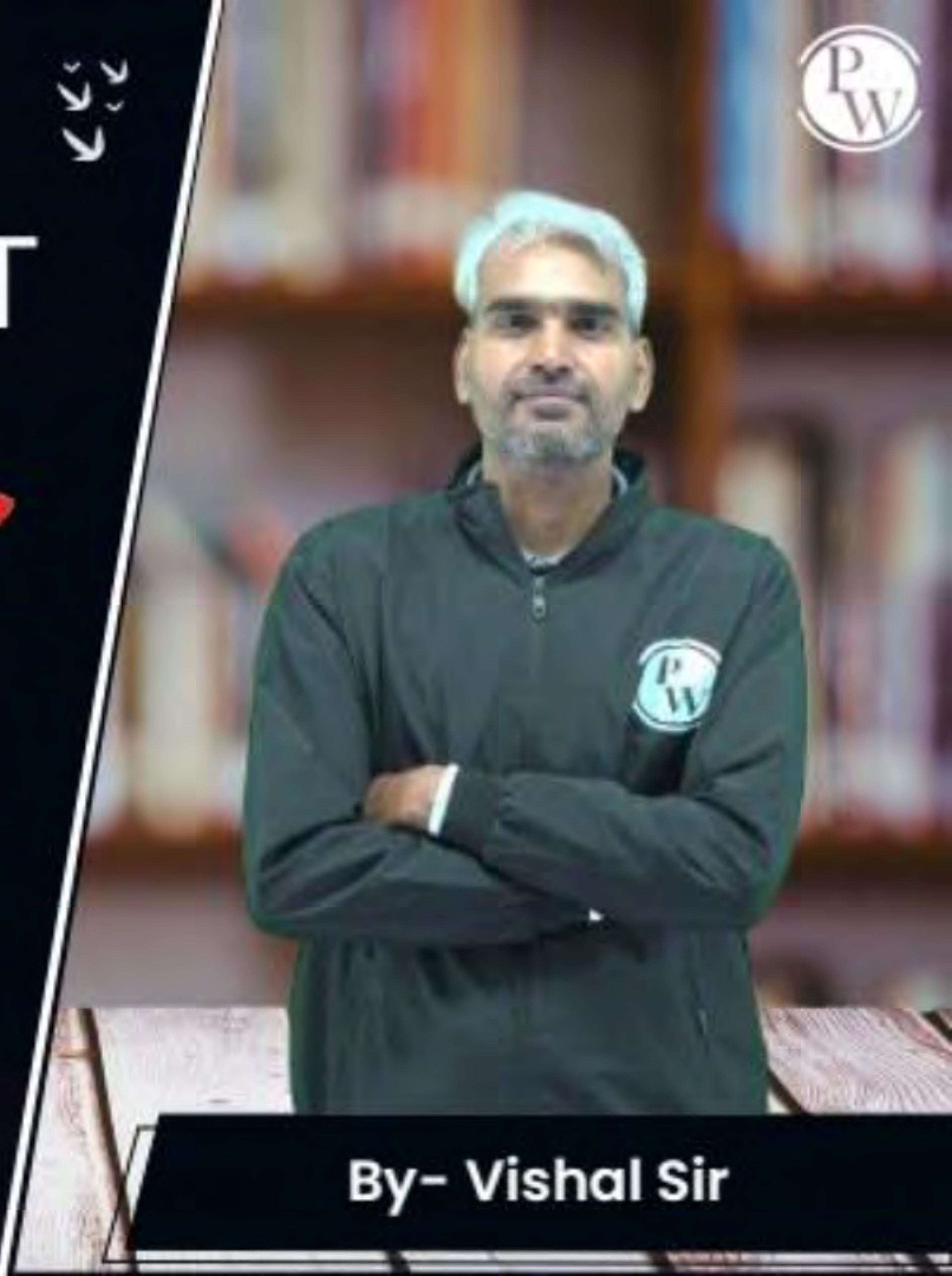
# Computer Science & IT

## Discrete Mathematics

**Set Theory & Algebra**

**Lecture No. 06**



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# Recap of Previous Lecture

- **Topic** Questions based on Multi-set
- **Topic** Cartesian Product
- **Topic** Relation
- **Topic** Types of Relation

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# Topics to be Covered

**Topic** Types of Relations

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Topic : Diagonal Relation / Identity Relation

Diagonal relation on set A is denoted by $\Delta_A$.

And it is defined as,

$$\Delta_A = \{(a,a) \mid \forall a \in A\}$$

eg: $\boxed{A = \{1,2,3\}}$

$$\Delta_A = \{(1,1),(2,2),(3,3)\}$$

$$\cancel{R_1 = \{(1,1),(2,2)\}}$$

not a diagonal Rel$^n$ on set A bcoz $(3,3) \notin R_1$

$$R_2 = \{(1,1),(2,2),(3,3),(1,2)\}$$

$(1,2) \in R_2$

$\therefore R_2$ is not diagonal Rel$^n$.

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# Topic : Reflexive Relation

A relation R on set A is said to be reflexive only if $x \, R \, x \;\; \forall x \in A$

i.e. $(x, x) \in R \;\; \forall x \in A$

$x$ relates to $x$
w.r.t. Rel^n R

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eg: Let $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ { Diagonal $Rel^n$ as well as Reflexive $Rel^n$ }

$R_2 = \{\underbrace{(1,1), (2,2), (3,3),}\ (1,2), (3,2)\}$ { It is a reflexive relation, but not a diagonal relation

$R_3 = \{(1,1), (2,2)\}$ { neither diagonal relation nor Reflexive Relation }

$R_4 = \underbrace{A \times A} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ → ∴ Reflexive Relation

Note:- (1) Every diagonal relation is reflexive relation, but reflexive Relation need not be diagonal relation

(2) Diagonal relation on set A is the smallest reflexive relation on set A

(3) $A \times A$ is the largest reflexive relation on set A.

Let $A = \{1, 2, 3, 4, \ldots\ldots\ldots n\}$

diagonal order pairs

$$A \times A = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots & (3,n) \\ \vdots & & & & \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{matrix} \right\}$$

Total number of elements in $A \times A = n^2$

Number of diagonal order pairs in $A \times A = n$

Number of non-diagonal order pairs in $A \times A = n^2 - n$

Q: How many reflexive relations are possible on a set A with 'n' elements ?

Reflexive Relation = Select all the diagonal order pairs and any number of non-diagonal order pairs may be selected

$n^2-n$

$\therefore$ Number of Reflexive Rel$^n$ $= \underbrace{{}^{n}C_{n}}_{\text{out of } n \text{ diagonal order pairs we need to choose all } n} * \left\{ {}^{n^2-n}C_0 + {}^{n^2-n}C_1 + {}^{n^2-n}C_2 + \cdots + {}^{n^2-n}C_{n^2-n} \right\}$

$= 1 * 2^{n^2-n}$

$= 2^{n^2-n}$ Ans

$\left\{ \text{we know } {}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \cdots {}^{n}C_n = 2^n \right\}$

eg: The relation "$\leq$" on any set of real number is reflexive

eg: let $A = \{0, 1, 2.5\}$

$$"\leq" = \{(0,0), (0,1), (0,2.5), \cancel{(1,0)}, (1,1), (1,2.5), \cancel{(2.5,0)}, \cancel{(2.5,1)}, (2.5,2.5)\}$$

$\therefore$ Reflexive

$(x, y) \in \leq$ if and only if $x \leq y$

## Topic : Irreflexive Relation

A relation R on set A is said to be irreflexive only if $x \not R x \ \forall x \in A$

i.e. $(x,x) \notin R \ \forall x \in A$

$x \not R x$ : $x$ is not related to $x$, w.r.t. relⁿ R

$x \not R x \equiv (x,x) \notin R$

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eg: Let $A = \{1, 2, 3\}$

$R_1 = \{\ \}$ [A relation on set A, such that it does not Contain any element in it is called an empty relation]

- not a diagonal $rel^n$ on set A
- Not a reflexive $rel^n$ on set A
- It is an irreflexive relation on set A

Smallest irreflexive $rel^n$ on set A is Empty relation

$R_2 = \{(1,2), (2,3), (3,1), (3,2)\}$ {no order pair of type $(x,x) \in R_2$ $\therefore$ It is irreflexive}

$R_3 = \{(1,2), (2,3), \cancel{(3,3)}\}$

because of Presence of $(3,3)$ $R_3$ is not an irreflexive Rel$^n$.

{Not reflexive as well}

**Note:** These may be relations on set A which are neither reflexive <u>nor</u> irreflexive

Q: let A be a set with n elements,
How many irreflexive relations are possible on set A.

Irreflexive Relation = None of the diagonal order pair should be present in the relation and Any number of non-diagonal order Pairs may be present.

Number of Irreflexive Rel$^n$ $= \underbrace{{}^{n}C_0}_{\text{Choose Zero out of 'n' diagonal order pairs}} * \left\{ {}^{n^2-n}C_0 + {}^{n^2-n}C_1 + \dots\dots + {}^{n^2-n}C_{n^2-n} \right\}$

$= 1 \times 2^{n^2-n}$

$= 2^{n^2-n}$

Q:: Let A is a set with 'n' elements,
How many relⁿ are possible on set A, which are neither Reflexive nor irreflexive

$$\text{Ans} = \underbrace{\left(2^{n^2}\right)}_{\substack{\text{total no. of} \\ \text{Rel}^n \text{ on} \\ \text{set } A}} - \underbrace{\left(2^{n^2-n} + 2^{n^2-n}\right)}_{\substack{\text{No. of relations on set } A \\ \text{which are either reflexive } \underline{\text{or}} \text{ irreflexive}}}$$

## Topic : Symmetric Relation

A relation R on set A is said to be symmetric only if,

if $\underbrace{(x, y)}_{} \in R$ then $(y, x) \in R$ $\forall x, y \in A$

i.e. if $x R y$ then $y R x$, $\forall x, y \in A$

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eg: let $A = \{1, 2, 3, 4\}$

$R_1 = \{ \quad \}$ Empty relation is the smallest symmetric relⁿ on set A

$R_2 = \{ \underset{y,\,x}{\overset{x,\,y}{(1, 1)}}, (2, 2) \}$ it is symmetric Relⁿ

Note: Presence or absence of any of the diagonal order pair will not matter for a symmetric relation

$R_3 = \{(1,1), \underbrace{(1,2), (2,1)}_{\text{both are Present}}, (3,2)\}$

$(3,2) \in R_3$

but $(2,3) \notin R_3$

$\therefore R_3$ is not Symmetric Relⁿ

$R_4 = \{(1,2), (2,1), (3,2), (2,3)\}$

$\therefore$ Symmetric Relⁿ.

Q: Let $A$ is a set with $n$ elements,
How many symmetric relations are possible
on set $A$.

Let $A = \{1, 2, 3, 4, \ldots\ldots\ldots n\}$

Pair of symmetric non-diagonal order pairs

$$A \times A = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots & (3,n) \\ \vdots & & & & \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{matrix} \right\}$$

Number of Pairs of symmetric non-diagonal Order Pairs $= \left(\dfrac{n^2 - n}{2}\right)$

$\# \text{Pairs} = \frac{n^2-n}{2}$

Symmetric Relation = Any number of diagonal order pairs may be present ✓ **and** Any number of **Pairs** of symmetric non-diagonal order pairs may be Present

Number of Symmetric Relations $= \left\{ {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots {}^nC_n \right\} * \left\{ {}^{\left(\frac{n^2-n}{2}\right)}C_0 + {}^{\left(\frac{n^2-n}{2}\right)}C_1 + \dots \dots {}^{\left(\frac{n^2-n}{2}\right)}C_{\left(\frac{n^2-n}{2}\right)} \right\}$

$= 2^n * 2^{\frac{n^2-n}{2}} = 2^{n + \frac{n^2-n}{2}}$

$= 2^{\frac{n^2+n}{2}}$

In a non-diagonal order pair $(x, y)$, we know $x$ & $y$ must be distinct

For two distinct elements $x$ & $y$,
Possible Choices of non-diagonal order pairs of $x$ & $y$ which are allowed w.r.t. Symmetric Relⁿ are

① both $(x,y)$ and $(y,x)$ are present in the Relation

and

② None of $(x,y)$ or $(y,x)$ is present in the Relation

For two distinct elements $x$ & $y$, there are four possible ways in which non-diagonal order pairs of $x$ & $y$ may be present in an arbitrary relation. they are

Case ① & Case ④ are allowed w.r.t Symmetric Relⁿ.

1. $(x,y)$ & $(y,x)$ both are present
2. $(x,y)$ is present and $(y,x)$ is not present
3. $(x,y)$ is not present & $(y,x)$ is present
4. $(x,y)$ is not present as well as $(y,x)$ is not present

Number of Symmetric Relation = Possible Choices w.r.t. diagonal order pairs and Possible Choices w.r.t. non-diagonal order pairs.

$= 2^{n} * 2^{{}^nC_2}$

two choices for each of n diagonal order pair

$^{n}C_{2}$ Pairs of distinct elements

{ Two distinct elements can be choosen in $^{n}C_{2}$ ways out of a set of n elements.

$\{1,2\}$

$\{(1,2)\ (2,1)\}$

$\{*, *\}$

each pairs has two Choices w.r.t Symmetric Relⁿ

$= 2^{n} * 2^{\frac{n(n-1)}{2}}$

$= 2^{(n^2+n)/2}$

# Topic : Anti-Symmetric Relation

A relation R on set A is said to be anti-symmetric only if,

$x \neq y$

if $(x,y) \in R$ and $(y,x) \in R$, then $x = y \ \forall x, y \in A$

i.e. if $x R y$ and $y R x$, then $x = y$, $\forall x, y \in A$

$x = y$ means diagonal order pair

i.e. diagonal order pairs may be present in Anti-Symm Relⁿ.

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eg Let $A = \{1, 2, 3\}$

- $R_1 = \{\ \}$ [Empty relation is smallest anti-symmetric Rel$^n$]

- $R_2 = \{(1,2), (3,1), (2,3)\}$ $R_2$ is anti-symmetric

$(1,2) \to (2,1) \notin R_2$ ✓ $\quad (3,1) \to (1,3) \notin R_2$ ✓ $\quad (2,3) \to (3,2) \notin R_2$ ✓

$R_3 = \{(1,3), (3,2), (1,1)\}$ $R_3$ is anti-symmetric

$(1,3) \to (3,1) \notin R_3$ ✓ $\quad (3,2) \to (2,3) \notin R_3$ ✓

$(1,1) \to$ No problem because of diagonal order pair

$\rightarrow R_4 = \{(1,2), (3,1), (2,3), (3,2)\}$

$(2,1) \notin R_4$ ✓ $(1,3) \notin R_4$ ✓

w.r.t $(2,3) \in R_4$

we have $(3,2) \in R_4$

<u>but</u> $3 \neq 2$

$\therefore R_4$ is not anti-symmetric Relⁿ.

For two distinct elements $x$ & $y$, there are four possible ways in which non-diagonal order pairs of $x$ & $y$ may be present in an arbitrary relation, they are

1. $(x,y)$ & $(y,x)$ both are present
2. $(x,y)$ is present and $(y,x)$ is not present
3. $(x,y)$ is not present & $(y,x)$ is present
4. $(x,y)$ is not present as well as $(y,x)$ is not present

Case 2, Case 3 & Case 4 are allowed in Anti-symmetric Reln

Q: Let $A$ is a set with $n$ elements,

How many anti-symmetric $rel^n$ are possible on set $A$.

# Antisymmetric Reln = Possible Choices w.r.t. diagonal Order pairs and Possible Choices w.r.t. non-diagonal Order pairs

i.e w.r.t distinct elements

$$= 2^n * 3^{\left({}^nC_2\right)}$$

${}^nC_2$ Pairs of distinct elements each with 3 choices

$$= 2^n * 3^{\frac{n^2-n}{2}}$$

# 2 mins Summary

**Topic** Types of Relations

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# THANK - YOU

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