Using recurrence relation show that under perfect ordering of leaf nodes, the alpha-beta pruning time complexity is $O(b^{m/2})$, where b is the effective branching factor and m is the depth of the tree.

Ans: let's first consider recursive equations which give the number of states to be considered. From the equations, we will determine a rough bound on the branching factor To determine the exact value of a state, we need the exact value of one of its children, and bounds on the rest of its children. To determine a bound of a state's value, we need the exact value of one of its children. Based on these observations, let S(m) be the minimum number of states to be considered m ply from a given state when we need to know the exact value of the state. Similarly, let R(m) be the minimum number of states to be considered m ply from a given state when we need to know a bound on the state's value. As usual, let b be the branching factor and m is the depth of the tree.. Thus, we have:

$$S(m) = S(m-1) + (b-1)R(m-1)$$

i.e., the exact value of one child and bounds on the rest, and

$$R(m) = S(m-1)$$

i.e., the exact value of one child. The base case is $S(0) = R(0) = 1$

When we expand the recursive equation, we get:

$$\begin{aligned} S(m) &= S(m-1) + (b-1)R(m-1) \\ &= (S(m-2) + (b-1)R(m-2)) + (b-1)S(m-2) \\ &= bS(m-2) + (b-1)R(m-2) \\ &= bS(m-2) + (b-1)S(m-3) \end{aligned}$$

It is obvious that $S(m-3) < S(m-2)$, so:

$$\begin{aligned} S(m) &< (2b-1)S(m-2) \\ &< 2bS(m-2) \end{aligned}$$

That is, the branching factor every two levels is less than $2b$, which means the effective branching factor is less than (2b)^1/2.

So, for even k, we derive S(m)≤(√2b)ˆm which is not too far off the asymptotic upper bound of (√b+1/2)^m+1 . In effect, alpha-beta pruning can nearly double the depth that a game tree can be searched in comparison to straightforward minimax. Therefore under perfect ordering of leaf nodes the time complexity is $O(b^{m/2})$