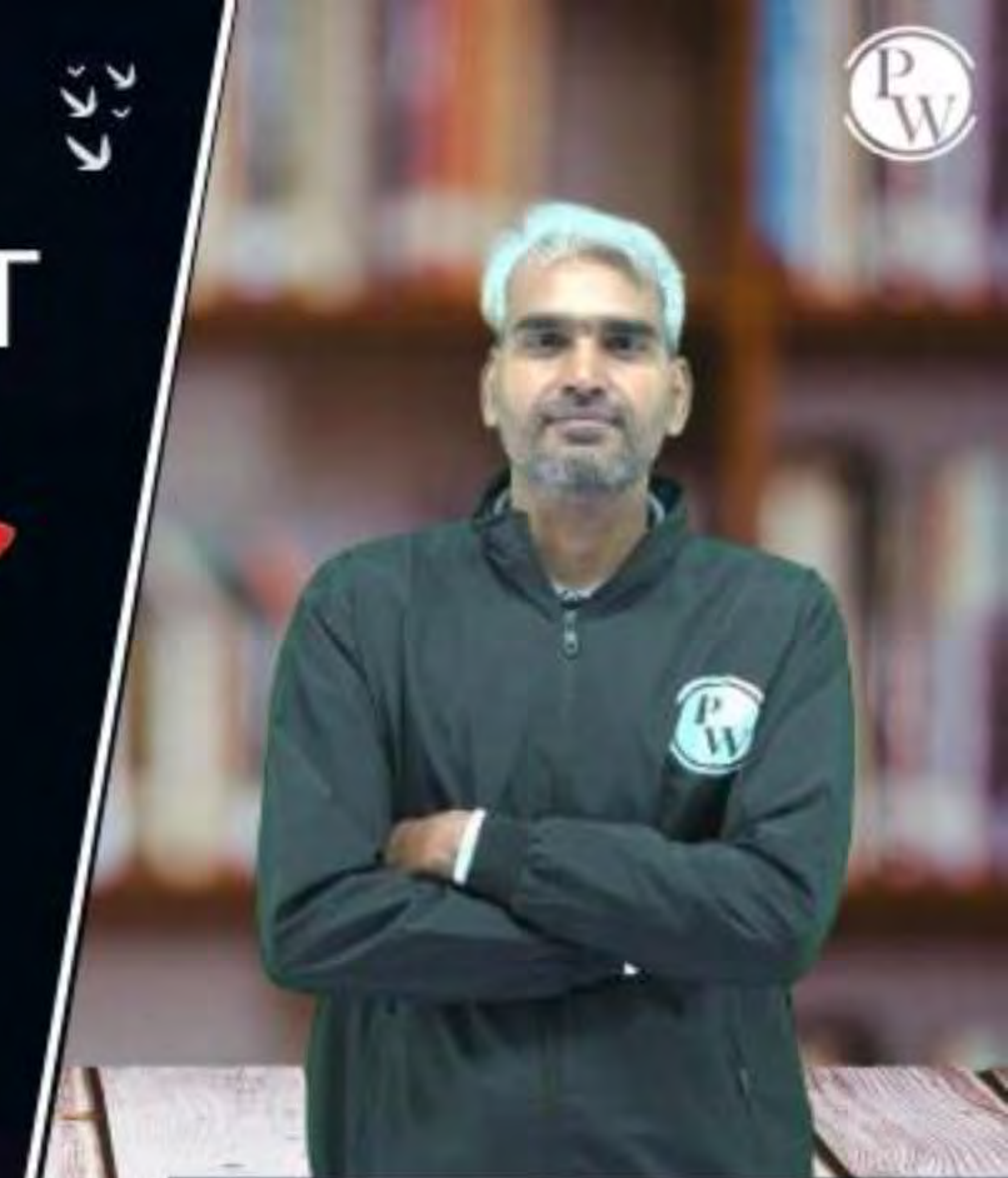
Computer Science & IT

Discrete Mathematics

Mathematical Logic

Lecture No. 03



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Recap of Previous Lecture

- Topic: Logical implications and logical equivalences
- Topic: Important equivalences
- Topic: Important statements
- Topic: Argument / Inference

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Topics to be Covered

- **Topic** Rules of inference
- **Topic** Practice questions
- **Topic** Proof by contradiction
- **Topic** Conditional proof rule

Slide

Topic : Logical Implication / Implication

Let P and Q are any two propositional functions.

* Whenever P is true if Q is also true then P logically implies Q. is true.

If there exist any case for which P is true but Q is false, then P logically implies Q is invalid

{i.e, P does not logically implies Q}

P logically implies Q if and only if $P \rightarrow Q$ is a tautology.

Topic : Logical Equivalence / Equivalence

- If P and Q are any two propositional functions, then P equivalent to Q is written as $P \equiv Q$. or $P \cong Q$

- P and Q are said to be equivalent if and only if they have same truth table. let $P = \sim a \vee b$ & $Q = a \to b$ then $P \equiv Q$

- $P \equiv Q$ if and only if $P \leftrightarrow Q$ is a tautology.

- $P \equiv Q$ if and only if. P logically implies Q and Q logically implies P.

- $P \longleftrightarrow Q$ is a tautology if and only if $P \to Q$ is a tautology as well as $Q \to P$ is a tautology.

Topic : Some important equivalences

① $\sim(\sim P) \equiv P$ {Double negation}

② $P \vee Q \equiv Q \vee P$
$P \wedge Q \equiv Q \wedge P$ {Commutative}

③ $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ {Associative}

Topic : Some important equivalences

(4) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

$\Big\}$ Distributive.

(5) $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$

$\sim(P \vee Q) \equiv \sim P \wedge \sim Q$

$\Big\}$ De' Morgan's

Topic : Some important equivalences

*⑥ $P \wedge (P \vee Q) \equiv P$
$P \vee (P \wedge Q) \equiv P$ } Absorption law

⑦ $P \wedge P \equiv P$
$P \vee P \equiv P$

⑧ $P \vee T \equiv T$
$P \wedge F \equiv F$

⑨ $P \wedge T \equiv P$
$P \vee F \equiv P$

⑩ $T \wedge T \equiv T$
$T \vee T \equiv T$
$F \wedge F \equiv F$
$F \vee F \equiv F$
$T \wedge F \equiv F$
$T \vee F \equiv T$

Some important Equivalences :

1. $P \rightarrow Q \equiv \sim P \vee Q$
2. $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$
3. $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

Topic : Some important statements

1. P implies Q $\equiv P \rightarrow Q$
2. If P then Q $\equiv P \rightarrow Q$
3. P only if Q $\equiv P \rightarrow Q$
4. P is sufficient condition for Q $\equiv P \rightarrow Q$
5. Q is necessary condition for P $\equiv P \rightarrow Q$

Continuity is necessary for differentiability $\equiv$ Differentiability $\rightarrow$ Continuity

* $\boxed{Q \text{ follows from } P \equiv P \rightarrow Q}$

Topic : Some important statements

6. P if Q $\equiv$ if Q then P $\equiv$ $Q \to P$

7. P when Q $\equiv$ $Q \to P$

   if

8. P follows from Q $\equiv$ $Q \to P$

9. P unless ~Q $\equiv$ $\sim(\sim Q) \to P \equiv Q \to P$

   simply replace unless by $\vee$ $\equiv P \vee \sim Q \equiv \sim Q \vee P \equiv Q \to P$

10. $\boxed{P \text{ unless } Q \equiv \sim Q \to P \equiv \sim P \to Q}$

$\overbrace{\text{you can not crack gate}}^{P}$ unless $\overbrace{\text{you appear for gate}}^{Q}$

|||

If $\underbrace{\text{you do not appear for gate}}_{\sim Q}$ then $\underbrace{\text{you can not crack gate}}_{P}$

$$\sim Q \xrightarrow{|||} P$$

$$\boxed{P \text{ unless } Q \equiv \sim Q \rightarrow P}$$

$\overbrace{\text{you can not crack gate}}^{P}$ unless $\overbrace{\text{you appear for gate}}^{Q}$

|||

If $\underbrace{\text{you cracked}}_{(\sim P)}$ gate then you $\underbrace{\text{appeared}}_{(Q)}$ for gate $\equiv \sim P \rightarrow Q$

Topic : Argument / Inference

The statement that,
"A set of premises { P1, P2, P3,……,Pn} yields another proposition Q" is called an argument.

Q is called conclusion of the argument.

Set of Premises (or) Hypothesis (or) given statements

P_1
P_2
$\vdots$
P_n

Propositions in the premises

$\therefore Q$ ← Conclusion

it is called argument

given statements

(i) If today is Sujay's B'day then today is 13^{th} Aug.

(ii) Today is 13^{th} Aug.

$\therefore$ Today is Sujay's B'day

Conclusion

Argument { it may be a valid argument (or) an invalid argument }

Topic : Argument / Inference

- Argument may be valid or invalid.
- The process of reasoning whether the argument is valid or invalid is called inference.

If conclusion Q can be inferred from the set of premises by applying some rules of inference and equivalences, then argument is said to be valid otherwise invalid.

i.e, whenever $P_1, P_2, P_3, \ldots P_n$ are true if Q is guaranteed to be true

Topic : Argument / Inference

Following statements are equivalent,

- Argument {P1, P2, P3,.......,Pn} $\vdash Q$ is valid.

  yields

- {P1, P2, P3,.......,Pn} Logically implies Q is true/valid.
- {P1 ∧ P2 ∧ P3 ∧ ...... ∧ Pn} → Q is a tautology.

Topic : Rules of inference

Any valid reasoning is rule of inference.

Topic : Rules of inference

1. Simplification

* $\dfrac{P \wedge Q}{\therefore P}$ is valid   similarly $\dfrac{P \wedge Q}{\therefore Q}$ is valid.

* $(P \wedge Q)$ logically implies P is valid

* $(P \wedge Q) \rightarrow P$ is a tautology.

Topic : Rules of inference

2. Addition

$$\frac{P}{\therefore P \vee Q}$$ is valid

* P logically implies $P \vee Q$

. $P \rightarrow (P \vee Q)$ is a tautology.

$$\frac{P}{\therefore P \wedge Q}$$ invalid

x

3. Conjunction

$$\frac{\begin{array}{c} P \\ Q \end{array}}{\therefore P \wedge Q}$$ is valid.

Topic : Rules of inference

4. Disjunctive Syllogism

$$\begin{array}{l} P \vee Q \\ \sim P \\ \hline \therefore Q \end{array}$$ is valid

$$\begin{array}{l} P \vee Q \\ \sim Q \\ \hline \therefore P \end{array}$$ is valid

Topic : Rules of inference

5. Conjunctive Syllogism

$$\dfrac{\begin{matrix}\sim(P \wedge Q) \\ P\end{matrix}}{\therefore \sim Q} \text{ is valid} \quad \equiv \quad \dfrac{\begin{matrix}\sim P \vee \sim Q \\ P\end{matrix}}{\therefore \sim Q} \text{ is valid}$$

Topic : Rules of inference

6. Modus Ponens

$$\begin{array}{l} P \rightarrow Q \\ \quad\quad P \\ \hline \therefore Q \end{array}$$

is valid

Mistake

7. Fallacy of affirming the consequent

$$\begin{array}{l} P \rightarrow Q \\ \quad\quad Q \\ \hline \therefore P \end{array}$$

is invalid

Topic : Rules of inference

8. Modus Tollen's

$$\frac{\begin{array}{r} P \rightarrow Q \\ \sim Q \end{array}}{\therefore \sim P}$$ is valid

9. Fallacy of denying the antecedent

$$\frac{\begin{array}{r} P \rightarrow Q \\ \sim P \end{array}}{\therefore \sim Q}$$ is invalid

invalid

10. Transitivity

$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore P \rightarrow R \end{array}$$ is valid

Topic : Rules of inference

11. Dilemma

$$\begin{array}{l} P \rightarrow R \\ Q \rightarrow R \\ P \vee Q \\ \hline \therefore R \end{array}$$ is valid

Topic : Rules of inference

12. Constructive Dilemma

$$\begin{array}{l} P \rightarrow R \\ Q \rightarrow S \\ P \vee Q \\ \hline \therefore R \vee S \end{array}$$ is valid

13. Destructive Dilemma

$$\left.\begin{array}{l} P \rightarrow R \\ Q \rightarrow S \\ \sim R \vee \sim S \end{array}\right\}$$

$$\frac{}{\therefore \sim P \vee \sim Q}$$ is valid

Topic : Rules of inference

14. Resolution

$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$$

is valid

15. Non Sequitur

$$\begin{array}{l} P \\ Q \\ \hline \therefore R \end{array}$$ is invalid

Q: Simplify the following statement

$$\{\sim p \wedge (\sim q \wedge r)\} \vee \{(q \wedge r) \vee (p \wedge r)\}$$

$$\{(\sim p \wedge \sim q) \wedge r\} \vee \{(p \vee q) \wedge r\}$$

$$\{\sim(p \vee q) \wedge r\} \vee \{(p \vee q) \wedge r\}$$

$$\{\underbrace{\sim(p \vee q) \vee (p \vee q)}_{T}\} \wedge r$$

$$T \wedge r$$

$$\equiv r$$

(a) True

(b) False

(c) p

(d) q

(e) r

Q: Simplify the Following Expression

$$[(P \vee Q) \wedge \sim\{\sim P \wedge (\sim Q \vee \sim R)\}] \vee [(\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)]$$

$$[(P \vee Q) \wedge \{P \vee (Q \wedge R)\}] \vee [\sim P \wedge (\sim Q \vee \sim R)]$$

$$[(P \vee Q) \wedge \{(P \vee Q) \wedge (P \vee R)\}] \vee \sim[P \vee (Q \wedge R)]$$

$$[(P \vee Q) \wedge (P \vee R)]$$

$$[P \vee (Q \wedge R)] \vee \sim\{P \vee (Q \wedge R)\}$$

$$\equiv \underline{\text{True}}$$

MSQ Q: Which of the following is/are valid ?

(a) $(P \wedge Q)$ logically implies $(P \leftrightarrow Q)$

$T \leftrightarrow T \equiv T$ } ∴ valid

(b) $(P \leftrightarrow Q)$ logically implies $(P \rightarrow Q)$

$F \rightarrow F = T$

$T \rightarrow T = T$ } ∴ valid

(c) $(P \leftrightarrow Q)$ logically implies $(P \rightarrow \sim Q)$

$T \rightarrow F \equiv F$ ← ∴ invalid

$F \rightarrow T = T$

(d) Q logically implies $(P \rightarrow Q)$

$? \rightarrow T \equiv T$ } ∴ Valid

Does $(P \leftrightarrow Q)$ logically implies $(P \wedge Q)$

$F \leftrightarrow F$ is T; $F \wedge F$ is F

← NO

MSQ Q: Which of the following propositional formula is/are tautology

(a) $(P \wedge Q) \longrightarrow (P \longleftrightarrow Q)$

(b) $(P \longleftrightarrow Q) \longrightarrow (P \longrightarrow Q)$

(c) $(P \longleftrightarrow Q) \longrightarrow (P \longrightarrow \sim Q)$

(d) $Q \longrightarrow (P \longrightarrow Q)$

MSQ Q: Which of the following logical implications are valid.

(a) $(P \wedge Q) \longrightarrow (P \leftrightarrow Q)$

(b) $(P \leftrightarrow Q) \longrightarrow (P \rightarrow Q)$

(c) $(P \leftrightarrow Q) \longrightarrow (P \rightarrow \sim Q)$

(d) $Q \longrightarrow (P \rightarrow Q)$

Q: Which of the following is/are tautology.

✗ (a) $(a \vee b) \longrightarrow (b \wedge c)$

$T \vee F = T \qquad F \wedge F = F$

✓ (b) $(a \wedge b) \longrightarrow (b \vee c)$

$T \quad T \qquad T \vee ? = T$

✗ (c) $(a \vee b) \longrightarrow (b \rightarrow c)$

$? \vee T = T \qquad T \longrightarrow F = F$

✗ (d) $(a \rightarrow b) \longrightarrow (b \rightarrow c)$

$? \longrightarrow T = T \qquad T \longrightarrow F = F$

Q. Consider two statements

$S_1: \{(A \wedge B) \longrightarrow C\} \equiv \{(A \rightarrow C) \wedge (B \rightarrow C)\}$

$S_2: \{(A \vee B) \longrightarrow C\} \equiv \{(A \rightarrow C) \vee (B \rightarrow C)\}$

Which of the following is true

(a) Both S_1 & S_2 are valid

(b) S_1 is valid, S_2 is invalid

(c) S_1 is invalid, & S_2 is valid

~~(d)~~ Both S_1 & S_2 are invalid.

$P \equiv Q$ ✓

iff

P logically implies Q

and

Q logically implies P.

Q Consider two statements

$S_1: \{(A \wedge B) \rightarrow C\} \equiv \{(A \rightarrow C) \wedge (B \rightarrow C)\}$

$T \quad F \quad F = T$; $T \rightarrow F = F \wedge F \rightarrow F = T$; $= F$

$S_2: \{(A \vee B) \rightarrow C\} \equiv \{(A \rightarrow C) \vee (B \rightarrow C)\}$

$(F \vee T) = T \rightarrow F = F$; $(F \rightarrow F = T) \vee (T \rightarrow F = F) = T$

L.H.S. $= (A \wedge B) \rightarrow C$

$\sim(A \wedge B) \vee C$

$\sim A \vee \sim B \vee C$

$\sim A \vee \sim B \vee C \vee C$

$(\sim A \vee C) \vee (\sim B \vee C)$

$(A \rightarrow C) \vee (B \rightarrow C)$

$\not\equiv$ R.H.S.

L.H.S. $= (A \vee B) \rightarrow C$

$\sim(A \vee B) \vee C$

$(\sim A \wedge \sim B) \vee C$

$(\sim A \vee C) \wedge (\sim B \vee C)$

$(A \rightarrow C) \wedge (B \rightarrow C)$

$\not\equiv$ R.H.S.

Q: Which of the following is/are true?

(a) $\{a \rightarrow (b \vee c)\} \equiv \{(a \wedge \sim b) \rightarrow c\}$

(b) $\{(P \rightarrow Q) \wedge (R \rightarrow Q)\} \equiv \{(P \vee R) \rightarrow Q\}$

(c) $\{(P \rightarrow Q) \vee (R \rightarrow Q)\} \equiv \{(P \wedge R) \rightarrow Q\}$

from Previous question

L.H.S. $= a \rightarrow (b \vee c)$

$\sim a \vee b \vee c$

$(\sim a \vee b) \vee c$

$\sim (a \wedge \sim b) \vee c$

$(a \wedge \sim b) \rightarrow c \equiv$ RHS.

Q. The propositional statement

$$\{P \rightarrow (Q \vee R)\} \longrightarrow \{(P \wedge Q) \rightarrow R\}$$ is

✓ (a) Satisfiable but not a tautology $\equiv$ Contingency

✗ (b) Valid $\equiv$ Always true $\equiv$ Tautology

✗ (c) Contradiction $\equiv$ Always false

(d) Can't be defined

Q. The propositional statement

$$\{P \rightarrow (Q \vee R)\} \longrightarrow \{(P \wedge Q) \longrightarrow R\} \text{ is}$$

$$\{T \rightarrow (F \vee F)\} = F \longrightarrow ? \qquad \equiv T$$

True at $P = T$, & $Q = R = F$,

$\therefore$ can not be contradiction

R.H.S. can be false only when $P = Q = T$ & $R = F$

When $P = Q = T$ & $R = F$

$\downarrow$

L.H.S. is $T \rightarrow (T \vee F) = T$

$\therefore$ not a tautology ✓

2 mins Summary

Topic Rules of inference

Topic Practice questions

Topic Proof by contradiction

Topic Conditional proof rule

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THANK - YOU

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