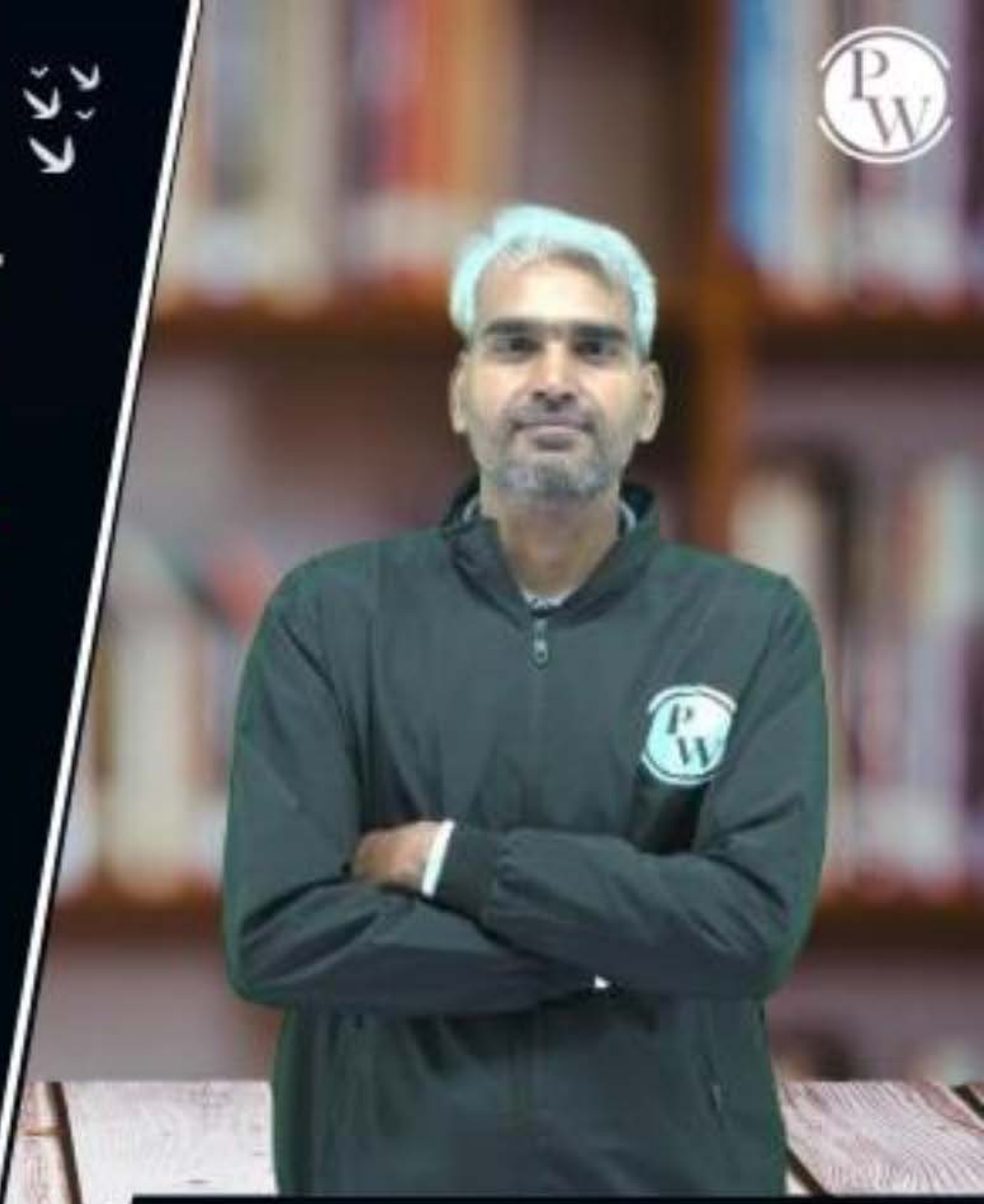
Computer Science & IT

Discrete Mathematics

Set Theory & Algebra

Lecture No. 11



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Recap of Previous Lecture

Topic Equivalence Class

Topic Number of equivalence relation on a set

Topic Bell Number

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Topics to be Covered

Topic Partial Order Relation

Topic Partially Ordered Set

Topic Total Order Relation

Topic Some Important terminologies

Topic Lattice

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Topic : Partial Order Relation

A relation R on set A is called a Partial order relation iff relation R is

1. Reflexive
2. Anti-symmetric
3. Transitive

eg: $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$

Reflexive ✓
Anti-symmetric ✓
Transitive ✓

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eg: 1 Relation "$\leq$" on any set of real numbers is a partial order relation

$$\text{"}\leq\text{"} \equiv (x, y) \in \text{"}\leq\text{"} \text{ iff } x \leq y$$

let $A = \{1, 2.5, 3, 4\}$

$\therefore$ "$\leq$" $= \{(1,1), (1,2.5), (1,3), (1,4),$
$\cancel{(2.5,1)}, (2.5,2.5), (2.5,3), (2.5,4)$
$\cancel{(3,1)}, \cancel{(3,2.5)}, (3,3), (3,4)$
$\cancel{(4,1)}, \cancel{(4,2.5)}, \cancel{(4,3)}, (4,4)\}$

$\downarrow$
is Partial order Relation

eg: 2 Let "$\div$" represents the relation divides,
Relation "$\div$" is a partial order relation on any set of (non-zero) Positive integers

let $A = \{1, 2, 3, 4\}$

$$\text{"}\div\text{"} = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$$

is a Partial order Relation

eg: 3 :- Let S be any collection of sets, then relation "$\subseteq$" on set S is a Partial order relation

let $S = \{ \overset{A}{\{\}}, \overset{B}{\{1,2\}}, \overset{C}{\{1,2,3\}}, \overset{D}{\{1,4,5\}} \}$

"$\subseteq$" $= \{ (A,A), (A,B), (A,C), (A,D),$
(B,B) (B,C)
(C,C)
$(D,D) \}$

if $X \subseteq Y$ & $Y \subseteq Z$ then $X \subseteq Z$

is a Partial order Relation

Topic : Partially Ordered Set

(P.O. SET)

Set A along with the partial order relation R defined on set A is called partially ordered set. Generally a POSET is denoted using (A, R).

<u>Where</u> A : Name of the set.

R : Partial order relation on set A.

They may denote a POSET using $(A, \leq)$, and they will specify the relation represented by "$\leq$"

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eg:

① If A is any set of real numbers, then $(A, \leq)$ is a POSET.

② let A is any set of Positive integers, then $(A, \div)$ is a POSET.

divides

(Or)

let A is any set of Positive integers, then $(A, \leq)$ is a POSET, and "$\leq$" represent the relation divides.

③ Let S is any Collection of sets, then

$(S, \subseteq)$ is a POSET.

Topic : Comparability

- Let R is a Partial order relation on set A.

For any two elements $a, b \in A$, a and b are said to be comparable w.r.t. relation R if and only if $a R b$ or $b R a$

i.e. $(a, b) \in R$ or $(b, a) \in R$

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Topic : Total Order Relation

* Let R is a partial order relation on set A,

R is called a total order relation if every pair of elements of set A are Comparable w.r.t. relation R

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* eg: let $A = \{1, 2, 3, 4\}$
and $(A, \div)$ is a POSET.

In the given example neither 2 divides 3, nor 3 divides 2
$\therefore$ 2 & 3 are not Comparable with each other

$\therefore$ Not a Total order Relation

* In the above POSET
  * 1 is Comparable with all the elements of set A
  * 2 is Comparable with 1, 2, & 4
  * 3 is Comparable with 1 and 3
  * 4 is Comparable with 1, 2, 4

Topic : Totally Ordered Set

* Let (A, R) be a POSET, it is called a totally ordered set (TOSET) if and only if every pair of element of set A is comparable w.r.t. relation R.

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Note:- ① Every total order relation is a partial order relation, but every partial order relation need not be a total order relation.

Similarly ② Every TOSET is a POSET, but every POSET need not be a TOSET

eg. let $A = \{1, 2, 3, 4\}$
and $(A, \leq)$ is POSET.

- w.r.t. relation "$\leq$" all the pair of elements of set A are Comparable, $\therefore$ relation "$\leq$" on set A is a total order relation, and hence POSET $(A, \leq)$ is a TOSET.

eg: let $A = \{1, 2, 3, 4\}$
and $(A, \div)$ is a POSET.

* In the above eg. $2, 3 \in A$,
  and 2 & 3 are not comparable w.r.t. relation "$\div$", $\therefore$ $(A, \div)$ on the given set A is just a POSET but not a TOSET.

eg: let $A = \{1, 2, 4, 8\}$

and $(A, \div)$ is a POSET.

- Every pair of elements of given set A are comparable w.r.t. relation "$\div$", $\therefore \div$ is a total order relation on given set A, and hence $(A, \div)$ is a TOSET

* Rel^n "$\leq$" on any set of real numbers is always a partial order relation as well as total order relation

* Rel^n "$\div$" on any set of +ve integers is always a POSET, but may or may not be a TOSET.

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Least upper bound / Lub / Join / Supremum

Let $(A, \leq)$ be a POSET, for any two elements $a, b \in A$, $\boxed{\text{if there exists an element } c \in A \text{ such that } a \leq c \text{ and } b \leq c \text{ then } c \text{ is called upper bound of } a \,\&\, b}$

and if there exists no element $d \in A$ such that $a \leq d$ and $b \leq d$ and $d \leq c$

then c is the least upper bound of $a \,\&\, b$.

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* If $a \, R \, b$ or $(a, b) \in R$

then a → lower side, b → upper side

* Least upper bound of elements a & b is denoted by $lub(a,b)$ or $a \vee b$

$\vee$ ← symbol used to represent lub.

Least upper bound / Lub / Join / Supremum

Let (A, R) be a POSET, for any two elements $a, b \in A$, if there exists an element $c \in A$ such that $a \overset{R}{} c$ and $b \overset{R}{} c$

i.e. $(a, c) \in R$ and $(b, c) \in R$

then c is called upper bound of a & b

and if there exists no element $d \in A$ such that $a \overset{R}{} d$ and $b \overset{R}{} d$ and $d \overset{R}{} c$

i.e. $(a, d) \in R$ & $(b, d) \in R$ & $(d, c) \in R$

then c is the least upper bound of a & b.

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Greatest lower bound / GLB / Meet / Infimum

Let $(A, \leq)$ be a POSET, for any two elements $a, b \in A$, if there exists an element $c \in A$ such that $c \leq a$ and $c \leq b$

then c is called lower bound of a & b

and if there exists no element $d \in A$ such that $d \leq a$ and $d \leq b$ and $c \leq d$

then c is the greatest lower bound of a & b

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Greatest lower bound / GLB / Meet / Infimum

Let (A, R) be a POSET, for any two elements $a, b \in A$, if there exists an element $c \in A$ such that $c \, R \, a$ and $c \, R \, b$

i.e. $(c, a) \in R$ & $(c, b) \in R$

then c is called lower bound of a & b

and if there exists no element $d \in A$ such that $d \, R \, a$ and $d \, R \, b$ and $c \, R \, d$

i.e. $(d, a) \in R$ and $(d, b) \in R$ and $(c, d) \in R$

then c is the greatest lower bound of a & b

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* Greatest Lower bound of a & b is denoted by Glb(a, b) <u>or</u> $a \wedge b$

symbol used to represent greatest lower bound eg: Let $A = \{1, 2, 3, 4, 5\}$

and $(A, \leq)$ is a POSET.

(i) Find lub(2,4)

lub(2,4) = 4

$2 \leq 2$ $\quad 2 \leq 3$ $\quad 2 \leq 4$ $\quad 2 \leq 5$

$4 \nleq 2$ $\quad 4 \nleq 3$ $\quad 4 \leq 4$ $\quad 4 \leq 5$

× ×

4 & 5 are the upper bounds of 2 & 4

among them 4 is the least upper bound

② Find glb(2,4)

glb(2,4) = 2

$1 \leq 2$ $\quad 2 \leq 2$ $\quad 3 \nleq 2$ $\quad 4 \nleq 2$ $\quad 5 \nleq 2$

× × ×

$1 \leq 4$ $\quad 2 \leq 4$ $\quad 3 \leq 4$ $\quad 4 \leq 4$ $\quad 5 \nleq 4$

1 & 2 are the lower bounds of 2 & 4 } among them 2 is the greatest lower bound

Note :- ① Let A is any set of real numbers, and $(A, \leq)$ is a POSET,

for any two elements $a, b \in A$ w.r.t rel^n "$\leq$"

$$\text{lub}(a,b) = \text{Max}(a,b)$$

$$\text{glb}(a,b) = \text{Min}(a,b)$$

Note :- 2 Let A is a set of all +ve integers and $(A, \div)$ is a POSET.

$\{1, 2, 3, 4, 5, 6, \ldots\}$ for any two elements $a, b \in A$ w.r.t rel^n "$\div$"

$$lub(a, b) = LCM(a, b)$$

$$glb(a, b) = GCD(a, b)$$

2 mins Summary

Topic Partial Order Relation / Partially Ordered Set (POSET)

Topic Total Order Relation / TOSET

Topic Some Important terminologies

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THANK - YOU

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