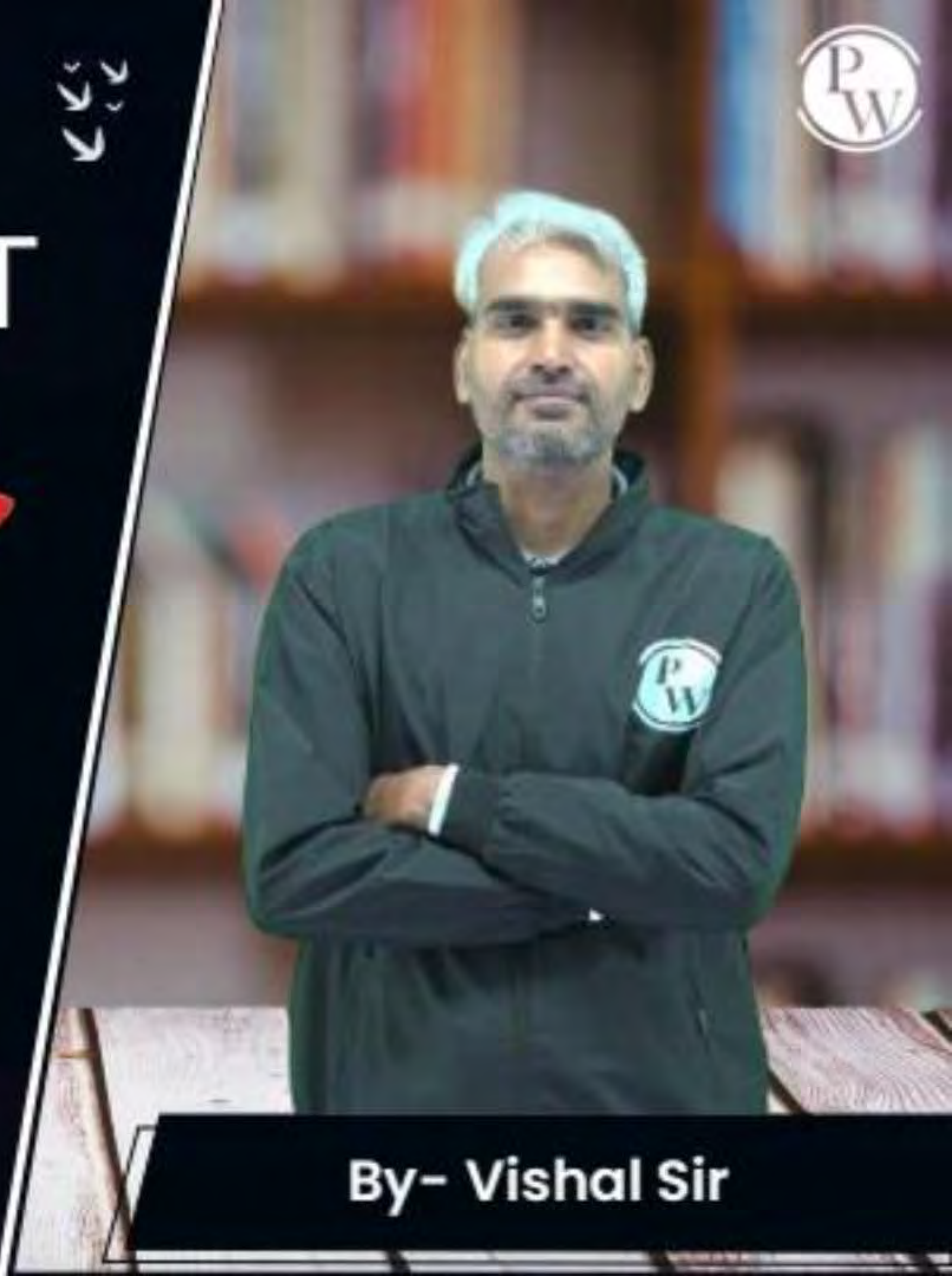
Computer Science & IT

Discrete Mathematics

Graph Theory

Lecture No. 14



By- Vishal Sir

Recap of Previous Lecture

Topic: Connectivity

Slide

Topics to be Covered

- **Topic** Euler trail , Euler Circuit and Traversable graph
- **Topic** Hamiltonian path, and Hamiltonian Circuit
- **Topic** Distance, Eccentricity, Diameter, Radius and Girth

Slide

H.W.

#Q. A simple graph with n vertices is necessarily connected if number of edges are more than E, then find the value of E.

i.e. if no. of edges are more than 'E' value then the graph with n- vertices can never be a disconnected graph

A graph with n- vertices may be a connected graph with just 'n-1' edges, but with 'n-1' edges the graph may be disconnected as well.

Ans $E = {}^{(n-1)}C_2$

Consider a Complete graph with $(n-1)$ vertices

it will have ${}^{(n-1)}C_2$ edges

and

an isolated vertex

It is a disconnected graph with n-vertices

i.e. upto ${}^{(n-1)}C_2$ edges a graph with n-vertices may be disconnected

But if the number of edges are more than ${}^{(n-1)}C_2$ then the graph is necessarily connected $\therefore E = {}^{(n-1)}C_2 = \frac{(n-1)(n-2)}{2}$

H.W.

#Q. Maximum number of edges possible in an undirected simple graph with n-vertices and k components?

i.e. We are looking for a value, s.t. if no. of edges are more than that value then a simple graph with n vertices can never have 'k' components

i.e. if no. of edges are more than that value then no. of components will become less than 'k'

Single connected component with remaining $(n-(k-1)) = (n-k+1)$ vertices

$\underbrace{\bullet \quad \bullet \quad \bullet \; - - - \; \bullet}_{(k-1) \text{ isolated vertices}}$

it is a graph with n-vertices & k components.

Maximum no. of edges that can be consumed by this simple graph with $(n-k+1)$ vertices are $(n-k+1)_{C_2}$

$\therefore$ Upto $(n-k+1)_{C_2}$ edges it may be possible to have a graph with n-vertices & 'k' components

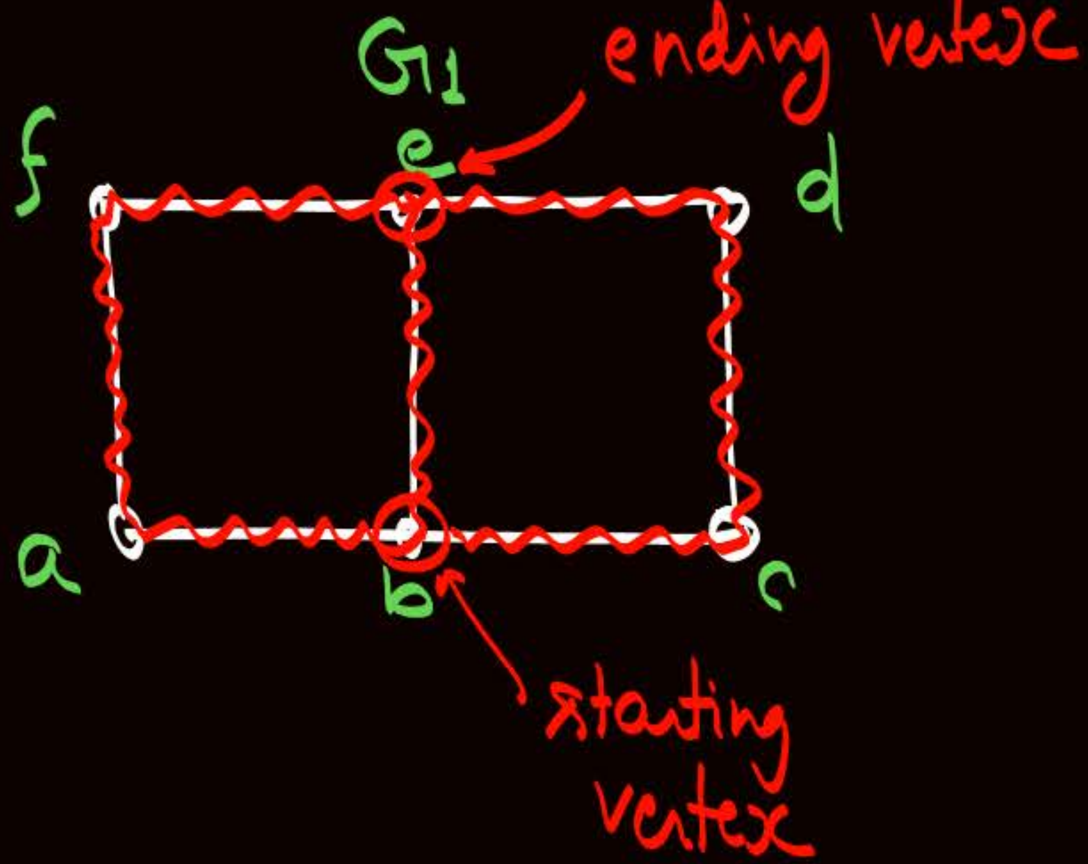
If no. of edges are more than $(n-k+1)_{C_2}$ then there can not be 'k' components in a simple graph with n-vertices

Topic : Euler trail / Euler path

i.e. Starting vertex & Ending vertex must be different

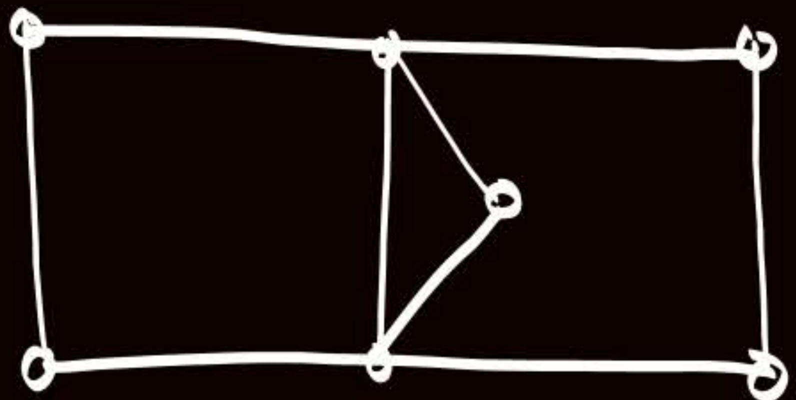
Let G be a connected graph, An open trail that visit every edge of the graph Exactly once {vertices may be visited more than once} is called an Euler trail or Euler path.

G_1



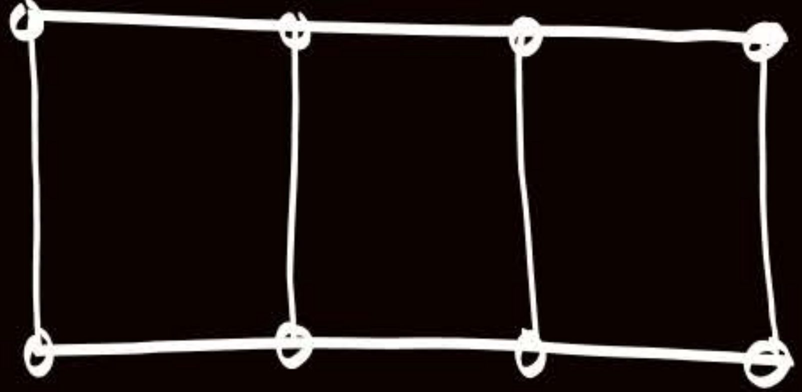
Euler trail Exists in graph G_1

G_2



No open Euler trail Exists in graph G_2

G_3



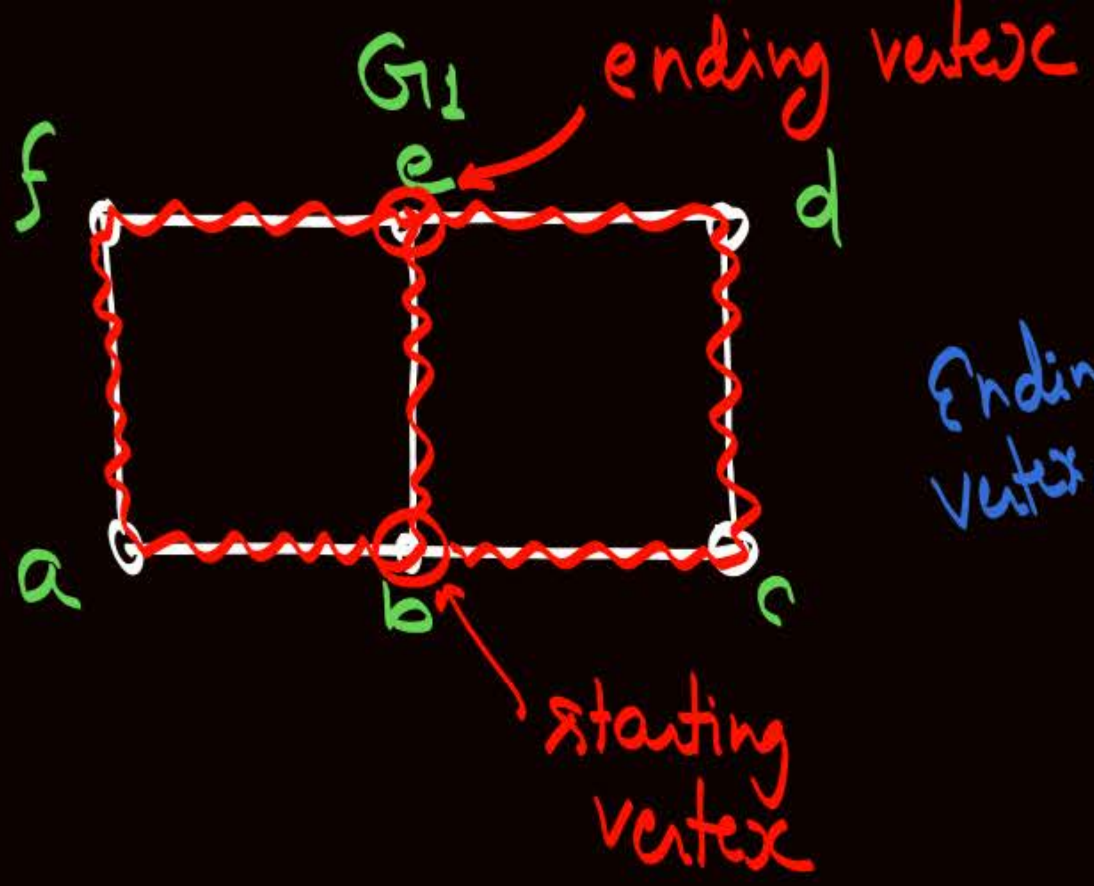
No Euler trail (open/close) Exists in graph G_3

Topic : Euler circuit

Closed

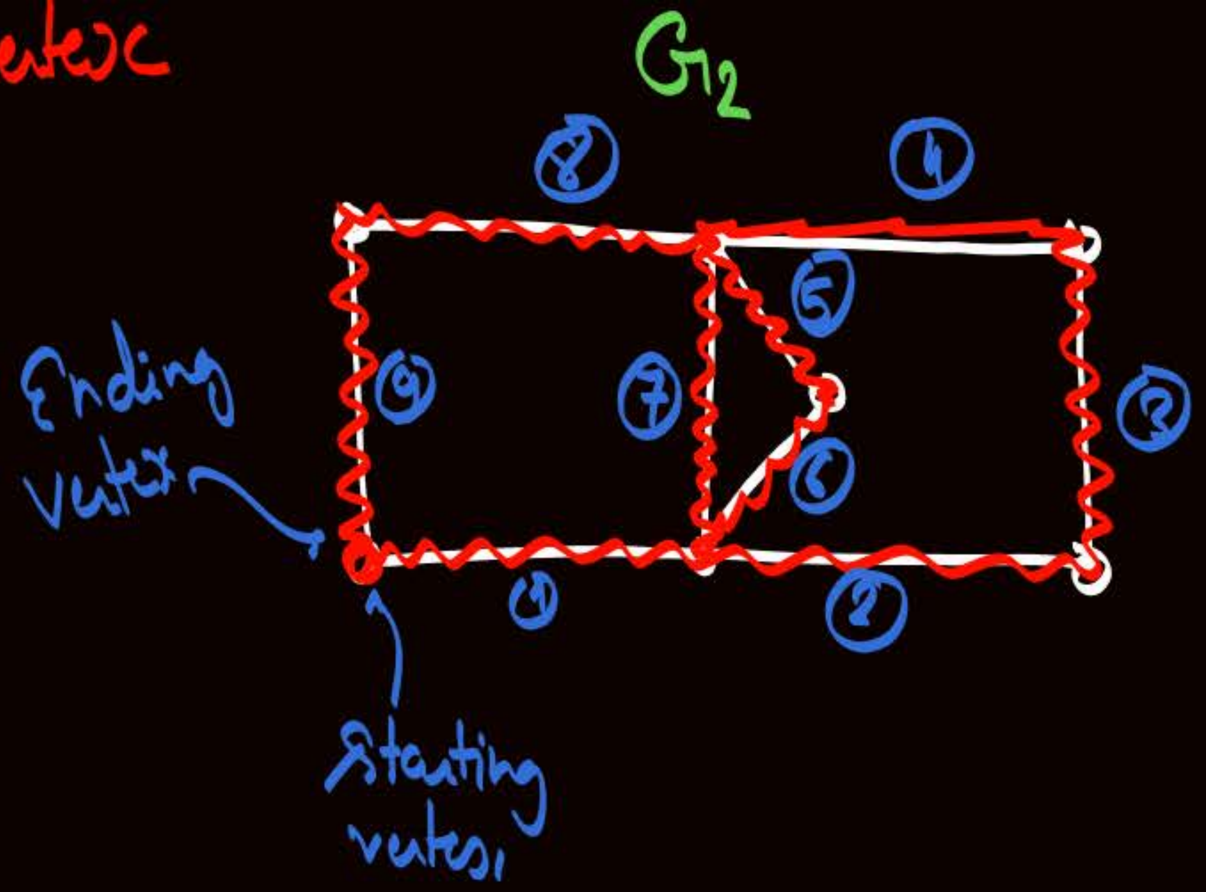
i.e, Start & End vertices are same

In a Connected graph G a Closed Euler trail that visit every edge of the graph Exactly once is Called an Euler Circuit.



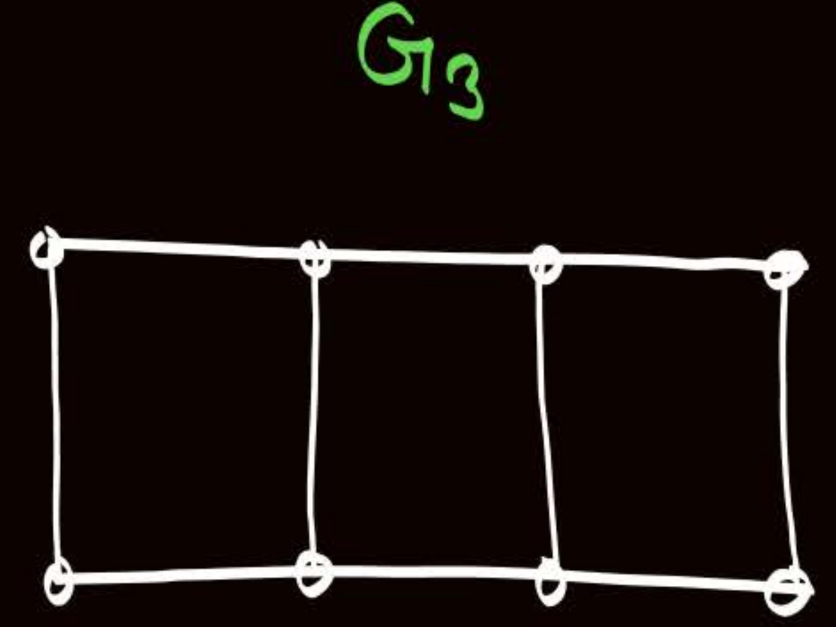
Euler trail Exists in graph G_1

but, No Euler Circuit exists in graph G_1



Euler Circuit exists in graph G_2

But no open Euler trail exists.



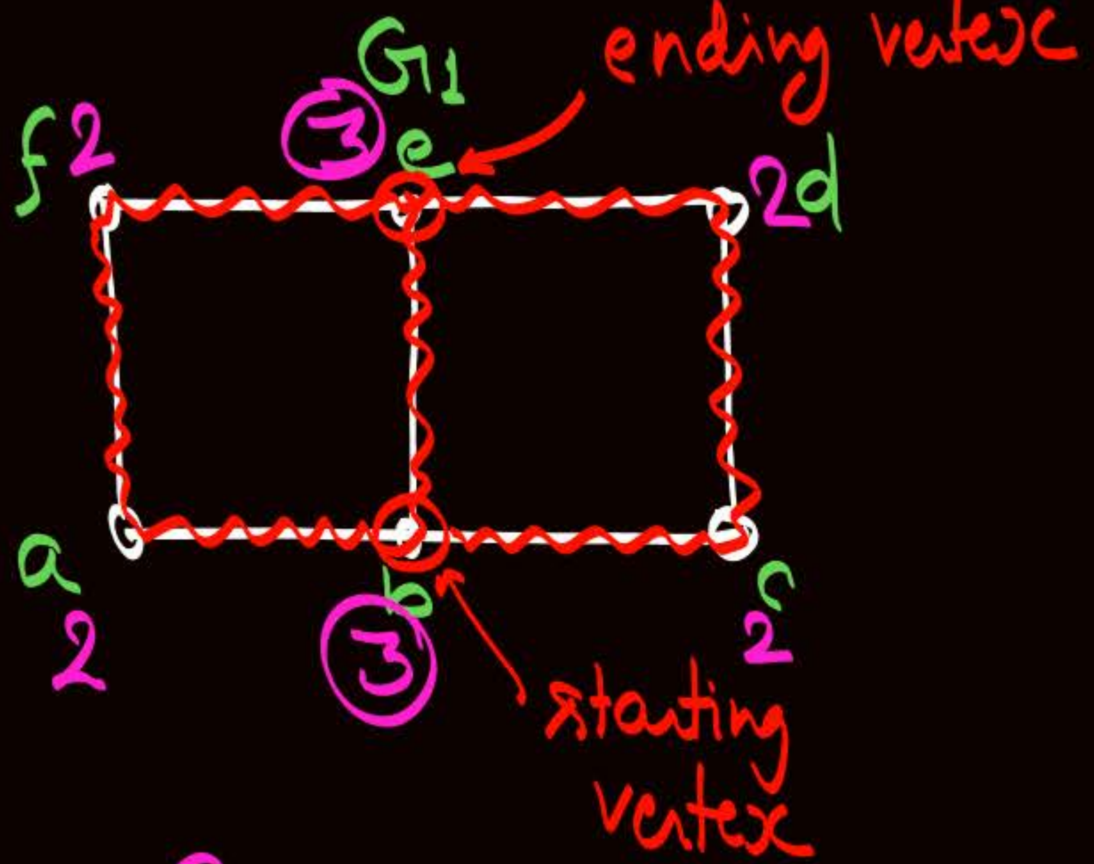
No Euler trail & No Euler Circuit in graph G_3

Topic : Eulerian graph / Euler graph

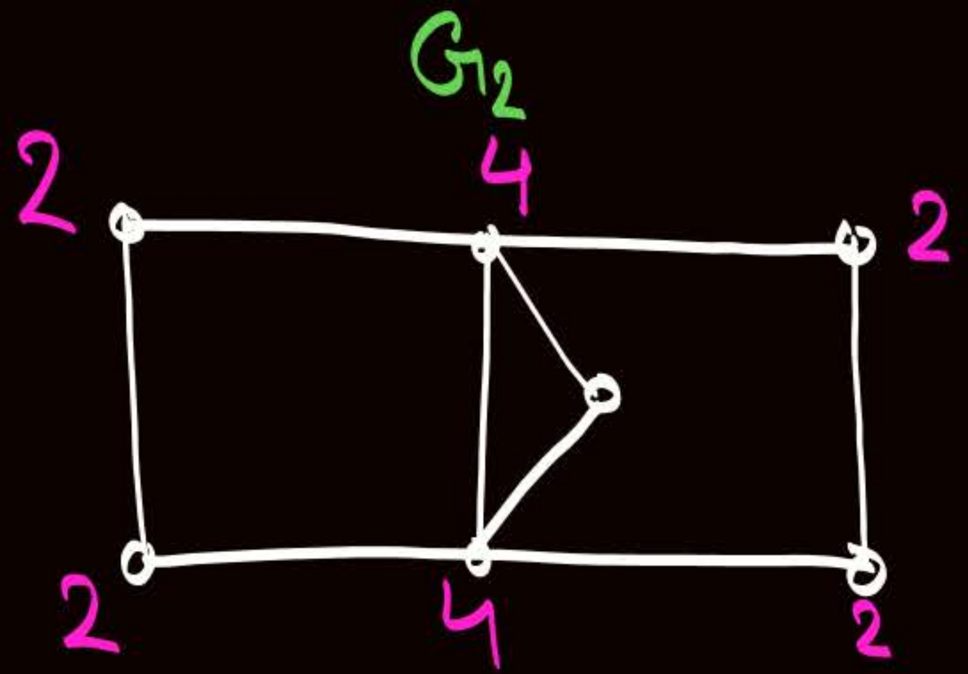
A connected graph G in which an Euler Circuit exists is called an Euler graph or Eulerian graph

(or)

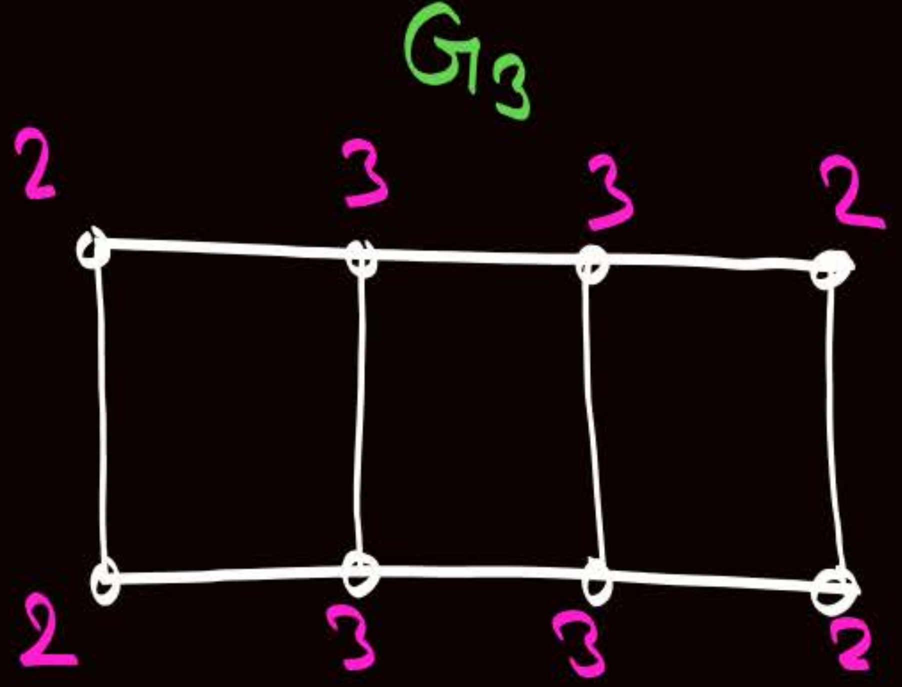
A graph G is called an Euler graph, if graph G is a Connected graph and no. of vertices with odd degree in graph G is zero.



Connected
+
No. of vertices with odd degree = 2
↓
then No Euler circuit
but Euler trail exist



Connected
+
No. of vertices with odd degree = 0
↓
then Euler Circuit exist
but no open Euler trail.



Connected
+
No. of vertices with odd degree > 2
↓
then No Euler Circuit
&
No Euler trail.

Note :- ✓ 1. If graph is a disconnected graph, then no Euler trail and no Euler Circuit exist in that graph.

2. If graph G is a Connected graph & number of vertices with odd deg. are '0', then Euler Circuit exist but no Euler trail

3. If graph G is a Connected graph and number of vertices with odd degree are exactly '2', then Euler trail exist but no Euler Circuit

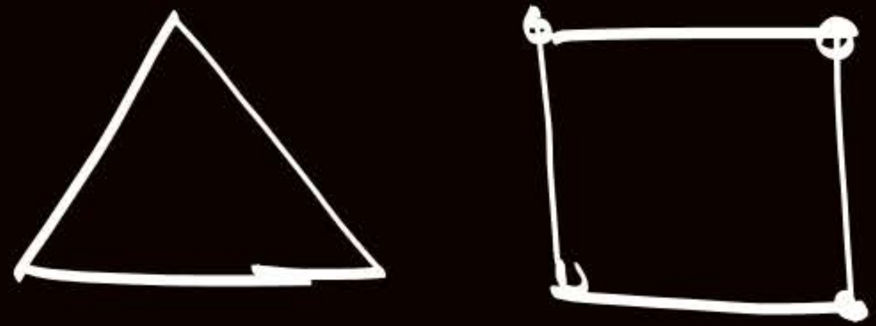
4. If graph G is a Connected graph & no. of vertices with odd degree are more than '2' {>2} then no Euler trail and no Euler Circuit

The statement that,

If all the vertices in a graph are of even degree then graph will contain an Euler Circuit.

is false, because graph may not be connected

eg:



G

Degree of each vertex is even but no Euler Circuit.

Topic : Traversable graph

- A graph G is called traversable if Euler Circuit <u>or</u> Euler trail exist in that graph

(Or)

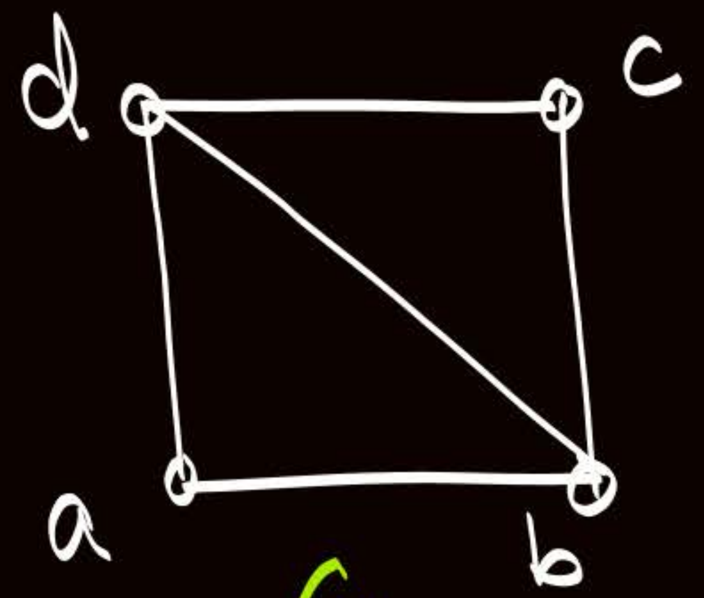
A graph G is traversable if and only if graph G is a connected graph <u>and</u> no. of vertices with odd degree in graph G <u>are</u> '0' <u>or</u> '2'

Topic : Hamiltonian path

An open path in graph G that visit every vertex of the graph exactly once {may skip some edges} is called an Hamiltonian Path

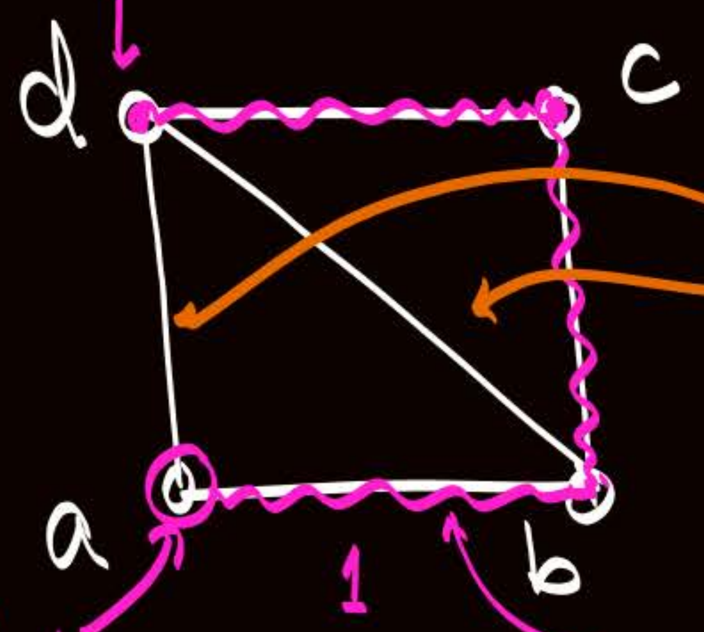
Starting & Ending vertex are different

eg:



G

End



We have skipped this edges.

Start

Hamiltonian path exist in graph G

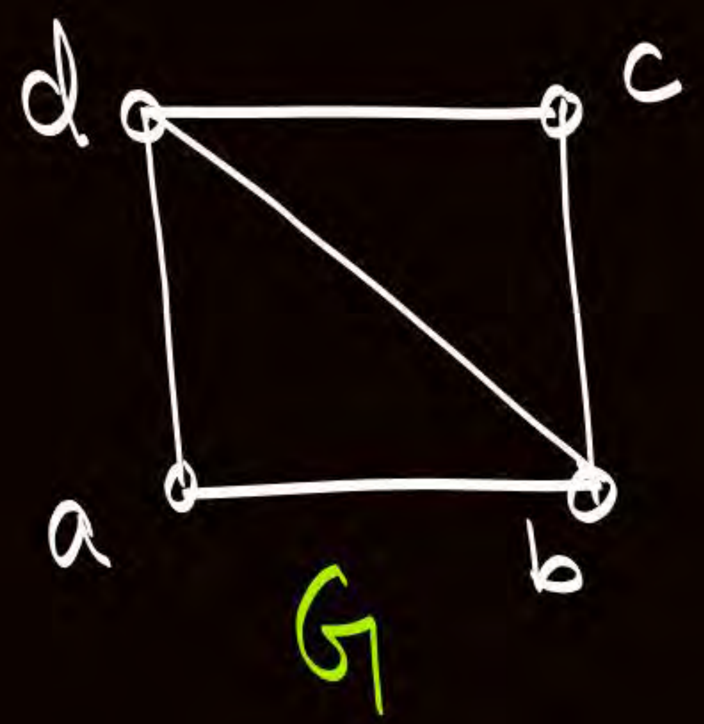
Hamiltonian path

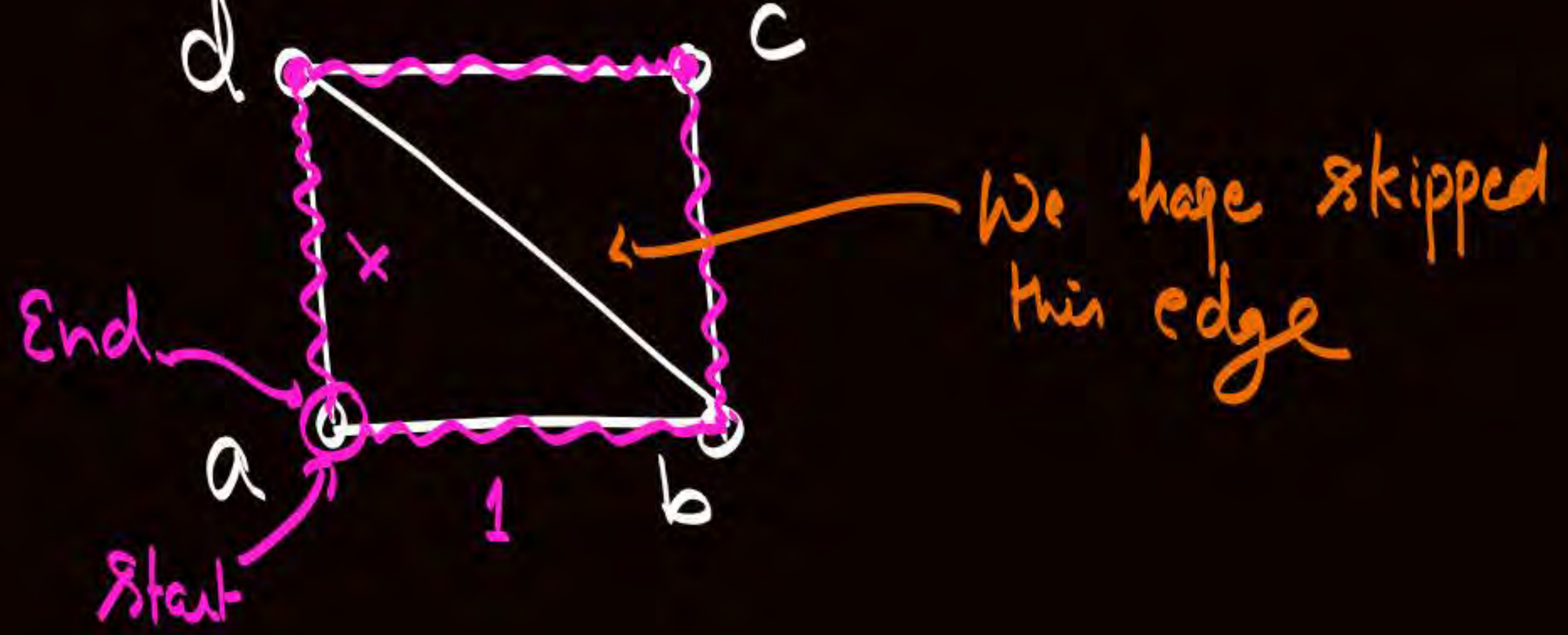
Topic : Hamiltonian circuit / Hamiltonian Cycle

A $\underbrace{\text{Closed Path}}$ in graph G that visit Every vertex of graph G exactly once {Except starting vertex} is called Hamiltonian Circuit.

it will be visited twice start & at End

i.e, Start & End vertex must be same

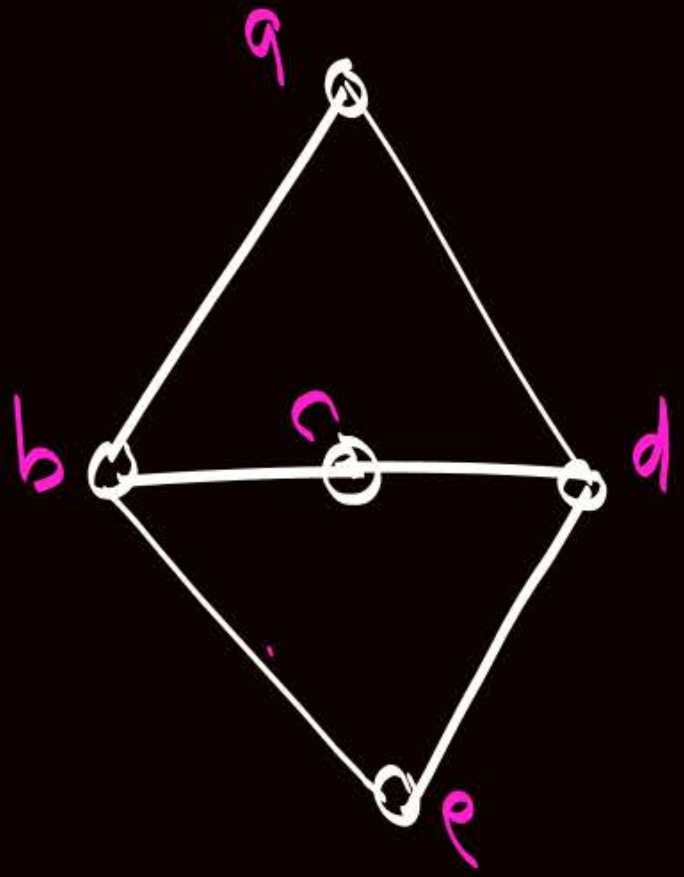


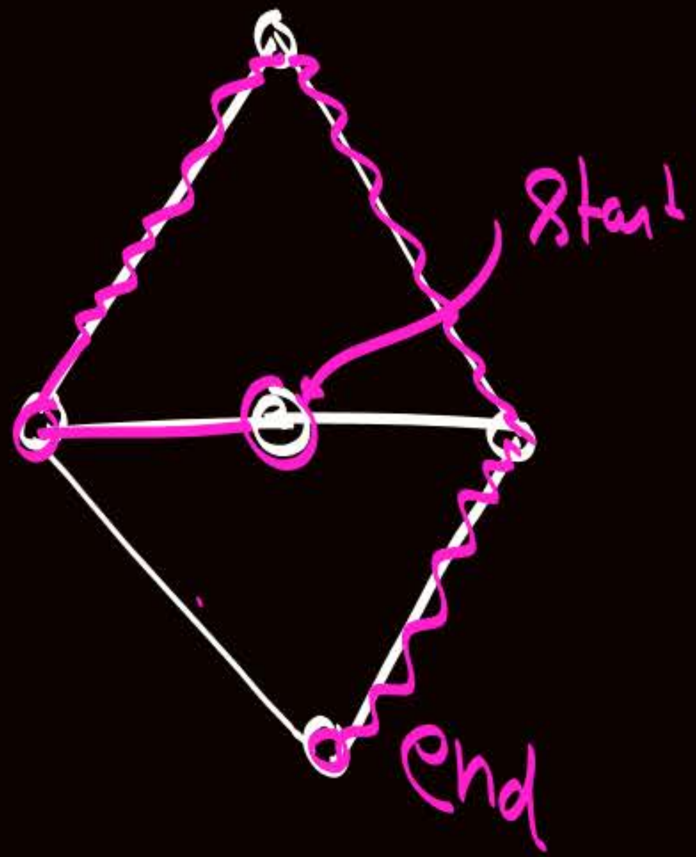


Hamiltonian Circuit exist in graph G

Note: If Hamiltonian Circuit exists in graph G, then Hamiltonian Path will also exist in that graph G,

But if Hamiltonian Path exists in graph G, then Hamiltonian Circuit may or may not exist.





Hamiltonian Path Exists in the graph, but No Hamiltonian Circuit

Topic : Hamiltonian graph

A graph G is said to be Hamiltonian graph, if Hamiltonian Circuit exist in that graph

Note: There is no trick to check whether the given Connected graph is a Hamiltonian graph or not.

We always need to analyse the graph

Topic : Dirac's theorem

connected

Let G = (V,E) be a graph with n vertices in which each vertex has degree at least n/2, Then G has a Hamiltonian cycle.

Q: Let K_n be a complete graph with n-vertices $(n \geq 3)$, then how many Hamiltonian cycles exist in graph K_n.

↳ All Hamiltonian Cycles will contain all n-vertices
i.e. We are looking different cycles of length-'n'

(i) If vertices are not labeled, then all cycles of length 'n' are same.

∴ If vertices are not labeled, then no. of Hamiltonian cycles = 1

(ii) If vertices are labeled then

No. of Hamiltonian Cycles = $\dfrac{(n-1)!}{2}$

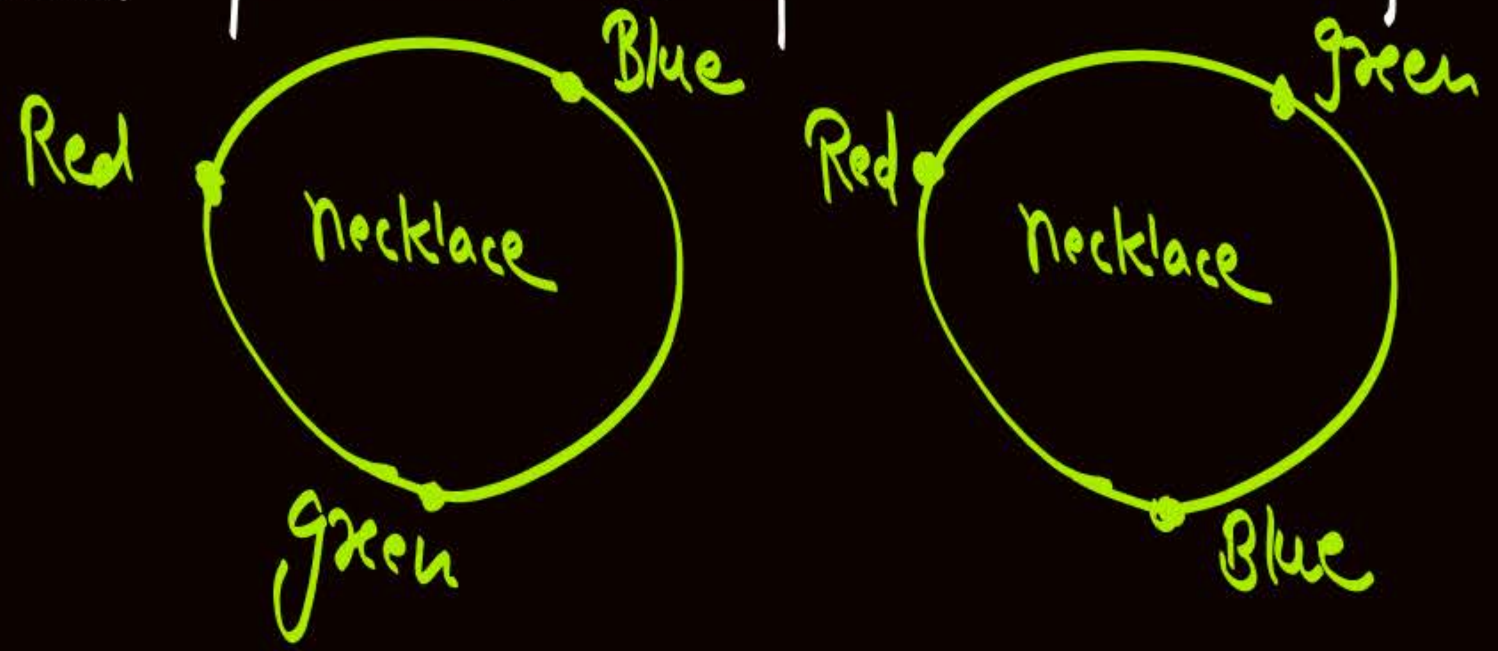
① No. of linear permutations of n-distinct objects = $n!$

② No. of Circular permutations of n-distinct objects = $(n-1)!$

When there is a difference b/w Clockwise & anticlockwise arrangement

③ No. of Circular permutations of n-distinct objects = $\dfrac{(n-1)!}{2}$

When clock wise & anticlockwise are same



Topic : Distance

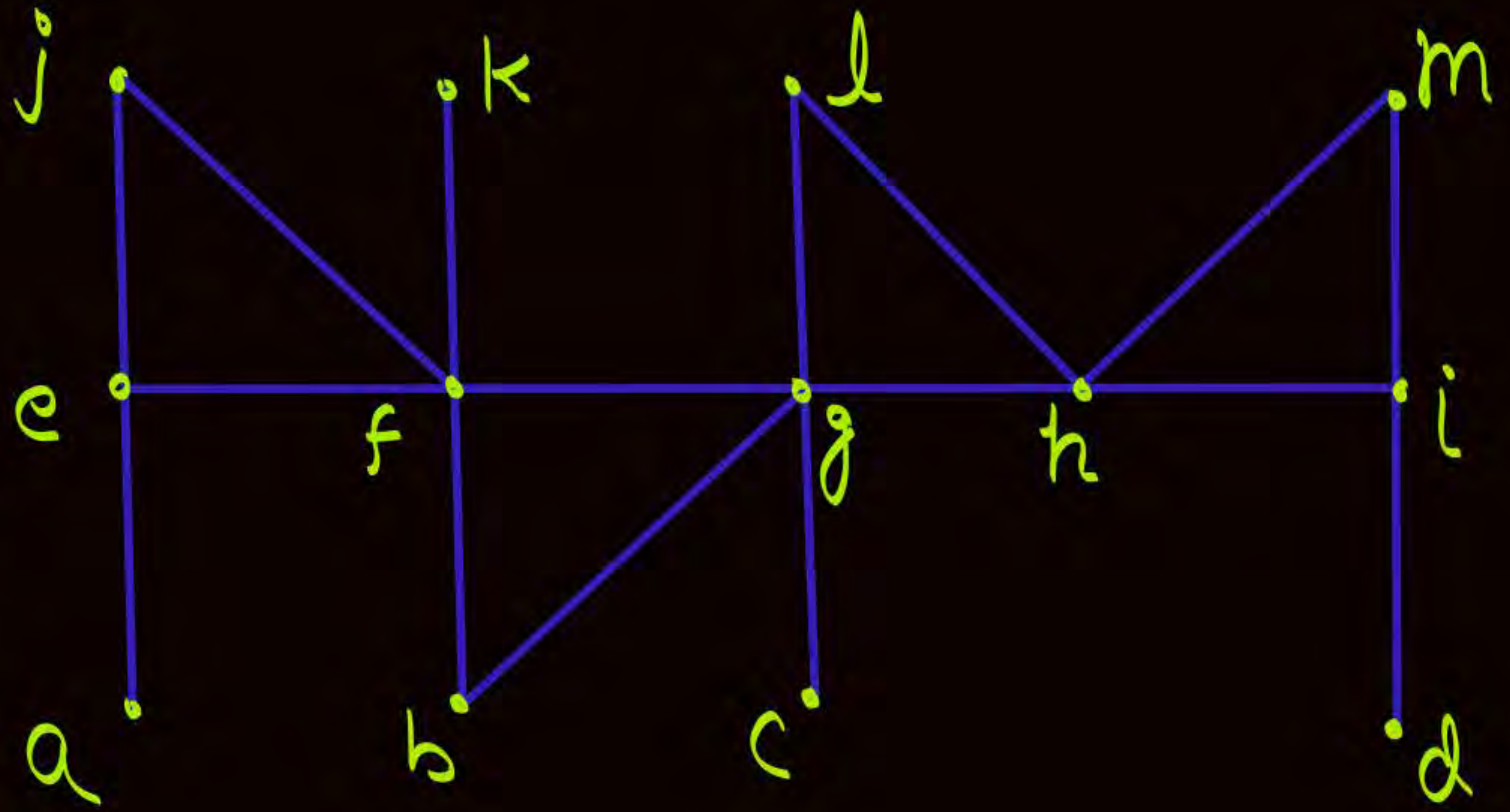
The distance between two vertices of graph G is defined as number of edges on the shortest path between those vertices.

Find distance of each vertex of graph G from vertex 'a'

a - b = 3
a - c = 4
a - d = 6
a - e = 1
a - f = 2
a - g = 3
a - h = 4
a - i = 5

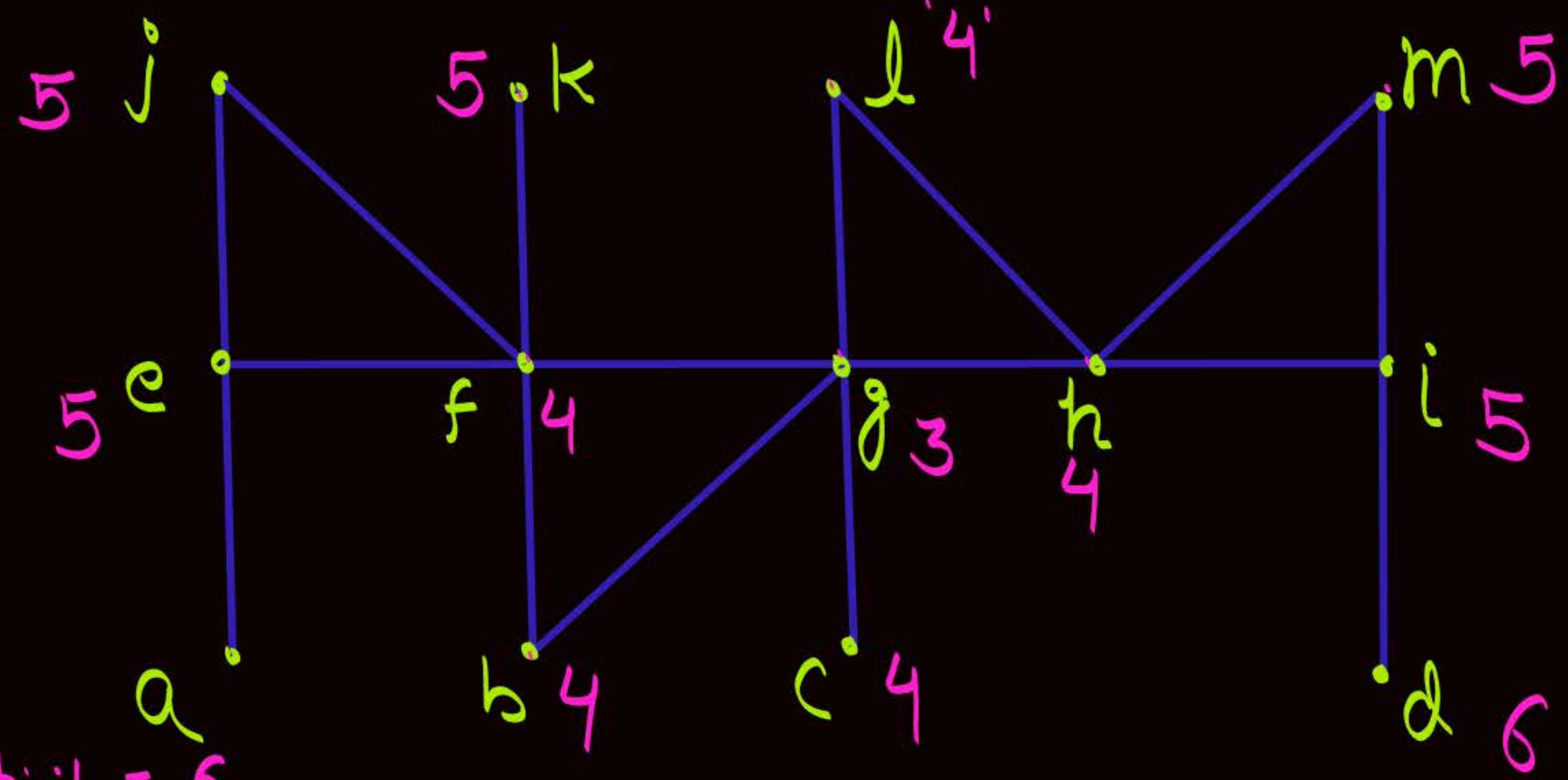
a - j = 2
a - k = 3
a - l = 4
a - m = 5

"G"



Topic : Eccentricity

Eccentricity of vertex $V \in G$ is the maximum distance of vertex 'V' from a Vertex of graph G

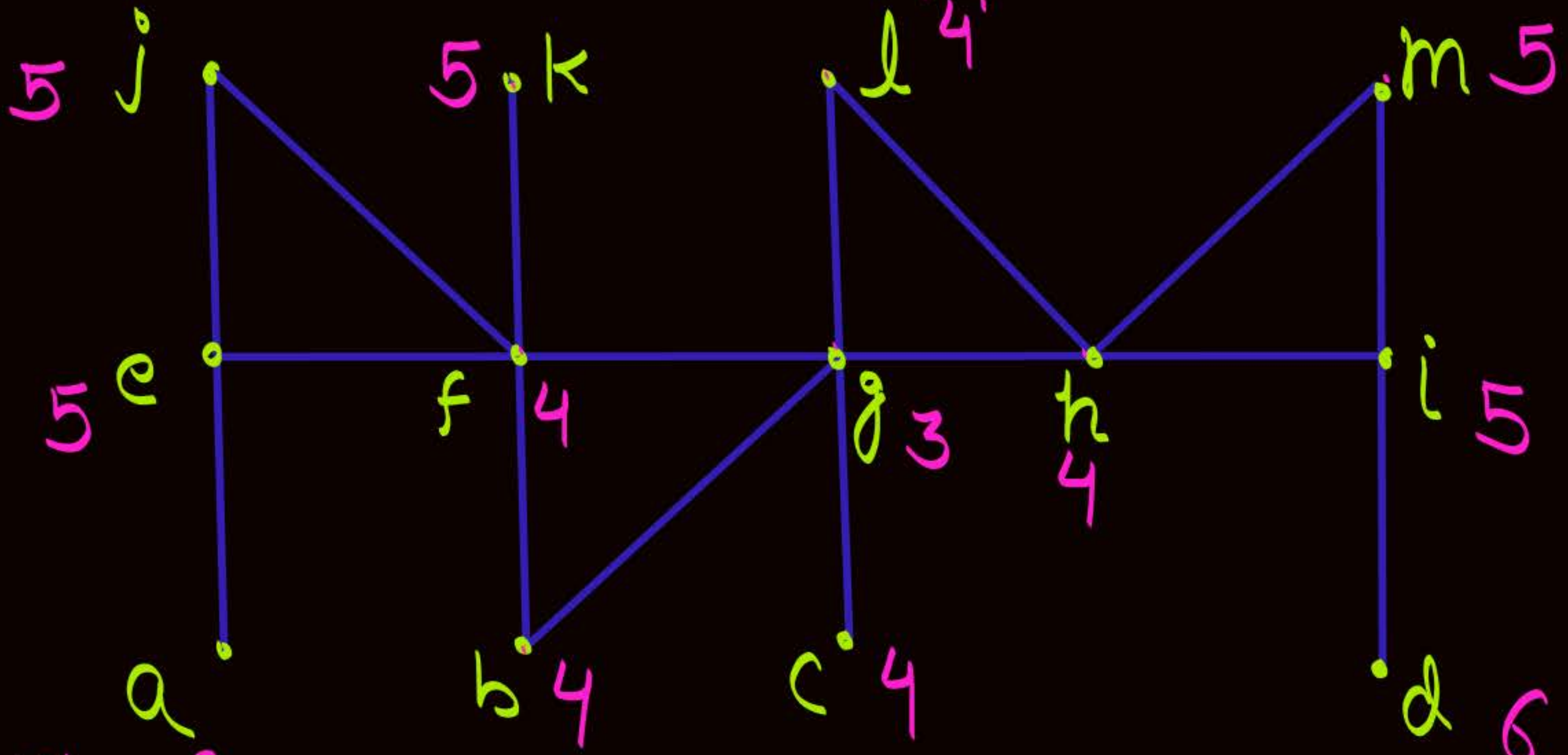


eccentricity = 6
{ w.r.t. vertex 'd' }

Topic : Diameter

{ Diameter of graph G is the distance between two farthest vertices of graph G }

Diameter of graph G is defined as maximum of the eccentricities of all the vertices of graph G.



eccentricity = 6
{ w.r.t. vertex 'd' }

Maximum eccentricity = 6

$\therefore$ Diameter of above graph = 6

Topic : Radius

Radius of graph G is defined as minimum of the eccentricities of all the vertices of graph G.

For the above graph in example

Radius = 3

Topic : Centre

The set of all the vertices with minimum eccentricity is defined as Centre of the graph

In the above example, Centre of the graph = $\{g\}$

There may be more than one vertex with minimum eccentricity value, centre will be defined as set of all those vertices

Topic : Girth

In graph theory, the girth of an undirected graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (that is, it is a forest), its girth is defined to be infinity.

2 mins Summary

- **Topic** Euler trail , Euler Circuit and Traversable graph
- **Topic** Hamiltonian path, and Hamiltonian Circuit
- **Topic** Distance, Eccentricity, Diameter, Radius and Girth

THANK - YOU

Slide