ENGINEERING MATHEMATICS

ALL BRANCHES

Probability and Statistics

DPP 01 Discussion Notes
(Part-03)

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TOPICS TO BE COVERED

01 Question

02 Discussion

Q. Questions

$\uparrow$ coin Tossed

$\rightarrow n = 3$ times

A fair coin is tossed 3 times. Let the random variable X denote the number of heads in 3 tosses of the coin. Find the sample space, the space of the random variable, and the probability density function of X.

X = No. of HEADS.

Probability Density FUNCTION

(a) $(3/x)\,(1/2)^{2x}\,(1/2)^{(2-x)}$

(b) $(3/2x)\,(1/2)^{x}\,(1/2)^{(1-x)}$

(c) $(3/x)\,(1/2)^{x}\,(1/2)^{(3-x)}$

(d) $(3/x)\,(1/2)^{x}\,(1/2)^{(4-x)}$

$n = 3$ coins Toss.

$X = 0, 1, 2, 3$ HEADS

$P(X=0) = \frac{1}{8}$ $P(X=2) = \frac{3}{8}$

$P(X=1) = \frac{3}{8}$ $P(X=3) = \frac{1}{8}$

HEADS = SUCCESS

TAIL = failure.

$X = 0, 1, 2, 3$

X	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

category
- SUCESS (HEAD) $p = \frac{1}{2}$
- FAILURE (Tailure) $q = \frac{1}{2}$

$n = 3$

^nC_r

n = No. of Trials

r = SUCCESS

0 SUCESS 1 SUCCESS 2 3

$P(X = x \text{ SUCCESS})$

$$\boxed{P(X = x \text{ success}) = {}^3C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}}$$

$X = 0$
$X = 1$
$X = 2$
$X = 3$

$x = 0, 1, 2, 3$ HEAD

^nC_x n Total
x SUCESS

Q. Questions

$$\sum_{x=1}^{12} x = \frac{x \times (x+1)}{2} \qquad \sum_{x=1}^{\infty} P[X=x]=1$$

$$1+2+3+4+\dots+12 = \frac{12\times 13}{2} \qquad \text{Total} = 1$$

If the probability of a random variable X is given by $f(x) = k\,(2x - 1), x = 1, 2, 3 \ldots, 12$. Find k.

$$\boxed{k = \frac{1}{144}}$$

$\boxed{f(x) = k(2x-1)}$ $x = 1, 2, 3 \dots 12$ find k

Total prob $= 1$ density function $= 1$

$$\sum_{x=1}^{12} k(2x-1) = 1$$

$$\sum_{x=1}^{12} 2kx - k = 1$$

$$2k\sum_{x=1}^{12} x - k\sum_{x=1}^{12} 1 = 1$$

Terms

$$2k \times \frac{12\times 13}{2} - k\times 12 = 1$$

$$\boxed{k = \frac{1}{144}}$$

Q. Questions

$X \le 2 \quad X > 1$

The density function for the continuous random variable X is

$$f_X(x) = \begin{cases} e^{-x} \text{ for } x > 0 \\ 0 \text{ for } x \le 0 \end{cases}$$

Find the Probability P [X ≤ 2 | X > 1].

$f(x) = \begin{cases} e^{-x} \text{ for } x > 0 \\ 0 \text{ for } x \le 0 \end{cases}$

Intersection

$P\left[\frac{X \le 2}{X > 1}\right] = \frac{P\left[\boxed{X \le 2 \cap X > 1}\right]}{P[X > 1]}$

$\Rightarrow \int_1^2 \quad x \le 2 \cap x > 1$

$\int_1^\infty e^{-x} dx$

Happening

$P\left[\frac{A}{B}\right]$ = Conditional Prob

B is already Happened $= \frac{P\boxed{A \cap B}}{P(B)}$

$\Rightarrow \frac{\int_1^2 e^{-x} dx}{\int_1^\infty e^{-x} dx}$

$$P(X\le 2 / X\ge 1) = \frac{\int_1^2 e^{-x}\,dx}{\int_1^\infty e^{-x}\,dx} = \frac{\left[-e^{-x}\right]_1^2}{\left[-e^{-x}\right]_1^\infty} = \frac{-e^{-2}+e^{-1}}{+e^{-1}}$$

$$\frac{P(X\le 2 \wedge X\ge 1)\ \text{common}}{(X\ge 1)}$$

$$= \frac{e^{-1}-e^{-2}}{e^{-1}}$$

$$P\left(\frac{X\le 2}{X\ge 1}\right) = 1-e^{-1} \text{ Ans}$$

Ans ✓

Q. Questions

#

A continuous random variable X has density function

$$f(x) = \begin{cases} 2x & 0 < x < \frac{1}{2} \\ \frac{4-2x}{3} & \frac{1}{2} \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find P [0.25 < x ≤ 1.25]

$\rightarrow \frac{3}{4}$

$P[0.25 < x \le 1.25]$

$\Rightarrow \int_{0.25}^{0.50} 2x\,dx + \int_{0.50}^{1.25} \left(\frac{4-2x}{3}\right) dx$

$\Rightarrow$ $\boxed{P[0.25 < x \le 1.25] = \frac{3}{4}}$ Ans

3 elsewhere

2

1.25 $\rightarrow \left(\frac{4-2x}{3}\right)$

0.5

$2x$

0.25

0

Q. Questions

Let X be a continuous random variable with probability function

$$f(x) = \frac{1}{2} e^{-|x-1|}, -\infty < x < \infty$$

Find the value of $P(1 < |X| < 2)$

Q. Questions

$-\frac{1}{2}$ -1 0 1 $\frac{1}{4}$ $\frac{1}{4x^2}$

The probability function of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{4} & \boxed{|x| \le 1} \\ \frac{1}{4x^2} & \text{otherwise} \end{cases}$$

Then $P\left(-\frac{1}{2} \le X \le 2\right) =$ $\frac{1}{2}$ ________.

$$f(x) = \begin{cases} \frac{1}{4} & \boxed{|x| \le 1} \quad \boxed{-1 \le x \le 1} \\ \frac{1}{4x^2} & \text{otherwise} \end{cases}$$

$$P\left(-\frac{1}{2} \le X \le 2\right) = \int_{-1/2}^{1} \frac{1}{4}\,dx + \int_{1}^{2} \frac{1}{4x^2}\,dx$$

$$= \frac{1}{2}$$

Q. Questions

Let X be a continuous random variable with the probability density function

$\int_a^b f(x)\,dx = 1$ valid pdf $= 1$

k constant

$\int_0^2 \frac{x}{8}dx + \int_2^4 \frac{K}{8}dx + \int_4^6 \frac{(6-x)}{8}dx = 1$

$\boxed{K=2}$

$$f(x) = \begin{cases} \frac{x}{8} & 0 < x < 2 \\ \frac{k}{8} & 2 \le x < 4 \\ \frac{6-x}{8} & 4 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{P[1<x<5] = 0.875}$

$P[1<x<5] = \int_1^2 \frac{x}{8}dx + \int_2^4 \frac{(2)}{8}dx + \int_4^5 \frac{(6-x)}{8}dx$

$K=2$

where k is a real constant. Then P $(1 < X < 5)$ equals ____________.

$= \int_1^5 f(x)\,dx$

Q. Questions

Suppose the random variable X has a probability density function

$$f(x) = \begin{cases} \dfrac{|x|}{4}, & -c \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

The value of c is

(a) 0.5

(b) 1

$\boxed{\text{(c) } 2}$ Option c

(d) 4

$$\int_{-c}^{c} \frac{|x|}{4}\,dx = 1$$

$$f(x) = \begin{cases} \dfrac{|x|}{4} & -c \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

Then the value of c:

If this valid pdf $\int_{-c}^{c} f(x)\,dx = 1$

$|x|$ is Even FUNCTION

$$\begin{cases} f(x) = |x| \\ f(-x) = |-x| = x \end{cases}$$

$$f(x) = f(-x)$$

This means $|x|$ is Even FUNCTION

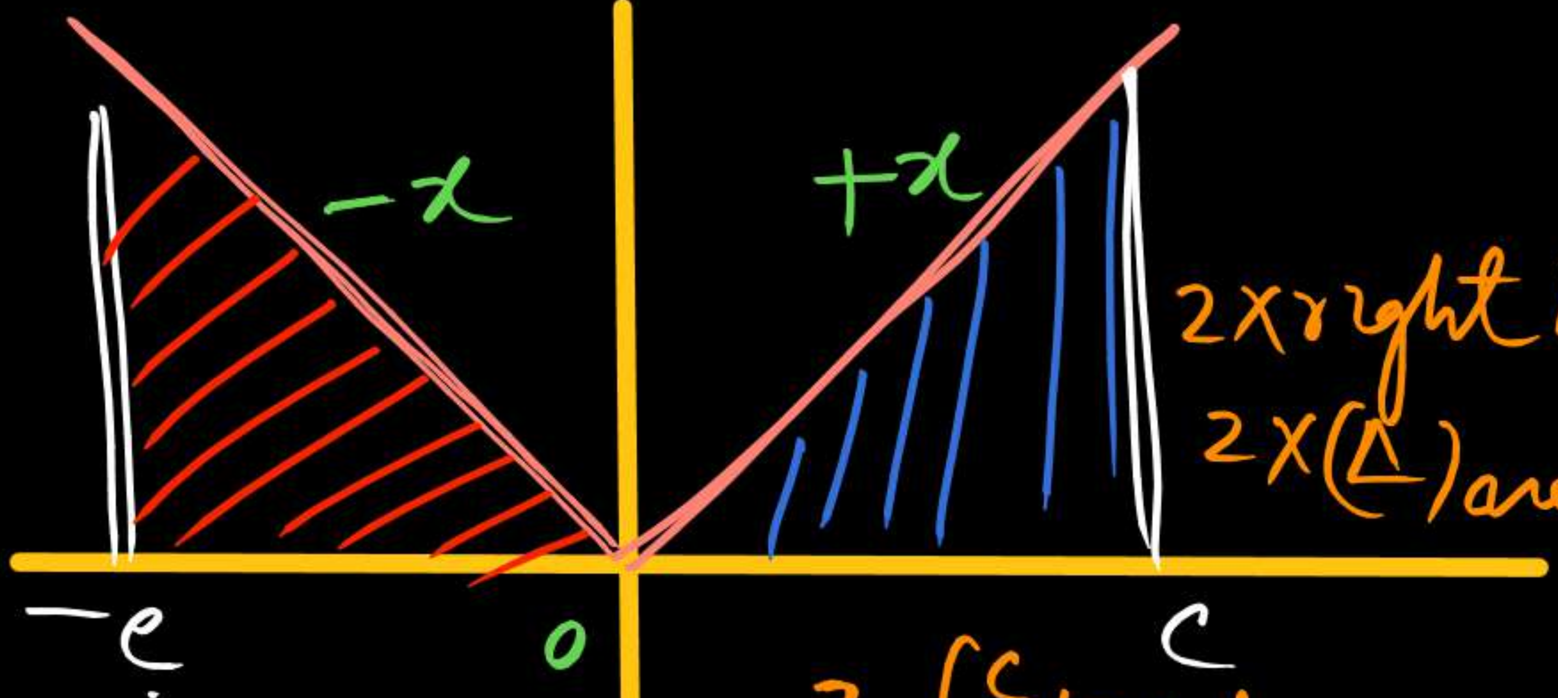
$$= 2\int_0^c \frac{x}{4}\,dx = 1$$

$$= 2\left[\frac{x^2}{8}\right]_0^c = 1$$

$$= 2\left[\frac{c^2}{8}\right] = 1$$

$$= c^2 = 4$$

$$c = \pm 2$$



$2\times$ right half

$2\times(\Delta)$ area

$$2\int_0^c \frac{|x|}{4}\,dx$$

$$= 2\int_0^c \frac{x}{4}\,dx$$

$$\int_{-c}^{0} \frac{-x}{4}\,dx + \int_0^c \frac{x}{4}\,dx$$

$$\left.\begin{aligned} c &= +2 \\ c &= -2 \end{aligned}\right]$$

Ans

Q. Questions

A random variable X has probability density function

$$f(x) = \begin{cases} kx(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

valid pdf

$$\int_0^1 kx(1-x) = 1$$

$$\checkmark \boxed{k = 6}$$

The value of k is

$$\int_0^1 kx(1-x)\,dx = 1$$

$$k = 6$$

(a) 2

(b) 6

(c) 5

(d) 4

Q. Questions

The probability distribution of a discrete random variable X is given in the table below.

$P(1 < X \leq 4) = 0.55$

The $P(1 < X \leq 4)$ is

(a) 0.55

(b) 0.85

(c) 0.70

(d) 0.40

x	0	1	2	3	4	5
P (X = x)	0.1	0.3	0.15	0.25	0.15	0.05

Q. Questions

Suppose the random variable X has a probability density function

value $\left\{ f(x) = \begin{cases} kx^3e^{-x/2}, & x > 0 \\ 0 & \text{otherwise} \end{cases} \right.$

$f(x) = kx^3 e^{-\frac{x}{2}}$ $x > 0$

The value of K

If this FUNCTION is valid pdf.

The vale of k is

$\int_a^b f(x)\,dx = 1$

(a) $\boxed{1/96}$ ✓

(b) 96

(c) 8/3

(d) 1/4

$\int_0^\infty kx^3 e^{-x/2}\,dx = 1$

Dummy vari $\begin{cases} \checkmark x \\ \checkmark t \end{cases}$

$$\int_0^\infty K x^3 e^{-\frac{x}{2}}\, dx = 1$$

$$\int_{t=0}^\infty K (2t)^3 e^{-t} \cdot 2\,dt = 1$$

$\frac{x}{2} = t$ (Transformation)

both sides Differentiate

$\frac{dx}{2} = dt$ $\boxed{dx = 2dt}$ [Transformation]

$\boxed{x = 2t}$

$x = 0 \quad t = 0$

$x = \infty \quad t = \infty$

$$\Rightarrow K \int_0^\infty 8 t^3 e^{-t} \cdot 2\,dt = 1$$

$$\Rightarrow K \cdot 16 \boxed{\int_0^\infty t^3 e^{-t}\,dt = 1}$$

GAMMA FUNCTION $\int_0^\infty e^{-t} t^{n-1}\,dt = \Gamma n$

$\Gamma n = (n-1)!$

$n - 1 = 3$

$n = 4$

$$\Rightarrow k \cdot 16\, \Gamma 4 = 1$$

$\Gamma 4 = 3! = 6$

$3 \times 2 \times 1 = 6$

$$\Rightarrow k \cdot 16 (6) = 1$$

$$\boxed{k = \frac{1}{96}}$$ ✓ Ans

Q. Questions

Let X be a continuous random variable with pdf

$[_0 \quad {}^{1}/_{3} \quad \overset{cx^2}{} \quad {}^{2}/_{3} \quad]_1$

$$f(x) = \begin{cases} \boxed{cx^2, \quad \text{for } 0 < x \leq 1,} \\ 0 \quad \text{otherwise} \end{cases}$$

For some positive constant c. The value of P $\left(X \leq \frac{2}{3} \middle| X > \frac{1}{3}\right)$ is

(a) 3/26

(b) 5/26

$\boxed{\text{(c) } 7/26}$ ✓

(d) 11/26

$P\left(X \leq \frac{2}{3} \middle| X > \frac{1}{3}\right)$

common event

$= \frac{P\left(X \leq \frac{2}{3} \cap X > \frac{1}{3}\right)}{P\left(X > \frac{1}{3}\right)}$

$\Rightarrow \frac{\int_{1/3}^{2/3} cx^2\, dx}{\int_{1/3}^{1} cx^2\, dx} \Rightarrow \frac{7}{26}$

Q. Questions

Suppose the random variable X has the probability density function

$$f(x)=\begin{cases} ce^{x/3}, & x\le 0,\\ ce^{-x/3}, & x>0,\end{cases}$$

For some positive constant c. The value of P [X > 6/X > 0] is

(a) $\boxed{e^{-2}}$ $= \underline{\underline{e^{-2}}}$

(b) ce^{-2}

(c) 0

(d) $1-e^{-2}$

$f(x)=\begin{cases} ce^{-x/3} & x\le 0\\ ce^{-x/3} & x>0\end{cases}$

$P\left[\frac{X>6}{X>0}\right]$

$= \frac{P[X>6 \cap X>0]}{P[X>0]}$

$= \frac{P(X>6)}{P(X>0)} = \frac{\int_6^\infty ce^{-\frac{x}{3}}dx}{\int_0^\infty ce^{-\frac{x}{3}}dx} = e^{-2}$

Q. Questions

Let X be a discrete random variable with probability function $P(X = x) = \frac{2}{3^x}$, for $x = 1, 2, 3, \ldots\ldots$ What is the probability that X is even?

(a) $\frac{1}{4}$

(b) $\frac{2}{7}$

(c) $\frac{1}{3}$

(d) $\frac{2}{3}$

$P(X=x) = \frac{2}{3^x}$

What is The Prob That x is even

$\underbrace{P(X=2) + P(X=4) + P(X=6) + P(X=8) \ldots}_{\text{Even}}$

$P(X=x)=\frac{2}{3^x}$ $P(X=2)=\frac{2}{3^2}$ $P(X=4)=\frac{2}{3^4}$ $P(X=6)=\frac{2}{3^6}$ $P(X=8)=\frac{2}{3^8}$ ---

$P(X=x \text{ is Even}) = P(X=2)+P(X=4)+P(X=6)+P(X=8)+ \cdots$

$P(X=x \text{ is Even}) = \frac{2}{3^2}+\frac{2}{3^4}+\frac{2}{3^6}+\frac{2}{3^8}+ \cdots$

$= \frac{2}{9}\left[\underbrace{1+\left(\frac{1}{9}\right)+\left(\frac{1}{9}\right)^2+\left(\frac{1}{9}\right)^4+\cdots}_{}\right]$

$\underbrace{1+\frac{1}{9}+\left(\frac{1}{9}\right)^2+\left(\frac{1}{9}\right)^4+\cdots}_{}$

G.P

G.P $\longrightarrow$ Infinite Terms - Sum

$S_\infty = \frac{a}{1-r}$

$= \frac{2}{9}\left[\frac{1}{1-\frac{1}{9}}\right]$

$= \frac{2}{9}\left[\frac{1}{\frac{8}{9}}\right] = \frac{1}{4}$

Q. Questions

$P[0 \le e^x \le 4] = P[e^x \le 4]$

Let X be a random variable with cumulative distribution function

$$\boxed{F_X(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-x} & \text{for } x > 0 \end{cases}}$$, What is P $(0 \le e^x \le 4)$?

$\rightarrow \text{Exp}(\lambda)$ $\lambda = 1$

$F_X(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-x} & x > 0 \end{cases}$

Exponetial Distribution

$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$

(a) e^{-4}

(b) $\frac{1}{4}$

(c) $\frac{1}{2}$

(d) $\frac{3}{4}$

$\text{Exp}(\lambda) \sim 1$

$\boxed{\lambda = 1}$

$P[0 \le e^x \le 4] = P[e^x \le 4]$

$= P[x \le \ln 4] = 1 - e^{-\lambda x} \quad \lambda = 1,\ \ln 4$

$= 1 - e^{-1 \times \ln 4}$

$= 1 - e^{-\ln 4}$

$=$

$= \frac{3}{4}$ Ans

Q. Questions

Let X is a random variable with density

$$f(x) = \frac{1}{4}e^{-\frac{|x|}{2}}, \quad -\infty < x < \infty$$

Then E(| X |) = 2 ________.

$f(x) = \frac{1}{4} e^{-\frac{|x|}{2}}$

$-\infty \le x \le \infty$

$E[|X|] = \int_{-\infty}^{\infty} |x| \frac{1}{4} e^{-\frac{|x|}{2}}$

$E[|X|] = \frac{2}{4}\int_0^{\infty} x\, e^{-\frac{x}{2}}\, dx$

$= \frac{1}{2}\int_0^{\infty} x e^{-\frac{x}{2}}\, dx \qquad \frac{x}{2} = t$

$\qquad x = 2t$

$\qquad dx = 2dt$

$= \frac{1}{2}\int_0^{\infty} (2t) \cdot e^{-t} \cdot 2dt$

$= 2\int_0^{\infty} \underline{t e^{-t} dt}$

$= 2 \times \Gamma 2$

$n - 1 = 1$

$n = 1 + 1$

$= \underline{2}$

$\Gamma n = (n-1)!$

$= 2 \times 1!$

$= 2$

Q. Questions

$$E[X] = \int_a^b x f(x)\,dx$$

If X is a random variable with density function

$$f(x) = \begin{cases} 1.4e^{-2x} + 0.9e^{-3x}, & x > 0, \\ 0 & \text{elsewhere} \end{cases}$$

Then E [X] =

(a) $\frac{9}{20}$

(b) $\frac{5}{6}$

(c) 1

(d) $\frac{230}{126}$

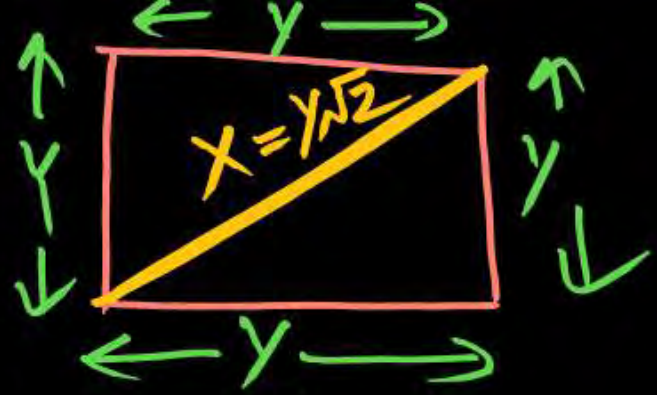
$f(x) = 1.4e^{-2x} + 0.9e^{-3x}$

$E[X] = \int_0^\infty x f(x)\,dx$

$\Rightarrow \int_0^\infty x \cdot [1.4e^{-2x} + 0.9e^{-3x}]\,dx$

$\Rightarrow \frac{9}{20} = 0.45$

Q. **Questions**



You are given a random variable X such that its density is

$$f_{x(}(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

A square with diagonal of length X is constructed. Find the expected value of the area of that square.

(a) 0.1

(b) 0.25

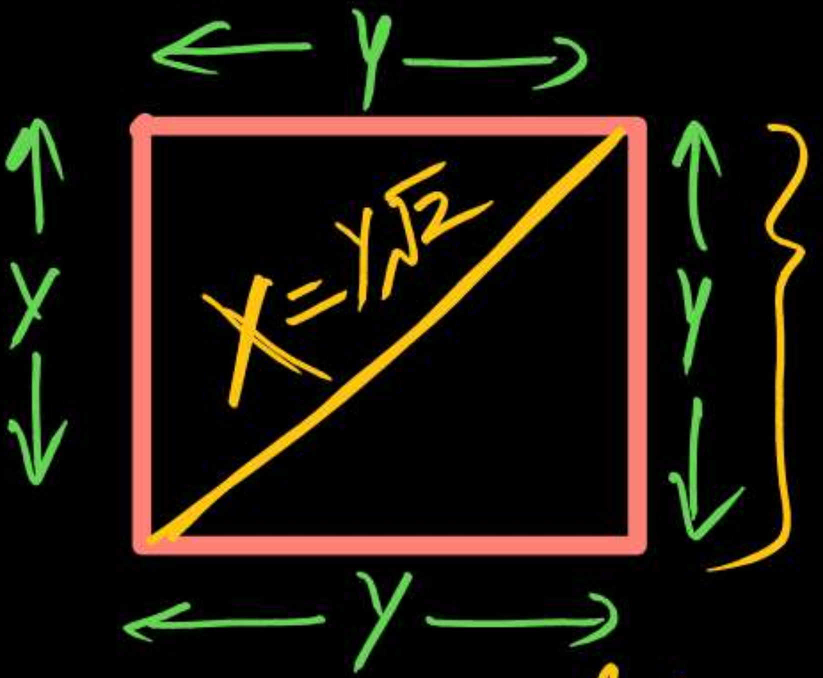
(c) $\frac{4}{7}$

(d) 0.3

$x = y\sqrt{2}$

$y = \left(\dfrac{x}{\sqrt{2}}\right)$

$y^2 = \dfrac{x^2}{2}$



Expected value $E[y^2] = E\left[\dfrac{x^2}{2}\right]$

$$E\left[\frac{x^2}{2}\right] = \int_0^1 \frac{x^2}{2} \cdot 3x^2\,dx$$

$$= \frac{3}{2}\int_0^1 x^4\,dx$$

$$= \frac{3}{2} \times \left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{2} \times \frac{1}{5} = \frac{3}{10}$$

$$f_X(x) = \begin{cases} 3x^2 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

mean

$$E[p^2] = E\left[\frac{x^2}{2}\right] = 0.3$$

Q. Questions

X has a distribution which is ~~partly continuous and partly discrete~~

$$f(x) = \begin{cases} \frac{1-p}{2}, & 0<x<1 \text{ continuous} \\ p & x=1 \text{ Discrete} \\ \frac{1-p}{2}, & 1<x<2 \text{ continuous} \\ 0 & \text{otherwise} \end{cases}$$

Find the variance of X in terms of p

$$\text{Var}(X) = E[X^2] - \left(E[X]\right)^2$$

(a) $\frac{1-p}{3}$ ✓ A

(b) $\frac{2-p}{3}$

(c) $\frac{1-p}{2}$

(d) $\frac{2-p}{2}$

$$Var(X) = E[X^2] - [E[X]]^2$$

$$E[X^2] = \int_0^1 x^2\left(\frac{1-p}{2}\right)dx + (1)^2 \cdot p + \int_1^2 x^2\left(\frac{1-p}{2}\right)dx$$

Discrete

$$f(x) = \begin{cases} \frac{1-p}{2} & 0 < x < 1 \\ p & x = 1 \\ \frac{1-p}{2} & 1 < x < 2 \end{cases}$$

$$\boxed{E[X^2] \Rightarrow \frac{4}{3} - \frac{p}{3}}$$

$$E[X] = \int_0^1 x\left(\frac{1-p}{2}\right)dx + 1\cdot(p) + \int_1^2 x\left(\frac{1-p}{2}\right)dx$$

$$\boxed{E[X] = 1}$$

$$= \frac{1}{3} - \frac{p}{3} = \frac{1-p}{3}$$

$$Var(X) = E[X^2] - [E[X]]^2$$

$$= \frac{4}{3} - \frac{p}{3} - (1)^2$$

Thank you