Unit - 4

VECTOR SPACES & LINEAR MAPS

Vector Space

Definition :- Let $(F, +, \cdot)$ be a field whose elements are called scalars. Let V be a non-empty set whose elements are called vectors. Then V is called a vector space over F if the following postulates hold good:

V_1. V is an abelian group with respect to an internal composition called addition of vectors to be denoted by '+' and defined in V, i.e.,

(i) V is closed with respect to '+', i.e.,

$$\alpha + \beta \in V \quad , \forall \alpha, \beta \in V$$

(ii) '+' is associative in V, i.e.,

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \ \forall \alpha, \beta, \gamma \in V.$$

(iii) '+' is commutative in V, i.e.,

$$\alpha + \beta = \beta + \alpha, \ \forall \alpha, \beta \in V$$

iv) There exists an unique vector 0 in V, called the zero vector (or Origin of Vectors) such that

$$0+\alpha = \alpha = \alpha+0, \ \forall\ \alpha \in V.$$

v) Corresponding to every element $\alpha \in V$, there exists an unique vector $-\alpha \in V$, such that

$$(-\alpha)+\alpha = 0 = \alpha+(-\alpha)$$

V_2. There is defined an external Composition in V over F called scalar multiplication and denoted multiplicatively i.e.,

$$a\alpha \in V, \forall\ a \in F \text{ and } \forall\ \alpha \in V.$$

In other words, say that V is closed with respect to scalar multiplication.

V_3. The two Composition defined above satisfy the following four postulates:-

1. $a(\alpha+\beta) = a\alpha + a\beta \ \forall\ a \in F$ and $\forall\ \alpha, \beta, \in V$
2. $(a+b)\alpha = a\alpha + b\alpha \ \forall\ a,b \in F$ and $\forall\ \alpha, \in V$
3. $(ab)\alpha = a(b\alpha) \ \forall\ a,b \in F$ and $\forall\ \alpha \in V$
4. $1\alpha = \alpha$, ~~,~~ $\forall\ \alpha \in V$ and 1 is the unity element of F.

Ex-1) Let F be an arbitary field. Prove that the set of all ordered n-tuples of the element of F with vector addition and Scalar multiplication defined by :-

$(a_1, a_2 \cdots a_n) + (b_1, b_2, \cdots b_n) = (a_1+b_1, a_2+b_2 \cdots a_n+b_n)$
and $K(a_1, a_2 \cdots a_n) = (Ka_1, Ka_2, \cdots Ka_n)$ where $a_i, b_i, K \in F$ is a vector Space over F.

Sol^n^ Let V be the Set of all ordered n-tuples over F, i.e, let

$$V = \{(a_1, a_2, \cdots a_n) : a_1, a_2, \cdots a_n \in F\}$$

Now we have to prove that V(F) is

To prove that (V, +) is abelian group

① Closure property—

Let $\alpha = (a_1, a_2, \cdots a_n)$; $\beta = (b_1, b_2, \cdots b_n) \in V$

$\alpha + \beta = (a_1, a_2, \cdots a_n) + (b_1, b_2, \cdots b_n)$

$(a_1+b_1, a_2+b_2, \cdots a_n+b_n) \in V$

∴ V is closed with respect to '+'.

② Associative property –

$\alpha = (a_1, a_2 \cdots a_n)$, $\beta = (b_1, b_2, \cdots b_n)$, $\gamma = (c_1, c_2 \cdots c_n) \in V$

$\alpha + (\beta + \gamma) = (a_1, a_2, \cdots a_n) + (b_1, b_2, \cdots b_n) + (c_1, c_2, \cdots c_n)$

$= (a_1, a_2 \cdots a_n) + (b_1 + c_1, b_2 + c_2, + \cdots b_n + c_n)$

$= ((a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \cdots, a_n + (b_n + c_n))$

$= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \cdots (a_n + b_n) + c_n))$

$= ((a_1 + b_1), (a_2 + b_2) \cdots (a_n + b_n)) + (c_1, c_2, \cdots c_n)$

$= \{(a_1, a_2 \cdots a_n) + (b_1, b_2, \cdots b_n)\} + (c_1, c_2, \cdots c_n)$

$= (\alpha + \beta) + \gamma$ $\quad \therefore$ V is associative with respect to V

③ existence of additive Inverse

$0 \in F \Rightarrow 0 = (0, 0, \cdots 0) \in V$

Now for all $\alpha = (a_1, a_2, \cdots a_n) \in V$

$0 + \alpha = (0, 0, \cdots 0) + (a_1, a_2, \cdots a_n)$

$= (0 + a_1, 0 + a_2 \cdots 0 + a_n)$

$= (a_1, a_2, \cdots a_n)$

$= \alpha$

Similarly If $\alpha + 0 = \alpha$

$\therefore$ 0 is identity element

(4) existence of additive Inverse

$\alpha = (a_1, a_2 \cdots a_n) \in V$ then $-\alpha = (-a_1, -a_2, -a_n)$

Then

$\alpha + (-\alpha) = (a_1, a_2 \cdots a_n) + (-a_1, -a_2, \cdots -a_n)$

$= (a_1 - a_1, a_2 - a_2, \cdots a_n - a_n)$

$= (0, 0, \cdots 0)$

$= 0$

Similarly $-\alpha + \alpha = 0$

(5) Commutative property :-

$\alpha = (a_1, a_2, \cdots a_n)$, $\beta = (b_1, b_2, \cdots b_n) \in V$

$\alpha + \beta = (a_1, a_2, \cdots a_n) + (b_1, b_2, \cdots b_n)$

$= (a_1 + b_1, a_2 + b_2, \cdots a_n + b_n)$

$= (b_1 + a_1, b_2 + a_2, \cdots b_n + a_n)$

$= (b_1, b_2, \cdots b_n) + (a_1, a_2 \cdots a_n)$

$= \beta + \alpha$

$\therefore$ V is Commutative respect to '+'.
Hence V of all ordered n-tuples is an additive abelian group.

② Scalar multiplication.

$a \in F, \alpha = (a_1, a_2, \dots a_n) \in V$

$a\alpha = a(a_1, a_2 \dots a_n)$

$= (aa_1, aa_2, \dots aa_n) \in V$

$\therefore$ V is closed with respect to scalar multiplication

3① $a \in F, \alpha = (a_1, a_2, \dots a_n)$

$a(\alpha+\beta) = a\{(a_1, a_2, \dots a_n) + (b_1, b_2 \dots b_n)\}$

$= a(a_1+b_1, a_2+b_2 \dots a_n+b_n)$

$= (a(a_1+b_1), a(a_2+b_2) \dots a(a_n+b_n))$

$= (aa_1+ab_1 + aa_2+ab_2 + \dots aa_n+ab_n)$

$= (aa_1 + aa_2 \dots aa_n) + (ab_1, ab_2 \dots ab_n)$

$= a(a_1, a_2 \dots a_n) + a(b_1, b_2, \dots b_n)$

$= a\alpha + b\beta.$

② $a, b \in F, \alpha = (a_1, a_2 \dots a_n) \in V$

General properties of a Vector Space :-

Theorem :- Let V(F) be a vector space, 0 be the zero vector V and 0 be the zero scalar of F, then

1) $a0 = 0$ $\quad a \in F$
2) $0\alpha = 0$ $\quad \forall \alpha \in V$
3) $a(-\alpha) = -(a\alpha)$ $\quad \forall a \in F, \forall \alpha \in V$
4) $(-a)\alpha = -(a\alpha)$ $\quad \forall a \in F, \forall \alpha \in V$
5) $a(\alpha - \beta) = a\alpha - b\beta$ $\quad \forall a \in F, \forall \alpha, \beta, \in V$
6) $a\alpha = 0 \Rightarrow \alpha = 0$ or $\alpha = 0$
7) $a\alpha = a\beta \Rightarrow \alpha = \beta$ if $0 \neq a \in F$

Proof :- (1) We know that

$0 + 0 = 0$

$a(0+0) = a0$ $\quad \forall a \in F$

$a0 + a0 = a0$ $\quad \forall a \in F$ ($\because$ V is a Vector Space)

$a0 + a0 = a0 + 0$ $\quad \forall a \in F$ {$\because$ 0, unit vector in V)

$a0 = 0$ $\quad \forall a \in F$

(2) We know that

$0 + 0 = 0$

$(0+0)\alpha = 0\alpha$ $\quad$ where $0 \in V$

$0\alpha + 0\alpha = 0\alpha$ $\quad \forall \alpha \in V$

$0\alpha + 0\alpha = 0\alpha + 0$ $\quad \forall \alpha \in V$

$0\alpha = 0$

$(a+b)\alpha = (a+b)(a_1, a_2, \dots a_n)$

$= \{(a+b)a_1, (a+b)a_2, \dots (a+b)a_n\}$

$= (aa_1 + ba_1, aa_2 + ba_2, \dots aa_n + ba_n)$

$= a(a_1, a_2, \dots a_n) + b(a_1, a_2, \dots a_n)$

$= a\alpha + b\alpha$

(3) $a, b \in F$, $\alpha = (a_1, a_2, \dots a_n) \in V$

$(ab)\alpha = (ab)(a_1, a_2, \dots, a_n)$

$= \{(ab)a_1, (ab)a_2, \dots (ab)a_n\}$

$= \{a(ba_1), a(ba_2) \dots a(ba_n)\}$

$= a(ba_1, ba_2 \dots ba_n)$

$= a\{b(a_1, a_2, \dots a_n)\}$

$= a(b\alpha)$

(4) $1 \in F$ and $\alpha = (a_1, a_2, \dots a_n) \in V$

$1\alpha = 1(a_1, a_2, \dots a_n)$

$= 1(a_1, 1a_2, \dots 1a_n)$

$= (a_1, a_2, \dots a_n)$

Hence V is a vector Space.

(3) $\alpha+(-\alpha)=0 \quad \forall \alpha\in V$

$a[\alpha+(-\alpha)]=a0 \quad \forall a\in F, \alpha\in V$

$a\alpha+a(-\alpha)=a0 \quad \forall a\in F, \alpha\in V$

$a\alpha+a(-\alpha)=0 \quad \forall a\in F, \alpha\in V$

The additive inverse of $a\alpha$ is $a(-\alpha)$

$\therefore a(-\alpha)=-a\alpha$

(4) $a+(-a)=0 \quad \forall a\in F$

$[a+(-a)]\alpha=0\alpha \quad \forall a\in F, \forall \alpha\in V$

$a\alpha+(-a)\alpha=0 \quad \forall a\in F, \forall \alpha\in V$

The additive inverse of $(-a)\alpha$ is $a\alpha$

$\therefore (-a)\alpha=(-a\alpha)$

(5) $a(\alpha-\beta)=a[\alpha+(-\beta)]$

$=[a\alpha+a(-\beta)]$

$=a\alpha+[-a\beta] \quad \{\because a(-\beta)=-a\beta\}$

$=a\alpha-a\beta$

(6) Let $a\alpha=0$ and $a\neq 0$ then $a^{-1}\in F$ exist

$a\alpha=0$

$a^{-1}(a\alpha)=a^{-1}0 \quad \{\because a^{-1}0=0$

$(a^{-1}a)\alpha=0 \quad \{\because a^{-1}a=1$

$1\alpha=0 \quad \{\because 1\alpha=\alpha$

$\alpha=0$

Hence the condition ① and ② hold.

To the condition are sufficient :- Now suppose that W is a non-empty subset of the vector space V(F) such that Condition ① and ⓘⓘ hold. We are to prove that W is a subspace of V(F).

Let $\alpha \in W$. If 1 is the unity element of F, then $-1 \in F$
Now $-1 \in F$, $\alpha \in W \Rightarrow (-1)\alpha \in W$.

{Since W is closed under Scalar multiplication}

$\Rightarrow -(1\alpha) \in W$ $\{\because (-1)\alpha = -(1\alpha)\}$

$\Rightarrow -\alpha \in W$

Therefore, additive inverse of each element of W exists in W.

Now $\alpha \in W, -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W$

{$\because$ W is closed under vector addition}

$\Rightarrow 0 \in W$, 0 being the zero vector of V. Thus the zero vector of V is also the zero vector of W.

Since $W \subseteq V$, Consequently the vector addition is commutative as well as associative in W. Therefore (W+) is an abelian group.

Vector Subspace :-

The vector subspaces are called simply subspaces.

Definition :- Let $V(F)$ be a vector space over the field $(F,+,\cdot)$ and W be a non-empty subset of V. Then $W(F)$ is called a subspace of $V(F)$ if W itself is vector space over the field $(F,+,\cdot)$ with respect to the operation and scalar multipli--cation in V.

Criterian for a subset to be a sub-space :-

Theorem 1 :- The necessary and sufficient condition for a non-empty subset W of a vector space $V(P)$ to be a subspace of V is that W is closed under vector addition and scalar multiplication in V i.e.,

$$\alpha, \beta \in W \Rightarrow \alpha+\beta \in W \ \forall \alpha, \beta \in W$$

&

$$\alpha \in F, \alpha \in W \Rightarrow a\alpha \in W \ \forall a \in F, \forall \alpha \in W$$

Proof :- The Conditions are necessary :- Suppose W is a vector subspace of $V(F)$ then we are to prove.

① $\alpha, \beta \in W \Rightarrow \alpha+\beta \in W \ \forall \alpha, \beta \in W$

② $\alpha \in F, \alpha \in W \Rightarrow a\alpha \in W \ \forall a \in F, \ \forall \alpha \in W.$

By the definition of vector subspace W itself is a vector space over F with respect to vector addition and scalar multiplication in V.

Again let $a\alpha = 0$ and $\alpha \neq 0$, Now we prove that $a = 0$
If $a \neq 0$ then $a^{-1} \in F$ exist.

$$\therefore a\alpha = 0$$
$$a^{-1}(a\alpha) = a^{-1}0$$
$$(a^{-1}a)\alpha = 0$$
$$1\alpha = 0$$
$$\alpha = 0$$

which is Contradiction

Hence $a = 0$

(7) $a\alpha = a\beta$

$$a\alpha + \{-(a\beta)\} = a\beta + \{-(a\beta)\}$$
$$a\alpha - a\beta = 0$$
$$a(\alpha - \beta) = 0$$
$$\alpha - \beta = 0$$
$$\alpha - \cancel{\beta} + \cancel{\beta} = 0 + \beta$$
$$\alpha = \beta$$
$$\{\because a \neq 0\}$$

Again from condition (ii) W is closed under scalar multi-plication. The remaining postulates of a vector space will hold in W, since they hold in $V \supseteq W$. Therefore, W itself is a vector space for these two compositions.

Hence W is a vector subspace of V. This proves the sufficient condition.

Hence Proved.

Q. Let $R_2(F)$ be the vector space of all 2×2 matrices over the field F. Let W denote the collection of all elements from the space $R_2(F)$ of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that W is subspaces of $R_2(F)$.

Solⁿ Let $\alpha = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$ and $\beta = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$ be any two elements of W.

Then we have

$$\alpha+\beta = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ -(b_1+b_2) & a_1+a_2 \end{pmatrix} \in W$$

Again if $c \in F$ and $\gamma = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in W$, then

$$c\gamma = c\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} ca & cb \\ -cb & ca \end{pmatrix} \in W$$

Hence W is a subspace of $R_2(F)$

Algebra of Vector Subspaces :-

Theorem 1 :- The intersection of any two subspece of a vector space $V(F)$ is also a subspaces of $V(F)$.

Or

If W_1 and W_2 are two subspaces of vector $V(F)$ then $W_1 \cap W_2$ is also a vector subspace of $V(F)$.