exists two or more minimal elements in a POSET, the no "minimum" element will exist in that POSET.

③ If there exists a unique minimal element in the POSET, then same minimal element will also be the minimum element of the POSET.

④ There can be at most one minimum element in a POSET.

Topic : NOTE

there exists at least one maximal element

① In a POSET, maximal elements need not be unique

② If there exists two or more maximal elements, then maximum element will not exist in that POSET.

③ If there exists a unique maximal element in a POSET, then same element will also be the maximum element of the POSET.

④ There can be at most one maximum element in a POSET.

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Q: let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

How many symmetric Relⁿ are possible with exactly 4 order Pairs.

Case ①

$\{(a,a)(b,b)(c,c)(d,d)\}$

$\underbrace{}_{{}^8C_4}$

Case ②

$\{(x,y),(y,x),(a,a),(b,b)\}$

${}^8C_2 * {}^6C_2$

Case ③

$\{(x,y)(y,x),(p,q),(q,p)\}$

Case ④

$\{(x,y),(y,x),(x,z),(z,x)\}$

$\{(x,y)(y,x),(y,z),(z,y)\}$

$\{(x,z)(z,x),(y,z),(z,y)\}$

2 mins Summary

Topic Least Upper Bound / Greatest Lower Bound

Topic Minimal / Maximal elements in a POSET

least element

Topic Minimum / Maximum element of a POSET

Greatest element

Topic Lattice

Slide

THANK - YOU

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