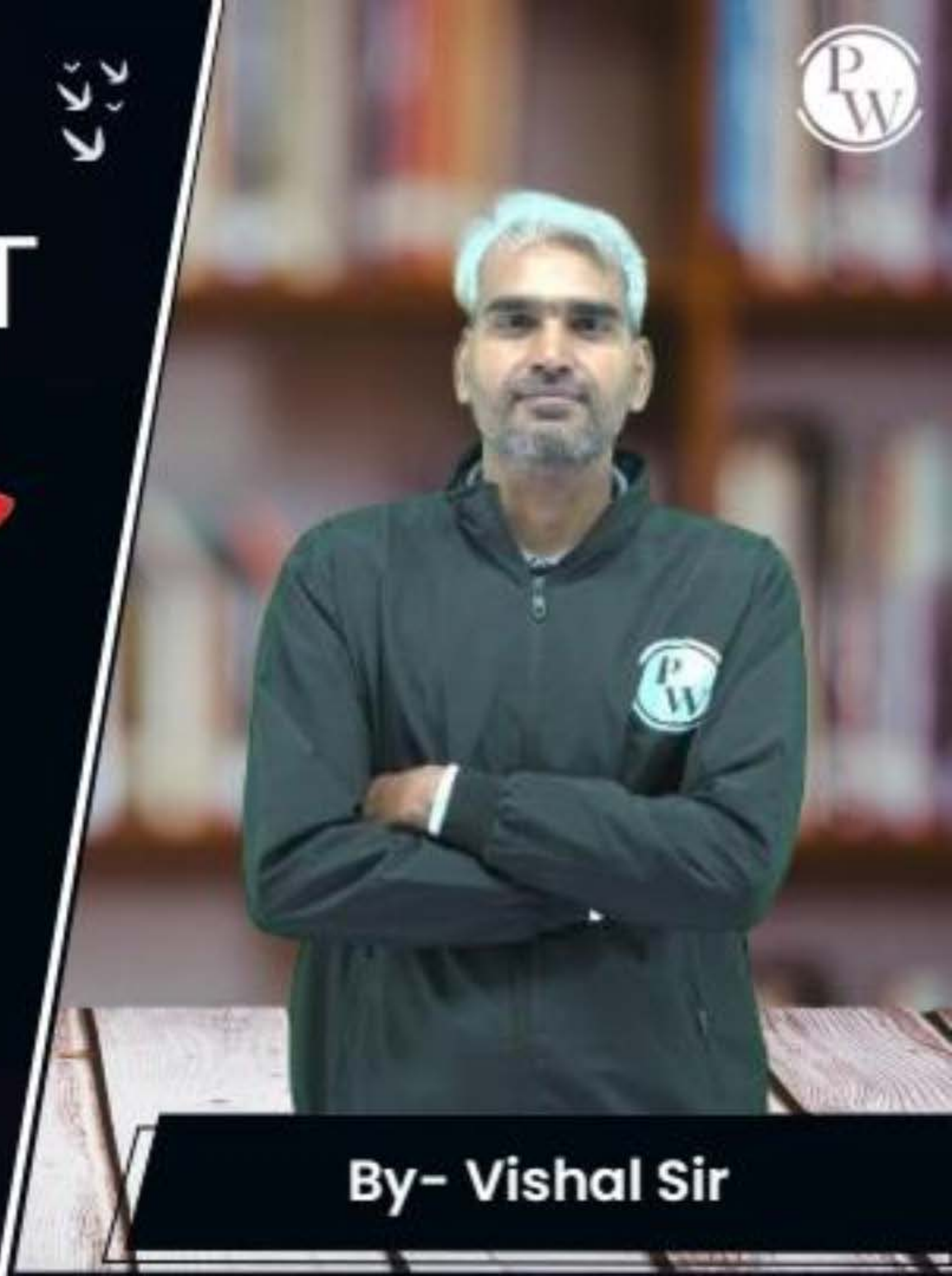
Computer Science & IT

Discrete Mathematics

Mathematical Logic

Lecture No. 05



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Recap of Previous Lecture

- **Topic** Practice questions
- **Topic** Proof by contradiction
- **Topic** Conditional proof rule
- **Topic** Predicate

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Topics to be Covered

- **Topic** Quantifiers
- **Topic** Scope of the quantifier
- **Topic** Negation of a statement formula

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Topic : Predicate

- It is part of a sentence or clause stating something about the subject.

$\underbrace{\text{Ram}}_{\text{Subject}}$ $\underbrace{\text{is a politician}}_{\text{Predicate}}$ → { This statement may be true or may be false }

Consider, P : is a politician { i.e., P is used to denote the predicate "is a politician" }

∴ P(Ram) : Ram is a politician

Subject

Predicate 'P' is applied over the subject 'Ram'

{ P(Ram) may be true
may be false }

let Predicate, S: is a sportsman

S(Ram): Ram is a sportsman. { May be true or false }

S(Mohan): Mohan is a sportsman. { May be true or false }

P: is a politician
S: is a sportsman

- $P(x)$: x is a politician
- $S(x)$: x is a sportsman

- $P(x) \longrightarrow S(x)$: if x is a politician then x is a sportsman

if 'x' is not a politician, then $P(x)$ will return false, and implication will return true irrespective of the truth value of Predicate $S(x)$.

If x is a politician {i.e. $P(x)$ is true}, then x must also be a sportsman for implication to be true

{if $P(x) = T$ & $S(x) = F$ then $P(x) \rightarrow S(x)$ is false.}

* $P(x) \longrightarrow S(y)$

  - For this predicate to be true,
    either x should not be a politician
    (or)
    if x is a politician then y must be sportsman

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* $P(x) \vee S(x)$:
  For this statement to be true,
  either x should be a politician or x should be sportsman

$P(x) \land S(x)$

$\hookrightarrow$ x must be politician as well as Sportsman

$P(x) \longleftrightarrow S(y)$: x is a politician iff y is a sportsman

$\hookrightarrow$ It will return true in two cases

(i) $P(x) = T$ & $S(y) = T$

& (ii) $P(x) = F$ & $S(y) = F$

Topic : Predicate with multiple subjects

Let Predicate, F : is a friend of

$F(x, y)$: x is a friend of y

Predicate

applied over x & y

F(Krishna, Sudama).

Consider Predicate, G: is greater than

$G(a, b)$: a is greater than b.

$G(2,3)$: $\underbrace{\text{2 is greater than 3}}_{\text{it is false}}$

$\therefore$ $G(2,3)$ will return false

$G(5,1)$: it returns true.

Let Predicate $F(x, y, t)$ denotes that

Person 'x' can Fool, person 'y', at time 't'

Topic : Quantifiers

- In predicate logic, predicates are used alongside quantifiers to express the extent to which a predicate is true over a range of elements → over the elements in the universe of discourse
- There are two types of quantifiers

1. Universal Quantifier (for all)
2. Existential Quantifier (for some / for at least one)

Topic : Universal Quantifier ($\forall$)

$\forall$: for all

$\forall_x$: for all x in the universe of discourse.

it represent the domain of the elements/objects on which Predicate will be applied

$\underbrace{\forall x \{P(x)\}}$: for all x, x is politician

it will return true only if all Person in the domain are Politician

{ Truth value of this Predicate formula will depend on the truth value produced by $\forall$ }

* W.r.t. $\forall_x \{P(x)\}$, $\forall$ will return true only if Predicat 'P' is not false for any element of the domain

Note: $\forall$ will return false if and only if Predicat is false for at least one element of domain.

Note:- If universe (domain) is empty, then for all ($\forall$) will always return true.

Note:

$$\underbrace{\forall x}_{\text{quantifier}} \; \underbrace{\{\, P(x) \,\}}_{\text{Scope of the quantifier}}$$

P: is a politician, S: is a sportsman

① $\underbrace{\forall x \{ P(x) \rightarrow S(x) \}}$

For this predicate formula to be true '$\forall$' must return true. i.e. $\forall x$, implication must be true

* For this predicate formula to be true, $P(x) \rightarrow S(x)$ should be true for all x.
* If there is any 'x' in the domain for which 'P' is true but 'S' is false then predicate formula will return false, otherwise it will always return true.

② $\underbrace{\forall x \{ P(x) \} \longrightarrow \forall x \{ S(x) \}}$

for this predicate formula to be true, implication must return true

$F \longrightarrow T/F \quad = T$

$T \longrightarrow T \quad = T$

① If P is false for any x in domain then $\forall x \{ P(x) \}$ will be false and hence implication will return true.

② If P is true for all x, then S must also be true for all x, for implication to be true

(1) $\forall x \{ P(x) \rightarrow S(x) \}$

<u>for all</u> x, if P is true then S must be true

(2) $\forall x \{ P(x) \} \rightarrow \forall x \{ S(x) \}$

if P is true for all x, then S must also be true for all x

Which of the following Predicate formula is $\underbrace{\text{Valid}}_{\text{always true.}}$?

(a) $\left(\forall x\{P(x) \rightarrow S(x)\}\right) \longrightarrow \left(\forall x\{P(x)\} \rightarrow \forall x\{S(x)\}\right)$

(b) $\boxed{\underbrace{\left(\forall x\{P(x)\} \rightarrow \forall x\{S(x)\}\right)}_{\text{LHS}} \longrightarrow \underbrace{\left(\forall x\{P(x) \rightarrow S(x)\}\right)}_{\text{R.H.S.}}}$

it is not valid because its truth value may be false

LHS = True

$U = \{1, 2, 3\}$

$\left.\begin{array}{l} P(1) = F \\ P(2) = T \\ P(3) = T \end{array}\right\} \forall x\{P(x)\} = F$ | $\left.\begin{array}{l} S(1) = F \\ S(2) = F \\ S(3) = F \end{array}\right\} \forall x\{S(x)\} = F$

R.H.S.

$\left.\begin{array}{l} \underset{F}{P(1)} \longrightarrow \underset{F}{S(1)} = T \\ \underset{T}{P(2)} \longrightarrow \underset{F}{S(2)} = F \\ \underset{T}{P(3)} \longrightarrow \underset{F}{S(3)} = F \end{array}\right\}$

$\forall x\{P(x) \rightarrow S(x)\}$ is false

Topic : Existential Quantifier (∃)

$\exists$: for some (or) for at least one

$\exists_x$: for some x in the universe of discourse
(or)
for at least one 'x' in the universe of discourse.

P: is a Politician

$\underbrace{\exists x \{P(x)\}}$: Some x are politician
At least one x is politician

It will return true if at least one person in the set of domain is a Politician

Note: $\exists$ will return true if and only if predicate is true for at least one element of domain

Note: If universe (domain) is Empty, then there exists ($\exists$) will always return false

Note

$$\underbrace{\exists x}_{\text{quantifier}} \underbrace{\{ P(x) \}}_{\text{scope of the quantifier}}$$

$$\underbrace{\exists x}_{\text{quantifier}} \underbrace{\{ P(x) \to Q(x) \}}_{\text{Scope of the quantifier}}$$

Topic : Scope of the Quantifier

- The part of the logical expression to which a quantifier can be applied is called the scope of the quantifier.

- Scope of the quantifier is either represented explicitly using bracket or comma, or The Scope of a quantifier is the shortest full sentence/predicate formula which follows it. Everything inside this shortest full sentence is said to be in the scope of the quantifier.

quantifier

Topic : Scope of the Quantifier

- A variable whose occurrence is bounded by a quantifier is called a bounded variable. Variables not bounded by any quantifiers are called free variables.

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$
- $\exists y(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$
- $(\forall y\, H(y) \rightarrow \exists z\, G(z)) \rightarrow \exists z\, W(z, y)$
- $(\forall z\, H(y) \rightarrow \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$
- $(\forall y\, H(y) \rightarrow \exists z \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$
- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists x\, G(z)$

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists z\, G(z)$

bounded Variable

bounded Variable

free Variable

bounded

Topic : Scope of the Quantifier

- $\exists y\,(\underbrace{\forall y\, H(y)}_{\text{bounded Variable}} \rightarrow \overset{\text{bounded}}{\exists z\, W(z, y)}) \rightarrow \underbrace{\exists z\, G(z)}_{\text{bounded variable}}$

all 'y' which are not bounded by any quantifier will be quantified using $\exists y$

bounded Variable

Note: No variable can be quantified by multiple quantifier

$\Downarrow$ We can re-write the predicate formula as

$$\exists x \left\{ \forall y\, H(y) \longrightarrow \exists z\, W(z, x) \right\} \longrightarrow \exists z\, G(z)$$

Topic : Scope of the Quantifier

- $$(\forall y\, H(y) \rightarrow \exists z\, G(z)) \rightarrow \exists z\, W(z, y)$$

bounded bounded bounded

free variable

Topic : Scope of the Quantifier

- $(\forall z\ H(y) \rightarrow \exists y\ W(z, y)) \rightarrow \exists z\ G(z)$

$\forall$ is trying to quantify variable 'z'

free variable

bounded variable

free variable

bounded

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z \exists y\, W(z, y)) \rightarrow \exists z\, G(z)$

bounded

bounded

bounden

bounded

Topic : Scope of the Quantifier

- $(\forall y\, H(y) \rightarrow \exists z\, W(z, y)) \rightarrow \exists x\, G(z)$

bounded

bounded

free variable

free variable

Note:-

We can not quantify a single variable using multiple quantifiers, even if we see something like that then variable will be quantified using inner-most quantifier

e.g. $\exists y\ \forall y\ H(y) \rightarrow \exists z\ W(z, y)$

'y' of predicate H is under the scope of $\exists y$ as well as $\forall y$ } $\Rightarrow$ but it will be quantified using innermost quantifier i.e, $\forall y$

Topic : Note

- Let, P(x): x is true.

~P(x): x is false

{Consider the universe as a set of some boolean expression}

	Form	Meaning
1	$\forall x\, P(x)$	All are true / Not at least one is false
2	$\exists x\, P(x)$	Some are true (At least one is true) / Not all are false
3	$\sim \forall x\, P(x)$	Not all are true / At least one is false
4	$\sim \exists x\, P(x)$	Not at least one true / All are false
5	$\forall x \sim P(x)$	All are false / Not at least one true
6	$\exists x \sim P(x)$	At least one false / Not all are true
7	$\sim \forall x \sim P(x)$	Not all are false / At least one true
8	$\sim \exists x \sim P(x)$	Not at least one false / All are true

Topic : Equivalences

1. $\forall x\, P(x) \equiv \sim \exists x(\sim P(x))$

2. $\exists x\, P(x) \equiv \sim \forall x(\sim P(x))$

3. $\sim \forall x\, P(x) \equiv \exists x(\sim P(x))$

4. $\sim \exists x\, P(x) \equiv \forall x(\sim P(x))$

Topic : Negation of a statement formula

In order to negate a given statement,

We need to replace $\exists$ by $\forall$ and $\forall$ by $\exists$

and finally we need to negate the scope of the corresponding quantifier.

Q: Write the negation of the statement formula

$$\forall x \{ P(x) \rightarrow Q(x) \}$$

Negation of statement formula $\forall x \{ P(x) \rightarrow Q(x) \} \equiv \sim \forall x \{ P(x) \rightarrow Q(x) \}$

$$\equiv \exists x \{ \sim (P(x) \rightarrow Q(x)) \}$$

$$\equiv \exists x \{ \sim (\sim P(x) \vee Q(x)) \}$$

$$\equiv \exists x \{ P(x) \wedge \sim Q(x) \}$$

Q. The statement formula

$\forall x \{B(x) \wedge I(x)\}$ is/are equivalent to

(a) $\exists x \{B(x) \rightarrow \sim I(x)\}$

(b) $\sim \exists x \{B(x) \rightarrow \sim I(x)\}$ ✓

(c) $\exists x \{\sim B(x) \rightarrow I(x)\}$

(d) $\sim \exists x \{\sim B(x) \rightarrow I(x)\}$

$$A \equiv \sim(\sim A)$$

$$\forall x \{B(x) \wedge I(x)\} \equiv \sim\{\sim \{\forall x \{B(x) \wedge I(x)\}\}\}$$

$$\equiv \sim\{\exists x \{\sim(B(x) \wedge I(x))\}\}$$

$$\equiv \sim \exists x (\sim B(x) \vee \sim I(x))$$

$$\equiv \sim \exists x \{B(x) \rightarrow \sim I(x)\}$$

2 mins Summary

Topic Quantifiers

Topic Scope of the quantifier

Topic Negation of a statement formula

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THANK - YOU

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