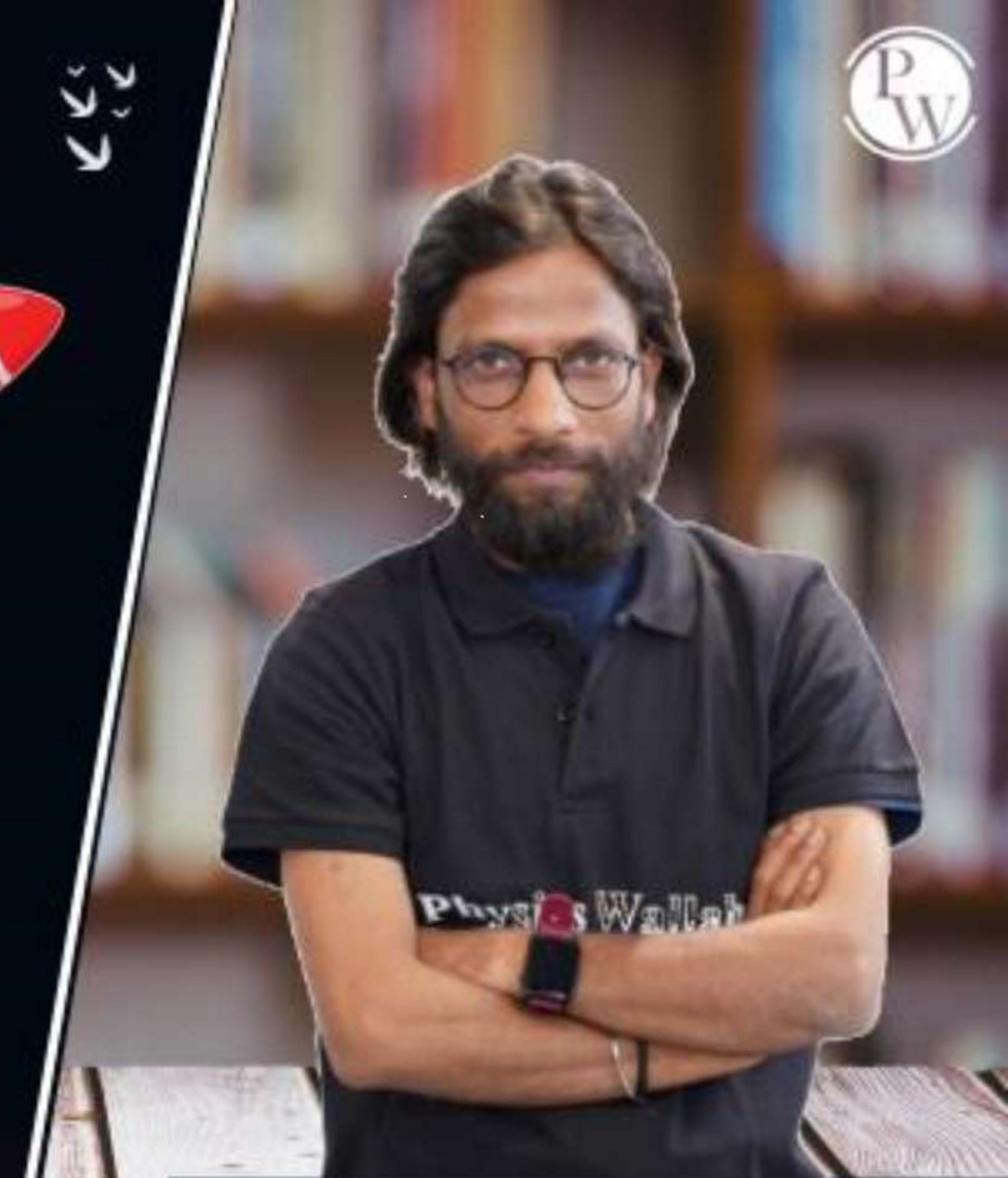
GATE-ALL BRANCHES
ENGINEERING MATHEMATICS

SINGLE VARIABLE CALCULUS

Lecture No.- 10



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Recap of Previous Lecture

Topic Problems based on definite integrals

Topics to be covered

Topic Concepts based on definite integrals

Topic Area based problems

Topic Improper integral

#Q. Let $f : [-1, 2] \to (0, \infty)$ be a continuous function such that $\boxed{f(x) = f(1 - x)}$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)dx$ and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and the x-axis. Then

$\int_{-1}^{2} f(x)\,dx$

(A) $R_1 = 2R_2$

(B) $R_1 = 3R_2$

(C) $2R_1 = R_2$

(D) $3R_1 = R_2$

$$\boxed{f(x)=f(1-x)}$$

$$R_1=\int_{-1}^{2}x\,f(x)\,dx$$

$$R_2=\int_{-1}^{2}f(x)\,dx$$

$$R_1=\int_{-1}^{2}x\,f(x)\,dx$$

$$R_1=\int_{-1}^{2}(-1+2-x)\,f(-1+2-x)\,dx$$

$$=\int_{-1}^{2}\underline{(1-x)}\,f(1-x)\,dx$$

Using Default

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$R_1=\int_{-1}^{2}(1-x)\,f(x)\,dx$$

$$R_1=\int_{-1}^{2}f(x)-\int_{-1}^{2}x\,f(x)\,dx$$

$$R_1=R_2-R_1$$

$$\boxed{2R_1=R_2}$$

Topic : Definite Integrals

The area bounded by the curve $y = \sqrt{4 - x}$, X-axis and Y-axis

(a) $\frac{8}{3}$

(b) $\frac{16}{3}$

(c) $\frac{32}{3}$

(d) None of these

H.W

go to NEST

Slide

Special Integrals

$$\int_0^{\pi/2} \sin^n x\,dx \quad \text{OR} \quad \int_0^{\pi/2} \cos^n x\,dx$$

Case (A)

$$\frac{(\text{Power}-1)}{\text{Power}}$$

$$I = \int_0^{\frac{\pi}{2}} \sin^6 x$$

$$= \frac{(5\times 3\times 1)}{6\times 4\times 2}\text{odd} \times \frac{\pi}{2}$$

$$= \frac{15\pi}{96} \quad \text{Ans}$$

Rule:

- ✓ Limit always 0 to $\frac{\pi}{2}$
- ✓ even Power question = multiply via $\frac{\pi}{2}$
- ✓ odd power → Multiply via 1.

$$I = \int_0^{\pi/2} \cos^{12} x\,dx = \frac{11\times 9\times 7\times 5\times 3\times 1}{12\times 10\times 8\times 6\times 4\times 2} \times \frac{\pi}{2}$$

$$I = \int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$$

Wallis Product

$$I = \int_0^{\frac{\pi}{2}} \sin^7 x \, dx = \frac{6 \times 4 \times 2}{7 \times 5 \times 3} \times 1 = \frac{48}{105}$$

$$I = \int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3 \times 1} \times 1 = \frac{2}{3}$$

$$I = \int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$I = \int_0^{\pi/2} \cos^8 x \, dx = \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$I = \int_0^{\frac{\pi}{2}} \sin^n x \cos^m x \, dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^8 x \, dx$$

m even
n even
m+n even $\rightarrow \frac{\pi}{2}$

$(n-1)\underline{(m-1)}$
$(n-1)_{odd}$ ———
$(n-1)_{even}$ ———

$$= \frac{(5 \times 3 \times 1) \times (7 \times 5 \times 3 \times 1)}{14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$I = \int_0^{\frac{\pi}{2}} \sin^3 \underline{x} \cos^4 \underline{x} \, dx$$

m even
n odd
m+n = odd ———

$$= \frac{2 \times \cancel{3} \times 1}{7 \times 5 \times \cancel{3} \times 1} \times 1 = \frac{2}{35} \text{ Ans}$$

Rule:

A) Limit must 0 to $\frac{\pi}{2}$

B) If $\begin{cases} m \text{ even} \\ n \text{ even} \\ m+n \text{ even} \end{cases}$ Working Together

= multiply by $\frac{\pi}{2}$

Any other condition

= No multiply

OR

multiply via 1

$$I = \int_0^{\frac{\pi}{2}} \sin^7 x \cos^3 x \, dx = \frac{\cancel{6 \times 4 \times 2} \times 2}{10 \times 8 \times \cancel{6 \times 4 \times 2}} = \frac{1}{40}$$

$$I = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^2 x \, dx = \frac{5 \times 3 \times 1 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$\left\{ \begin{array}{l} m \text{ even} \\ n \text{ even} \\ m+n \text{ even} \end{array} \right]$

All are simultaneously occur $= \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sin^8 x \cos^9 x \, dx$$

✓ $m = \text{even}$

✗ $n = \text{odd}$

✗ $m+n = \text{odd}$

$$= \frac{7 \times 5 \times 3 \times 1 \times 8 \times 6 \times 4 \times 2}{17 \times 15 \times 13 \times 11 \times 9 \times 7 \times 5 \times 3 \times 1} \times 1$$

Questions

The solution for $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ is

(a) 0 (b) $\frac{1}{15}$

(c) 1 (d) $\frac{8}{3}$

$I = \int_0^{\pi/6} \cos^4 3\theta \; \sin^3 6\theta \, d\theta$

$3\theta = t \quad t = 0 \times 3$

$3\,d\theta = dt \quad 3 \times \frac{\pi}{6} = t \quad t = 0$

$d\theta = \frac{dt}{3}$

$t = \frac{\pi}{2}$

$I = \int_0^{\frac{\pi}{2}} \cos^4 t \; \sin^3 2t \cdot \frac{dt}{3}$

$= \int_0^{\frac{\pi}{2}} \cos^4 t \, (2\sin t \cos t)^3 \frac{dt}{3}$

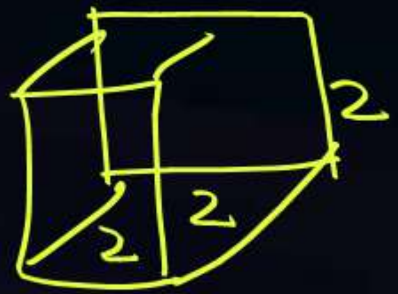
$= \frac{8}{3} \int_0^{\pi/2} \sin^3 t \cos^7 t \, dt$

$(2\sin t \cos t)^3 = 8\sin^3 t \cos^3 t$

$2^3 = 2 \times 2 \times 2 = 8$

$= \frac{\cancel{8}}{3} \times \frac{\cancel{2} \times \cancel{6 \times 4 \times 2}}{\cancel{10}_{5} \times \cancel{8} \times \cancel{6 \times 4 \times 2}}$

$= \frac{1}{15}$



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Improper Integrals: Improper Integral of First Kind:

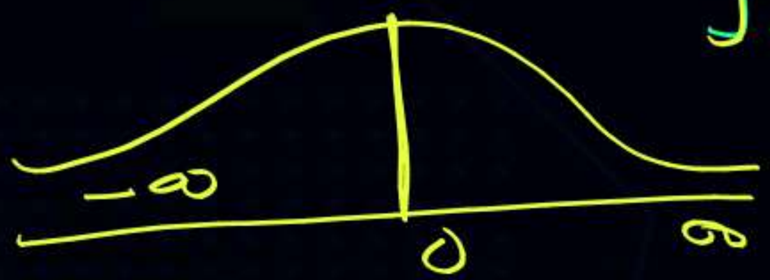
Range of Interval is Infinite

1) $\int_a^{\infty} f(x)\,dx \Rightarrow \lim_{t\to\infty} \int_a^t f(x)\,dx$ — Finite (convergent) / Infinite (Divergent)

2) $\int_{-\infty}^{a} f(x)\,dx \Rightarrow \lim_{t\to-\infty} \int_t^a f(x)\,dx$ — Finite (convergent) / Infinite (divergent)

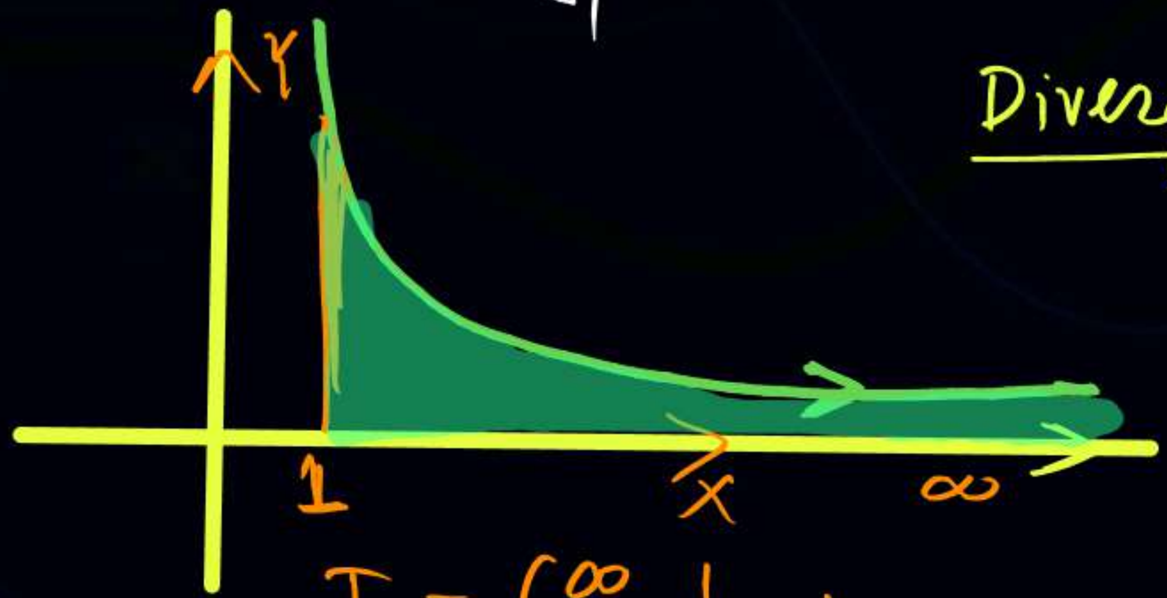
3) $\int_{-\infty}^{\infty} f(x)\,dx \Rightarrow \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$ — finite convergent / Infinite (divergent)

$= \lim_{t\to-\infty} \int_t^a f(x)\,dx + \lim_{t\to\infty} \int_a^t f(x)\,dx$



$a = 0$

$$I_1 = \int_1^{\infty} \frac{1}{x}\,dx \qquad \frac{1}{2}$$



Divergent

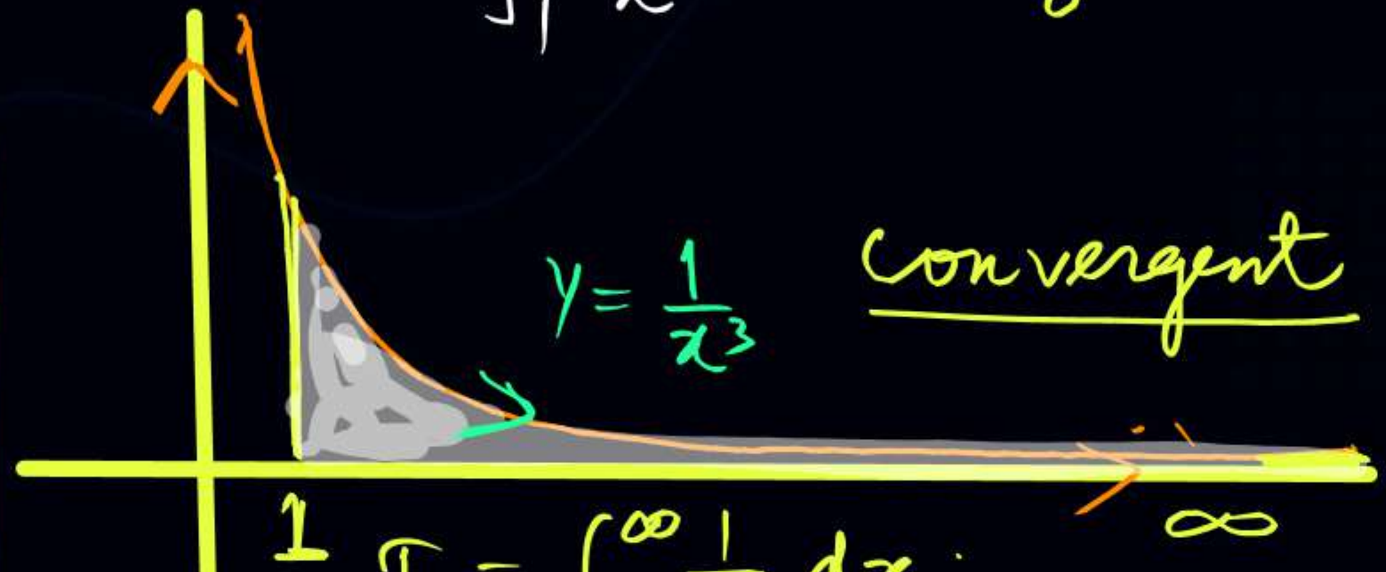
$$I = \int_1^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left[\ln x\right]_1^t = \lim_{t\to\infty} \left[\ln t - \underbrace{\ln 1}_{\to 0}\right]$$

$$= \infty = \text{Divergent}$$

$$I = \int_1^{\infty} \frac{1}{x^3}\,dx \qquad \frac{1}{8}$$



Convergent

$$I = \int_1^{\infty} \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\underbrace{-\frac{1}{2t^2}}_{\to 0} + \frac{1}{2}\right]$$

$$= \frac{1}{2} = \text{Convergent}$$

Improper Integral of second Kind →

$$\int_a^b f(x)\,dx =$$

- If function $x=a$ undefined (Discontinuity)
- If Function $x=b$ undefined (discontinuous)

✓ $I=\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$f(x)=\frac{1}{\sqrt{x}}$

$f(0)=\infty$

✓ $I=\int_0^{\frac{\pi}{2}} \tan x\,dx$

$f(x)=\tan x$

$f\left(\frac{\pi}{2}\right)=\tan\frac{\pi}{2}=\infty$

Rule-No. 1 If $y=f(x)$ at $x=a$ undefined

(right hand) b

RHL

$(a+h)$

forward

a

$$\boxed{\int_a^b f(x)\,dx = \lim_{h\to 0}\int_{a+h}^{b} f(x)\,dx}$$

$I=\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$I=\int_0^1 \frac{1}{\sqrt{x}}\,dx = \underset{h\to 0}{\mathrm{lt}}\int_{0+h}^{1}\frac{1}{\sqrt{x}}\,dx$$

$$\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

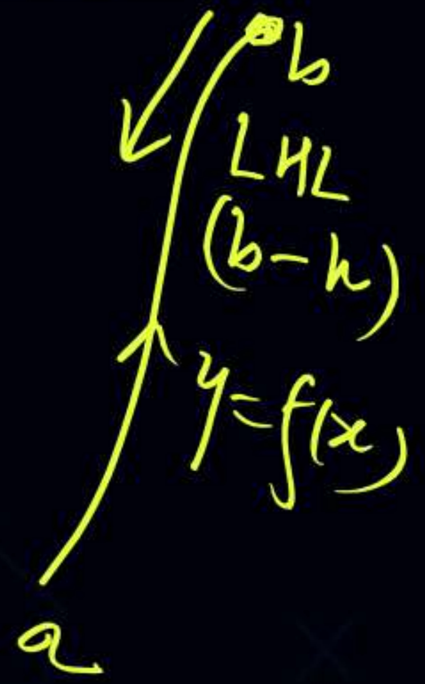
$$2x^{\frac{1}{2}}$$

$$=\underset{h\to 0}{\mathrm{lt}}\left[2\sqrt{x}\right]_{0+h}^{1}$$

$$=\underset{h\to 0}{\mathrm{lt}}\left[2\sqrt{1}-2\sqrt{0+h}\right]$$

$$=\underline{2}$$

CASE 02



$\int_a^b f(x)\,dx$ at $x=b$ (disconti/undefined)

$$=\underset{h\to 0}{\mathrm{lt}}\int_a^{b-h} f(x)\,dx$$

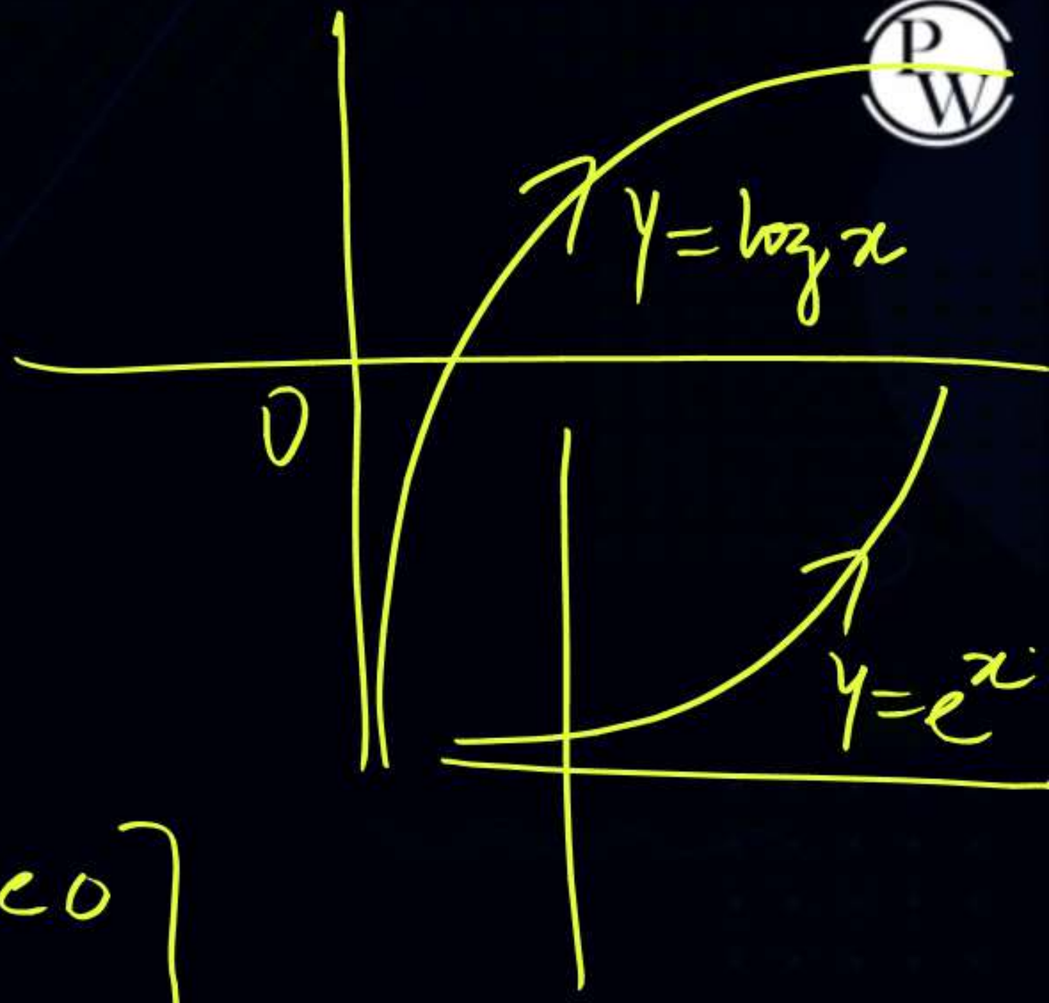
$$I = \int_0^{\frac{\pi}{2}} \tan x \, dx$$

$$= \lim_{h \to 0} \int_0^{\frac{\pi}{2}-h} \tan x \, dx$$

$$= \lim_{h \to 0} \left[\log \sec x \right]_0^{\frac{\pi}{2}-h}$$

$$= \lim_{h \to 0} \left[\log \sec\left(\frac{\pi}{2}-h\right) - \log \sec 0 \right]$$

$= \underline{\text{Undefined} = \text{divergent}}$



Questions

$I = \int_1^\infty \frac{1}{x^3} dx = \frac{1}{2}$

If $S = \int_1^\infty x^{-3}\, dx$ then S has the value

$S = \int_1^\infty x^{-3} dx = \frac{1}{2}$

(a) $\frac{-1}{3}$

(b) $\frac{1}{4}$

(c) $\frac{1}{2}$

(d) 1

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Questions

H.W

The value of the following improper integral is $\int_0^1 x \log x \, dx =$ ____.

(a) $\frac{1}{4}$

(b) 0

(c) $\frac{-1}{4}$

(d) 1

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Questions

The value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is

(a) $-\pi$

(b) $\frac{-\pi}{2}$

(c) $\frac{\pi}{2}$

(d) π

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{t \to -\infty} \left[\tan^{-1} 0 - \tan^{-1}(t)\right] + \lim_{t \to \infty} \left[\tan^{-1} t - \tan^{-1} 0\right]$$

$$= \tan^{-1} 0 - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1} 0$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \pi$$

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Questions

The integral $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$ is equal to ____.

$$\lim_{h\to 0} \left[-2\sqrt{1-x}\right]_0^{1-h} \Rightarrow \lim_{h\to 0}\left[-2\sqrt{1-(1-h)} + 2\sqrt{1-0}\right]$$

$$= \underline{\underline{2}} \ \underline{\underline{\text{Ans}}}$$

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THANK - YOU