GATE-All BRANCHES
Engineering Mathematics

Linear Algebra

Lecture No.- 10

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Recap of previous lecture

Topic Problems based on eigen values

Topics to be Covered

Topic Properties of eigen values

Topic Problems based on eigen values

Topic LU decomposition — video 19 min

Topic : Linear Algebra

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#Q. The product of the non-zero eigen values of the matrix
$$\begin{array}{ccccc} + & - & + & - & + \end{array}$$
$$\begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$
is________.

λ = Eigen value (Non-ZERO eigen value)

$$|A - I\lambda| = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & (1-\lambda) \end{vmatrix} + 1\begin{vmatrix} 0 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= (1-\lambda)^2\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda \\ \cancel{1} & 1 & 1-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \left[(1-\lambda)^2-1\right]=0$$

$$= \lambda^3(\lambda-2)(\lambda-3)=0$$

$$= \lambda = 0, 0, 0, 2, 3$$

Product of Non-ZERO eigen value $= 2 \times 3$

$$= \boxed{6}$$

$\because$

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H.W

#Q. The matrix $A=\begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$ has three distinct eigen values and one of its eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Which of the following can be another eigen vector of A ?

A $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

C $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

D $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

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$$\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ right

#Q. For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, one of the normalized eigen vectors is given as

A $\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

B $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

C $\begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{bmatrix}$

D $\begin{bmatrix} \frac{1}{5} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda^2 - 6\lambda - 2\lambda + 12 = 0$

$\lambda(\lambda - 6) - 2(\lambda - 6) = 0$

$\lambda = 2 \quad \lambda = 6$

$\lambda = 2, 6$

$\lambda = 2 \rightarrow X_N = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$\begin{bmatrix} 5-2 & 3 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 + 3x_2 = 0$

$x_1 = -x_2$

$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

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#Q. The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the from $\begin{bmatrix} 1 \\ a \end{bmatrix}$ & $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is a +b ?

A 0

B 1/2

C 1

D 2

$\lambda^2 - (3\lambda) + (2) = 0$

$\lambda^2 - 2\lambda - \lambda + 2 = 0$

$\lambda(\lambda-2) - 1(\lambda-2) = 0$

$\underline{\lambda = 1, \lambda = 2}$

$\lambda_1 = 1$

$\begin{bmatrix} 1-1 & 2 \\ 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\underline{x_2 = 0}$ $x_1 = \text{any const}$

$X_2 = 0$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\lambda_2 = 2}$ $\begin{bmatrix} 1-2 & 2 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 = \frac{1}{2}$

$X_2 = \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ $b = 1/2$

$a = 0$ $b = 1/2$ $a + b = 1/2$

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#Q. Which of the following is an eigen vector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ is-

A. $[1\ -2\ \ 0\ \ 0]^T$

B. $[0\ \ 0\ \ 1\ \ 0]^T$

C. $[1\ \ 0\ \ 0\ -2]^T$

D. $[1\ -1\ \ 2\ \ 1]^T$

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Ans} = [1\ \ -2\ \ 0\ \ 0]^T$$

eigen value + eigen vector | Lengthy Process | Two complex Two real

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EE

$\rightarrow$ max $a = f(\lambda)$

#Q. The maximum value of 'a' such that the matrix $\begin{bmatrix} -3 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & a & -2 \end{bmatrix}$ three linearly independent real eigenvectors is

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

A $\frac{2}{3\sqrt{3}}$

B $\frac{1}{3\sqrt{3}}$

C $\frac{1+2\sqrt{3}}{3\sqrt{3}}$

D $\frac{1+\sqrt{3}}{3\sqrt{3}}$

$$A = \begin{bmatrix} -3-\lambda & 0 & -2 \\ 1 & -1-\lambda & 0 \\ 0 & a & -2-\lambda \end{bmatrix}$$

$$a = -\frac{1}{2}\left(\lambda^3 + 6\lambda^2 + 11\lambda + 6\right)$$

$$(-3-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ a & -2-\lambda \end{vmatrix} + 0 - 2\begin{vmatrix} 1 & -1-\lambda \\ 0 & a \end{vmatrix} = 0$$

$$(-3-\lambda)(-1-\lambda)(-2-\lambda) - 2(a) = 0$$

$$2a = (-3-\lambda)(1-\lambda)(-2-\lambda)$$

$$a = -\frac{1}{2}(\lambda+3)(\lambda+1)(\lambda+2)$$

$$a = -\frac{1}{2}\left(\lambda^3 + 6\lambda^2 + 11\lambda + 6\right)$$

$a = f(\lambda) \longrightarrow$ Local max/min

Function – max value.

$$\frac{da}{d\lambda} = 0 \;(\text{Max/min})$$

$$\frac{da}{d\lambda} = -\frac{1}{2}\left(3\lambda^2 + 12\lambda + 11\right)$$

$$-\frac{1}{2}\left(3\lambda^2 + 12\lambda + 11\right) = 0$$

$$3\lambda^2 + 12\lambda + 11 = 0$$

$$\lambda = \frac{-12 \pm \sqrt{144 - 4\times 11\times 3}}{2\times 3} = -2 \pm \frac{1}{\sqrt{3}}$$

$$\lambda = -2 \pm \frac{1}{\sqrt{3}}$$

$\lambda = -2 + \frac{1}{\sqrt{3}}$ max $f''(x) < 0$

$\lambda = -2 - \frac{1}{\sqrt{3}}$ (min) $f''(x) > 0$

Max value

$$a = f(\lambda)$$

$$a = -\frac{1}{2}\left(\lambda^3 + 6\lambda^2 + 11\lambda + 6\right)$$

$$= -\frac{1}{2}\left[\left(-2+\frac{1}{\sqrt{3}}\right)^3 + 6\left(-2+\frac{1}{\sqrt{3}}\right)^2 + 11\left(-2+\frac{1}{\sqrt{3}}\right) + 6\right]$$

$$a = \frac{1}{3\sqrt{3}}$$

$$\frac{da}{d\lambda} = -\frac{1}{2}\left(3\lambda^2 + 12\lambda + 11\right)$$

$$\frac{d^2a}{d\lambda^2} = -\frac{1}{2}\left(6\lambda + 12\right)$$

$$\left[\frac{d^2a}{d\lambda^2}\right]_{\lambda = -2+\frac{1}{\sqrt{3}}}$$

$$= -\frac{1}{2}\left(6 \times \left(-2+\frac{1}{\sqrt{3}}\right) + 12\right)$$

$$= -\frac{1}{2}\left[\cancel{-12} + \cancel{12} + \frac{6}{\sqrt{3}}\right]$$

$$= -\frac{3}{\sqrt{3}} < 0 \text{ max}$$

$$f''(x) < 0$$

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#Q. The set of eigenvalues of which one of the following matrices is NOT equal to the set of eigen values of $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$?

A

A^T

A $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ $\lambda^2 - 4\lambda + (3-8) = 0$
$\lambda^2 - 4\lambda - 5 = 0$

B $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ $\lambda^2 - 4\lambda + (3-8) =$

C $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ $\lambda^2 - 4\lambda + (3-4) = 0$
$\lambda^2 - 4\lambda - 1 = 0$

D $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\lambda^2 - \cancel{6\lambda} + (8-3) =$

#Q. The eigenvalues of an orthogonal matrix are

$\nvdash$ unit modulus

A zero

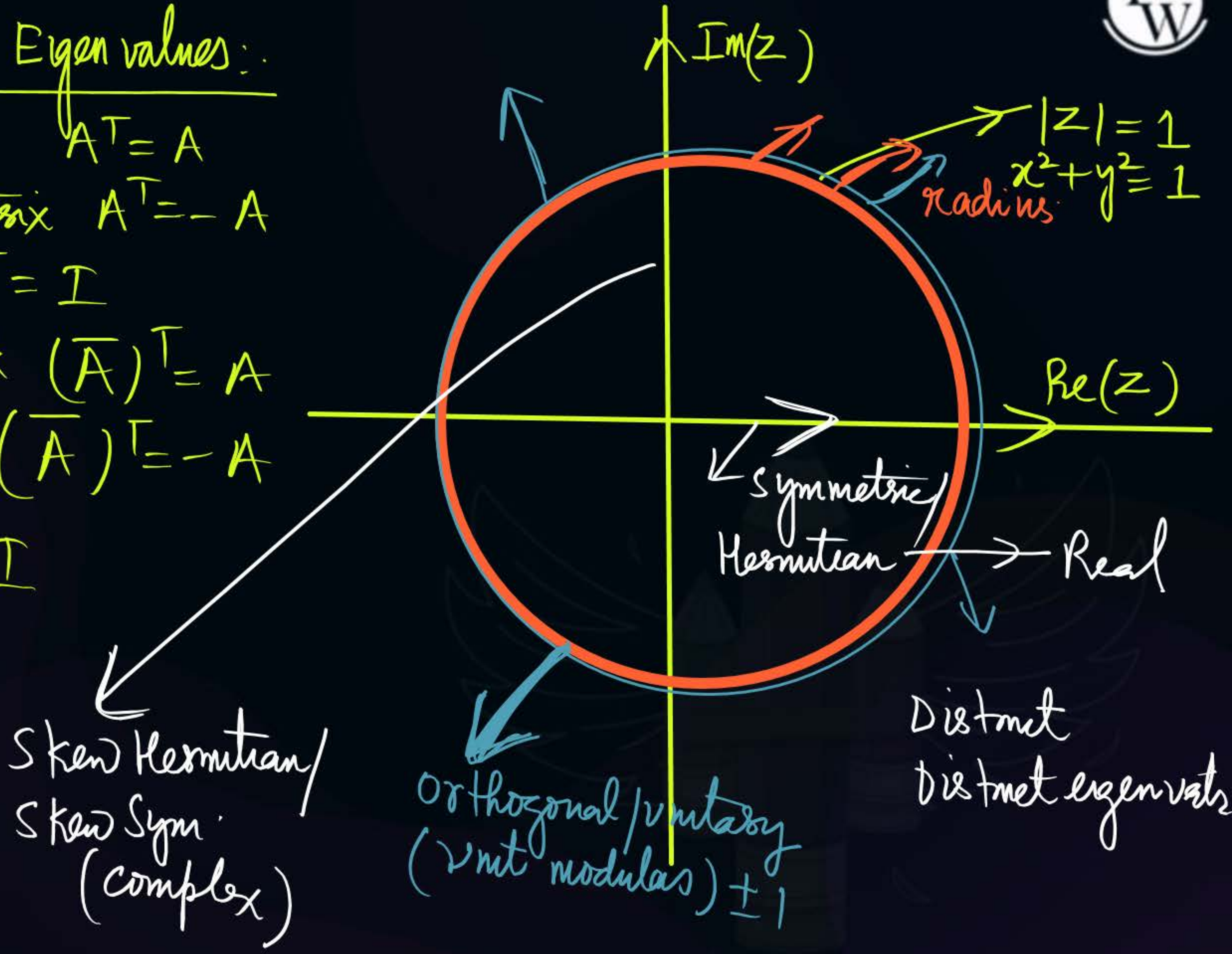
B imaginary

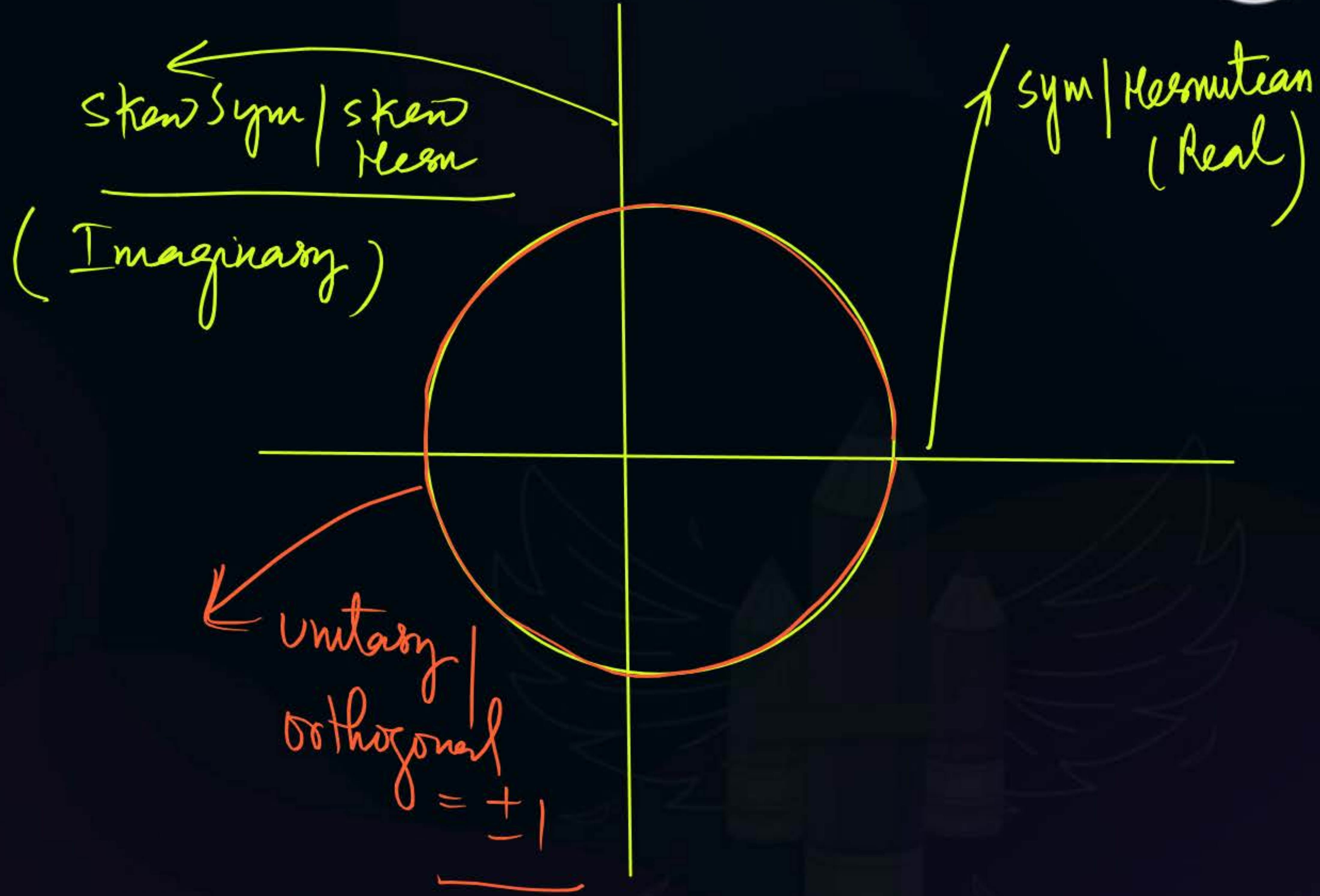
C always negative

D of unit modulus

Behaviour of Eigen values:

- ✓ symmetric matrix $A^T = A$
- ✓ skew symmetric matrix $A^T = -A$
- ✓ orthogonal $AA^T = I$
- ✓ Hermitian matrix $(\bar{A})^T = A$
- ✓ Skew Hermitian $(\bar{A})^T = -A$
- ✓ Unitary $A(\bar{A})^T = I$




Skew Sym | Skew Herm
(Imaginary)
Sym | Hermitean
(Real)
Unitary | orthogonal
= ±1

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#Q. Let $A = \begin{bmatrix} 2 & -\frac{1}{20} \\ 0 & 5 \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$ than a + b =

$AA^{-1} = I$

$\begin{bmatrix} 2 & -\frac{1}{20} \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a = \frac{1}{5}$

$b = \frac{1}{200}$

$a + b = \frac{1}{5} + \frac{1}{200}$

$= \frac{40+1}{200}$

$= \frac{41}{200}$ Ans

A $\frac{41}{100}$

B $\frac{31}{200}$

C $\frac{51}{100}$

D $\frac{41}{200}$

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#Q. The sum of the eigenvalues of the given matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1$

$= 7$ Ans

A 4

B 5

C 7

D None of these

Slide 14

$\lambda^3 - (\text{Trace})\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - \det A = 0$

#Q. If $\begin{pmatrix} 2 \\ y \\ z \end{pmatrix}$, y z ∈ ℝ, is an eigen vector corresponding to a real eigen value of the matrix $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -4 \\ 0 & 1 & 3 \end{pmatrix}$ the z– y is equal to ....

$X = \begin{bmatrix} 2 \\ y \\ z \end{bmatrix}$

real

$y, z \in R$

y, z unknown

$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -4 \\ 0 & 1 & 3 \end{bmatrix}$

eigen value $= 1, 1 \pm i$

$\lambda^3 - (\text{Trace})\lambda^2 + [A_{11}+A_{22}+A_{33}]\lambda - \det A = 0$

x
y
z

$$\begin{bmatrix} 0-1 & 0 & 2 \\ 1 & 0-1 & -4 \\ 0 & 1 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -4 \\ 0 & 1 & +2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$= -x_1 + 2x_3 = 0$

$x_1 - x_2 - 4x_3 = 0$

$x_2 + 2x_3 = 0$

$-x_1 = -2x_3$

$x_1 = 2x_3$

$2x_3 - 4x_3 = x_2$

$-2x_3 = x_2$

$x_3 = k \quad \underline{x_2 = -2k}$

$x_1 - x_2 - 4x_3 = 0$

$x_1 = 4x_3 + x_2 = 4k - 2k$

$= 2k$

$$\Rightarrow \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$z - y$

$= 1 + 2 = 3$

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#Q. Let $\alpha,\beta,\gamma,\delta$ the eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

H.W

Then $\alpha^2+\beta^2+\gamma^2+\delta^2 = \ldots..$

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eigen value

$\lambda_1, \lambda_2, \lambda_3 \rightarrow |\lambda_1|, |\lambda_2|, |\lambda_3|$

#Q. A system matrix is given as follows : $A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$ The absolute value of the ratio of the maximum eigen value to the minimum eigen value is ________.

[absolute value]

$\boxed{|A - I\lambda| = 0}$

$\frac{\text{Max absolute}}{\text{min absolute}}$

$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$= (\lambda+1)(\lambda+2)(\lambda+3) = 0$

$\lambda_1 = -1$ absolute value $|-1| = 1$

$\lambda_2 = -2$ $|-2| = 2$

$\lambda_3 = -3$ $|-3| = 3$

$\text{Ratio} = \frac{\text{max}}{\text{min}} = \frac{3}{1} = \underline{\underline{3}}$

THANK - YOU