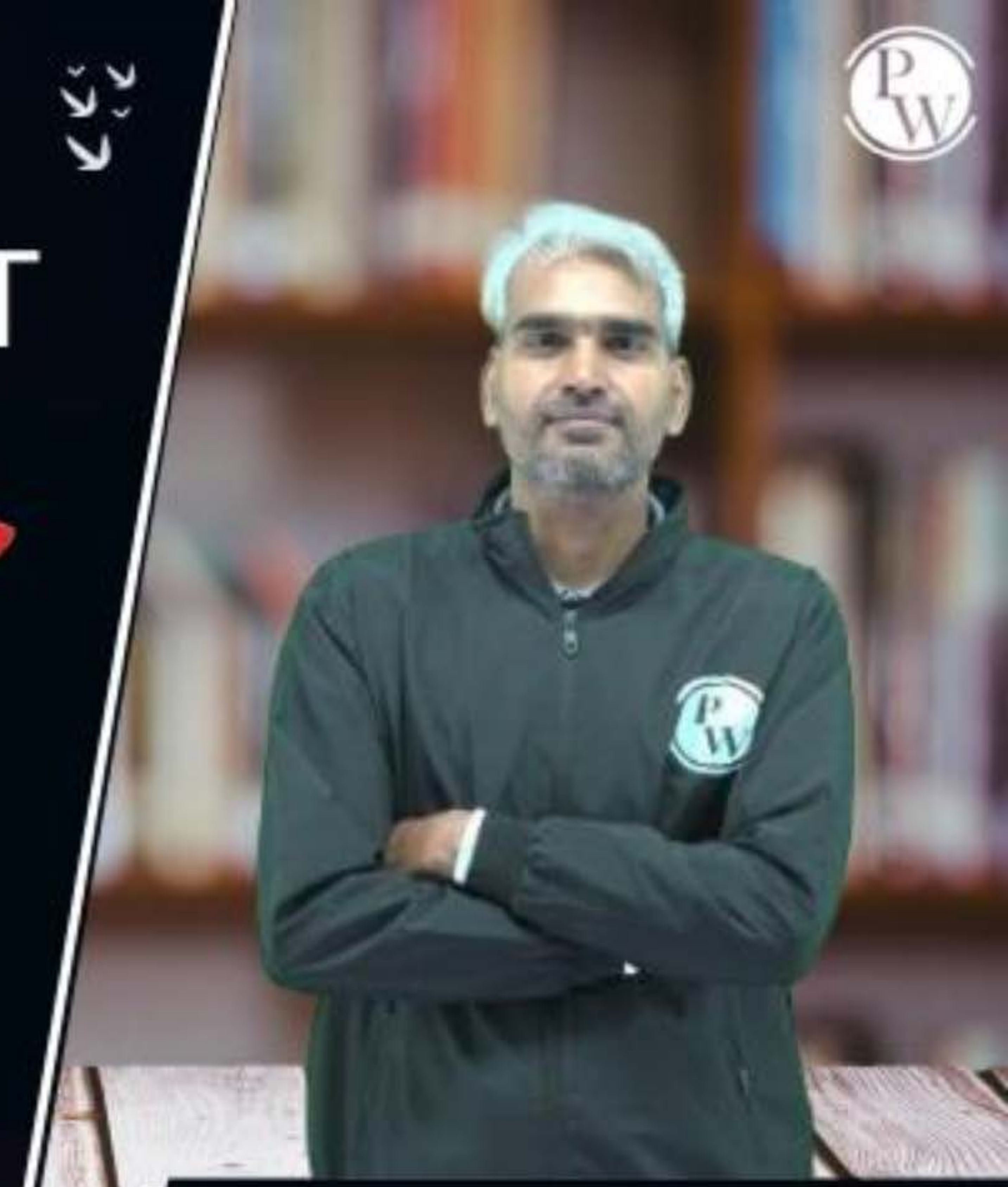
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 13



By- Vishal Sir

Recap of Previous Lecture

Topic Least Upper Bound / Greatest Lower Bound

Topic Minimal / Maximal elements in a POSET

Topic Minimum(Least) element of a POSET

Topic Maximum(Greatest) element of a POSET

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Topics to be Covered

Topic Lattice

Topic Hasse Diagram

Topic Types of Lattices

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Topic : Join Semi Lattice

Join (Least Upper bound)

A POSET in which least upper bound exists for every pair of elements is called a "join semi lattice".

eg: let $A = \{2, 3, 4, 6, 12\}$ and $(A, \div)$ is a POSET.

[least upper bound exists for every pair of elements.
$\therefore$ Given POSET is Join Semi lattice]

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Topic : Meet Semi Lattice

Meet = greatest lower bound

A POSET in which greatest lower bound exists for every Pair of elements is called a Meet-semi lattice.

eg: let, $A = \{1, 2, 3, 4\}$ and $(A, \div)$ is a POSET

Greatest lower bound exists for every pair of elements

$\therefore$ Given POSET is a Meet semi lattice.

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A POSET in which both least upper bound as well as greatest lower bound exists for every pair of elements is called a lattice

(Or) A POSET which is Join semi lattice as well as a Meet semi lattice is called a lattice.

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eg: ① If A is any set of real numbers, then POSET $(A, \leq)$ is a lattice

② If A is a set of all +ve integers, then POSET $(A, \div)$ is a lattice

③ If A is any finite set, and $P(A)$ is Power set of A. then POSET $(P(A), \subseteq)$ is a lattice.

(4) If A is any set of +ve integers, then POSET $(A, \div)$ may or may not be a lattice

e.g. let $A = \{2, 3, 4, 6\}$, then POSET $(A, \div)$ is not a lattice, as $lub(3, 4)$ does not exist

e.g. let $A = \{1, 2, 3, 4, 6, 12\}$, then POSET $(A, \div)$ is a lattice.

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<u>Note:-</u> Let 'n' is any +ve integer, then

"D_n" denotes the set of all +ve divisors of "n",

and for any positive integer 'n'

POSET $(D_n, \div)$ is always a lattice

eg: $n = 4$, then $D_4 = \{1, 2, 4\}$ and $(\{1, 2, 4\}, \div)$ is a lattice

eg: $n = 12$, then $D_{12} = \{1, 2, 3, 4, 6, 12\}$, and $(D_{12}, \div)$ is a lattice

Topic : Lattice

A lattice is an algebraic structure denoted by $[L, \vee, \wedge]$ where

L is the underlying set, and

$\vee$ and $\wedge$ are the binary operation representing the lub and glb respectively

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Topic : Lattice

In a lattice $[L, \vee, \wedge]$ following properties holds true

① Commutative Property

$$\left.\begin{aligned} a \vee b &= b \vee a \\ a \wedge b &= b \wedge a \end{aligned}\right\} \forall a, b \in L$$

② Associative Property

$$\left.\begin{aligned} (a \vee b) \vee c &= a \vee (b \vee c) \\ (a \wedge b) \wedge c &= a \wedge (b \wedge c) \end{aligned}\right\} \forall a, b, c \in L$$

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③ Idempotent Law :-

$$\left.\begin{aligned} a \vee a &= a \\ a \wedge a &= a \end{aligned}\right\} \forall a \in L$$

④ Absorption Law :-

$$\left.\begin{aligned} a \vee (a \wedge b) &= a \\ a \wedge (a \vee b) &= a \end{aligned}\right\} \forall a, b \in L$$

Note:- Distributive property need not hold true in all the lattices,

Any lattice in which distributive Property also holds true for every triple of elements is called a distributive Lattice

$\Rightarrow$ i.e., Every lattice need not be a distributive lattice

Topic : Hasse Diagram / POSET Diagram

In a Hasse diagram of a POSET,

1. There is a vertex corresponding to every element of set.
2. There is an edge from vertex a to vertex b only if a is related to b and there is no element x in the set such that a is related to x, and x is related to b.(Transitivity is implied in the Hasse diagram not represented explicitly)
3. No self-loop on the vertices (i.e. reflexivity is implied in the Hasse diagram not represented explicitly).
4. It is not directed but it uses implied upward orientation. → i.e. if $a R b$, then $(a,b) \in R$

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Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSETs

1) $(\{-1, 0, 2.5, 4, 6\}, \leq)$
2) $(D_6, \div)$
3) $(D_{12}, \div)$
4) $(\{2, 3, 4, 6\}, \div)$
5) $(\{2, 3, 6, 12\}, \div)$
6) $(\{1, 2, 3, 4, 6, 9\}, \div)$

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Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{-1, 0, 2.5, 4, 6\}, \leq)$

↓

it is a POSET as well as a TOSET

Hasse diagram for a totally ordered set will always be a linear chain.

∴ Totally ordered set are also known as linearly ordered set

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Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{-1, 0, 2.5, 4, 6\}, \leq)$

↓

it is a POSET as well as a TOSET

maximal element as well as maximum element

join lub(0,4)=4

Meet glb(4,0)=0

Minimal element as well as Minimum element.

Hasse diagram for a totally ordered set will always be a linear chain.

∴ Totally ordered set are also known as linearly ordered set

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Draw the hasse diagram for the following POSET

$(D6, \div)$

$D_6 = \{1, 2, 3, 6\}$

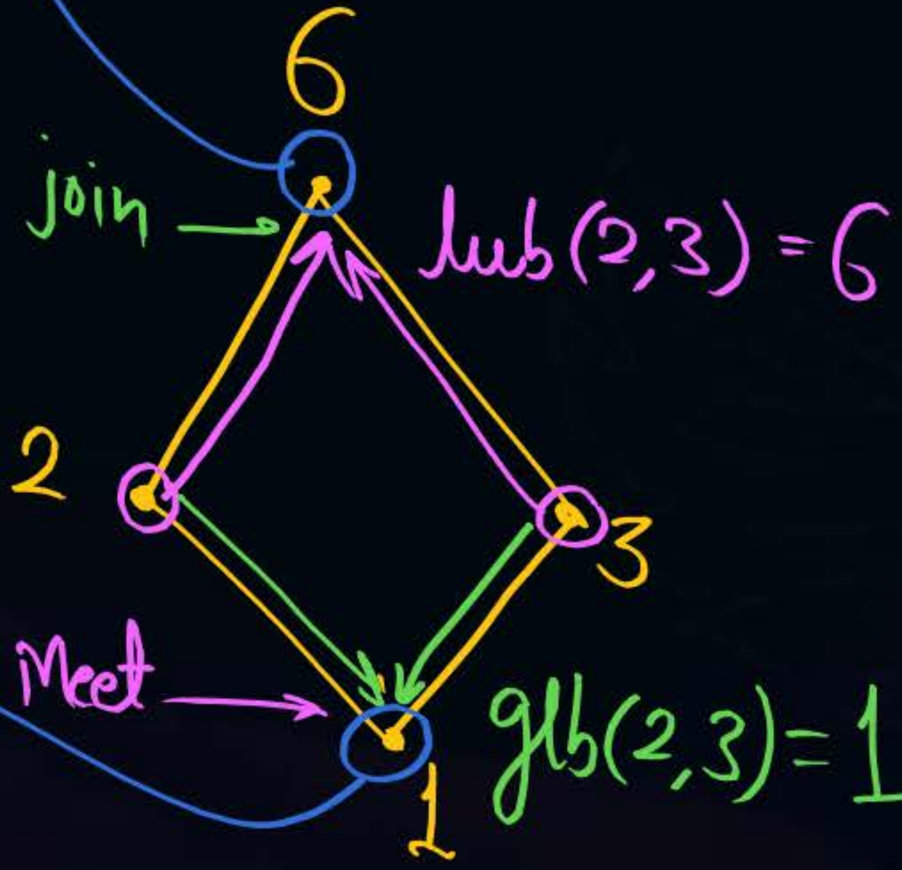
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(D6, \div)$

$D_6 = \{1, 2, 3, 6\}$

Maximal as well as Maximum element



Minimal as well as Minimum element.

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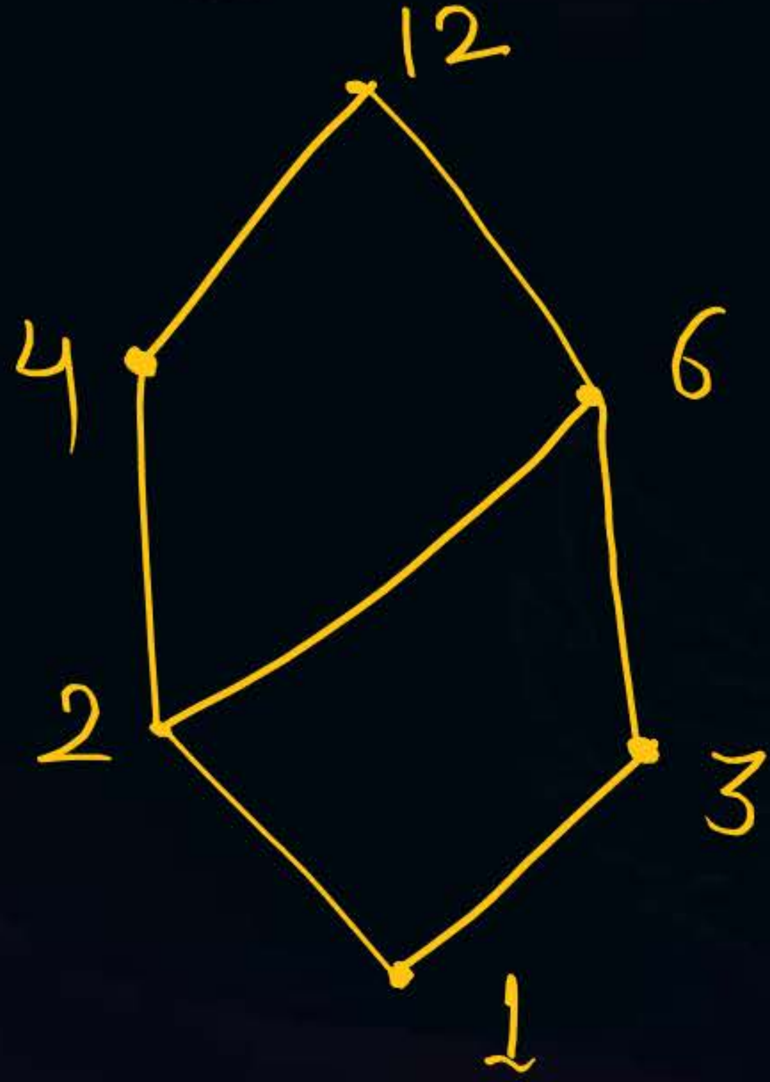
Draw the hasse diagram for the following POSET

$(D12, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$



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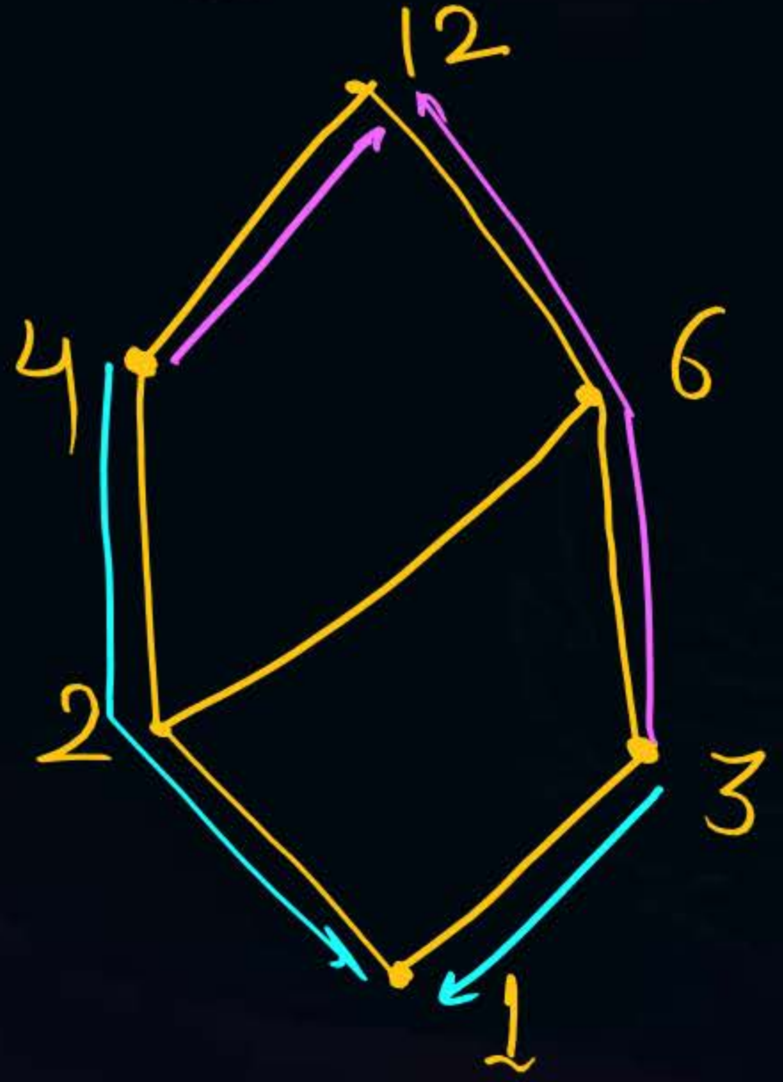
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(D12, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$



$lub(4,3) = 12$

$glb(4,3) = 1$

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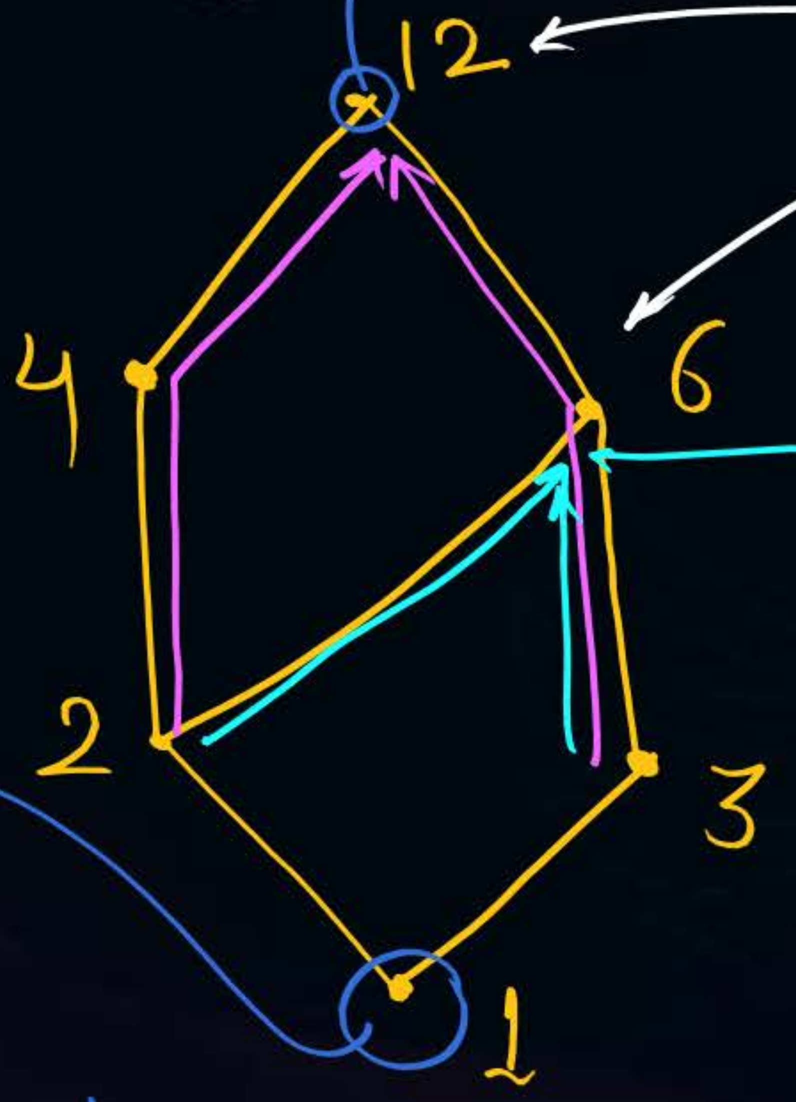
Draw the hasse diagram for the following POSET

$(D12, \div)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

Maximal as well as maximum element

12 & 6 are upper bound of 2 & 3



lub(2,3) = 6

Minimal as well as Minimum element

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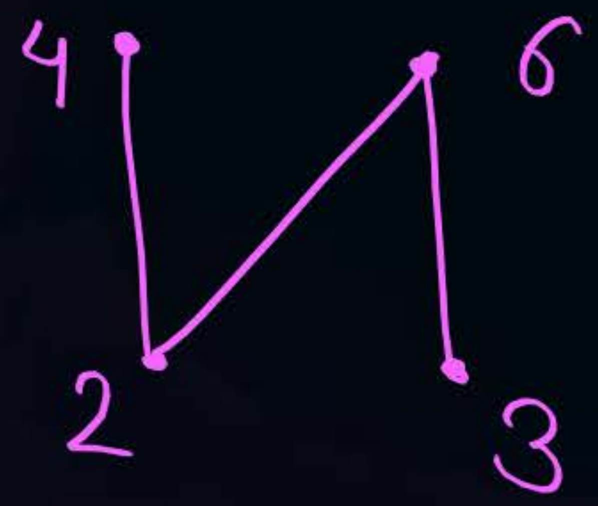
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{2,3,4,6\},\div)$



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Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

✓ $(\{2,3,4,6\},\div)$

If there exists two or more maximal elements, then that POSET can not be a Join semi lattice.
* Because lub does not exists for any pair of maximal elements

lub(4,6) = does not exist

Two maximal elements
∴ No Maximum element

If there exists two or more minimal elements, then that POSET can not be a Meet semi-lattice, As glb does not exists for any pair of minimal elements.

Two minimal elements
∴ No minimum element

Neither a Join-semi lattice Nor a Meet semi lattice

glb(2,3) = does not exist

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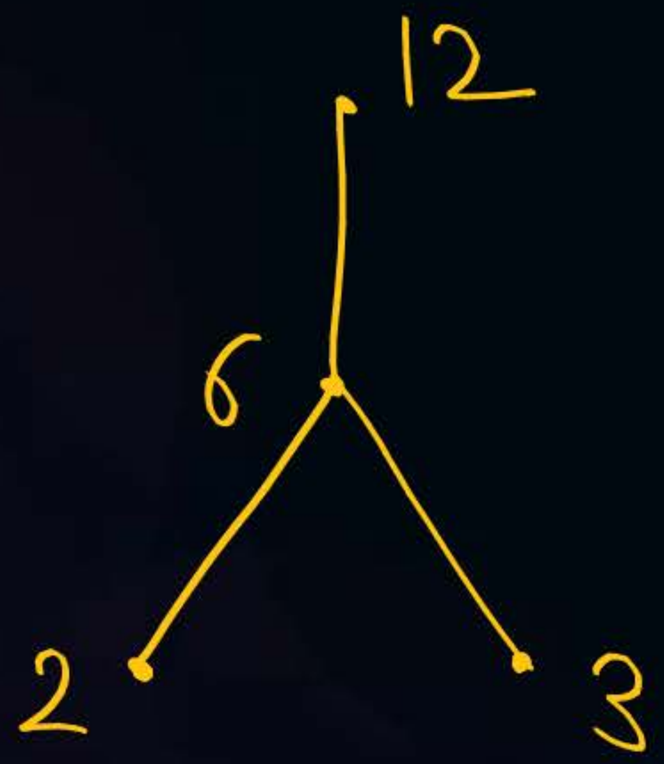
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{2,3,6,12\},\div)$



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Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{2,3,6,12\}, \div)$

lub exists for every pair of elements, ∴ Join Semi-lattice

There are two minimal elements, ∴ Not a Meet-semi lattice

glb(2,3) = does not exist

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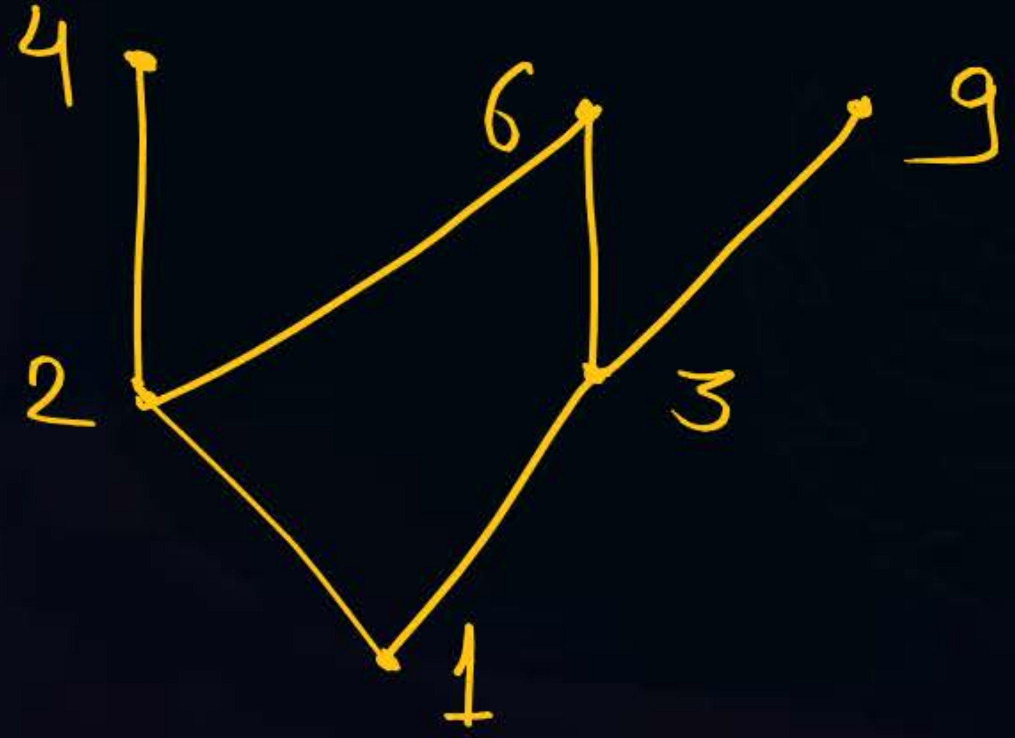
Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{1,2,3,4,6,9\},\div)$



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Topic : Hasse Diagram / POSET Diagram

Draw the hasse diagram for the following POSET

$(\{1,2,3,4,6,9\}, \div)$

lub(4,6) = does not exist

lub(4,9) = does not exist

More than one maximal elements,
∴ Not a Join-semi lattice

glb exists for every pair of elements,
∴ It is a Meet-semi lattice

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2 mins Summary

Topic Lattice

Topic Hasse Diagram

Topic Different Types of Lattices

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THANK - YOU

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