GATE
ALL BRANCHES

ENGINEERING MATHEMATICS

Single Variable Calculus

Lecture No. 03

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01 How to calculate limits

- ✓ Limits defination
- ✓ How to evaluate The Limits

  Plug-in Limits

<u>L-Hospital Rule</u>

2 marks L-Hospital Rule:

$$\boxed{L = \lim_{x \to a} \frac{f(x)}{g(x)}} = \lim_{x \to a} \frac{\text{Num}}{\text{Deno}}$$

$$= \lim_{x \to a} \underbrace{f(x) \times g(x)}$$ = convert the

$$\lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}} = \frac{\text{Num}}{\text{Den}}$$

Rule:

$$\left[\begin{array}{l} \text{If } \lim_{x \to a} \frac{f(x)}{g(x)} \left(\frac{\to 0}{\to 0}\right) \text{form} \\ \text{If } \lim_{x \to a} \frac{f(x)}{g(x)} \left(\frac{\to \infty}{\to \infty}\right) \text{form} \end{array}\right]$$

Apply L-Hospital Rule

Indeterminate forms $\left(\frac{\to 0}{\to 0}\right)$ OR $\left(\frac{\to \infty}{\to \infty}\right)$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \text{Plug in Limits}$$

$\boxed{\text{Ans} = \text{Limit}}$

Remove It

$\left(\frac{\to 0}{\to 0}\right)$ $\left(\frac{\to \infty}{\to \infty}\right)$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \lim_{x\to a}\frac{f'''(x)}{g'''(x)} = \text{----}$$

Again apply L-Hospital Rule

Again Differentiation Again L-Hospital Rule.

Till remove The $\frac{\to 0}{\to 0}$ OR $\frac{\to\infty}{\to\infty}$

Plug in Limit

$\boxed{\text{Ans} = L}$

Some Fundamentals (Limits)

✓ Template method : (A) $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to a} \frac{f(x)}{g(x)}$ Using L-Hospital Rule

$\left(\frac{\to 0}{\to 0}\right)$ $\lim_{x \to 0} \frac{\cos x}{1} = 1$

$\lim_{x \to 0} \frac{\sin 2x}{2x} = 1$

$\lim_{f(x) \to 0} \frac{\sin[f(x)]}{[f(x)]} = 1$

$\left(\frac{\to 0}{\to 0}\right) = 1$

$f(x) = x^2$

$f(0) = 0$

$\lim_{x \to 0} \frac{\sin(x^2)}{x^2}$ $\left(\frac{\to 0}{\to 0}\right)$

$= 1$

$\frac{\sin 0}{0}$

$f(x) \to 0$

$$\# \quad \underset{x\to 0}{Lt}\frac{\sin x}{x} = \underset{x\to 0}{lt}\frac{x}{\sin x} = \underset{x\to 0}{lt}\frac{\sin^{-1}x}{x} = \underset{x\to 0}{lt}\frac{x}{\sin^{-1}x} = \underset{x\to 0}{lt}\frac{\tan x}{x} = \underset{x\to 0}{lt}\frac{\tan^{-1}x}{x}$$

Using L-Hospital Rule $\longrightarrow = 1$

$$\# \quad \underset{x\to 0}{lt}\frac{e^x-1}{x} = \left(\frac{\to 0}{\to 0}\right) \text{ Form}$$

$$\boxed{\underset{x\to 0}{Lt}\frac{e^x-1}{x} = 1}$$

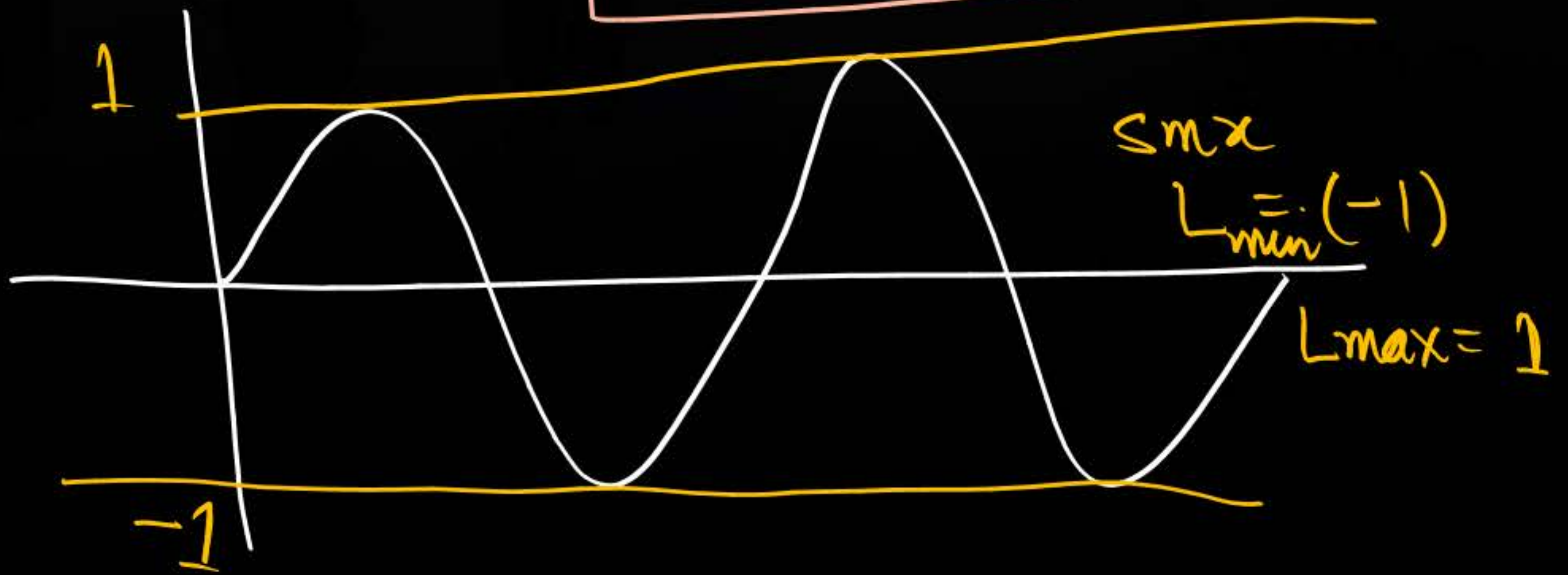
Using L-Hospital Rule

$$\underset{x\to 0}{lt}\frac{e^x}{1} = e^0 = 1$$

$$\boxed{\underset{x\to \infty}{Lt}\frac{\sin x}{x} = 0}$$

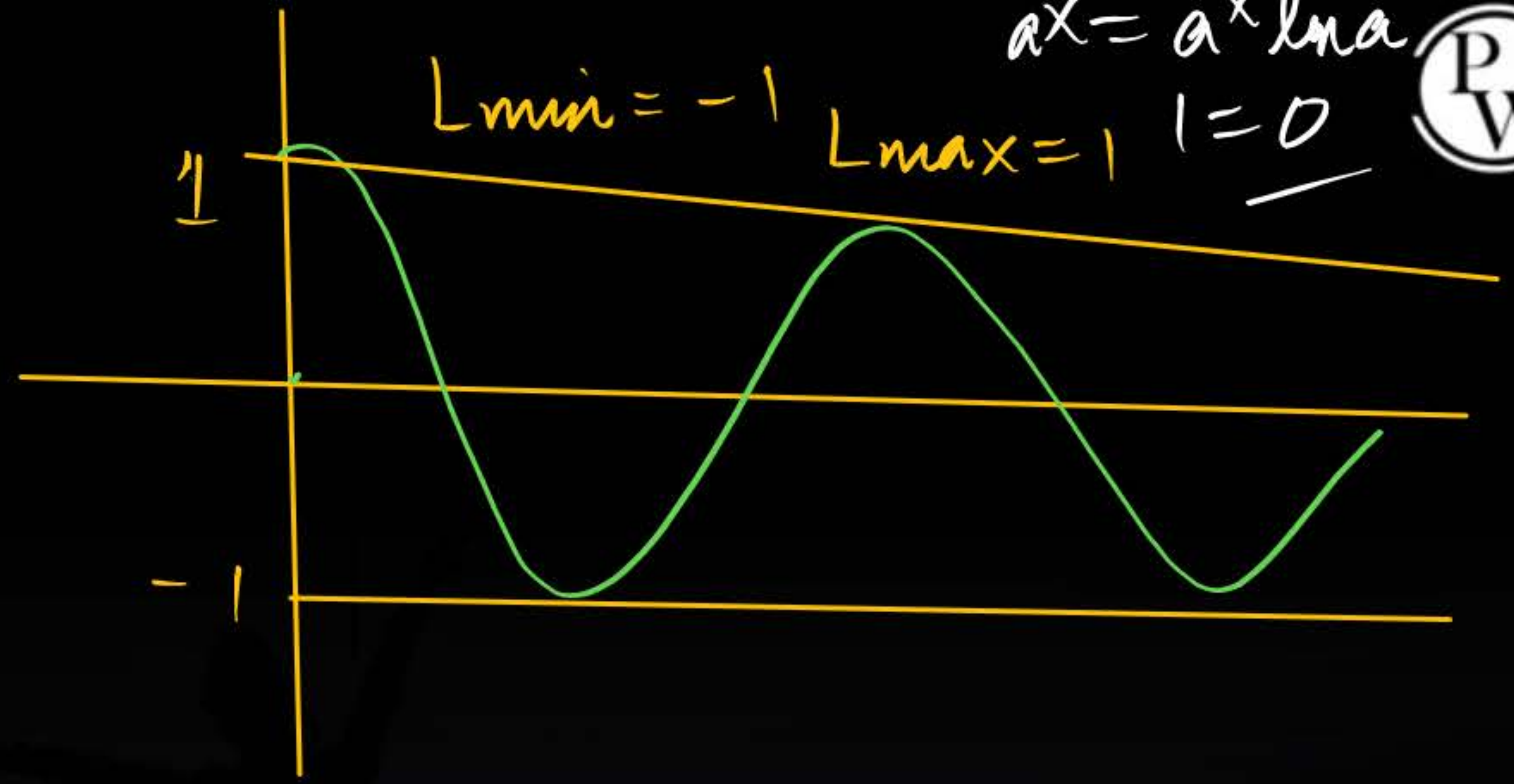
$$\# \quad \underset{x\to \infty}{lt}\frac{\sin x}{x}\left(\frac{\to \infty}{\to \infty}\right)$$

$$= \underset{x\to \infty}{lt}\frac{[-1,1]}{x} = \to 0$$



$\lim_{x\to\infty} \frac{\cos x}{x} = \lim_{x\to\infty} \frac{[-1,1]}{x}$

$$\boxed{\lim_{x\to\infty} \frac{\cos x}{x} = 0}$$



$a^x = a^x \ln a$

$1 = 0$

$\lim_{x\to 0} \frac{x}{\tan x} \left(\frac{\to 0}{\to 0}\right)$ form

Using L-Hospital Rule

$$= \lim_{x\to 0} \frac{1}{\sec^2 x} = 1$$

$$\boxed{\lim_{x\to 0} \frac{\tan x}{x} = 1}$$

$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to a} \frac{f(x)}{g(x)} \left(\frac{\to 0}{\to 0}\right)$

$$= \lim_{x\to 0} \frac{a^x \ln a - 0}{1}$$

$$= \ln a$$

$$\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a}$$

Q. Questions

The value of $\lim_{x \to 1} \frac{x^{P+1} - (P+1)x + P}{(x-1)^2}$ is:

(i) P

(ii) P–1

(iii) P(P+1)

(iv) $\frac{P(P+1)}{2}$

Q. Questions

The value of limit $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ is:

Do yourself

(i) 1

(ii) 0

(iii) –1

(iv) 2

Q. Questions

Using L-Hospital Rule.

The value $\lim_{x\to 0} \frac{\sin\left(\pi\cos^2 x\right)}{x^2}$ is:

(i) π

(ii) $-\pi$

(iii) 2π

(iv) -2π

L-Hospital Rule. $\lim_{x\to 0} \frac{\boxed{\sin(\pi\cos^2 x)}}{x^2}$

$\cos^2 x = 1-\sin^2 x$

Using Template method

$\sin(\pi-\theta)$

S (+)	A
T	C

$= \lim_{x\to 0} \frac{\sin[\pi(1-\sin^2 x)]}{x^2}$

$= \lim_{x\to 0} \frac{\sin[\pi-\pi\sin^2 x]}{x^2}$

$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2}$

$\lim_{x\to 0} \frac{\sin[x]}{x} = 1$

$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{(\pi\sin^2 x)} \frac{\pi\sin^2 x}{x^2}$

$= \lim_{x\to 0} \underbrace{\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}}_{1} \pi \underbrace{\left(\frac{\sin x}{x}\right)^2}_{1} = \pi$

$\Rightarrow \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \pi$

$1 \times 1 \times \pi = \pi$

Q. Questions

The value of $\lim_{x\to\infty} e^x \tan \frac{a}{e^x}$ is:

Do yourself

(i) a

(ii) 0

(iii) 1

(iv) None of these

Q. Questions

Evaluate : $\lim_{x\to 1}\frac{2^x-2}{x-1}$

$\frac{2^1-2}{1-1} = \left(\frac{\to 0}{\to 0}\right)$

$\underset{x\to 1}{\mathrm{lt}}\ \frac{2^x-2}{(x-1)}\ \left(\frac{\to 0}{\to 0}\right)$ form

Using L-Hospital Rule

$= \underset{x\to 1}{\mathrm{lt}}\ \frac{2^x \ln 2 - 0}{1-0}$ $\left(\frac{\to 0}{\to 0}\right)$ X

$= \underline{2\ln 2}$ $\underline{\text{Ans}}$ Direct Plug-in

$= 2\ln 2$

$\frac{d}{dx}(2^x)$

$= 2^x \ln 2$

$\boxed{L = 2\ln 2}$

* 2) $\underset{x\to 1}{\mathrm{lt}}\ \frac{2^x-2}{(x-1)} = \underset{x\to 1}{\mathrm{lt}}\ 2\boxed{\frac{(2^{x-1}-1)}{(x-1)}}$

$\underset{x\to 0}{\mathrm{lt}}\ \frac{a^x-1}{x}$

$= \underline{2\ln 2}$ Template $\boxed{\underset{x\to 0}{\mathrm{lt}}\ \frac{a^x-1}{x} = \ln a}$

Q. Questions

Evaluate :

$$\lim_{x \to a} \frac{e^{\sqrt{x}} - e^{\sqrt{a}}}{x - a}$$

$$\underset{x \to a}{Lt} \frac{e^{\sqrt{x}} - e^{\sqrt{a}}}{(x - a)}$$

Using L Hospital Rule $\left(\frac{\to 0}{\to 0}\right)$ form

$$\underset{x \to a}{lt} \frac{e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} - 0}{1 - 0}$$

$$= \frac{e^{\sqrt{a}}}{2\sqrt{a}} \text{ Ans}$$

Remove The $\left(\frac{\to 0}{\to 0}\right)$ OR $\left(\frac{\to \infty}{\to \infty}\right)$

Plug in

Template method $= \underset{x \to a}{lt} \frac{e^{\sqrt{x}} - e^{\sqrt{a}}}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})} = \underset{x \to a}{lt} \frac{e^{\sqrt{a}}}{(\sqrt{x}+\sqrt{a})} \left(\frac{e^{\sqrt{x}-\sqrt{a}} - 1}{\sqrt{x}-\sqrt{a}}\right)$

$$= \frac{e^{\sqrt{a}}}{2\sqrt{a}} \text{ Ans}$$

Q. Questions

Evaluate :

$$\lim_{x \to 0} \frac{6^x - 2^x - 3^x + 1}{\sin^2 x}$$

Do yourself

Q. Questions

The value of $\lim\limits_{e^{x \to a}} \frac{\log\{1+(x-a)\}}{(x-a)}$ is:

Do yourself

(i) 1

(ii) e

(iii) e^a

(iv) None of these

Q. Questions

The value of $\lim_{x \to 0} \frac{e^{\tan x} - e^{x}}{\tan x - x}$ is:

(i) 1

(ii) 2

(iii) 1/2

(iv) 0

$= \underset{x \to 0}{\text{lt}} \frac{e^{\tan x} - e^{x}}{\tan x - x}$

$= \underset{x \to 0}{\text{lt}} e^{x} \left\{ \frac{\left[e^{\tan x - x} - 1 \right]}{(\tan x - x)} \right\}$

Template

$= 1$ ✓ Ans

Q. Questions

Evaluate: $\lim_{x \to a} \frac{x^a - a^x}{x^x - a^a}$ $\left(\frac{0}{0} \text{ type of indeterminate form}\right)$

Do yourself

Q. Questions

The value of $\lim_{x \to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ is: [$\infty - \infty$ Type of in determinate form]

(i) 1

(ii) 0

(iii) 1/2

(iv) 3/2

Do yourself

M. Imp. **SOME Important Limit:**

$$L = \lim_{x \to a} [f(x)]^{g(x)}$$

Taking log both sides

$$\log L = \lim_{x \to a} \underbrace{g(x) \log_e [f(x)]}_{\text{Using factor, Rationalize / L-Hospital Rule / Template}}$$

$$\boxed{L = e^{\lim_{x \to a} g(x) \log f(x)}}$$

$\underbrace{\to \infty^{\to \infty} \text{ OR } \to 0^{\to 0} \text{ OR } \boxed{\to 1^{\to \infty}}}_{\text{THREE types (Indeterminate form)}}$ Trick

Q. Questions

Evaluate :

$$\lim_{x \to 0} (1-2x)^{1/x}$$

$\boxed{L = e^{-2}}$

$$L = \underset{x \to 0}{Lt} (\underbrace{1-2x}_{f(x)})^{\frac{1}{x}\, g(x)} \rightarrow (\rightarrow 1)^{\rightarrow \infty}$$

Taking log both sides

$$\log L = \underset{x \to 0}{lt} \frac{1}{x} \log(1-2x)$$

$$= \underset{x \to 0}{lt} \frac{\log_e(1-2x)}{x} \quad \left(\frac{\rightarrow 0}{\rightarrow 0}\right) \text{ form}$$

$$= \underset{x \to 0}{lt} \frac{\frac{1}{(1-2x)}(-2)}{1}$$

$$= \underset{x \to 0}{lt} \frac{-2}{(1-2x)} = -2$$

$$\log L = -2$$

$$\boxed{L = e^{-2}}$$

Trick :

If

$$L = \lim_{x \to a} [f(x)]^{g(x)} = (\to 1)^{\to \infty}$$

$$\text{Ans} = e^{A}$$

Where $A = \lim_{x \to a} [f(x) - 1]\, g(x)$

$\to \infty^{\to \infty}$

$\to 0^{\to 0}$

$\to 1^{\to \infty}$

log both sides

$\to 1^{\to \infty} = e^{A}$

Q. Questions

Evaluate :

$$\lim_{x\to 1} x^{\cot \pi x}$$

$$\lim_{x\to 1} x^{\cot(\pi x)}$$

$(\to 1^{\to\infty})$ form

$$L = e^{A}$$

$$A = \lim_{x\to 1}\left[f(x)-1\right] g(x)$$

$$= \lim_{x\to 1}\left[(x-1)\right]\cot(\pi x)$$

$$= \lim_{x\to 1} \frac{(x-1)\pi}{\tan(\pi x) \times \pi}$$

Template

$$= \frac{1}{\pi}\lim_{x\to 1} \frac{(\pi-\pi x)}{\tan(\pi-\pi x)} = \frac{1}{\pi}$$

$$= \boxed{e^{1/\pi}}$$ Ans

Q. Questions

The value of $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ is: [1^∞ Type of indeterminate form]

M.W

Do Yourself

(i) e

(ii) e^{π}

(iii) $e^{-2/\pi}$

(iv) $e^{2/\pi}$

Q. Questions

The value of L= $\lim_{x\to 0}(1/x)^{\sin x}$ is:

$L = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$

[∞^0 Type of indeterminate form]

$\log L = \lim_{x\to 0} \sin x \log\left(\frac{1}{x}\right)$

$= \lim_{x\to 0} \frac{\log\left(\frac{1}{x}\right)}{\left(\frac{1}{\sin x}\right)} = \left(\frac{\to\infty}{\to\infty}\right)$ form

(i) 0

(ii) 1/2

(iii) 1

(iv) None of these

$e^0 = 1$ ✓

$\Rightarrow \lim_{x\to 0} \frac{x\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{\sin^2 x}\cos x}$

Again L-Hospital Rule

Self Assessment Test

Q. Self Assessment Test

Evaluate : 3-times L-Hospital Rule

$$\lim_{x\to 0}\frac{x(e^x-1)+2(\cos x-1)}{x(1-\cos x)}$$

L =

$$\lim_{x\to\infty}\frac{x^3-\cos x}{x^2+(\sin x)^2}$$

(i) ∞

(ii) 0

(iii) 2

(iv) Does not exist

$$\lim_{x\to\infty}\frac{x^3-\cos x}{x^2+(\sin x)^2}$$

$$\lim_{x\to\infty}\frac{x\left[x^2-\cancel{\frac{\cos x}{x}}\right]}{x^2\left[1+\cancel{\left(\frac{\sin x}{x^2}\right)^2}\right]}$$

0

0

$$L = \lim_{x\to 0} \frac{x(e^x-1)+2[\cos x-1]}{x[1-\cos x]}$$

$$= \lim_{x\to 0} \frac{xe^x - x + 2\cos x - 2}{x - x\cos x} \quad \left(\frac{\to 0}{\to 0}\right)$$

$$= \lim_{x\to 0} \frac{(x+1)e^x - 1 - 2\sin x}{1-[-x\sin x+\cos x]}$$

$$= \lim_{x\to 0} \frac{e^x(x+1) - 2\sin x}{1 + x\sin x - \cos x} \quad \left(\frac{\to 0}{\to 0}\right)$$

$$= \lim_{x\to 0} \frac{(x+1)e^x + e^x - 2\cos x}{x\cos x + \sin x + \sin x}$$

$$\lim_{x\to 0} \frac{(x+1)e^x + e^x - 2\cos x}{x\cos x + \sin x \times 2}$$

Again apply L-Hospital Rule.

$$\boxed{L = 3}$$

Q. Self Assessment Test

Evaluate :

$$x \times \left(\sin\left(\frac{1}{x}\right)\right)$$

$$x \times [\sin(\infty)]$$

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = x \cdot x\,[-1, 1] = \underline{\underline{0}}$$

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \left(\frac{\to\infty}{\to\infty}\right)$$

$$= \underline{0 \text{ (Ans)}}$$

(i) ∞

(ii) 0

(iii) 1

(iv) Does not exist

$\lim_{\theta\to 0} \frac{\sin m\theta}{\theta}$, where m is an integer, is one of the following:

$$\lim_{\theta\to 0} \frac{\sin m\theta}{m\theta} \times m$$

$$m$$

(i) m

(ii) $m\pi$

(iii) $m\theta$

(iv) 1

Q. Self Assessment Test

GO TO WEST H.W

Evaluate :

L-Hospital Rule

$$\lim_{x\to 0} \frac{1}{10}\frac{1-e^{-j5x}}{1-e^{-jx}} =$$

L-Hospital Rule

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) \text{is} \Rightarrow$$

L-Hospital Rule

$$\underset{x\to 0}{\mathrm{Lt}} \frac{x-\sin x}{1-\cos x} \text{ is} \Rightarrow$$

$$\underset{x\to 0}{\mathrm{Lt}}\left(\frac{e^{2x}-1}{\sin(4x)}\right) \Rightarrow$$

L-Hospital Rule

Q. Self Assessment Test

Evaluate :

$\underset{x\to\infty}{\mathrm{Lt}}\left(\frac{x+\sin x}{x}\right)$ is equal to

(i) $-\infty$

(ii) 0

(iii) 1

(iv) ∞

$\underset{x\to\infty}{\mathrm{lt}}\left[1+\frac{\sin x}{x}\right]$

$= 1$

The expression $\lim_{a\to 0}\frac{x^a-1}{a}$ is equal to

x – constant
a = variable

(i) $\log x$

(ii) 0

(iii) $x\log x$

(iv) ∞

$\underset{a\to 0}{\mathrm{lt}}\frac{x^a-1}{a}$ Diff. w.r.t a

$= \underset{a\to 0}{\mathrm{lt}}\frac{x^a\log_e x}{1}$

$= \log_e x$

Q. Self Assessment Test

Evaluate : $\to 1^{\vec{\infty}}$

$= e^{A}$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to

(i) e^{+2}

(ii) e

(iii) 1

(iv) e

$= \left[1+\frac{1}{x}-1\right]2x$

$= 2$

Ans $= e^{2}$

Do Yourself NEST

The value of $\lim_{x\to 0}\frac{1-\cos\left(x^2\right)}{2x^4}$ is H.W

(i) 0

(ii) 1/2

(iii) 1/4

(iv) undefined

Q. Self Assessment Test

Evaluate :

$$\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right) \text{ is } \_\_\_\_\_\_\_\_.$$

$$\lim_{x\to 0}\left(\frac{\tan x}{x^2 - x}\right) \text{ is } \_\_\_\_\_\_\_\_.$$

Do yourself

H.W

Q. Self Assessment Test

$$\underset{f(x)\to 0}{Lt} \frac{\sin[f(x)]}{[f(x)]} = 1$$

The value of $\lim\limits_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}$

Using - L-Hospital Rule

(i) is 0

(ii) is – 1

(iii) is 1

(iv) Does not exit

Compute $\lim\limits_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Using L-Hospital Rule

(i) 1

(ii) limit does not exit

(iii) 53/12

(iv) 108/7

Thank You!

GW Soldiers