

Master Theorem: Practice Problems and Solutions

Master Theorem

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with¹ $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.
Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n .

Practice Problems

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

1. $T(n) = 3T(n/2) + n^2$
2. $T(n) = 4T(n/2) + n^2$
3. $T(n) = T(n/2) + 2^n$
4. $T(n) = 2^n T(n/2) + n^n$
5. $T(n) = 16T(n/4) + n$
6. $T(n) = 2T(n/2) + n \log n$

¹most of the time, $k = 0$

$$7. T(n) = 2T(n/2) + n/\log n$$

$$8. T(n) = 2T(n/4) + n^{0.51}$$

$$9. T(n) = 0.5T(n/2) + 1/n$$

$$10. T(n) = 16T(n/4) + n!$$

$$11. T(n) = \sqrt{2}T(n/2) + \log n$$

$$12. T(n) = 3T(n/2) + n$$

$$13. T(n) = 3T(n/3) + \sqrt{n}$$

$$14. T(n) = 4T(n/2) + cn$$

$$15. T(n) = 3T(n/4) + n \log n$$

$$16. T(n) = 3T(n/3) + n/2$$

$$17. T(n) = 6T(n/3) + n^2 \log n$$

$$18. T(n) = 4T(n/2) + n/\log n$$

$$19. T(n) = 64T(n/8) - n^2 \log n$$

$$20. T(n) = 7T(n/3) + n^2$$

$$21. T(n) = 4T(n/2) + \log n$$

$$22. T(n) = T(n/2) + n(2 - \cos n)$$

Solutions

1. $T(n) = 3T(n/2) + n^2 \implies T(n) = \Theta(n^2)$ (Case 3)
2. $T(n) = 4T(n/2) + n^2 \implies T(n) = \Theta(n^2 \log n)$ (Case 2)
3. $T(n) = T(n/2) + 2^n \implies \Theta(2^n)$ (Case 3)
4. $T(n) = 2^n T(n/2) + n^n \implies$ Does not apply (a is not constant)
5. $T(n) = 16T(n/4) + n \implies T(n) = \Theta(n^2)$ (Case 1)
6. $T(n) = 2T(n/2) + n \log n \implies T(n) = n \log^2 n$ (Case 2)
7. $T(n) = 2T(n/2) + n/\log n \implies$ Does not apply (non-polynomial difference between $f(n)$ and $n^{\log_b a}$)
8. $T(n) = 2T(n/4) + n^{0.51} \implies T(n) = \Theta(n^{0.51})$ (Case 3)
9. $T(n) = 0.5T(n/2) + 1/n \implies$ Does not apply ($a < 1$)
10. $T(n) = 16T(n/4) + n! \implies T(n) = \Theta(n!)$ (Case 3)
11. $T(n) = \sqrt{2}T(n/2) + \log n \implies T(n) = \Theta(\sqrt{n})$ (Case 1)
12. $T(n) = 3T(n/2) + n \implies T(n) = \Theta(n^{\lg 3})$ (Case 1)
13. $T(n) = 3T(n/3) + \sqrt{n} \implies T(n) = \Theta(n)$ (Case 1)
14. $T(n) = 4T(n/2) + cn \implies T(n) = \Theta(n^2)$ (Case 1)
15. $T(n) = 3T(n/4) + n \log n \implies T(n) = \Theta(n \log n)$ (Case 3)
16. $T(n) = 3T(n/3) + n/2 \implies T(n) = \Theta(n \log n)$ (Case 2)
17. $T(n) = 6T(n/3) + n^2 \log n \implies T(n) = \Theta(n^2 \log n)$ (Case 3)
18. $T(n) = 4T(n/2) + n/\log n \implies T(n) = \Theta(n^2)$ (Case 1)
19. $T(n) = 64T(n/8) - n^2 \log n \implies$ Does not apply ($f(n)$ is not positive)
20. $T(n) = 7T(n/3) + n^2 \implies T(n) = \Theta(n^2)$ (Case 3)
21. $T(n) = 4T(n/2) + \log n \implies T(n) = \Theta(n^2)$ (Case 1)
22. $T(n) = T(n/2) + n(2 - \cos n) \implies$ Does not apply. We are in Case 3, but the regularity condition is violated. (Consider $n = 2\pi k$, where k is odd and arbitrarily large. For any such choice of n , you can show that $c \geq 3/2$, thereby violating the regularity condition.)