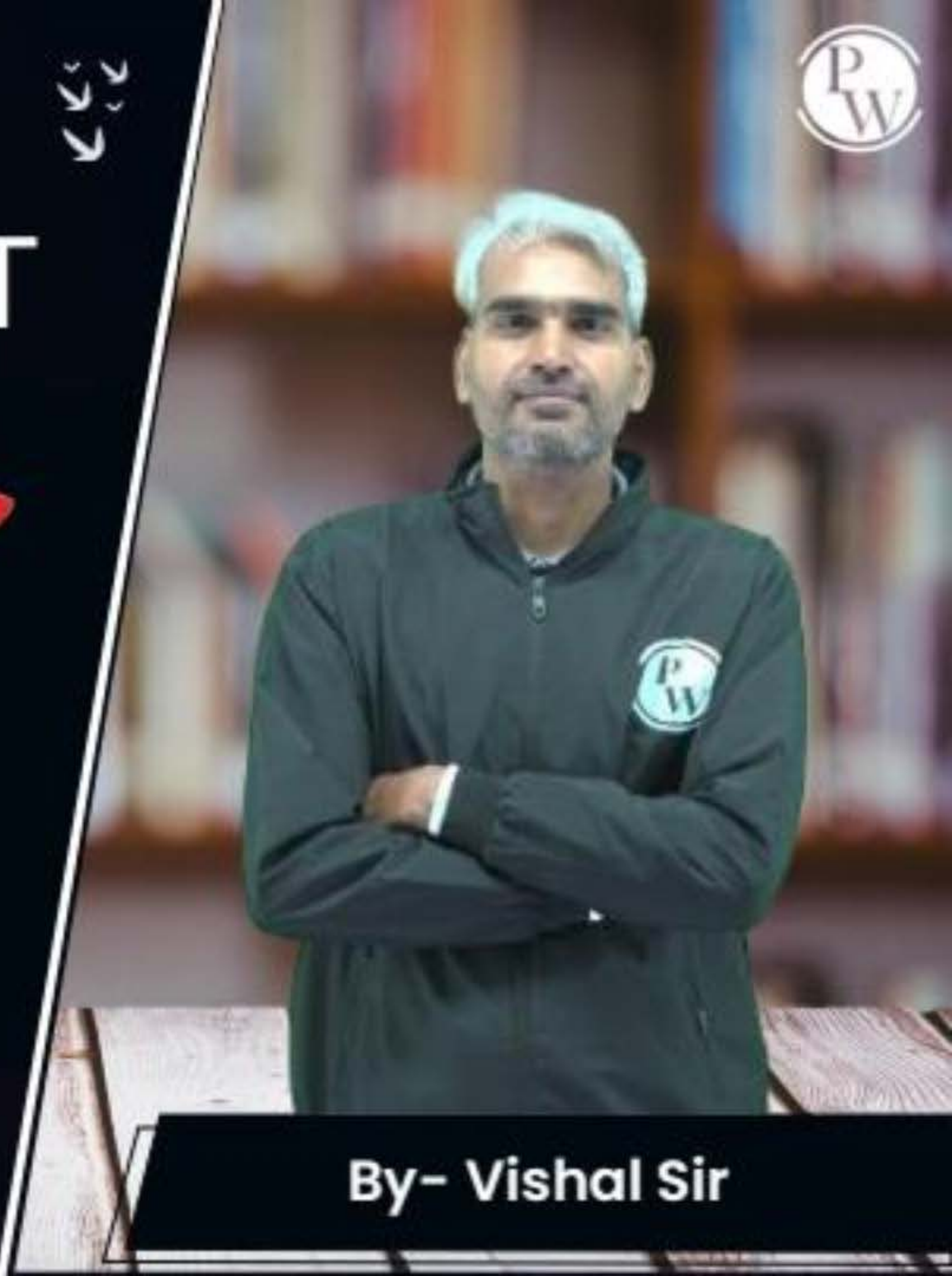
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 15



By- Vishal Sir

Recap of Previous Lecture

Topic Types of Lattices

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Topics to be Covered

Topic Types of Lattices

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Topic : Bounded Lattice

* Let $[L, \vee, \wedge]$ is a lattice.

→ If there exists an element $I \in L$
such that $a \vee I = I, \forall a \in L$,
then I is called Upper bound of lattice L
(or) universal upper bound.

→ If there exists an element $O \in L$
such that $a \wedge O = O, \forall a \in L$,
then O is called lower bound of lattice L
(or) universal lower bound

* Note:- If both I and O exists in lattice L, then L is called a bounded lattice

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eg: In lattice w.r.t. POSET $(D_{12}, \div)$

'12' is the universal upper bound

& '1' is the universal lower bound,

$\therefore$ Lattice is the bounded lattice

eg: let A is any finite set, and $(P(A), \subseteq)$ is the POSET, and lattice w.r.t. POSET $(P(A), \subseteq)$ is a bounded lattice

Where, $A \in P(A)$ is the universal upper bound

$\emptyset \in P(A)$ is the universal lower bound

Note:- ① In a lattice, greatest (Maximum) element of the POSET is the universal upper bound.

and ② In a lattice, least (Minimum) element of the POSET is the universal lower bound

Note:- ③ A lattice need not be a bounded lattice.

set of all natural numbers

eg: $\underbrace{(\mathbb{N}, \leq)}$ ✓

it is a lattice

universal lower bound is "1".

But, Universal upper bound does not exist in this lattice

$\therefore$ Not a bounded lattice

Note ③ :-

Set of all integers

eg. $\underbrace{(\mathbb{Z}, \leq)}$

it is a lattice

But in this lattice neither universal lower bound nor universal upper bound exists.

$\therefore$ Not a bounded lattice

Note ④ :- If lattice is not a bounded lattice, then underlying set will be an infinite set, but Converse of the statement need not be true

eg: let $A = \{x \mid x \in R \text{ and } 0 \le x \le 1\}$

it is an infinite set.

where $(A, \le)$ is a bounded lattice

universal lower bound = 0

universal upper bound = 1

Topic : Complement of an element in a lattice

{Complements of an element may exists only in a bounded lattice}

+ Let $[L, \vee, \wedge]$ be a bounded lattice with I & 0 as the universal upper bound and universal lower bound respectively }

* For an element $a \in L$
  if there exists any element $b \in L$
  such that

$$a \vee b = I \text{ \{universal upper bound\}}$$
$$a \wedge b = 0 \text{ \{universal lower bound\}}$$

* Then a & b are called Complement of each other

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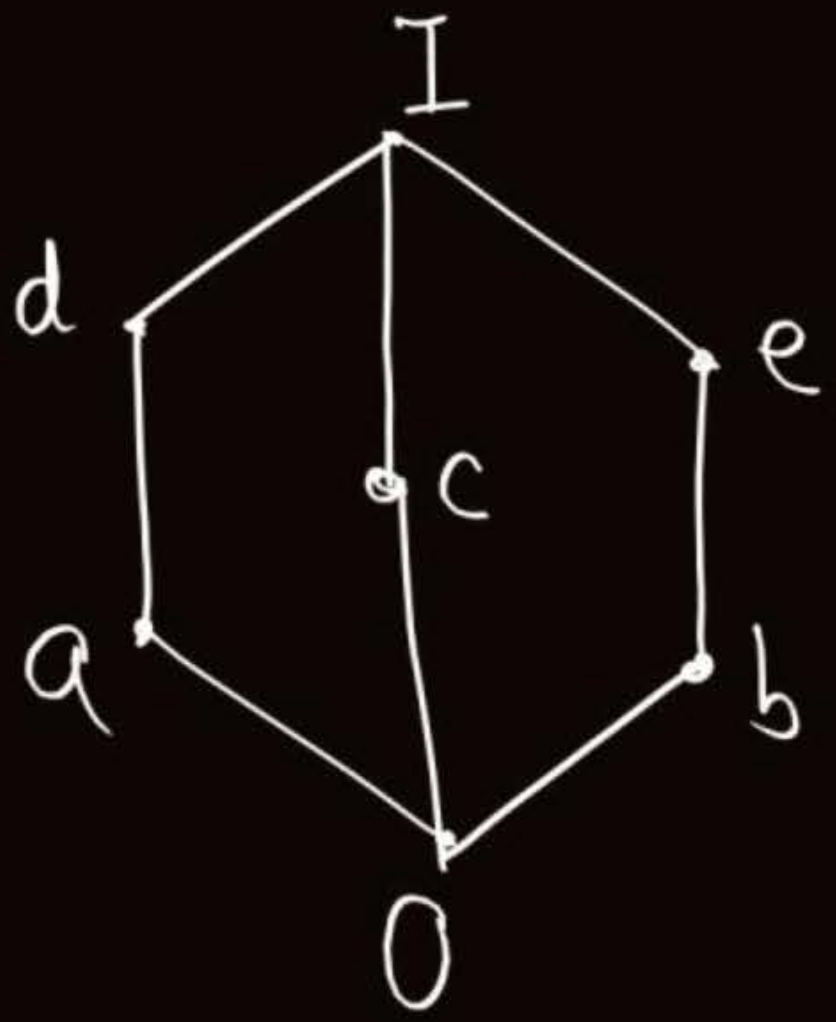
<u>Note</u>:- In a bounded lattice complement of an element need not exist, and if exists then it need not be unique

<u>Note</u>:- In any bounded lattice, Universal upper bound and universal lower bound are always Complement of each other, no other Complement exists for them.

eg: Find complement of every element of the lattice given below.



$\overline{I} = 0$

$\overline{0} = I$

$\overline{a} = b, c, e$

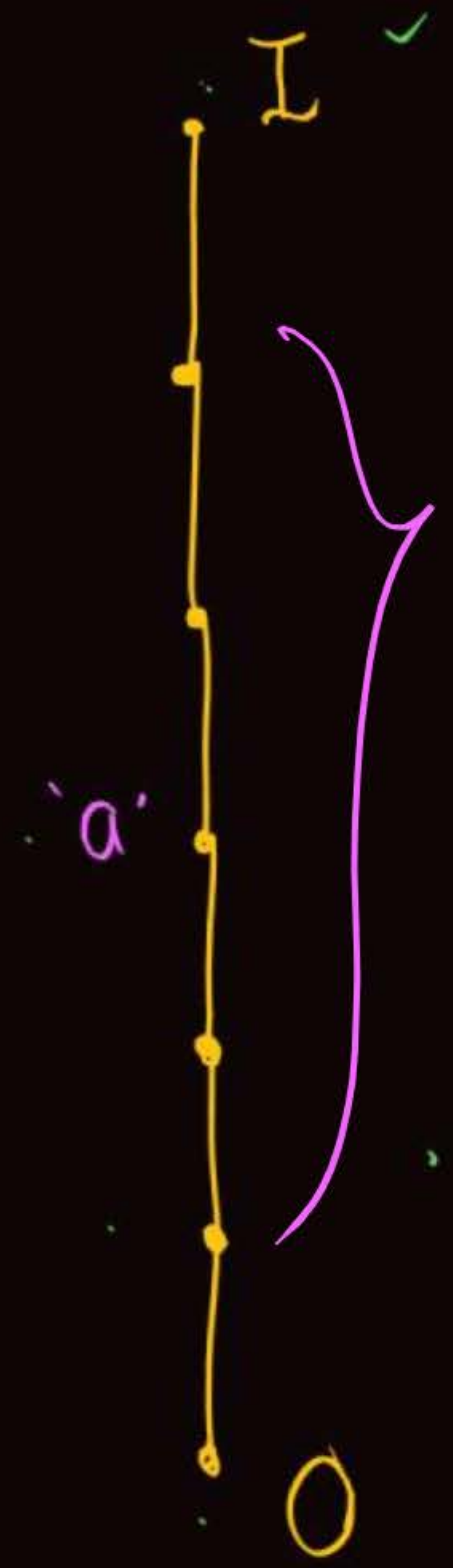
$\overline{b} = c, a, d$

$\overline{c} = a, b, d, e$

$\overline{d} = b, c, e$

$\overline{e} = a, d, c$

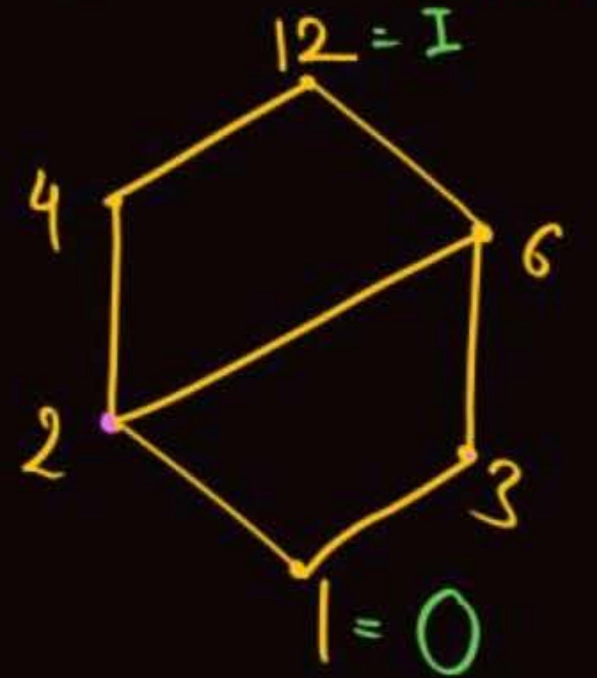
Note:-



- In a bounded totally ordered set, I & 0 are complement of each other, and complement does not exists for any other element of totally ordered set

$\boxed{\text{Two elements on the same line can not be Complement of each other, Except for I \& 0.}}$

Q: Find Complement of every element of lattice $(D_{12}, \div)$. (If exists)



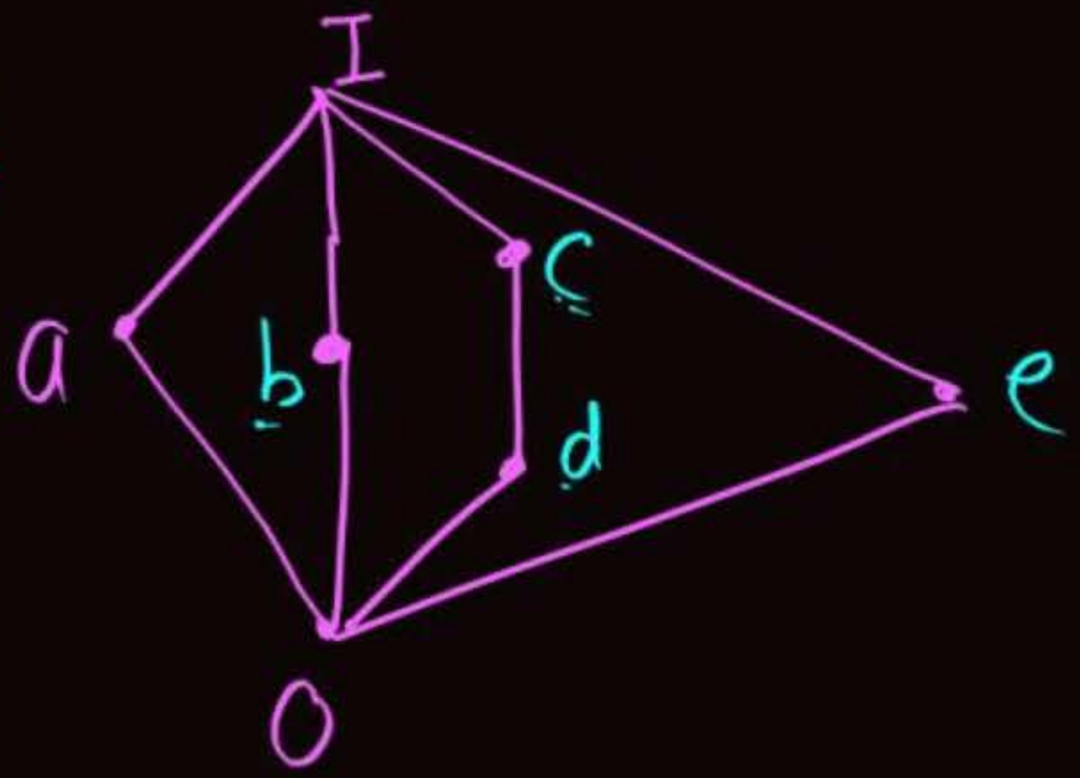
$\bar{1} = 12$
$\bar{12} = 1$
} 1 & 0 are complement of each other

$\bar{2}$ = $\bar{2}$ can not exist on line 1-2-4-12
$\bar{2}$ can not exist on line 1-2-6-12
Check w.r.t '3'
$2 \wedge 3 = 1 = 0$ {it is okay}
but $2 \vee 3 = 6 \neq I$ {$\therefore$ 2 & 3 can not be Complement of each other}
$\bar{2}$ = does not exists

$\bar{6} = ? \Rightarrow$ Can not exist on
1-3-6-12 or
1-2-6-12
Check w.r.t. 4.
$6 \wedge 4 = 2 \neq 0 \therefore$ 6 & 4 Can not be Complement of each other
$\bar{6}$ = does not exist

$\bar{3} = 4 \Rightarrow$
$\bar{4} = 3 \Rightarrow$
$3 \vee 4 = 12 = 1$
& $3 \wedge 4 = 1 = 0$
} $\therefore \bar{3} = 4$ & $\bar{4} = 3$

Q:



How many complements exist for element 'a'

$\bar{a} = b, c, d, e$

$\therefore$ No. of complements of $a = 4$

Topic : Complemented lattice

A lattice in which complement exists for every element is called Complemented lattice

→ In a complemented lattice every element has $\underbrace{\text{at least one}}_{\text{i.e. 1 or more}}$ complement

In a complemented lattice, complement of an element must exist and need not be unique

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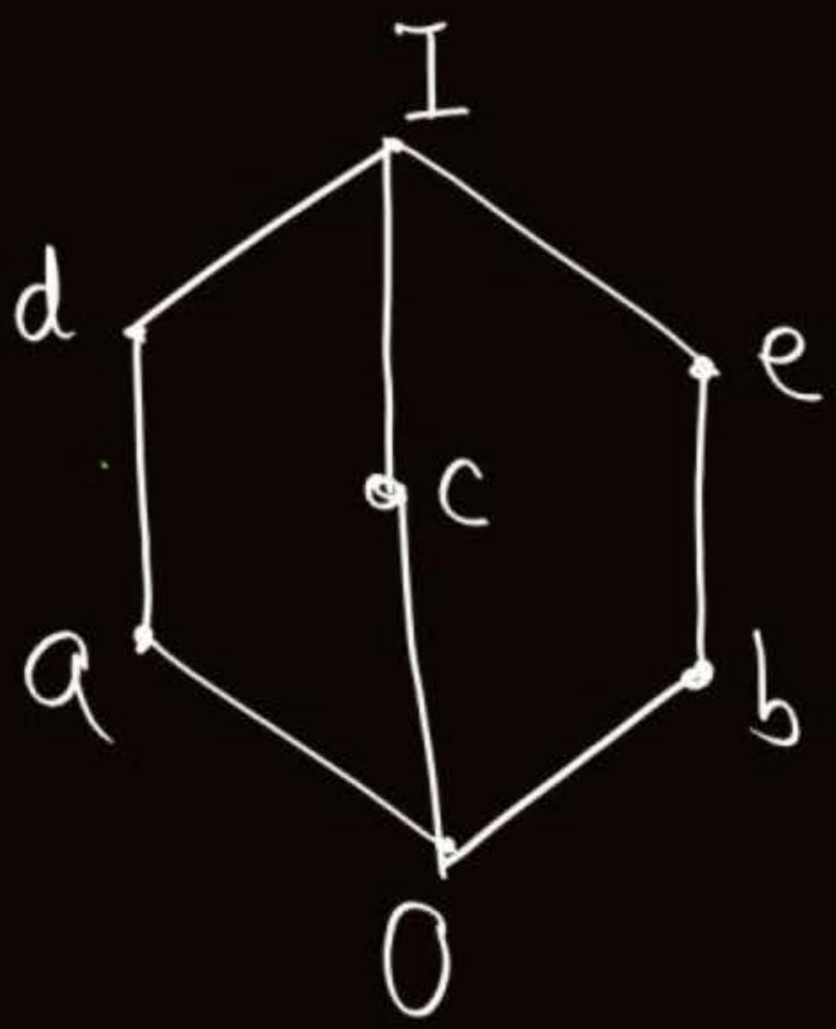
eg: Find complement of every element of the lattice given below.



$\overline{I} = O$

$\overline{O} = I$

$\overline{a} = b, c, e$

$\overline{b} = c, a, d$

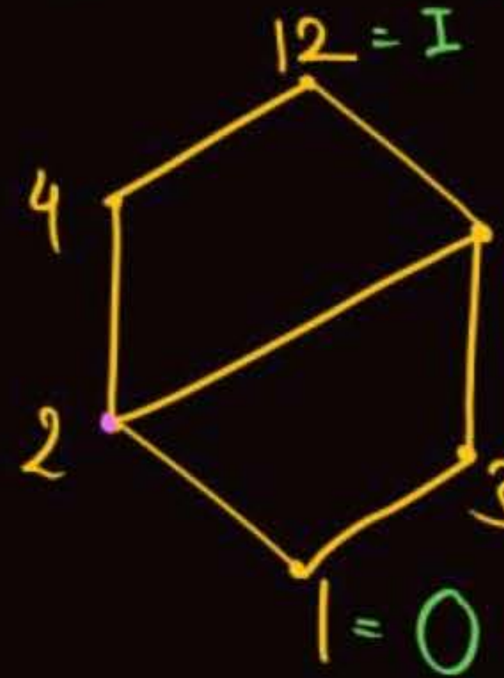
$\overline{c} = a, b, d, e$

$\overline{d} = b, c, e$

$\overline{e} = a, d, c$

Every element has at least one complement
∴ It is a Complemented lattice

Q: Find Complement of every element of lattice $(D_{12}, \div)$. (If exists)



$\bar{1} = 12$, $\overline{12} = 1$ } 1 & 0 are complement of each other

$\bar{2}$ = $\bar{2}$ can not exist on line 1-2-4-12

$\bar{2}$ can not exist on line 1-2-6-12

Check w.r.t '3'

$2 \wedge 3 = 1 = 0$ {it is okay}

but $2 \vee 3 = 6 \neq I$ {∴ 2 & 3 can not be complement of each other}

$\bar{2}$ = does not exists

$\bar{6}$ = ? ⇒ Can not exist on 1-3-6-12 or 1-2-6-12

Check w.r.t. 4.

$6 \wedge 4 = 2 \neq 0$ ∴ 6 & 4 can not be complement of each other

$\bar{6}$ = does not exist

$\bar{3} = 4 \Rightarrow$, $\bar{4} = 3 \Rightarrow$ $3 \vee 4 = 12 = I$ & $3 \wedge 4 = 1 = 0$ } ∴ $\bar{3} = 4$ & $\bar{4} = 3$

In given lattice, Complement does not exist for some elements, ∴ not a Complemented lattice

Topic : Distributive Lattice

* A lattice $[L, \vee, \wedge]$ is called distributive lattice, if and only if

$$\left.\begin{aligned} a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \\ a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \end{aligned}\right\} \forall a, b, c \in L$$

→ A lattice is called distributive lattice if distributive property holds true for every triple (three elements) belonging to the lattice

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Note:- In a distributive lattice every element has at most '1' Complement.

In a distributive lattice every element has 0 or 1 Complement.

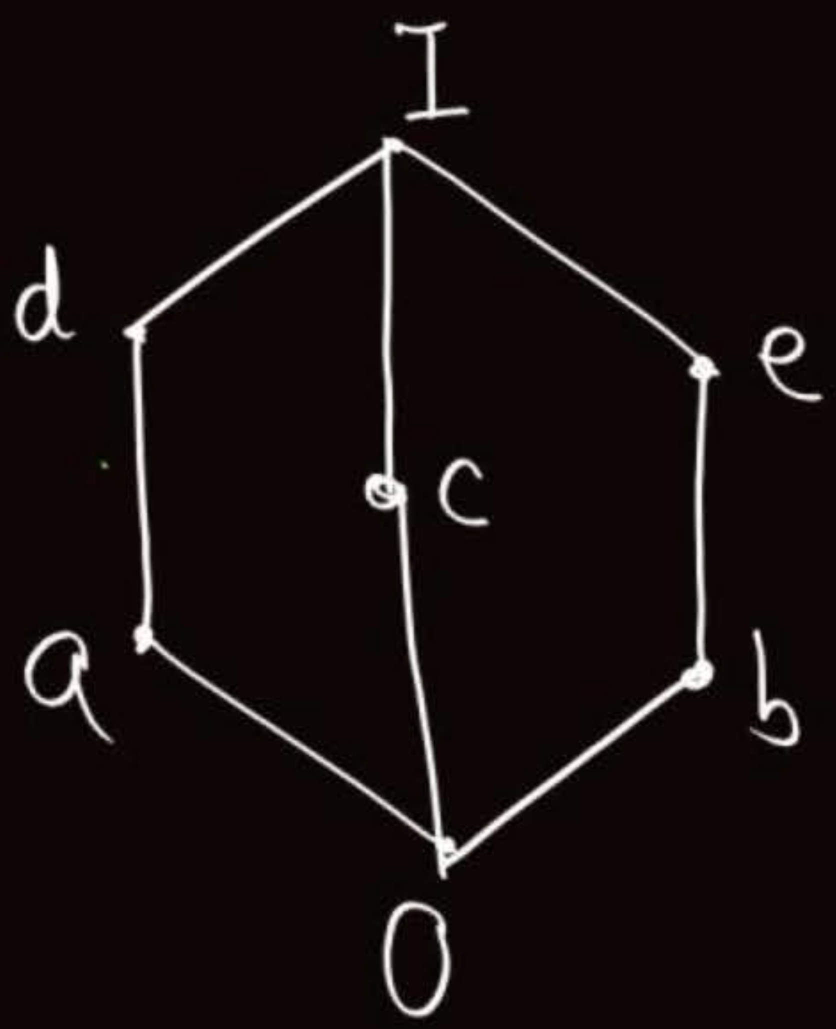
In a distributive lattice, Complement of an element need not exist, but if exists then it must be Unique

1. In a normal lattice an element can have any number of complements {Zero or more}

2. In a complemented lattice every element has at least one complement {i.e. 1 or more}

3. In a distributive lattice every element has at most one complement {i.e. 0 or 1}

eg: Find complement of every element of the lattice given below.



$\overline{I} = O$

$\overline{O} = I$

$\overline{a} = b, c, e$

$\overline{b} = c, a, d$

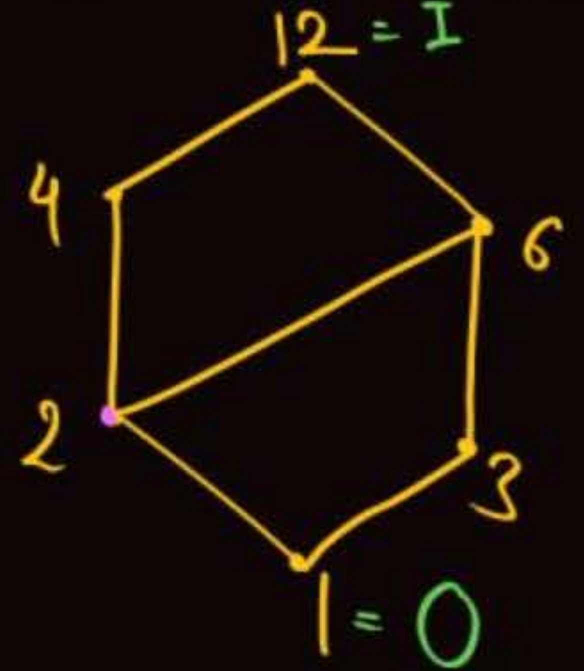
$\overline{c} = a, b, d, e$

$\overline{d} = b, c, e$

$\overline{e} = a, d, c$

More than one complement for some elements, ∴ Not a distributive lattice

Q: Find Complement of every element of lattice $(D_{12}, \div)$. (If exists)



$\bar{1} = 12$, $\overline{12} = 1$ } 1 & 0 are complement of each other

$\bar{2}$ = $\bar{2}$ Can not exist on line 1-2-4-12
$\bar{2}$ Can not exist on line 1-2-6-12
Check w.r.t '3'
$2 \wedge 3 = 1 = 0$ {It is okay}
but $2 \vee 3 = 6 \neq I$ {$\therefore$ 2 & 3 Can not be Complement of each other}
$\bar{2}$ = does not exists

$\bar{6}$ = ? ⇒ Can not exist on 1-3-6-12 or 1-2-6-12
Check w.r.t. 4.
$6 \wedge 4 = 2 \neq 0$ $\therefore$ 6 & 4 Can not be complement of each other
$\bar{6}$ = does not exist

$\bar{3} = 4$ ⇒ $3 \vee 4 = 12 = 1$
$\bar{4} = 3$ ⇒ & $3 \wedge 4 = 1 = 0$ } $\therefore$ $\bar{3} = 4$ & $\bar{4} = 3$

Every element has Zero or one Complement, $\therefore$ It is a distributive lattice

H.W.

Which of the following statements is/ are not true

$\overline{\{\}} = A$

$\overline{\{1\}} = \{2,3\}$

$\overline{\{2\}} = \{1,3\}$

a) If A is any finite set then [P(A),⊆] is distributive lattice (True)

b) Every sub lattice of a distributive lattice is also a distributive lattice

c) Every totally ordered set is a distributive lattice

d) Every totally ordered set is bounded

e) Every distributive lattice is bounded

f) Every distributive lattice is a complemented lattice

let $A = \{1,2,3\}$

$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

& $(P(A), \subseteq)$ is lattice

for any set $X \in P(A)$ there is a unique complement of X i.e. $\overline{X} = A - X$

i.e. No element has more than one complement

i.e. Distributive

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Which of the following statements is/ are not true

a) If A is any finite set then [P(A),⊆] is distributive lattice
b) Every sub lattice of a distributive lattice is also a distributive lattice (True)
c) Every totally ordered set is a distributive lattice
d) Every totally ordered set is bounded
e) Every distributive lattice is bounded
f) Every distributive lattice is a complemented lattice

let $[M, \vee, \wedge]$ be a sub-lattice of lattice $[L, \vee, \wedge]$

if $\forall a, b, c \in M$ then $a, b, c \in L$

lub & glb in sublattice M will be exactly same as lattice L
∴ This distributive will hold true in lattice M as well

We know L is a distributive lattice

$$\left.\begin{aligned} \therefore\ & a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \\ \& \ & a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \end{aligned}\right\} \text{holds in } L\ \forall a, b, c \in L$$

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Which of the following statements is/ are not true

a) If A is any finite set then [P(A),⊆] is distributive lattice
b) Every sub lattice of a distributive lattice is also a distributive lattice
c) Every totally ordered set is a distributive lattice (True)
d) Every totally ordered set is bounded
e) Every distributive lattice is bounded
f) Every distributive lattice is a complemented lattice

It is a bounded totally ordered set in which $\bar{I}=O$ & $\bar{O}=I$ and Complement does not exist for any other element
i.e. every element has at most one Complement ∴ lattice is distributive

If totally ordered set is not bounded then Complement does not exist for any element ∴ Distributive

Slide

Which of the following statements is/ are not true

a) If A is any finite set then $[P(A), \subseteq]$ is distributive lattice
b) Every sub lattice of a distributive lattice is also a distributive lattice
c) Every totally ordered set is a distributive lattice ✓
d) Every totally ordered set is bounded False for eg: $(N, \leq)$ is a totally ordered set but not bounded
e) Every distributive lattice is bounded
f) Every distributive lattice is a complemented lattice

Slide

Which of the following statements is/ are not true

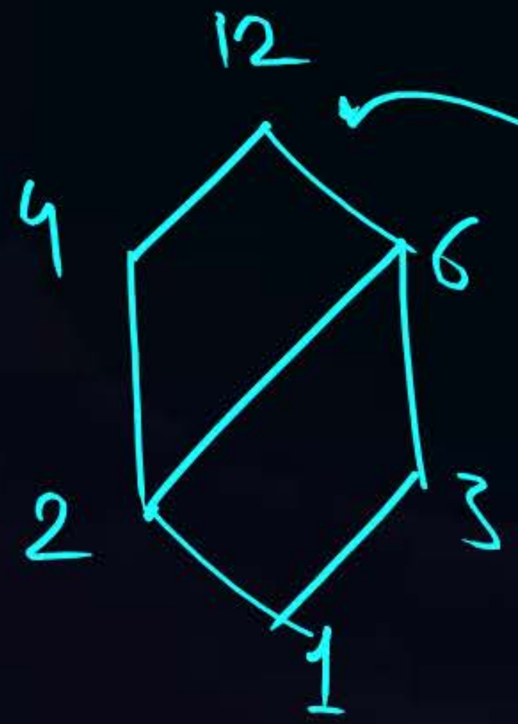
a) If A is any finite set then $[P(A), \subseteq]$ is distributive lattice
b) Every sub lattice of a distributive lattice is also a distributive lattice
c) Every totally ordered set is a distributive lattice
d) Every totally ordered set is bounded
e) Every distributive lattice is bounded False $\left\{ \text{eg. } (N, \leq) \text{ is distributive but not bounded} \right\}$
f) Every distributive lattice is a complemented lattice

Slide

Which of the following statements is/ are not true

a) If A is any finite set then [P(A),⊆] is distributive lattice
b) Every sub lattice of a distributive lattice is also a distributive lattice
c) Every totally ordered set is a distributive lattice
d) Every totally ordered set is bounded
e) Every distributive lattice is bounded
f) Every distributive lattice is a complemented lattice {False}

eg: $(D_{12}, \div)$ is distributive but not complemented



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Topic : NOTE

Lattice with single element

It is the only lattice with single element and it is distributive

Lattices Possible with two elements

It is the only POSET diagram Possible for a lattice with two elements, and it is distributive

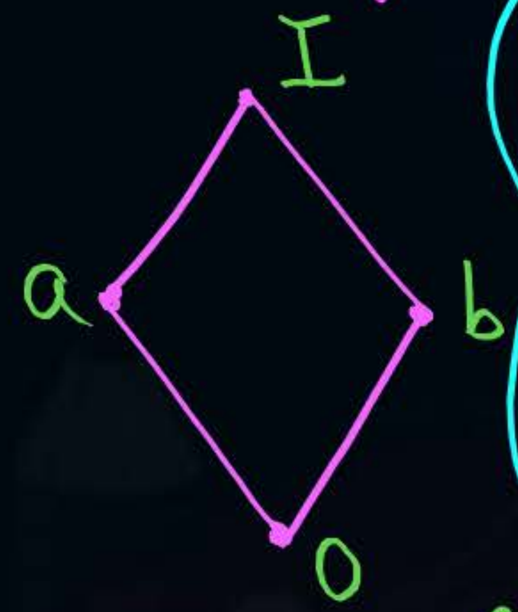
Lattices Possible with three elements

It is the only POSET diagram Possible for a lattice with three elements, and it is distributive

Lattices Possible with four elements ✓

✓

distributive



$\overline{I} = O$

$\overline{O} = I$

$\overline{a} = b$

$\overline{b} = a$

every element has at most one complement

∴ distributive

This are the only two POSET diagrams possible for lattices with four elements, and both are distributive

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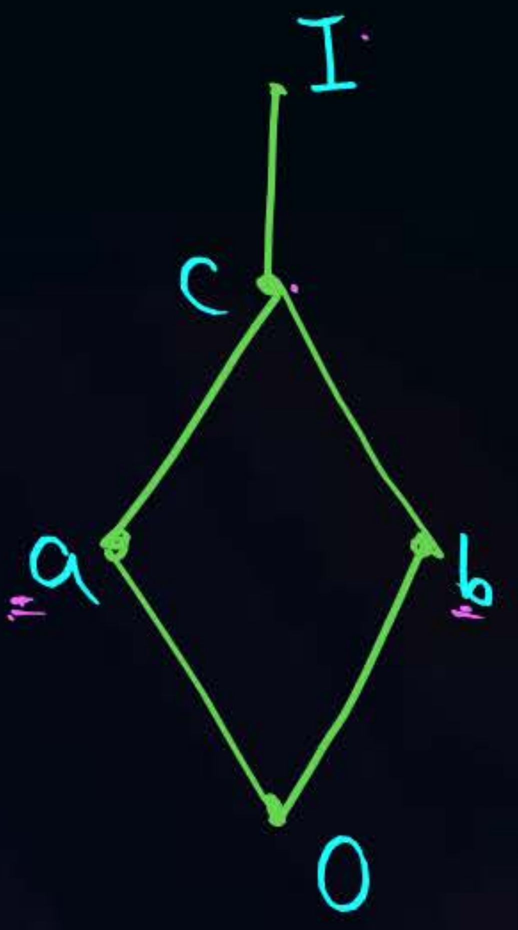
Note:- Any lattice with four or less element is always distributive

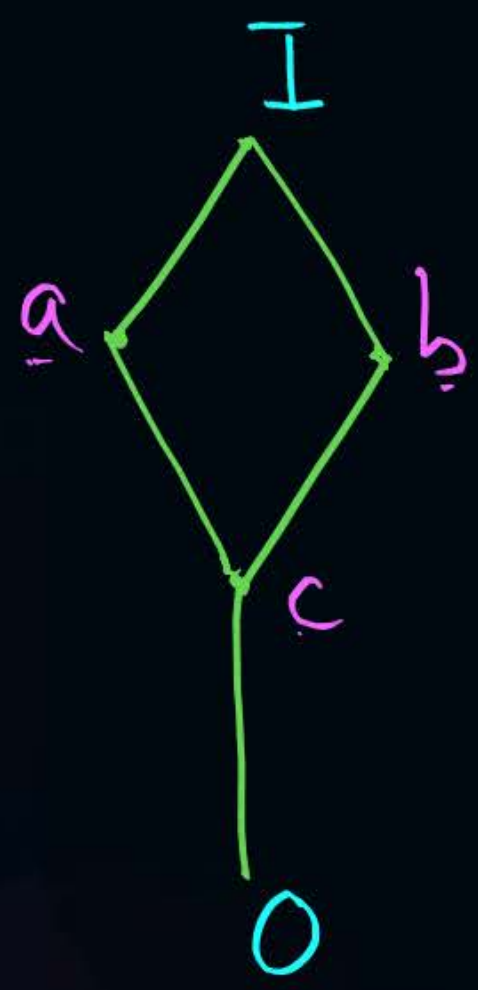
Topic : NOTE

POSET diagrams Possible w.r.t. lattices of five elements

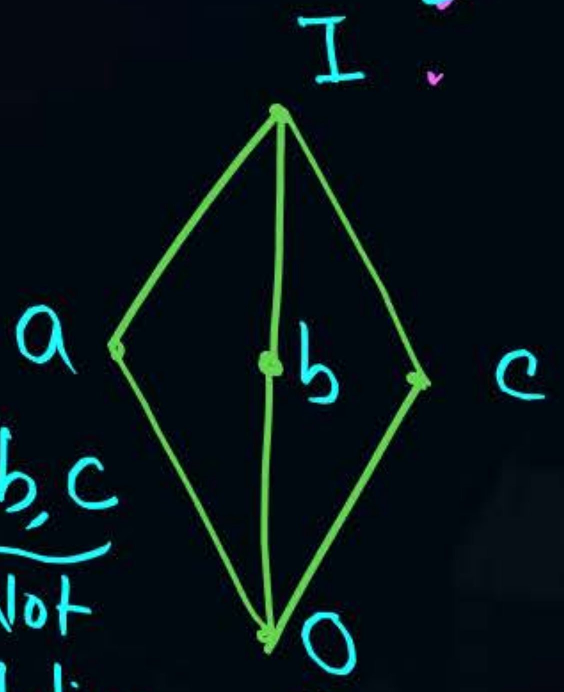
distributive



distributive



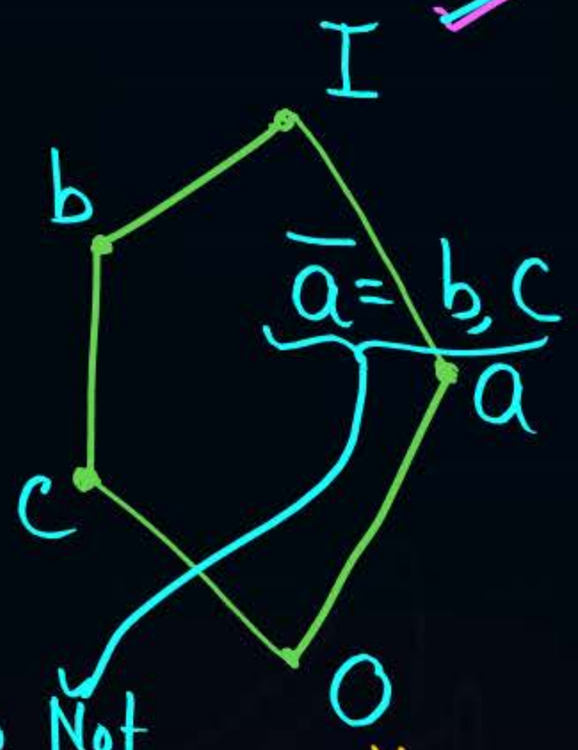
distributive



$\underbrace{\bar{a} = b, c}$

$\therefore$ Not distributive

L_1^*



$\underbrace{\bar{a} = b, c}$

$\therefore$ Not distributive L_2^*

L_1^* and L_2^* are the only two lattices with 5 elements which are not distributive

Slide

Topic : NOTE

A Lattice "L" is not distributive if and Only if "L" has a sub-lattice imp which is isomorphic {similar} to L_1^* or L_2^*.

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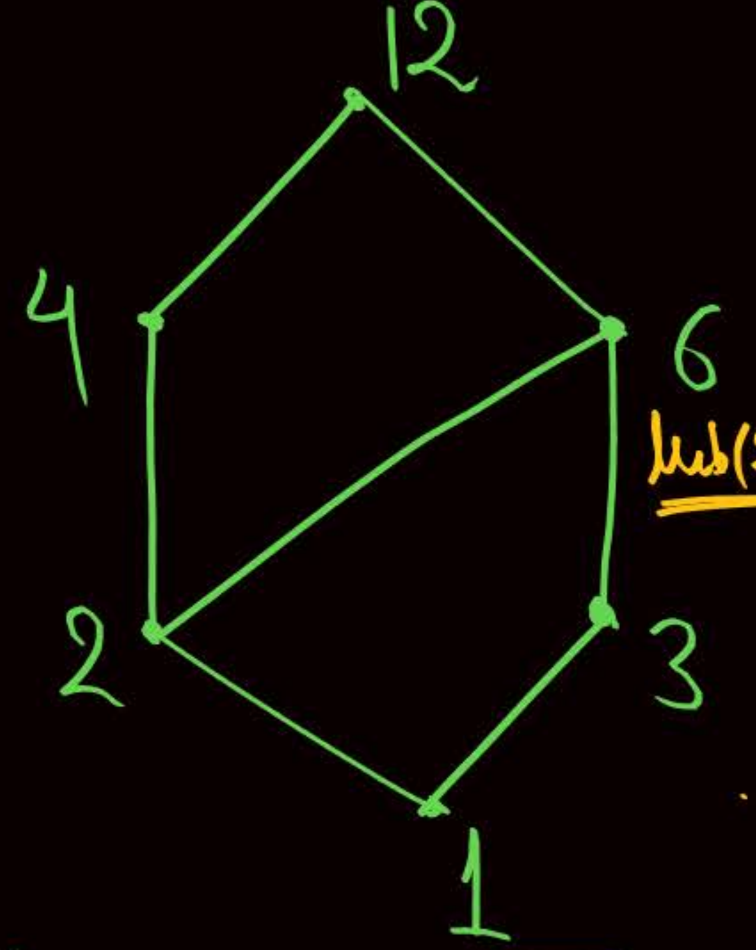
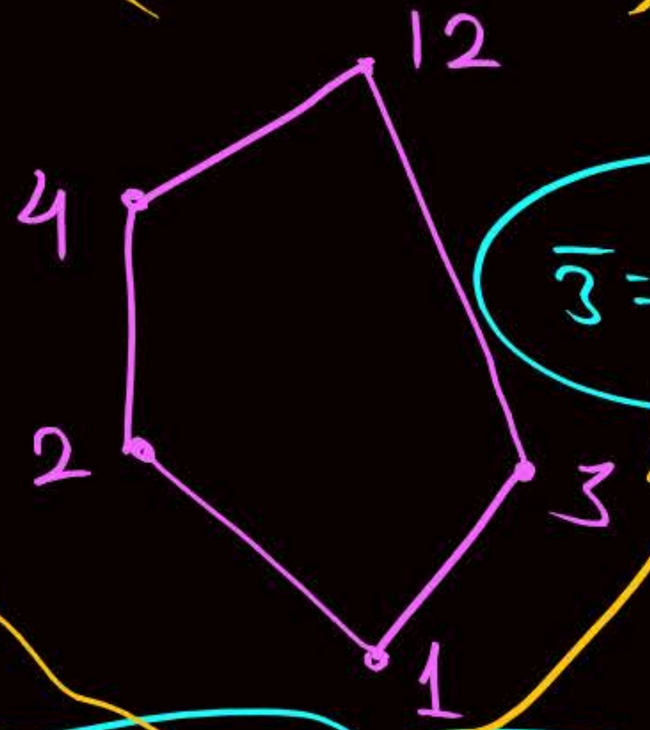
Q: Check whether lattice $(D_{12}, \div)$ is distributive or not?



$lub(2,3) = 6 \Longrightarrow$ lets take subset $\{1, 2, 3, 4, 12\}$ of set D_{12}

Poset diagram for $(\{1, 2, 3, 4, 12\}, \div)$



$\bar{3} = 2, 4$

$lub(2,3) = 12$

It is isomorphic to L_2^* but $(\{1,2,3,4,12\}, \div)$ is not a sub-lattice of $(D_{12}, \div)$

No sub-lattice of $(D_{12}, \div)$ is isomorphic to L_1^* or L_2^*

$\therefore (D_{12}, \div)$ is distributive

we already know $(D_{12}, \div)$ is distributive

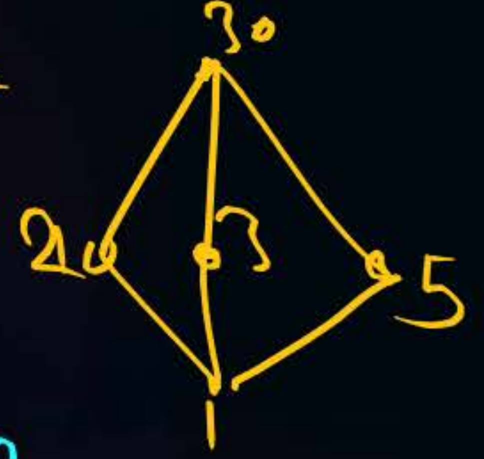
Which of the following lattice is /are not distributive?

(A) $(D_{125}, \div)$ $D_{125}: \underline{\{1, 5, 25, 125\}}$ 4 elements $\therefore$ distributive

(B) $(\underline{P(A), \subseteq})$ every element has exactly one complement, i.e. No element has more than one complement $\therefore$ distributive

(C) $(\underline{D_{12}}, \div)$ distributive

(D) $(\{1, 2, 3, 5, 30\}, \div)$ =



isomorphic to L_1^* $\therefore$ Not distributive

(E) $(\underline{\{1, 2, 3, 4, 12\}, \div})$

isomorphic to L_2^* $\therefore$ Not distributive

Slide

at least one complement

* A lattice which is Complemented as well as distributive is called a Boolean lattice or Boolean Algebra

at most one Complement

* In a Boolean lattice every element has exactly one Complement

Slide

Which of the following lattice ~~is~~ /are Boolean Algebra?

(A) $(D_{125}, \div)$
- distributive ✓
- Complemented ✗ {Complement does not exist for 5 & 25}

(B) $(P(A), \subseteq)$ ✓ → Every element has exactly one Complement

(C) $(D_{12}, \div)$
- distributive ✓
- Complemented ✗ {for 2 & 6 Complement does not exist}

(D) $(\{1, 2, 3, 5, 30\}, \div)$
- distributive ✗
- Complemented ✓ {Every element has at least one Complement}

(E) $(\{1, 2, 3, 4, 12\}, \div)$
- Distributive ✗
- Complemented ✓

Slide

Topic : NOTE

* Let n is any +ve integer, and D_n is the set of all +ve divisors of n.

If D_n does not contain any element which is a perfect square "except 1", then n is called a square free integer

* If n is a square free integer, then $(D_n, \div)$ is a boolean lattice.
Where for any $x \in D_n$, $\bar{x} = \frac{n}{x}$

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Which of the following lattice is /are Boolean algebra.

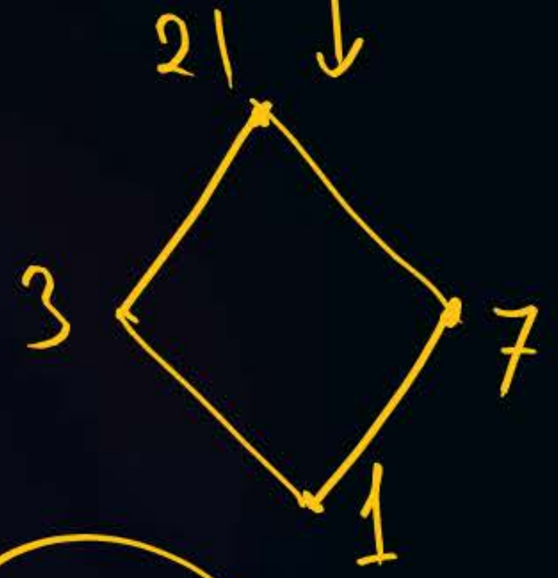
(A) $(D_{21}, \div)$ $\quad D_{21} = \{1, 3, 7, 21\}$ $\quad$ 21 is a square free integer

$\therefore$ it is a boolean lattice

(B) $(D_{110}, \div)$

(C) $(D_{24}, \div)$

(D) $(D_{91}, \div)$



$\bar{1} = \frac{21}{1} = 21$

$\bar{3} = \frac{21}{3} = 7$

$\bar{7} = \frac{21}{7} = 3$

$\overline{21} = \frac{21}{21} = 1$

$\bar{1} = 21$

$\overline{21} = 1$

$\bar{3} = 7$

$\bar{7} = 3$

Slide

Which of the following lattice is /are Boolean algebra.

(A) $(D_{21}, \div)$

(B) $(D_{110}, \div)$ $D_{110} = \{\overbrace{1, 2, 5, 10, 11, 22, 55, 110}^{\text{110 is a square free integer}}\}$

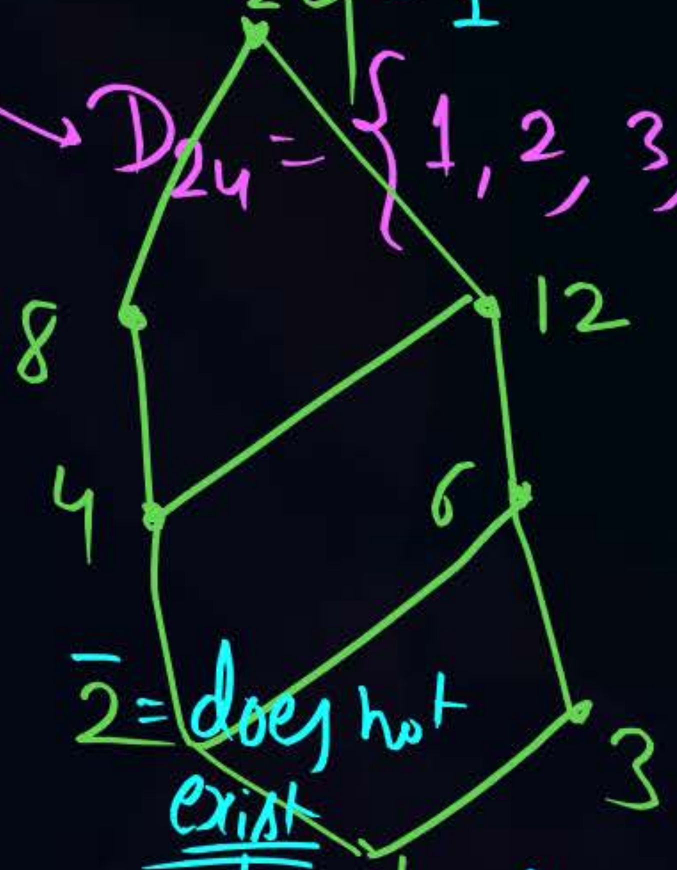
∴ $(D_{110}, \div)$ is a boolean lattice

(C) $(D_{24}, \div)$ $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

4 → Perfect square

∴ $(D_{24}, \div)$ is not a boolean lattice

$24' = 1$



$\bar{2}$ = does not exist

↓

∴ Not a Boolean lattice

$1' = 24$

(D) $(D_{91}, \div)$

$D_{91} = \{1, 7, 13, 91\}$

No perfect square

∴ Boolean lattice

Slide

What is the complement of element 5 in boolean algebra $(D_{110}, \div)$

$$\bar{5} = \frac{110}{5} = 22$$

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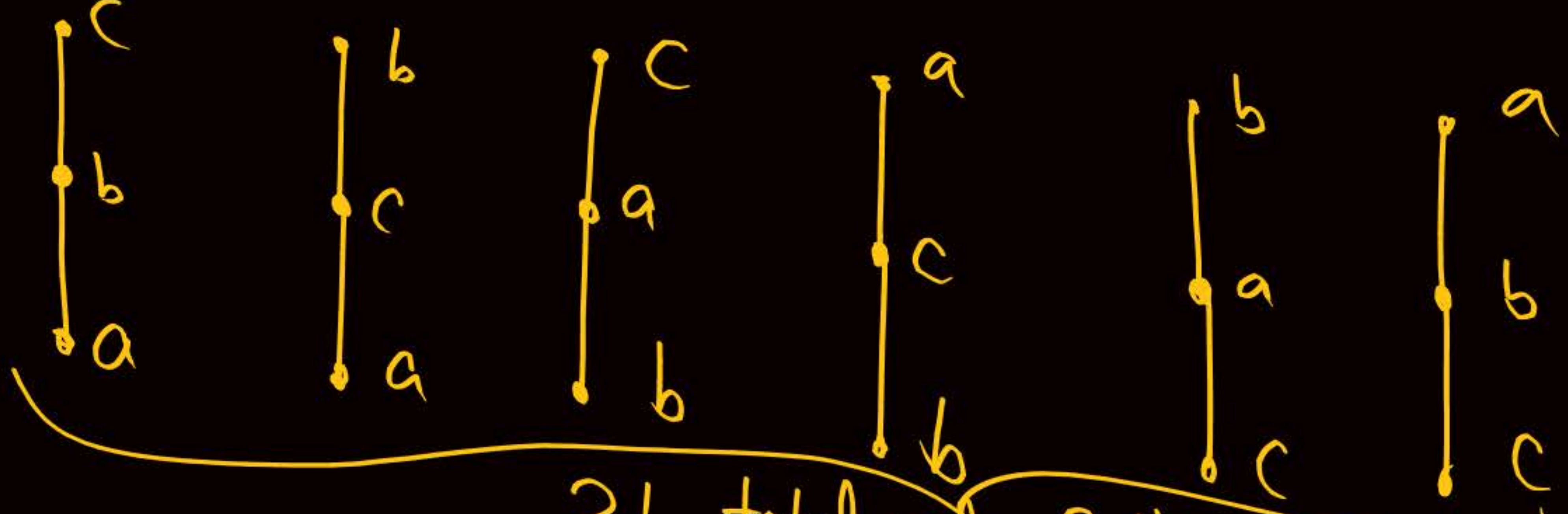
Q:- How many total order rel^n are possible on a set A with 'n' elements = Ans = $n!$

eg: Total order Rel^n Possible on set $A = \{a, b, c\}$



3! total order Rel^n

2 mins Summary

Topic Different Types of Lattices and Hasse Diagram

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THANK - YOU

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